Quasi-self-dual Skyrme model

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A modification of the Skyrme model has been recently proposed, which admits an exact self-dual sector by the introduction of six scalar fields assembled in a symmetric, positive, and invertible 3×3 matrix h. In this paper we study soft manners of breaking the self-duality of that model. The crucial observation is that the self-duality equations impose distinct conditions on the three eigenvalues of h, and on the three fields lying in the orthogonal matrix that diagonalizes h. We keep the self-duality equations for the latter, and break those equations associated to the eigenvalues. We perform the breaking by the addition of kinetic and potential terms for the h fields, and construct numerical solutions using the gradient flow method to minimize the static energy. It is also shown that the addition of just a potential term proportional to the determinant of h leads to a model with an exact self-dual sector and with self-duality equations differing from the original ones by just an additional coupling constant.

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I. INTRODUCTION

Self-duality plays a prominent role in many areas of physics, from condensed matter to high energy physics and cosmology. The key ingredient for the appearance of self-dual sectors in a given theory is the existence of a (homotopic) topological charge that admits an integral representation; i.e., there is a density of topological charge [1]. The invariance of that charge under any smooth variation of the fields leads, through the integral representation, to local identities which are, in general, second order partial differential equations satisfied by any smooth field configuration. Together with the self-duality equations, which are first order partial differential equations, those identities imply the second order (dynamical) Euler-Lagrange equations of a given field theory. In addition, in many cases the self-duality leads to a lower bound on the static energy (or Euclidean action) determined by the topological charge and that is saturated by the self-dual solutions. Therefore, on each topological sector such self-dual solutions have the minimum allowed energy and so they are very stable.

In this paper we want to study the partial breaking of the self-duality, and try to explore the consequences it has on the physics of the remaining quasi-self-dual sector. We shall do that in the context of a Skyrme model in $(3+1)$ dimensions. As it is well known, the original Skyrme model [2,3] does not possess an exact nontrivial self-dual sector [4]. Several modifications of the Skyrme model have been proposed to accommodate a self-dual sector [5–12]. We shall consider in this paper the model proposed in [13] defined, in $(3+1)$-dimensional Minkowski space-time, by the action

$$S_1 = \int d^4x \left[\frac{m_0^2}{2} h_{ab} R_\mu^a R^{b,\mu} - \frac{1}{4e_0^2} h_{ab}^{-1} H_{\mu\nu}^a H^{b,\mu\nu} \right] \quad (1.1)$$

where, like in the usual Skyrme model, R_μ^a are the components of the Maurer-Cartan form, i.e., $i\partial_\mu U U^\dagger \equiv R_\mu^a T_a$, with U being a group element of $SU(2)$, and T_a being a basis of its Lie algebra, satisfying

$$[T_a, T_b] = i\varepsilon_{abc} T_c; \qquad \mathrm{Tr}(T_a T_b) = \kappa \delta_{ab} \quad (1.2)$$

with κ being a constant depending upon the representation [$\kappa = 1/2$ for the spinor representation, and $\kappa = 2$ for the triplet (adjoint) representation]. $H_{\mu\nu}^a$ is the curl of that form, i.e., $H_{\mu\nu}^a \equiv \partial_\mu R_\nu^a - \partial_\nu R_\mu^a$, and m_0 and e_0 are coupling constants, of the dimension of mass and dimensionless, respectively. The model possesses, in addition to the three chiral fields (pions) parametrizing U, six extra scalar fields assembled in the symmetric and invertible matrix h_{ab}, a,

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$b = 1, 2, 3$. For the static energy associated to (1.1) to be positive it is required that the eigenvalues of matrix h must also be positive.

The properties of such a model have been studied in great detail in [14], and in [15] a modification of it has been applied to nuclear matter. By coupling it to a fluid theory, where the order parameter is a fractional power of the density of baryonic charge, it was possible to reproduce the bulk behavior of the binding energy and the radii of 265 nuclei. Such a list of nuclei contains all the stable nuclei up to ^{208}Pb, and above that, nuclei with a half-life greater than 10^3 years, up to ^{240}Pu. The values of such quantities are reproduced with an excellent accuracy (about 1% for both the radius and the binding energy) for the quasistable nuclei with mass number equal to 20 or greater [15]. The error increases for light nuclei with mass number below 20. The main properties of the model (1.1), studied in [14], can be summarized as follows:

(i) The self-dual sector is defined by the nine self-duality equations

$$\lambda h_{ab} R_i^b = \frac{1}{2}\varepsilon_{ijk} H_{jk}^a; \qquad \lambda \equiv \pm m_0 e_0 \tag{1.3}$$

where the indices $a, b = 1, 2, 3$, refer to the group indices, and $i, j, k = 1, 2, 3$, to the space coordinates x_i. The self-duality equations (1.3) imply the nine static Euler-Lagrange equations, three of them associated to the U fields and also the six equations associated to the scalar fields assembled in h_{ab}.

(ii) The self-dual sector, defined by (1.3), and the static sector of the theory (1.1) are equivalent; i.e., any static solution is self-dual and vice-versa.

(iii) The static Euler-Lagrange equations associated to the scalar fields h_{ab} imply the self-duality equations (1.3) and so indirectly imply also the static Euler-Lagrange equations associated to the U fields.

(iv) The introduction of the six scalars h_{ab} makes the static sector of (1.1) invariant under conformal transformations in $\mathbb{R}^3$, i.e., the self-duality equations (1.3), the nine static Euler-Lagrange equations associated to (1.1), as well as the static energy

$$E_1 = \int d^3x \left[\frac{m_0^2}{2} h_{ab} R_i^a R_i^b + \frac{1}{4e_0^2} h_{ab}^{-1} H_{ij}^a H_{ij}^b\right] \tag{1.4}$$

are all invariant under the conformal group $SO(3,2)$. The infinitesimal conformal transformations in $\mathbb{R}^3$ are given by $\delta x_i = \zeta_i$ with $\partial_i\zeta_j + \partial_j\zeta_i = 2D\delta_{ij}$, with D vanishing for translations and rotations. It is constant for dilatations, and it is linear in the x_i's for the special conformal transformations. The U fields are scalars under the conformal group, i.e., $\delta U = 0$, and the h fields have conformal weight -1, i.e., $\delta h_{ab} = -D h_{ab}$.

(v) The self-duality leads to a lower bound on the static energy (1.4), and for the self-dual solutions such a bound is saturated as

$$E_1^{\mathrm{BPS}} = 48\pi^2 \frac{|m_0|}{|e_0|} |Q| \tag{1.5}$$

where Q is the topological charge

$$Q = \frac{i}{48\pi^2} \int d^3x \varepsilon_{ijk} \widehat{\mathrm{Tr}}(R_i R_j R_k) \tag{1.6}$$

which gives the winding number of the maps $S^3 \to SU(2)$, where S^3 is $\mathbb{R}^3$ with the spatial infinity identified to a point. Remember that in order to have finite energy solutions the U field must go to a constant at infinity and so, for topological considerations, one can consider such an identification. In (1.6) we have used the normalized trace

$$\widehat{\mathrm{Tr}}(T_a T_b) = \frac{1}{\kappa}\mathrm{Tr}(T_a T_b) = \delta_{ab}. \tag{1.7}$$

In additional, the sign of Q and λ in (1.3) are related through

$$\mathrm{sign}(Q\lambda) = -1. \tag{1.8}$$

(vi) An important role is played by the real and symmetric matrix

$$\tau_{ab} \equiv R_i^a R_i^b. \tag{1.9}$$

If $\det\tau = 0$, then the only possible static solution is $U =$ constant. If $\det\tau \neq 0$, then the self-duality equations (1.3) imply that the matrix h is determined from the U-fields configuration by

$$h_{\mathrm{BPS}} = \frac{\sqrt{\det\tau}}{|m_0 e_0|} \tau^{-1}. \tag{1.10}$$

That means that the self-duality equations are satisfied by any nontrivial configuration of the U fields, and the h fields adjust themselves to solve the self-duality, taking the form (1.10). For the BPS (self-dual) field configurations (1.10) the quadratic and quartic terms in the space-time derivatives of (1.1) give exactly the same contribution to the total energy (1.4), i.e.,

$$\frac{m_0^2}{2}\int d^3x (h_{\mathrm{BPS}})_{ab} R_i^a R_i^b = \frac{1}{4e_0^2}\int d^3x (h_{\mathrm{BPS}})_{ab}^{-1} H_{ij}^a H_{ij}^b = 24\pi^2 \frac{|m_0|}{|e_0|}|Q| \tag{1.11}$$

and the topological charge (1.6), can be written in terms of the eigenvalues of $h_{\rm BPS}$ as

$$Q = -\frac{\lambda^3}{16\pi^2}\int d^3x \ \det\, h_{\rm BPS}. \tag{1.12}$$

(vii) As the matrices h and τ are and symmetric, they can be diagonalized by orthogonal transformations, i.e.,

$$\begin{aligned} &h = Mh_D M^T; \qquad MM^T = \mathbb{1}; \qquad (h_D)_{ab} = \varphi_a\delta_{ab} \\ &\tau = N\tau_D N^T; \qquad NN^T = \mathbb{1}; \qquad (\tau_D)_{ab} = \omega_a\delta_{ab} \end{aligned} \tag{1.13}$$

When the self-duality equations (1.3) hold true, and so (1.10) is valid, we have that the matrices h and τ commute and so can be diagonalized simultaneously; i.e., $M = N$, and the eigenvalues are related by

$$\varphi_a = \frac{1}{|m_0 e_0|}\sum_{b,c=1}^{3}\frac{|\varepsilon_{abc}|}{2}\sqrt{\frac{\omega_b\omega_c}{\omega_a}} \tag{1.14}$$

or equivalently

$$\frac{\omega_1}{\varphi_2\varphi_3} = \frac{\omega_2}{\varphi_1\varphi_3} = \frac{\omega_3}{\varphi_1\varphi_2} = m_0^2 e_0^2 \tag{1.15}$$

and so

$$\begin{aligned} &\frac{\omega_1\omega_2}{\varphi_3} = \frac{\omega_1\omega_3}{\varphi_2} = \frac{\omega_2\omega_3}{\varphi_1} = m_0^4 e_0^4 \varphi_1\varphi_2\varphi_3 \\ &\omega_1\varphi_1 = \omega_2\varphi_2 = \omega_3\varphi_3 = m_0^2 e_0^2 \varphi_1\varphi_2\varphi_3 \end{aligned} \tag{1.16}$$

(viii) The action (1.1) is invariant under the global symmetry $SU(2)_L \otimes SU(2)_R$ defined by the transformations

$$\begin{aligned} &U \to g_L U; \qquad R_\mu^a \to d_{ab}(g_L)R_\mu^b; \\ &\quad h_{ab} \to d_{ac}(g_L)h_{cd}d^T_{db}(g_L) \end{aligned} \tag{1.17}$$

and

$$U \to U g_R; \qquad R_\mu^a \to R_\mu^a; \qquad h_{ab} \to h_{ab} \tag{1.18}$$

with $g_{L/R} \in SU(2)_{L/R}$, and where $d(g)$ is the 3×3 matrix for the group element g in the adjoint (triplet) representation of $SU(2)$, i.e.,

$$gT_a g^{-1} = T_b d_{ba}(g); \ \ d(g_1)d(g_2) = d(g_1 g_2) \tag{1.19}$$

In the self-dual model described above, the h fields are not propagating as they enter into the action (1.1) through the coupling to the U fields by contracting the group indices, and there is not kinetic term for them. In this paper we want to break the self-duality by adding kinetic and potential terms for the h fields. Note that the 3×3 orthogonal matrix M, diagonalizing h in (1.13), has three independent components. Therefore, we can take the six scalar fields assembled in the matrix h to be those three components of M and the three eigenvalues φ_a, $a = 1, 2, 3$, introduced in (1.13). The eigenvalues of h are invariant under the transformations $SU(2)_L \otimes SU(2)_R$, given in (1.17) and (1.18), and the M fields are invariant under (1.18) and transform as $M \to d(g_L)M$ under (1.17). In addition, the M fields are scalars under the conformal group $SO(3,2)$, and the φ fields have conformal weight -1, i.e., $\delta\varphi_a = -D\varphi_a$ [see item (iv) above]. With such a decomposition of fields, a kinetic term for the h fields takes the form $\mathrm{Tr}(\partial_\mu h)^2 = (\partial_\mu\varphi_a)^2 + \mathrm{Tr}([M^T\partial_\mu M, h_D])^2$. Since φ_a and $M^T\partial_\mu M$ are invariant under the chiral transformations $SU(2)_L \otimes SU(2)_R$, given in (1.17) and (1.18), we shall take an arbitrary linear combination of those terms. So, we shall consider the theory

$$S = S_1 + S_2 \tag{1.20}$$

with S_1 given in (1.1) and

$$\begin{aligned} S_2 = \int d^4x \Bigg[&\frac{\mu_0^2}{2}\left[\sum_{a=1}^{3}\kappa_a(\partial_\mu\varphi_a)^2 + \kappa_4\mathrm{Tr}([M^T\partial_\mu M, h_D])^2\right] \\ &- \mathcal{V}(\varphi) - \frac{\beta_3^2}{2}\mathrm{Tr}(\mathbb{1} - U)\Bigg] \end{aligned} \tag{1.21}$$

where μ_0 and β_3 are coupling constants of dimension of mass and of mass^2, respectively, and κ_a, $a = 1, 2, 3, 4$ are dimensionless parameters. The β_3-term is a mass term for the (pion) U fields.

If $\beta_3 = 0$, the action (1.21) is invariant under (1.17) and (1.18). However, if $\beta_3 \neq 0$, the symmetry $SU(2)_L \otimes SU(2)_R$ is broken to the diagonal subgroup $g_R = g_L^{-1} \equiv g^{-1}$, i.e., $U \to gUg^{-1}$. The conformal symmetry of the static energy associated to (1.21), i.e.,

$$\begin{aligned} E_2 = \int d^3x \Bigg[&\frac{\mu_0^2}{2}\left[\sum_{a=1}^{3}\kappa_a(\partial_i\varphi_a)^2 + \kappa_4\mathrm{Tr}([M^T\partial_i M, h_D])^2\right] \\ &+ \mathcal{V}(\varphi) + \frac{\beta_3^2}{2}\mathrm{Tr}(\mathbb{1} - U)\Bigg] \end{aligned} \tag{1.22}$$

is broken by the β_3-term, as U is a scalar under $SO(3,2)$, and so it does not compensate the transformation of the volume d^3x. The kinetic, and possible mass terms in $\mathcal{V}$, for the φ_a fields also break the conformal symmetry, as the φ_a fields have conformal weight -1. The potential $\mathcal{V}$ does not break the conformal symmetry only if it is cubic in the φ_a fields.

In this paper we shall break the self-duality only partially, as we shall impose that the matrices h and τ still commute [see (1.13)]

$$[h,\tau]=0 \leftrightarrow M=N. \tag{1.23}$$

Since (1.10) ceases to be true, we have that the eigenvalues of h and τ will not be related by (1.14), (1.15), and (1.16) anymore.

From (1.9) we see that the entries of the matrix τ are functionals of the U fields and their first derivatives. Consequently, the entries of the orthogonal matrix N and the eigenvalues ω_a, introduced in (1.13), are also functionals of U fields and their first derivatives. Therefore, the condition (1.23) is saying that the three M fields are determined from the U fields, in a way similar to that in (1.10), where h is determined from the U fields, when the full self-duality equations are valid. For that reason we can consider (1.23) as a quasi-self-duality equation. As we explain in Sec. II such a condition introduces nice simplifications in the model.

There are some particular interesting cases of the full theory (1.20) that break the self-duality in a soft manner, as we discuss in Sec. III. In addition to the three quasi-self-duality equations (1.23), one can impose algebraic relations among the eigenvalues of the matrices h and τ, such that the variation of the energy functional E_1, with respect to the U fields, becomes proportional to the variation of the topological charge, and so vanishes identically. In other words, algebraic relations among the eigenvalues of h and τ solve the part of the Euler-Lagrange equations associated to the U fields, coming from E_1. A further consequence of such algebraic relations is that the matrix h becomes proportional to the self-dual matrix $h_{\rm BPS}$ given in (1.10). It then follows that the variation of density of the energy functional E_1 with respect to the φ fields becomes proportional to the variation of $\det h$ with respect to the same fields. Therefore, we can solve the Euler-Lagrange equations associated to the φ fields by restricting the energy functional E_2 to be proportional to $\det h$, i.e., by restricting (1.22) to the case where $\kappa_a=0$, $a=1,\ 2,\ 3,\ 4$, $\beta_3=0$, and $\mathcal{V}=\frac{\beta_{\mathcal{V}}^2}{2}\det h=\frac{\beta_{\mathcal{V}}^2}{2}\varphi_1\varphi_2\varphi_3$, with $\beta_{\mathcal{V}}$ a real dimensionless coupling constant. So, the static energy of such model becomes

$$E_{\rm quasi-sd}=\int d^3x\left[\frac{m_0^2}{2}h_{ab}R_i^aR_i^b+\frac{1}{4e_0^2}h_{ab}^{-1}H_{ij}^aH_{ij}^b+\frac{\beta_{\mathcal{V}}^2}{2}\det h\right]. \tag{1.24}$$

The model (1.24) has an exact self-dual sector where the self-duality equation differs from (1.3) by a multiplicative parameter, i.e., it is given by

$$\frac{\lambda}{\alpha}h_{ab}R_i^b=\frac{1}{2}\varepsilon_{ijk}H_{jk}^a;\qquad \lambda\equiv\pm m_0e_0 \tag{1.25}$$

where α is a monotonically decreasing function of the strength $\beta_{\mathcal{V}}$ of the potential, with $\alpha=1$ for $\beta_{\mathcal{V}}=0$ (the details are given in Sec. III).

Note that $\int d^3x\det h$ is invariant under conformal transformations in $\mathbb{R}^3$. Therefore, such a soft manner of breaking the self-duality preserves all the symmetries of the self-dual Skyrme model (1.1), namely, the global symmetries $SU(2)_L\otimes SU(2)_R$ defined by the transformations (1.17) and (1.18), as well as the conformal symmetry in the three-dimensional space. In addition, note that since $h=\alpha h_{\rm BPS}$, with $h_{\rm BPS}$ given in (1.10), the h fields still act as spectators of the U fields, which in turn remain totally free. Therefore, such a theory also leads to an infinite number of exact topological solutions for any value of Q and extends the results obtained in [14]. The total energy $E=E_1+E_2$ is proportional to $|Q|$, but the proportionality constant is a monotonic increasing function of $\beta_{\mathcal{V}}$, i.e., the strength of the potential $\mathcal{V}$.

In order to construct solutions for the full theory (1.20), subjected to the quasi-self-duality equations (1.23), we shall work with the so-called rational map ansatz [16–18] for the U fields, which is described in Sec. IV A. In such an ansatz, spheres of radius r, in the spatial submanifold $\mathbb{R}^3$, are stereographically projected on a plane parametrized by a complex coordinate w. The U fields are then given by a profile function f, depending only on the radial distance r, and a complex field u which is a holomorphic function of w and a map between two-spheres. In such an ansatz, the first two eigenvalues of the matrix τ, defined in (1.9), become equal, and the third one is a function of the radial variable only, i.e., $\omega_1=\omega_2$ and $\omega_3=\omega_3(r)$.

As a consequence of the quasi-self-duality equations (1.23), the Euler-Lagrange equations for the M fields become differential equations to be satisfied by the U fields, in addition to their own Euler-Lagrange equations. That would be a too restrictive condition on the U fields. However, we observe that by imposing that the eigenvalues of the matrix h depend only on the radial distance r, i.e., $\varphi_a=\varphi_a(r)$, and in addition that the first two eigenvalues are equal, i.e., $\varphi_1=\varphi_2$, we solve the Euler-Lagrange equations for the M fields automatically. Under such conditions the Euler-Lagrange equations of u and $\bar{u}$ fields are also automatically satisfied.

The drawback of that procedure is that only some special configurations of the u field with unity topological degree can be solutions of our quasi-self-dual model, inside the holomorphic ansatz. Such fixing of the u field imposes radial symmetry to the topological charge density and restricts the construction of topological solutions with large values of Q by choosing properly the boundary conditions of the profile f function, which may lead to unstable static solutions for $|Q|\geq 2$. However, the advantage of the procedure is that we are left to solve only three ordinary differential equations, which correspond to the Euler-Lagrange equations for the profile function $f(r)$, and for

the $\varphi_1(r)$ (equal to φ_2) and $\varphi_3(r)$ fields. Those equations are solved numerically using the gradient flow method to minimize the static energy of the system.

Such an analysis of the static sector of the full theory (1.20) is very important to study the effect of allowing the scalar fields in the matrix h to be propagating fields. From the results of Sec. IV C one observes, as the strength of the kinetic and potential terms for the h fields increases, the eigenvalues φ_a of the matrix h tend to grow at the origin and to fall exponentially faster at large distances.

The paper is organized as follows. In Sec. II we obtain all nine Euler-Lagrange equations for the static version of (1.20) inside the quasi-self-dual ansatz (1.23). In Sec. III we show how some special algebraic relations among the eigenvalues of the matrices h and τ lead to an exact self-dual sector of the model (1.24). In Sec. IV we consider the static version of full theory (1.20), and in Sec. IV A we construct a holomorphic ansatz for it, compatible with (1.23). In Sec. IV B we analyze the Euler-Lagrange equations of (1.20) within the holomorphic and (1.23) ansatz. The numerical solutions of those equations are constructed in Sec. IV C for a quadratic potential for the h fields. Our conclusions are presented in Sec. V. Appendix A presents the proof of the algebraic relations used in Sec. III, and Appendix B shows why only the solutions with unity baryonic charge satisfy the conditions of Sec. IV B. Appendix C presents some details of our numerical methods.

II. THE QUASI-SELF-DUALITY

The static energy E_1, defined in (1.4), can be written as

$$E_1 = \int d^3x \left[\frac{m_0^2}{2}\mathrm{Tr}(h\tau) + \frac{1}{4e_0^2}\mathrm{Tr}(h^{-1}\sigma)\right] \tag{2.1}$$

where τ is defined in (1.9), and where we have introduced the matrix

$$\sigma_{ab} \equiv H_{ij}^a H_{ij}^b. \tag{2.2}$$

The quantities $R_i \equiv i\partial_i U U^\dagger \equiv R_i^a T_a$ satisfy the Maurer-Cartan equation $\partial_i R_j - \partial_j R_i + i[R_i, R_j] = 0$, and so we have that

$$H_{ij}^a \equiv \partial_i R_j^a - \partial_j R_i^a = -i\widehat{\mathrm{Tr}}([R_i, R_j]T_a) = \varepsilon_{abc} R_i^b R_j^c. \tag{2.3}$$

Conjugating both sides of the commutation relations (1.2) with an $SU(2)$ group element g, and using (1.19), one gets that

$$\varepsilon_{abc} d_{dc}(g) = \varepsilon_{def} d_{ea}(g) d_{fb}(g). \tag{2.4}$$

The adjoint representation of $SU(2)$ is a real and unitary representation, and so the matrices $d(g)$ are orthogonal. In fact, any orthogonal matrix with determinant 1 (-1) can be identified with a given matrix $d(g)$ $(-d(g))$ for some $g \in SU(2)$. Therefore, the orthogonal matrices M and N (as well as their transposes) satisfy (2.4) with a sign given by $\det M = \pm 1$, i.e.,

$$\varepsilon_{abc} M_{dc}(g) = \det M \varepsilon_{def} M_{ea}(g) M_{fb}(g) \tag{2.5}$$

and a similar relation for N. So, using that fact, (2.3), and (1.13) we get that

$$\sigma_{ab} = \varepsilon_{acd}\varepsilon_{bef}\tau_{ce}\tau_{df} = \varepsilon_{cde}\varepsilon_{cdf}\omega_c\omega_d N_{ae} N_{fb}^T = (N\sigma_D N^T)_{ab} \tag{2.6}$$

where we have defined the matrix

$$(\sigma_D)_{ab} = \sum_{c,d=1}^{3} \varepsilon_{cda}\varepsilon_{cdb}\omega_c\omega_d \tag{2.7}$$

which is diagonal

$$\sigma_D = 2\,\mathrm{diag.}\,(\omega_2\omega_3, \omega_1\omega_3, \omega_1\omega_2). \tag{2.8}$$

So, σ is diagonalized by the same orthogonal matrix N as τ, and as a consequence of the condition (1.23) we have that

$$[h, \tau] = [h, \sigma] = [\tau, \sigma] = 0. \tag{2.9}$$

Therefore, when considering variations with respect to the M fields we have that

$$\delta^{(M)} h = [\delta^{(M)} M M^T, h]; \quad \text{and} \quad \delta^{(M)} h^{-1} = [\delta^{(M)} M M^T, h^{-1}] \tag{2.10}$$

and so

$$\delta^{(M)} E_1 = \int d^3x \mathrm{Tr}\left[\delta^{(M)} M M^T \left(\frac{m_0^2}{2}[h,\tau] + \frac{1}{4e_0^2}[h^{-1},\sigma]\right)\right] = 0. \tag{2.11}$$

We then conclude that the Euler-Lagrange equations associated to the M fields, coming from E_1, are automatically satisfied due to the condition (1.23) which leads to (2.9). Therefore, the nontrivial Euler-Lagrange equations associated to the M fields come from the κ_4-term in E_2, defined in (1.22), and it is given by

$$\partial_i[h_D, [h_D, M^T\partial_i M]] + [M^T\partial_i M, [h_D, [h_D, M^T\partial_i M]]] = 0. \tag{2.12}$$

In addition, using (2.1) and (2.8) and the fact that we are assuming that $M = N$, we get that the variation of E_1 with respect to φ_a is given by

$$\delta^{(\varphi_a)}E_1 = \frac{1}{2e_0^2}\int d^3x\left[m_0^2e_0^2\omega_a - \frac{1}{2}\sum_{b,c=1}^{3}|\varepsilon_{abc}|\frac{\omega_b\omega_c}{\varphi_a^2}\right]\delta\varphi_a. \tag{2.13}$$

Now, when considering variations with respect to the U fields we have that $\delta^{(U)}\tau = [\delta^{(U)}NN^T,\tau] + N\delta^{(U)}\tau_D N^T$ and a similar relation for σ. Therefore, using (1.23), (2.8), and (2.9), we get

$$\begin{aligned}\delta^{(U)}E_1 &= \int d^3x\left[\frac{m_0^2}{2}\mathrm{Tr}(h_D\delta^{(U)}\tau_D) + \frac{1}{4e_0^2}\mathrm{Tr}(h_D^{-1}\delta^{(U)}\sigma_D)\right]\\ &= \frac{1}{2e_0^2}\int d^3x\left[m_0^2e_0^2\sum_{a=1}^{3}\varphi_a\delta^{(U)}\omega_a\right.\\ &\quad\left.+\frac{1}{2}\sum_{a,b,c=1}^{3}|\varepsilon_{abc}|\frac{\delta^{(U)}(\omega_b\omega_c)}{\varphi_a}\right]\end{aligned} \tag{2.14}$$

In addition, we have, from (1.22), that

$$\delta^{(U)}E_2 = -\frac{\beta_3^2}{2}\int d^3x\delta^{(U)}\mathrm{Tr}(U). \tag{2.15}$$

Consequently, the Euler-Lagrange equations associated to the U fields, coming from E_1 and E_2, do not involve the M fields.

Since M is a 3×3 orthogonal matrix, it follows that $M^T\partial_i M$ is a matrix in the adjoint representation of the $SU(2)$ Lie algebra. So, we can write $M^T\partial_i M = i\mathcal{M}_i^a d(T_a)$ with $d_{ab}(T_c) = i\varepsilon_{acb}$. In addition one can show that

$$[h_D,[h_D,d(T_a)]] = \frac{1}{2}\sum_{b,c=1}^{3}|\varepsilon_{abc}|(\varphi_b-\varphi_c)^2 d(T_a). \tag{2.16}$$

Therefore

$$\mathrm{Tr}([M^T\partial_i M,h_D])^2 = -\mathrm{Tr}(M^T\partial_i M[h_D,[h_D,M^T\partial_i M]]) \tag{2.17}$$

$$= \sum_{a,b,c=1}^{3}|\varepsilon_{abc}|(\varphi_a-\varphi_b)^2\mathcal{M}_i^c\mathcal{M}_i^c \tag{2.18}$$

where we have used the fact that $\mathrm{Tr}(d(T_a)d(T_b)) = 2\delta_{ab}$. Consequently, the variation of E_2, given in (1.22), with respect to φ_a is

$$\begin{aligned}\delta^{(\varphi_a)}E_2 &= \int d^3x\left[\mu_0^2\left[-\kappa_a\partial_i^2\varphi_a\right.\right.\\ &\quad\left.\left.+\kappa_4\sum_{b,c=1}^{3}|\varepsilon_{abc}|(\varphi_a-\varphi_b)\mathcal{M}_i^c\mathcal{M}_i^c\right] + \frac{\delta\mathcal{V}}{\delta\varphi_a}\right]\delta\varphi_a.\end{aligned} \tag{2.19}$$

In some of our applications it will be useful to treat the quantities R_i^a as a 3×3 matrix with the following ordering of rows and columns: $R_i^a = (R)_{ia}$, $i = 1, 2, 3$, and $a = 1, 2, 3$. Therefore

$$\varepsilon_{ijk}R_i^aR_j^bR_k^c = \varepsilon_{abc}\varepsilon_{ijk}R_{i1}R_{j2}R_{k3} = \varepsilon_{abc}\ \det R. \tag{2.20}$$

Then, from (1.7) we have that

$$\varepsilon_{ijk}\widehat{\mathrm{Tr}}(R_iR_jR_k) = i3\ \det\ R. \tag{2.21}$$

But from (1.9) we have that $\det\tau = (\det R)^2$. Therefore, the topological charge (1.6) can be written as

$$Q = -\frac{\varepsilon}{16\pi^2}\int d^3x\sqrt{\det\tau};\qquad \det R = \varepsilon\sqrt{\det\tau};\qquad \varepsilon = \pm 1. \tag{2.22}$$

Note that the eigenvalues of the matrix τ, given in (1.9), are all non-negative since if v_a is an arbitrary real vector then

$$v^T\tau v = \sum_{i=1}^{3}(v_aR_i^a)^2 \geq 0. \tag{2.23}$$

The topological charge is invariant under any (homotopic) smooth variation of the U fields, i.e., $\delta^{(U)}Q = 0$, and consequently the eigenvalues of τ have to satisfy

$$\int d^3x\delta^{(U)}\sqrt{\omega_1\omega_2\omega_3} = 0. \tag{2.24}$$

Such a relation will be very useful, in Sec. III, in the construction of models that break the self-duality in a soft manner.

III. THE FIRST TYPE OF QUASI-SELF-DUAL MODEL

By considering the coefficient of $\delta^{(U)}\omega_a$ in (2.14), for each value of $a = 1, 2, 3$, we observe that if we impose

$$\begin{aligned}m_0^2e_0^2\varphi_1 + \frac{\omega_3}{\varphi_2} + \frac{\omega_2}{\varphi_3} &= \Lambda\sqrt{\frac{\omega_2\omega_3}{\omega_1}}\\ m_0^2e_0^2\varphi_2 + \frac{\omega_3}{\varphi_1} + \frac{\omega_1}{\varphi_3} &= \Lambda\sqrt{\frac{\omega_1\omega_3}{\omega_2}}\\ m_0^2e_0^2\varphi_3 + \frac{\omega_2}{\varphi_1} + \frac{\omega_1}{\varphi_2} &= \Lambda\sqrt{\frac{\omega_1\omega_2}{\omega_3}}\end{aligned} \tag{3.1}$$

with Λ being an arbitrary constant with dimension of mass, which is non-negative since the eigenvalues φ_a and ω_a, $a = 1, 2, 3$, are non-negative, then, as a consequence of (2.24), (2.14) becomes

$$\delta^{(U)}E_1 = \frac{\Lambda}{e_0^2}\int d^3x\,\delta^{(U)}\sqrt{\omega_1\omega_2\omega_3} = 0. \tag{3.2}$$

In other words, the algebraic relations (3.1) imply that the part of the Euler-Lagrange equations associated to the U fields, coming from E_1, are satisfied. Therefore, if we drop the pion mass term from (1.22), i.e., take $\beta_3 = 0$, we are left to consider only the Euler-Lagrange equations associated to the h fields. So, as far as the U fields are concerned, the algebraic relations (3.1) play the same role, in the theory (1.20), as the (differential) self-duality equations (1.3) in the self-dual Skyrme model (1.1).

The solutions of the algebraic equations (3.1) are constructed in Appendix A. There are basically three types of solutions, but since we need the eigenvalues of the matrix h to be positive, only one type is adequate for our applications. It is given by

$$\varphi_a = \frac{\alpha}{|m_0e_0|}\sum_{b,c=1}^{3}\frac{|\varepsilon_{abc}|}{2}\sqrt{\frac{\omega_b\omega_c}{\omega_a}} \tag{3.3}$$

with α being related to Λ by

$$\alpha = \frac{1}{2}\left(\frac{\Lambda}{m_0e_0} \pm \sqrt{\frac{\Lambda^2}{m_0^2e_0^2} - 8}\right) \quad \text{with} \quad \Lambda \geq 2\sqrt{2}|m_0e_0|;\ \ \alpha \geq 0. \tag{3.4}$$

Note that (3.3) differs from (1.14) only by the factor α, and one can check that (3.3), together with (1.23), imply that the matrix h has the form

$$h = \alpha h_{\rm BPS}; \quad \text{with} \quad h_{\rm BPS} = \frac{\sqrt{\det\tau}}{|m_0e_0|}\tau^{-1}. \tag{3.5}$$

Using the definition of τ in (1.9), one gets that (3.5) leads to

$$|m_0e_0|h_{ac}R_i^cR_i^b = \alpha\sqrt{\det\tau}\,\delta_{ab} \rightarrow \quad |m_0e_0|h_{ac}R_i^c = \alpha\sqrt{\det\tau}\,(R^{-1})_{ai}. \tag{3.6}$$

Using (2.20) and (2.22), one gets that

$$\frac{1}{2}\varepsilon_{abc}\varepsilon_{ijk}R_i^aR_j^b = \pm\sqrt{\det\tau}\,(R^{-1})_{ck}. \tag{3.7}$$

Combining (3.6) and (3.7) one gets that the h fields must satisfy a generalized version of the self-dual equations (1.3) given by

$$\lambda h_{ab}R_i^b = \frac{\alpha}{2}\varepsilon_{ijk}H_{jk}^a; \qquad \lambda \equiv \pm m_0e_0 \tag{3.8}$$

where we have used (2.3).

Using (3.3) one gets that (2.13) becomes

$$\delta^{(\varphi_a)}E_1 = \frac{m_0^2}{2}\int d^3x\left[1 - \frac{1}{\alpha^2}\right]\omega_a\delta\varphi_a = \frac{m_0^4e_0^2}{2}\int d^3x\left[1 - \frac{1}{\alpha^2}\right]\frac{1}{\alpha^2}\frac{\delta\det h}{\delta\varphi_a}\delta\varphi_a \tag{3.9}$$

where, in the last equality, we have used the fact that (3.3) implies that

$$\frac{\omega_1}{\varphi_2\varphi_3} = \frac{\omega_2}{\varphi_1\varphi_3} = \frac{\omega_3}{\varphi_1\varphi_2} = \frac{m_0^2e_0^2}{\alpha^2}. \tag{3.10}$$

Therefore, if we choose all the terms in E_2, given in (1.22), to vanish except for the potential term which we take to be proportional to $\det h$, i.e., we assume (1.24), we solve the Euler-Lagrange equations associated to the φ fields. Using the notation of (1.24) we then get that[1]

$$\beta_{\mathcal{V}}^2 = m_0^4e_0^2\left[\frac{1}{\alpha^4} - \frac{1}{\alpha^2}\right]; \quad \text{or} \quad \alpha(\vartheta) = \sqrt{\frac{2}{1+\sqrt{1+4\vartheta}}}; \quad \text{with} \quad \vartheta \equiv \frac{\beta_{\mathcal{V}}^2}{m_0^4e_0^2}. \tag{3.11}$$

The Euler-Lagrange equations for the M fields, given in (2.12), comes from the κ_4-term in E_2, given in (1.22). Since we have dropped that term we do not have such an equation in this model.

Therefore, the solutions of the modified self-duality equations (3.8) are static solutions of the theory defined by the following static energy functional:

$$E_{\rm quasi-sd} = \int d^3x\left[\frac{m_0^2}{2}h_{ab}R_i^aR_i^b + \frac{1}{4e_0^2}h_{ab}^{-1}H_{ij}^aH_{ij}^b + \frac{\beta_{\mathcal{V}}^2}{2}\det h\right]. \tag{3.12}$$

We have then obtained an extension of the theory (1.4), by the addition of a potential proportional to $\det h$, which admits an exact self-dual sector. The self-duality equations for the two theories differ just by a multiplicative constant in one of its two terms. Such a particular extension of the BPS theory (1.4) preserves the conformal invariance in three spacial dimensions as well the global symmetry $SU(2)_L \otimes SU(2)_R$ defined by the transformations (1.17) and (1.18).

[1]Note that there two solutions for α, namely, $\alpha^2 = 2/(1 \pm \sqrt{1+4\vartheta})$. But since α and ϑ are non-negative parameters, α is reduced to (3.11).

Using (2.22), one gets that the static energy (3.12) evaluated on the solutions of the self-duality equations (3.8) becomes

$$E_{\text{quasi-sd}} = 24\pi^2 \frac{|m_0|}{|e_0|}\left(\alpha + \frac{1}{\alpha} + \frac{\vartheta}{3}\alpha^3\right)|Q|. \tag{3.13}$$

Writing (3.13) in terms of the BPS static energy E_1^{BPS}, given in (1.5), and using (3.11) we obtain

$$E_{\text{quasi-sd}} = E_1^{\text{BPS}} \frac{\sqrt{2}}{3} \frac{2+\sqrt{1+4\vartheta}}{\sqrt{1+\sqrt{1+4\vartheta}}}; \quad \text{with}$$
$$E_1^{\text{BPS}} = 48\pi^2 \frac{|m_0|}{|e_0|}|Q| \tag{3.14}$$

which is monotonic increasing on ϑ. In addition, on the weak coupling regime $\vartheta \ll 1$ the static energy (3.14) becomes $E \approx E_1^{\text{BPS}}(1 + \vartheta/6 - \vartheta^2/8 + \mathcal{O}(\vartheta^3))$, and on the strong coupling regime $\vartheta \gg 1$ we have $E \approx E_1^{\text{BPS}}(2\vartheta^{\frac{1}{4}}/3 + \vartheta^{-\frac{1}{4}}/2 - \vartheta^{-\frac{3}{4}}/16 + \mathcal{O}(\vartheta^{-5/4}))$.

Therefore, the addition of a potential term proportional to $\det h$ to the theory (1.1) does not really break the self-duality. The self-duality equations (3.8), for the static theory (3.12), differ from the self-duality equations (1.3) for the static theory (1.4) by the replacement $m_0 e_0 \to m_0 e_0/\alpha$, with α given by (3.11). Note from (3.11) that $\alpha = 1$ implies $\beta_{\mathcal{V}} = 0$ and so the absence of a potential term. On the other hand, the limit $\alpha \to 0$ corresponds to strong coupling, i.e., $\beta_{\mathcal{V}} \to \infty$. In addition, the lower bound on the static energy, saturated by the self-dual solutions, grows monotonically with the increase of the potential strength. Note that the U fields are still totally free, as the h fields still act as spectators. Indeed, given a U-field configuration, and so a τ matrix, the h fields get determined in terms of U by Eq. (3.5).

IV. THE SECOND TYPE OF QUASI-SELF-DUAL MODEL

We now consider the static theory $E = E_1 + E_2$, with E_1 given by (1.4) and E_2 by (1.22), assuming only the quasi-self-duality condition (1.23). We shall construct a holomorphic ansatz for the U fields, involving a radial profile function $f(r)$ and a complex field u depending upon the angles of the spherical polar coordinates.

A. The holomorphic ansatz

In order to construct an ansatz for the full theory (1.20) we shall use the decomposition of the $SU(2)$ group element U in terms of a real scalar field f and a complex scalar field u, together with its complex conjugate $\bar{u}$, as follows [13,16,19]:

$$U = W^\dagger e^{ifT_3} W \quad \text{with} \quad W = \frac{1}{\sqrt{1+|u|^2}}\begin{pmatrix} 1 & iu \\ i\bar{u} & 1 \end{pmatrix}. \tag{4.1}$$

Through (4.1) the Maurer-Cartan can be written as

$$R_i = R_i^a T_a = i\partial_\mu U U^\dagger = -G\Sigma_i G^\dagger \quad \text{with} \quad G = W^\dagger e^{ifT_3/2} \tag{4.2}$$

with

$$\Sigma_i = \partial_i f T_3 + \frac{2\sin(f/2)}{1+|u|^2}[i\partial_i(u-\bar{u})T_1 - \partial_i(u+\bar{u})T_2]. \tag{4.3}$$

From (4.2) we have that the matrix τ, defined in (1.9), becomes

$$\begin{aligned}\tau_{ab} &= \widehat{\text{Tr}}(\Sigma_i G^\dagger T_a G)\widehat{\text{Tr}}(\Sigma_i G^\dagger T_b G) \\ &= d^T_{ac}(G^\dagger)\widehat{\text{Tr}}(\Sigma_i T_c)\widehat{\text{Tr}}(\Sigma_i T_d) d_{db}(G^\dagger)\end{aligned} \tag{4.4}$$

where $d^T(G^\dagger) = d(G) = d(W^\dagger)d(e^{ifT_3/2})$ is the adjoint representation of $G^\dagger$, which, using (1.19) and (4.2), gives

$$d(e^{ifT_3/2}) = \begin{pmatrix} \cos\frac{f}{2} & \sin\frac{f}{2} & 0 \\ -\sin\frac{f}{2} & \cos\frac{f}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

and

$$d(W^\dagger) = \frac{1}{1+|u|^2}\begin{pmatrix} \frac{1}{2}(2+u^2+\bar{u}^2) & \frac{1}{2}i(u^2-\bar{u}^2) & i(u-\bar{u}) \\ \frac{1}{2}i(u^2-\bar{u}^2) & \frac{1}{2}(2-u^2-\bar{u}^2) & -(u+\bar{u}) \\ -i(u-\bar{u}) & u+\bar{u} & 1-|u|^2 \end{pmatrix}. \tag{4.6}$$

We now use spherical coordinates, but instead of using the polar and azimuthal angles we stereographic project the two-sphere on a plane and parametrize that plane by a complex coordinate w together with its complex conjugate $\bar{w}$. So, we have the coordinate transformation

$$x_1 = r\frac{-i(w-\bar{w})}{1+|w|^2}; \quad x_2 = r\frac{(w+\bar{w})}{1+|w|^2}; \quad x_3 = r\frac{|w|^2-1}{1+|w|^2} \tag{4.7}$$

where r is the radial distance. The Euclidean space metric becomes

$$ds^2 = dr^2 + \frac{4r^2}{(1+|w|^2)^2}dwd\bar{w} \tag{4.8}$$

and so

$$d^3x = \sqrt{-g}drdwd\bar{w}; \qquad \sqrt{-g} = \frac{2r^2}{(1+|w|^2)^2}. \tag{4.9}$$

We now use the holomorphic ansatz for the $SU(2)$ fields defined by

$$f \equiv f(r); \qquad u \equiv u(w); \qquad \bar{u} \equiv \bar{u}(\bar{w}) \tag{4.10}$$

where $u(w)$ is a map between two-spheres (S^2). However, for the $u(w)$ field to be a well-defined map between two-spheres it has to be a ratio of two polynomials, p_1 and p_2, with no common roots, i.e., the so-called rational map [16–18]

$$u(w) = \frac{p_1(w)}{p_2(w)}. \tag{4.11}$$

A well-known feature of the rational map (4.11) is that its algebraic degree, defined as the highest power of w in either of the polynomials p_1 and p_2, corresponds exactly to its topological degree n, which can be writhen in the integral representation as

$$n = \frac{1}{4\pi}\int d\Omega q = \frac{i}{2\pi}\int dw \wedge d\bar{w}\frac{|p_2\partial_w p_1 - p_1\partial_w p_2|^2}{(|p_1|^2+|p_2|^2)^2} \tag{4.12}$$

where Ω is the solid angle, and we use $d\Omega = \frac{2idw\wedge d\bar{w}}{(1+|w|^2)^2}$ and the following definition:

$$q \equiv \frac{(1+|w|^2)^2}{(1+|u|^2)^2}\partial_w u \partial_{\bar{w}}\bar{u}. \tag{4.13}$$

The topological charge density ρ of (1.6) can be written using (4.1), (4.10), and (4.13) as

$$\rho \equiv \frac{i}{48\pi^2}\varepsilon_{ijk}\widehat{\mathrm{Tr}}(R_iR_jR_k) = -\frac{f'(r)}{4\pi^2}\frac{\sin^2(f(r)/2)}{r^2}q \tag{4.14}$$

and so due to (4.12) the topological charge (1.6) becomes

$$Q = \frac{[f-\sin f]_{f(\infty)}^{f(0)}}{2\pi}n. \tag{4.15}$$

Note that due to (4.14) we get that $\mathrm{sign}(Qf') = -1$ and so (1.8) leads to

$$\mathrm{sign}(Q) = -\mathrm{sign}(\lambda) = -\mathrm{sign}(f'). \tag{4.16}$$

As a consequence of the holomorphic ansatz (4.10), the matrix $\widehat{\mathrm{Tr}}(\Sigma_i T_a)\widehat{\mathrm{Tr}}(\Sigma_i T_b)$ becomes diagonal. Indeed, from (4.3), (4.8), and (4.10), one gets that

$$(\tau_D)_{ab} \equiv \widehat{\mathrm{Tr}}(\Sigma_i T_a)\widehat{\mathrm{Tr}}(\Sigma_i T_b) = \omega_a\delta_{ab} \tag{4.17}$$

with

$$\omega_1 = \omega_2 = \frac{4\sin^2(f/2)}{r^2}q \qquad \omega_3 = (f')^2 \tag{4.18}$$

where prime denotes derivatives with respect to r. Comparing (1.13) and (4.4) we then conclude that

$$N = d^T(G^\dagger) = d(G) = d(W^\dagger)d(e^{ifT_3/2}). \tag{4.19}$$

B. The Euler-Lagrange equations

We start the analysis of the Euler-Lagrange equations, in the holomorphic ansatz, by noticing that, if one considers u, $\bar{u}$, $\partial_w u$, and $\partial_{\bar{w}}\bar{u}$, as independent variables, then the quantity q defined in (4.13) satisfies

$$\frac{\delta q}{\delta u} - (1+|w|^2)^2\partial_w\left(\frac{1}{(1+|w|^2)^2}\frac{\delta q}{\delta\partial_w u}\right) = 0 \tag{4.20}$$

together with its complex conjugate.

From (4.18) we have that ω_3 depends only on the radial profile function $f(r)$, and ω_1 and ω_2 depend upon u and $\bar{u}$ through q only, and they are linear in q. Therefore, from (2.14) we observe that the variation of E_1 with respect to the u field is

$$\delta^{(u)}E_1 = \frac{1}{2e_0^2}\int d^3x\hat{\omega}(r)\Big[m_0^2e_0^2(\varphi_1+\varphi_2) + \omega_3(r)\left(\frac{1}{\varphi_1}+\frac{1}{\varphi_2}\right) + 2\hat{\omega}(r)\frac{q}{\varphi_3}\Big]\delta^{(u)}q \tag{4.21}$$

where, following (4.18), we have defined

$$\hat{\omega}(r) \equiv \frac{4\sin^2(f/2)}{r^2}. \tag{4.22}$$

Consequently, if we consider the ansatz

$$\varphi_1 = \varphi_1(r); \qquad \varphi_2 = \varphi_2(r); \qquad \varphi_3 = \hat{\varphi}_3(r)q \tag{4.23}$$

we get, using (4.9), that (4.21) vanishes as a consequence of (4.20). For the same reasons one gets that $\delta^{(\bar{u})}E_1 = 0$. From (4.1) we see that $\mathrm{Tr}U$ does not depend upon the fields u and $\bar{u}$. Therefore, from (4.26) we get that

$$\delta^{(u)}E_2 = \delta^{(\bar{u})}E_2 = 0. \tag{4.24}$$

So, the conditions (4.23) are sufficient for the Euler-Lagrange equations, associated to the u and $\bar{u}$ fields, to be satisfied within the holomorphic ansatz (4.10).

Using (4.22) and (4.23) one gets from (2.14) that

$$\delta^{(f)}E_1 = \frac{1}{2e_0^2}\int d^3x\Big[m_0^2e_0^2(\varphi_1+\varphi_2)\delta^{(f)}\hat{\omega} + \left(\frac{1}{\varphi_1}+\frac{1}{\varphi_2}\right)\delta^{(f)}(\hat{\omega}\omega_3) + m_0^2e_0^2\hat{\varphi}_3\delta^{(f)}\omega_3 + \frac{\delta^{(f)}\hat{\omega}^2}{\hat{\varphi}_3}\Big]q. \tag{4.25}$$

As q factors out, one observes that the variation E_1 with respect to the profile function f leads to a radial equation for it. However, from (4.26) and (4.1) one gets that

$$\delta^{(f)}E_2 = \frac{\beta_3^2}{2}\int d^3x\sin\left(\frac{f}{2}\right). \tag{4.26}$$

Therefore, for $\beta_3 \neq 0$, one has to impose that q must be a constant, in order to get a radial equation for f.

Let us now analyze the Euler-Lagrange equations for the M fields given in (2.12). Our quasi-self-dual condition (1.23) requires $M = N$, and so (2.12) becomes, in fact, equations for the U fields. We do not want the U fields to be subjected to additional equations, besides their own Euler-Lagrange equations. Therefore, we want (2.12) to be solved automatically by the holomorphic ansatz, supplemented by some extra conditions. From (4.19) and the holomorphic ansatz (4.10) for the U fields we obtain

$$\begin{aligned} N^T\partial_r N &= \frac{i}{2}f'd(T_3)\\ N^T\partial_w N &= \frac{1}{1+|u|^2}[-\bar{u}\partial_w u d(T_3) - ie^{-if/2}\partial_w u d(T_1+iT_2)]\\ N^T\partial_{\bar{w}} N &= \frac{1}{1+|u|^2}[u\partial_{\bar{w}}\bar{u}d(T_3) - ie^{if/2}\partial_{\bar{w}}\bar{u}d(T_1-iT_2)] \end{aligned} \tag{4.27}$$

where we have used the fact that $d_{ab}(T_c) = i\varepsilon_{acb}$. Therefore, using (2.16), we get that

$$\begin{aligned} [h_D,[h_D,N^T\partial_r N]] &= \frac{i}{2}f'(\varphi_1-\varphi_2)^2d(T_3)\\ [h_D,[h_D,N^T\partial_w N]] &= \frac{1}{1+|u|^2}[-\bar{u}\partial_w u(\varphi_1-\varphi_2)^2d(T_3) - ie^{-if/2}\partial_w u[(\varphi_2-\varphi_3)^2d(T_1)+i(\varphi_1-\varphi_3)^2d(T_2)]]\\ [h_D,[h_D,N^T\partial_{\bar{w}} N]] &= \frac{1}{1+|u|^2}[u\partial_{\bar{w}}\bar{u}(\varphi_1-\varphi_2)^2d(T_3) - ie^{if/2}\partial_{\bar{w}}\bar{u}[(\varphi_2-\varphi_3)^2d(T_1)-i(\varphi_1-\varphi_3)^2d(T_2)]]. \end{aligned} \tag{4.28}$$

It then follows that

$$\partial_r[h_D,[h_D,N^T\partial_r N]] + [N^T\partial_r N,[h_D,[h_D,N^T\partial_r N]]] = \frac{i}{2}(f'(\varphi_1-\varphi_2)^2)'d(T_3).$$

But that involves first and second derivatives of the profile function f, which cannot be canceled by the remaining terms of (2.12). Therefore, we shall impose, besides (4.23), the condition $\varphi_1(r) = \varphi_2(r)$. One can then check that all the terms in (2.12) vanish except for those involving w and $\bar{w}$ derivatives of the φ_3 field. Considering the form of φ_3, given in (4.23), those derivatives do not cancel each other unless we assume that q is constant. However, as shown in Appendix B the only rational maps (4.11) that lead to a constant value of q have the form $u = e^{i\alpha}w$ or $u = \frac{\beta(w-|\beta|^{-1}e^{i\alpha})}{w+|\beta|e^{i\alpha}}$, where α is a real constant contained in the interval $[0, 2\pi)$ and β is an arbitrary complex constant with $\beta \neq 0$. Note that both of these rational maps leads to $q = 1$. Therefore, we are lead to consider the following ansatz for the φ fields:

$$\varphi_1 = \varphi_2 \equiv \varphi_1(r); \qquad \varphi_3 = \varphi_3(r) \tag{4.29}$$

and for the u fields

$$u = e^{i\alpha}w \;\text{ or }\; u = \frac{\beta(w-|\beta|^{-1}e^{i\alpha})}{w+|\beta|e^{i\alpha}};\;\text{ and so }\; q = 1. \tag{4.30}$$

Note that imposing that q must be constant is equivalent to imposing that the topological charge density inside the holomorphic ansatz, as given in (4.14), must have radial symmetry. For the rational maps (4.30), which have topological degree $n = 1$, the topological charge (4.15) becomes

$$Q = \frac{[f-\sin f]_{f(\infty)}^{f(0)}}{2\pi}. \tag{4.31}$$

Summarizing, using the holomorphic ansatz (4.10) together with the conditions (4.29) and (4.30) we get that the Euler-Lagrange equations for the M, u, and $\bar{u}$ fields are automatically satisfied. We are then left with three radial equations which are the Euler-Lagrange equations for the profile function f and for the φ fields.

The Euler-Lagrange equation for f is given by

$$m_0^2\left[\frac{1}{r^2}\partial_r(r^2\varphi_3 f')-2\varphi_1\frac{\sin f}{r^2}\right]-\frac{\beta_3^2}{2}\sin(f/2)+\frac{1}{e_0^2}\left[\frac{1}{r^2}\partial_r\left(\frac{8\sin^2(f/2)}{\varphi_1}f'\right)-\frac{16\sin^3(f/2)\cos(f/2)}{r^4\varphi_3}-\frac{2\sin f(f')^2}{r^2\varphi_1}\right]=0. \tag{4.32}$$

The Euler Lagrange equations for the $\varphi_1=\varphi_2$ and φ_3 fields are, respectively,

$$\frac{\mu_0^2}{r^2}[\kappa_1\partial_r(r^2\varphi_1')-\kappa_4 2(\varphi_1-\varphi_3)]-\frac{\delta\mathcal{V}}{\delta\varphi_1}-m_0^2\frac{2\sin^2(f/2)}{r^2}\left[1-\frac{1}{m_0^2e_0^2}\frac{(f')^2}{\varphi_1^2}\right]=0 \tag{4.33}$$

and

$$\frac{\mu_0^2}{r^2}[\kappa_3\partial_r(r^2\varphi_3')+\kappa_4 4(\varphi_1-\varphi_3)]-\frac{\delta\mathcal{V}}{\delta\varphi_3}-\frac{m_0^2}{2}\left[(f')^2-\frac{16\sin^4(f/2)}{m_0^2e_0^2r^4\varphi_3^2}\right]=0. \tag{4.34}$$

Because of the condition (4.29) we had to assume that the potential $\mathcal{V}$ is symmetric under the exchange $\varphi_1\leftrightarrow\varphi_2$, and that the coupling constants κ_1 and κ_2, introduced in (1.22), are the same.

In the next section we show how to solve numerically those three radial equations.

C. Numerical solutions for a quadratic potential $\mathcal{V}$

Consider the static sector of the theory (1.20) with $\mathcal{V}=\frac{\beta_1^2}{2}\mathrm{Tr}h^2=\frac{\beta_1^2}{2}\sum_{a=1}^3\varphi_a^2$, and $\kappa_\alpha=1$, $\alpha=1,\ 2,\ 3,\ 4$. As we are working with the ansatz (1.23), (4.10), (4.29), and (4.30), we shall be concerned with configurations of unity topological charge only. Therefore, we shall measure the energy in units of $48\pi^2\frac{|m_0|}{|e_0|}$. That means that the BPS energy (1.5) of the self-dual configurations (1.10) becomes $E_1^{\mathrm{BPS}}=1$, for $Q=1$. We shall measure length in units of $\mu_0^2/(m_0^3e_0)$, and rescale the h fields, and so the φ fields, by the dimensionless factor m_0^2/μ_0^2. Therefore, using (1.4) and (1.22) the total static energy can be rewritten, in terms of the new units, as

$$E=\frac{1}{96\pi^2}\int d^3x\left[h_{ab}R_i^aR_i^b+\frac{1}{2}h_{ab}^{-1}H_{ij}^aH_{ij}^b+\mathrm{Tr}(\partial_ih)^2+\sigma_1\mathrm{Tr}(h^2)+\sigma_2\mathrm{Tr}(\mathbb{1}-U)\right] \tag{4.35}$$

with

$$\sigma_1=\frac{\mu_0^2\beta_1^2}{m_0^6e_0^2};\qquad \sigma_2=\frac{\mu_0^2\beta_3^2}{m_0^6e_0^2}. \tag{4.36}$$

The Euler-Lagrange equations (4.32), (4.33), and (4.34) become, respectively,

$$\Delta E_f\equiv\frac{1}{r^2}[\partial_r(r^2A)+2B\sin(f)]+\frac{\sigma_2}{2}\sin(f/2)=0, \tag{4.37}$$

$$\Delta E_{\varphi_1}\equiv\varphi_1''+\frac{2}{r}\varphi_1'-\frac{2}{r^2}(\varphi_1-\varphi_3)-\sigma_1\varphi_1-\frac{2}{r^2}\sin^2(f/2)\left[1-\frac{(\partial_rf)^2}{\varphi_1^2}\right]=0, \tag{4.38}$$

$$\Delta E_{\varphi_3}\equiv\varphi_3''+\frac{2}{r}\varphi_3'+\frac{4}{r^2}(\varphi_1-\varphi_3)-\sigma_1\varphi_3-\frac{1}{2}(f')^2\left[1-\frac{16\sin^4(f/2)}{r^4\varphi_3^2(\partial_rf)^2}\right]=0, \tag{4.39}$$

where

$$A\equiv-\partial_rf\left[\varphi_3+\frac{8\sin^2(f/2)}{r^2\varphi_1}\right],$$
$$B\equiv\varphi_1\left[1+\frac{(\partial_rf)^2}{\varphi_1^2}+\frac{4\sin^2(f/2)}{r^2\varphi_1\varphi_3}\right]. \tag{4.40}$$

Inside ansatz (1.23), (4.10), (4.29), and (4.30) the static energy (4.35) is reduced to

$$E=E_1+E_2;\quad E_1=\mathcal{E}_2+\mathcal{E}_4;\quad E_2=\mathcal{E}_h+\mathcal{E}_{\sigma_1}+\mathcal{E}_{\sigma_2} \tag{4.41}$$

with

$$\mathcal{E}_2 \equiv \frac{1}{96\pi^2}\int d^3x h_{ab}R_i^a R_i^b = \frac{1}{12\pi}\int drr^2\left(\frac{\varphi_3(\partial_r f)^2}{2}+\varphi_1\frac{4\sin^2\frac{f}{2}}{r^2}\right)$$
$$\mathcal{E}_4 \equiv \frac{1}{192\pi^2}\int d^3x h_{ab}^{-1}H_{ij}^a H_{ij}^b = \frac{1}{12\pi}\int drr^2\left[\frac{4\sin^2\frac{f}{2}}{r^2}\left(\frac{(\partial_r f)^2}{\varphi_1}+\frac{2\sin^2\frac{f}{2}}{r^2\varphi_3}\right)\right]$$
$$\mathcal{E}_h \equiv \frac{1}{96\pi^2}\int d^3x \mathrm{Tr}(\partial_i h)^2 = \frac{1}{12\pi}\int drr^2\left[(\partial_r\varphi_1)^2+\frac{(\partial_r\varphi_3)^2}{2}+\frac{2(\varphi_1-\varphi_3)^2}{r^2}\right]$$
$$\mathcal{E}_{\sigma_1} \equiv \frac{\sigma_1}{96\pi^2}\int d^3x \mathrm{Tr}(h^2) = \frac{\sigma_1}{12\pi}\int drr^2\left(\varphi_1^2+\frac{\varphi_3^2}{2}\right)$$
$$\mathcal{E}_{\sigma_2} \equiv \frac{\sigma_2}{96\pi^2}\int d^3x \mathrm{Tr}(\mathbb{1}-U) = \frac{\sigma_2}{12\pi}\int drr^2\left(1-\cos\frac{f}{2}\right). \tag{4.42}$$

The stability of the solutions of (4.37)–(4.39) under the scale Derrick's argument [20,21] imposes relations only between the terms of E_2, since E_1 is conformal invariant in three spatial dimensions. Indeed, since the h fields have conformal weight -1 by the scaling transformation $x \to \alpha x$, these fields must transform as $h \to \alpha^{-1}h$, and so the E_2 terms of (4.41) transform as $\mathcal{E}_h \to \alpha^{-1}\mathcal{E}_h$, $\mathcal{E}_{\sigma_1} \to \alpha\mathcal{E}_{\sigma_1}$ and $\mathcal{E}_{\sigma_2} \to \alpha^3\mathcal{E}_{\sigma_2}$. Therefore, the stable solutions under the Derrick's argument need to satisfy

$$-\mathcal{E}_h + \mathcal{E}_{\sigma_1} + 3\mathcal{E}_{\sigma_2} = 0; \qquad \mathcal{E}_h + 3\mathcal{E}_{\sigma_2} > 0. \tag{4.43}$$

The inequality of (4.43) is automatically satisfied and the first relation imposes that the dimensionless quantity

$$\text{Derrick} \equiv \frac{|-\mathcal{E}_h + \mathcal{E}_{\sigma_1} + 3\mathcal{E}_{\sigma_2}|}{E_2} \tag{4.44}$$

must be zero. Note that the term E_2 in the denominator of the lhs of (4.44) prevents unstable solutions of (4.37)–(4.39) from leading to small values of the quantity (4.44) under the weak coupling regime $E_2 \ll E_1$, where all the terms $\mathcal{E}_h$, $\mathcal{E}_{\sigma_1}$, and $\mathcal{E}_{\sigma_2}$ are small.

The simplest topological solutions that we can construct are those with Skyrme charge $Q = 1$, and due to (4.31) we shall impose the boundary conditions $f(0) = 2\pi$ and $f(\infty) = 0$. So, expanding (4.37), (4.38), and (4.39) in Taylor series at $r = 0$ we obtain

TABLE I. The quantities (4.41), (4.42), and (4.44) associated with the solutions of Eqs. (4.37)–(4.39) with $Q = 1$ for some values of σ_1 and σ_2.

N_c	σ_1	σ_2	E	Derrick	$\mathcal{E}_2$	$\mathcal{E}_4$	$\mathcal{E}_h$	$\mathcal{E}_{\sigma_1}$	$\mathcal{E}_{\sigma_2}$
i	0.25	0.25	1.26924	2.48×10^{-3}	0.35860	0.70973	0.12606	0.04950	0.02535
ii	0.50	0.50	1.32700	3.11×10^{-3}	0.33712	0.75577	0.14222	0.06711	0.02479
iii	1.00	1.00	1.39771	8×10^{-5}	0.31419	0.81127	0.15982	0.08873	0.02370
iv	2.00	2.00	1.48360	3.3×10^{-4}	0.29045	0.87773	0.17945	0.11419	0.02179
v	4.00	4.00	1.58693	1.3×10^{-4}	0.26657	0.95620	0.20139	0.14342	0.01934
vi	0.25	4.00	1.37545	1.1×10^{-4}	0.34137	0.76872	0.18433	0.02934	0.05167
vii	4.00	0.25	1.56171	7×10^{-5}	0.26615	0.95056	0.17531	0.16689	0.00280

TABLE II. The thicknesses of solutions with $Q = 1$ of Eqs. (4.37)–(4.39), the size of the lattice $r_{\max}$, and the quantities $(E_1 - 1)$ and $\mathcal{E}_4/\mathcal{E}_2$ for the values of σ_1 and σ_2 from Table I.

N_c	*i*	*ii*	*iii*	*iv*	*v*	*vi*	*vii*
t_f	0.99040	0.77787	0.60444	0.46323	0.35029	0.53182	0.41235
t_{φ_1}	1.23552	0.96684	0.74737	0.56949	0.42818	0.71016	0.48892
t_{φ_3}	1.13680	0.89573	0.69698	0.53423	0.40373	0.64507	0.46551
$r_{\max}$	25	25	14	14	14	25	14
$E_1 - 1$	0.06832	0.09289	0.12547	0.16817	0.22277	0.11010	0.21671
$\mathcal{E}_4/\mathcal{E}_2$	1.9792	2.2418	2.5821	3.0220	3.5870	2.25188	3.5716

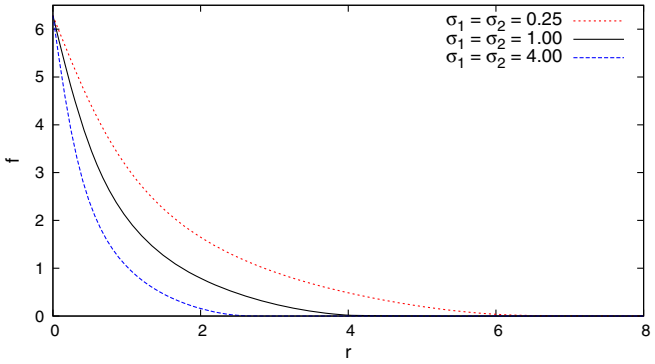

FIG. 1. The $f(r)$-field solution of (4.37)–(4.39) corresponding to $Q = 1$ for $\sigma_1 = \sigma_2 = 0.25, 1.00, 4.00$.

$$\partial_r^2 f(0) = \partial_r^4 f(0) = 0; \quad \partial_r \varphi_a(0) = \partial_r^3 \varphi_a(0) = 0; \quad \varphi_1(0) = \varphi_3(0) \tag{4.45}$$

with $a = 1, 3$. We use the gradient flow method with adaptive step size to minimize the static energy (4.41) and to get the solutions of (4.37)–(4.39) with $Q = 1$, as described in Appendix C. The coordinate r lies in the interval $[0, r_{\max}]$, where $r_{\max}$ is the size of the lattice. Table I shows the energies (4.42) corresponding to the solutions of (4.37)–(4.39) for some pairs of values of σ_1 and σ_2 labeled by an index N_c. The highest value of (4.44) is 3.11×10^{-3}, which means that on all numerical solutions presented in the Tables I and II the term $|-\mathcal{E}_h + \mathcal{E}_{\sigma_1} + 3\mathcal{E}_{\sigma_2}|$ of (4.43) is equal to or less than 0.311% of E_2. So, the relation (4.43) imposed by the Derrick's scale argument is satisfied with quite good precision.

The numerical solutions of $f(r)$ and $\varphi_a(r)$ obtained for all parameters of Table I are monotonically decreasing (see examples in Figs. 1–3). The amplitude $\varphi_a(0)$ of the eigenvalues of h and the thickness t for each of the fields, defined as the value of r for which the field reaches half of its value at $r = 0$, are given in Table II. On Table II and in Fig. 1 we can see that the thickness of f decreases when σ_2 grows and the U fields becomes more massive. The same

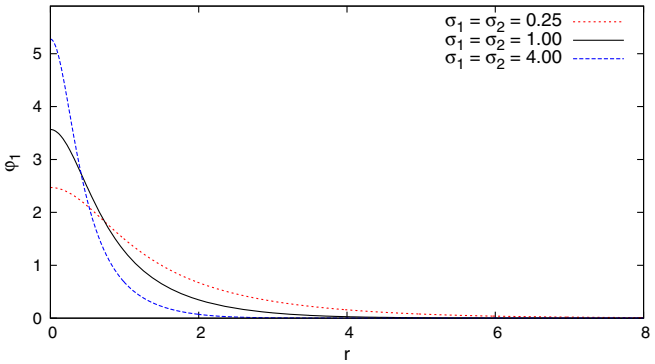

FIG. 2. The $\varphi_1(r)$-field solution of (4.37)–(4.39) corresponding to $Q = 1$ for $\sigma_1 = \sigma_2 = 0.25, 1.00, 4.00$.

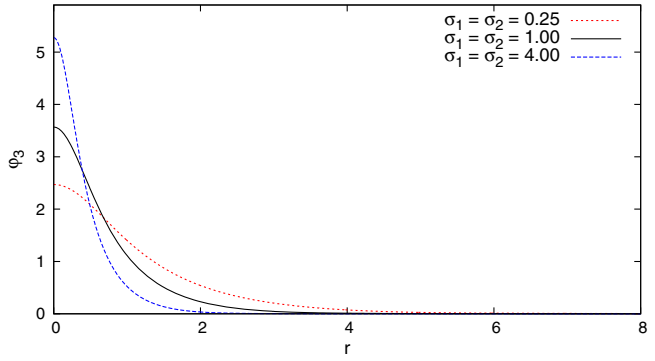

FIG. 3. The $\varphi_3(r)$-field solution of (4.37)–(4.39) corresponding to $Q = 1$ for $\sigma_1 = \sigma_2 = 0.25, 1.00, 4.00$.

follows for the φ_a fields when σ_1 grows and the h fields becomes more massive, but in contrast their amplitude increases (see Figs. 1 and 3).

The quadratic and quartic terms of (1.4) in the spatial derivatives become the same for the self-dual configurations (1.10) [see (1.12)], and so in the units defined above we must have $\mathcal{E}_2^{\rm BPS} = \mathcal{E}_4^{\rm BPS} = 1/2$. In additional, any solutions of (4.37)–(4.39) must satisfy the Bogomolny bound $E_1 \geq E_1^{\rm BPS} = 1$, which has its lower bound saturated by (1.10) (see Table II). The self-dual solutions (1.10) are conformally invariant in $\mathbb{R}^3$ and possesses an infinite number of exact solutions for each value of Q, and so we cannot directly compare the shape of the BPS configurations with the solutions of (4.37)–(4.39). However, we can use the quantity $(E_1 - 1)$ and the ratio $\mathcal{E}_4/\mathcal{E}_2$ to measure how far the static solutions of the full theory (4.35) are from the self-dual sector of the BPS Skyrme model (1.1). Indeed, from Table I we see that these two quantities tend to increase, getting farther and farther from 1, when either σ_1 or σ_2 grow and are more sensitive to σ_1 than σ_2.

V. CONCLUSION

We have proposed extensions of the Skyrme model (1.1) that allow the breaking of its self-dual sector in a soft manner. The self-duality equations (1.3) impose that the matrix h must be proportional to the matrix τ, as shown in (1.10). Therefore, the two matrices are diagonalized by the same orthogonal matrix M, see (1.13), and their eigenvalues are related by (1.14). We extend the theory (1.1) by introducing kinetic and potential terms for the h fields, and impose that the matrices h and τ should still be diagonalized by the same orthogonal matrix M. These are our conditions (1.23), which we call quasi-self-duality equations.

We study two distinct cases of the breaking of the self-duality equations. The first one comes from the observation that by imposing algebraic relations among the eigenvalues of the matrices h and τ, given in (3.1), one gets that the part of the Euler-Lagrange equations associated to the U fields, coming from E_1, given in (1.4), is automatically satisfied,

since the variation of E_1 becomes proportional to the variation of the topological charge, as shown in (3.2). The other observation is that the variation of E_1 with respect to the φ fields is proportional solely to the variation of $\det h$. Therefore, choosing E_2 to contain just a potential term proportional to $\det h$, one solves the Euler-Lagrange equations for the φ and U fields. We are then led to the theory (3.12), which is shown to possess an exact self-dual sector. The corresponding self-duality equations are given in (3.8), and they differ from the original self-duality equations (1.3) by a constant α which is a monotonically decreasing function of the strength of the potential proportional to $\det h$. The theory (3.12) has the same global symmetry $SU(2)_L \otimes SU(2)_R$, as (1.1), and it is also conformally invariant in the three-dimensional spatial submanifold $\mathbb{R}^3$. In addition, the static energy of the self-dual solutions is proportional to the topological charge Q, and the proportionality constant grows with the strength of the potential $\det h$. As in the original theory (1.1), the h fields act as spectators in the sense that, given a configuration for the U fields, they adjust themselves to solve the self-duality equations. It is remarkable that a theory like (3.12) possesses an exact self-dual sector. That may lead to new interesting applications, especially for nuclear matter as was done in [15] for the theory (1.1).

The second way of breaking the self-duality of the theory (1.1), but respecting the quasi-self-duality equations (1.23), is by introducing kinetic and potential terms for the φ fields. In order to study such a case we use the holomorphic ansatz for the U fields, given in (4.1) and (4.10). As a consequence of (1.23), the Euler-Lagrange equations for the M fields become extra conditions for the U fields to satisfy. In order to avoid such strongly restricting conditions, we solve the equations for the M fields by imposing conditions on the φ and u fields. We find that all three eigenvalues φ_a have to depend only on the radial distance r, φ_1 and φ_2 have to be equal, and the complex u field has to correspond to configurations of unity topological charge. Those conditions are given in (4.29) and (4.30). In order to construct the solutions we have to solve therefore just three ordinary differential equations, corresponding to the Euler-Lagrange for the profile function $f(r)$ and for the eigenvalues $\varphi_1(r)$ and $\varphi_3(r)$, of the matrix h. Those equations are solved numerically using the gradient flow method to minimize the static energy of the system. We perform the simulations for a potential which is quadratic in the φ fields, i.e., proportional to $\mathrm{Tr}h^2$. Qualitatively, the solutions look similar to the self-dual solutions. However, the profile function $f(r)$ and the fields $\varphi_1(r)$ and $\varphi_3(r)$ decay exponentially faster, at large distances, with the increase of the coupling constants associated to the kinetic and potential terms of the h fields. In addition, $\varphi_1(r)$ and $\varphi_3(r)$ grow at the origin with the increase of those same constant constants.

The results we have obtained may shed some light on the structures underlying the self-duality in models of type (1.1). It would be interesting to generalize our results by breaking completely the self-duality, i.e., by not imposing (1.23), and construct solutions with topological charges higher than unity, by performing three-dimensional numerical simulations to minimize the static energy. That could help to understand better the role of the h fields. In addition, it could help to improve the applications to nuclear matter done in [15], by performing the breaking of the self-duality with the introduction of kinetic and potential terms for the h fields.

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APPENDIX A: THE SOLUTIONS OF (3.1)

Let us introduce the quantities

$$\alpha_1 = m_0 e_0 \varphi_1 \sqrt{\frac{\omega_1}{\omega_2\omega_3}}; \qquad \alpha_2 = m_0 e_0 \varphi_2 \sqrt{\frac{\omega_2}{\omega_1\omega_3}};$$
$$\alpha_3 = m_0 e_0 \varphi_3 \sqrt{\frac{\omega_3}{\omega_1\omega_2}}; \qquad \beta = \frac{\Lambda}{m_0 e_0}. \tag{A1}$$

Then, Eq. (3.1) can be written as

$$\alpha_1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} = \beta; \quad \alpha_2 + \frac{1}{\alpha_1} + \frac{1}{\alpha_3} = \beta; \quad \alpha_3 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \beta. \tag{A2}$$

Subtracting Eq. (A2) in pairs we observe that

$$\alpha_1 - \frac{1}{\alpha_1} = \alpha_2 - \frac{1}{\alpha_2} = \alpha_3 - \frac{1}{\alpha_3}. \tag{A3}$$

In addition, we can write (A2) as

$$\begin{aligned}\beta\alpha_1\alpha_2 - \alpha_1 - \alpha_2 &= \beta\alpha_2\alpha_3 - \alpha_2 - \alpha_3 \\ &= \beta\alpha_1\alpha_3 - \alpha_1 - \alpha_3 = \alpha_1\alpha_2\alpha_3.\end{aligned} \tag{A4}$$

Again, subtracting the relations (A4) in pairs we get that

$$(\beta\alpha_1 - 1)(\alpha_2 - \alpha_3) = 0; \qquad (\beta\alpha_2 - 1)(\alpha_1 - \alpha_3) = 0;$$
$$(\beta\alpha_3 - 1)(\alpha_1 - \alpha_2) = 0. \tag{A5}$$

Such equations have three types of solutions:

(1) If we take $\alpha_1 = \alpha_2 = 1/\beta$, then all three Eqs. (A5) are satisfied. Then (A4) imposes that $\alpha_3 = -\beta$.

Doing cyclic permutations of the indices we get three solutions

$$\alpha_1 = \alpha_2 = \frac{1}{\beta} \quad \text{and} \quad \alpha_3 = -\beta$$
$$\alpha_1 = \alpha_3 = \frac{1}{\beta} \quad \text{and} \quad \alpha_2 = -\beta$$
$$\alpha_2 = \alpha_3 = \frac{1}{\beta} \quad \text{and} \quad \alpha_1 = -\beta \tag{A6}$$

(2) By taking three α_a's equal to $1/\beta$ we solve all three Eqs. (A5). Then (A4) imposes that $\beta^2 = -1$. So we get the solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \pm i \qquad \beta = \mp i \tag{A7}$$

which is a particular case of (A6).

(3) Finally by taking $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$ we solve all three Eqs. (A5). Then (A4) leads to

$$\alpha(\alpha^2 - \beta\alpha + 2) = 0. \tag{A8}$$

The solution $\alpha = 0$ should be discarded since from (A1), it would imply $\varphi_a = 0$, and so a vanishing h matrix. Therefore, we get two solutions

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 - 8}\right)$$
$$= \sqrt{2}\left(\gamma \pm \sqrt{\gamma^2 - 1}\right); \qquad \beta = 2\sqrt{2}\gamma. \tag{A9}$$

Note that, if we consider γ real, we have

$$\gamma \geq 1 \to \gamma + \sqrt{\gamma^2 - 1} \geq 1 \text{ and } 0 \leq \gamma - \sqrt{\gamma^2 - 1} \leq 1$$
$$\gamma \leq -1 \to \gamma - \sqrt{\gamma^2 - 1} \leq -1 \text{ and } -1 \leq \gamma + \sqrt{\gamma^2 - 1} \leq 0.$$

Therefore, α_a can take any non-negative real value when $\gamma \geq 1$, and any nonpositive real value when $\gamma \leq -1$. For $-1 \leq \gamma \leq 1$ we write $\gamma = \cos\theta$, with $0 \leq \theta \leq \pi$. Then

$$\gamma + \sqrt{\gamma^2 - 1} = e^{i\theta}; \text{ and } \gamma - \sqrt{\gamma^2 - 1} = e^{-i\theta}. \tag{A10}$$

Therefore, $\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{2}e^{i\theta}$, with $0 \leq \theta \leq 2\pi$. If we consider γ complex, then the α_a's can, in principle, be any complex number.

As argued in (2.23), all three eigenvalues ω_a of the τ matrix are non-negative. Therefore, the solutions (A6) impose that, if $\beta > 0$, one eigenvalue φ_a of the h matrix is negative, and, if $\beta < 0$, that two eigenvalues are negative. That implies that the energy E_1 is not positive definite. On the other hand, the solutions (A7) imply that the eigenvalues φ_a of the h matrix are pure imaginary and so E_1 is pure imaginary too.

APPENDIX B: THE RATIONAL MAPS THAT LEAD TO $q = \text{const}$.

In this section we will prove that the only rational map (4.11) for which the functional $q(w, \bar{w})$, defined in (4.13), is constant corresponds to (4.30). Using (4.11) in (4.13) we obtain

$$q(w, \bar{w}) = A(w, \bar{w})^2 |W(w)|^2 \geq 0, \quad \text{with}$$
$$A(w, \bar{w}) \equiv \frac{1 + |w|^2}{|p_1|^2 + |p_2|^2}, \tag{B1}$$

where we introduced the Wronskian $W \equiv p_2 d_w p_1 - p_1 d_w p_2$. The topological degree n of the u field corresponds exactly with the highest power of w in either of the polynomials p_1 and p_2, as mentioned in Sec. II, and so $n \geq 1$. Because of its definition, if both p_1 and p_2 are polynomials of degree n, then the term of order $n(n-1)$ of the Wronskian will vanish, and therefore W is a polynomial of degree $n(n-2)$ or less. In addition, the denominator of (B1) satisfies $|p_1|^2 + |p_2|^2 > 0$ for all values of w and $\bar{w}$, since p_1 and p_2 has no common roots. So, it follows from (B1) that $q(w, \bar{w})$ vanishes only for the values of w that corresponds with the roots of the Wronskian. Therefore, if W has roots, then q cannot be constant. On the other hand, if the polynomial W has no root, then it must be constant, i.e., the rational map (4.11) must satisfy the condition

$$W(w) = \text{const.} \tag{B2}$$

However, the constant of (B2) cannot be zero, otherwise q will vanish and the same goes for n, which is written in the integral representation in (4.12). So it follows from (B2) and (B1) that if $q = \text{const.}$ the function A, which is positive and finite due the definition (B1), must also be constant.

Suppose that m_1 and m_2 are, respectively, the degrees of $p_1(w)$ and $p_2(w)$, and let us define $m \equiv \max(m_1, m_2)$. Since the algebraic degree of u is equal to m, i.e., $m = n$, we get from (B1) that

$$A \sim |w|^{-2(n-1)} \quad \text{for } |w| \gg 1. \tag{B3}$$

Therefore, the function A cannot be constant for every value of $n \geq 2$. Since q is symmetric by exchange $p_1 \leftrightarrow p_2$, the most general rational map u that can be considered with $n = 1$ is given by

$$p_1 = \beta(w - a) \quad \text{and} \quad p_2 = (w - b)^l, \tag{B4}$$

where $l = 0$, 1, the parameters a, b, β are complex numbers, and $\beta \neq 0$. In addition, since p_1 and p_2 do not have common roots, so $a \neq b$ for $l = 1$. The quantity $A(w, \bar{w})$ of (B1) is the ratio between two polynomials in w and $\bar{w}$, with crossed terms. So, since A is constant we can write (B1) as a polynomial equation and we must consider two distinct cases:

(1) The rational map $u(w) = p_1(w)/p_2(w)$ constructed by the relation (B4) with $l = 0$. It then follows from (B1) and (B4) that A will be constant if and only if the following polynomial equation is satisfied:

$$1 + w\bar{w} = A(1 + |\beta|^2|a|^2 + |\beta|^2(w\bar{w} - a\bar{w} - \bar{a}w)). \tag{B5}$$

Note that the quantities w and $\bar{w}$ are independent. It results from (B5) that $a = 0$, $|\beta| = 1$. and $A = 1$, and therefore

$$u = e^{i\alpha}w, \quad \forall\ \alpha \in [0, 2\pi), \tag{B6}$$

with the phase α being constant in the physical space. Such a phase was already expected since the function (4.13) has a $U(1)$ global symmetry due to its invariance by the transformations $u \to e^{i\alpha}u$ and $\bar{u} \to e^{-i\alpha}\bar{u}$. Note that for the rational map (B6) the Wronskian is the phase itself, i.e., $W = e^{i\alpha}$, and therefore satisfies the condition (B2). In addition, due to the definition (B1), the rational map (B6) implies $q = 1$.

(2) The rational map $u(w) = p_1(w)/p_2(w)$ constructed by the relation (B4) with $l = 1$. It then follows from (B1) and (B4) that A will be constant if and only if the following polynomial equation is satisfied:

$$\begin{aligned} 1 + w\bar{w} = A[&(|b|^2 + |\beta|^2|a|^2) + (1 + |\beta|^2)w\bar{w} \\ &- (b + a|\beta|^2)\bar{w} - (\bar{b} + \bar{a}|\beta|^2)w], \end{aligned} \tag{B7}$$

where $a \neq b$, which can be written also as

$$\begin{aligned} 1 &= A(1 + |\beta|^2) \\ 0 &= |\beta|^2 a + b \end{aligned} \tag{B8}$$

$$1 = A(|\beta|^2|a|^2 + |b|^2). \tag{B9}$$

Note that since a, b, β are complex numbers and A is a real number, then the system of algebraic equations (B8) has only four real equations for fixing seven real variables. The first and second lines of (B8) lead to $A = \frac{1}{1+|\beta|^2}$ and $b = -a|\beta|^2$, respectively. Using such relations, the third line of (B8) leads to $|a| = |\beta|^{-1}$, or equivalently $|b| = |\beta|$. Writing a in the polar form $a = |\beta|^{-1}e^{i\alpha}$ we have that

$$p_1 = \beta(w - |\beta|^{-1}e^{i\alpha});\ \ p_2 = w + |\beta|e^{i\alpha};\ \ A = \frac{1}{1+|\beta|^2} \tag{B10}$$

with the phase α being again a constant in the physical space. Note that due to (B10) the polynomials p_1 and p_2 do not have common roots and the Wronskian is a complex constant of nonzero modulus, i.e., $W = \frac{\beta}{|\beta|}e^{i\alpha}(1 + |\beta|^2)$. However, due to (B1) all the rational maps of the form (B10) also lead to $q = 1$.

It is concluded that only the rational maps (B6) and (B10) satisfy the condition $q = \text{const}$, and for such maps this constant is determined and we obtain $q = 1$.

APPENDIX C: THE GRADIENT FLOW METHOD APPLIED TO MINIMIZE THE STATIC ENERGY (4.35)

In this appendix we will discuss the numerical method used in Sec. IV C. We use the gradient flow method with adaptive step size to minimize the static energy (4.41) and to get the solutions of the equations of motion (4.37)–(4.39) with $Q = 1$. The range of r considered is $[0, r_{\max}]$, where the value of $r_{\max}$, and so the size of the lattice, can depend of σ_1 and σ_2 and is chosen to ensure that $0 < \varphi_a(r_{\max}) < 8 \times 10^{-7}$, $f(r_{\max}) < 4 \times 10^{-8}$. The interval between neighboring points is $\Delta r = 0.005$ and the grid has $p = r_{\max}/\Delta r$ points parametrized by an integer k, where we replace $r \to \Delta r k$.

We use the discrete version of the equations of motion (4.37)–(4.39) and the energies (4.42), where the first and second derivatives are given by the central formula of fourth order and the integrals are computed with the trapezoidal rule. In particular, at $k = 1$ and $k = p - 2$ the derivatives are calculated with the central formula of second order and at $k = p - 1$ we use the first order backward difference formula.[2]

The gradient flow method will start with a field configuration with finite energy, called seed configuration, chosen as the discrete version of the self-dual configuration

$$\begin{aligned} &f(\zeta) = 8\ \arctan(e^{-\zeta}); \qquad \varphi_1(\zeta) = -f'; \\ &\varphi_3(\zeta) = \hat{\varphi}_3(\zeta) = -\frac{4\sin^2(f/2)}{\zeta^2 f'} \end{aligned} \tag{C1}$$

that satisfies (4.45), which will have its fields successively modified on each discrete point $j = 2, \ldots, p - 1$ and $k = 1, \ldots, p - 1$ of the grid by

$$\begin{aligned} &f_j \to f_j - \Delta\alpha_f \Delta E_{f,j}; \\ &\varphi_{b,k} \to \varphi_{b,k} - \Delta\alpha_{\varphi_b} \Delta E_{\varphi_b,k}; \quad b = 1, 3 \end{aligned} \tag{C2}$$

where $\Delta\alpha_a$ represent the finite step size. The field values f_1, $\varphi_{1,0}$, and $\varphi_{3,0}$ can be estimated from the neighbors points, once (C2) is done, using

[2]Because of the Jacobian term r^2 only the kinetic term of h gives a nontrivial contribution to the energy at $k = 0$, given by $2\frac{r_{\max}}{p}(\varphi_{1,0} - \varphi_{3,0})^2$, which by (4.45) can only be nonzero for a field configuration that is not a static solution of Eqs. (4.37)–(4.39). Therefore, we do not need to compute any derivative at $k = 0$.

$$f_1 = \frac{f_0 + f_2}{2}; \qquad \varphi_{a,0} = \varphi_{a,4} - 2(\varphi_{a,3} - \varphi_{a,1}) \tag{C3}$$

The first equality of (C3) is obtained using $f''(0) = 0$ where the second derivative is given by the second order forward formula. The second equality of (C3) is obtained taking the equality between the central formula of second and fourth order to the first derivative.[3] The method ends when the maximum value of each $|\Delta E_f|, |\Delta E_{\varphi_1}|$, and $|\Delta E_{\varphi_3}|$ on the grid, restricted to the points considered in (C2), is smaller than the Derrick $< 4 \times 10^{-4}$.

Once we obtain the solutions of (4.37)–(4.39), where the fields $f(r)$ and $\varphi_a(r)$ are monotonic, we can compute the thickness t_f, defined as the value of r that satisfies $f(r) = f(0)/2$. First, we get the value r_p, which is the numerical value of r on the grid that minimizes the function $|f(0)/2 - f(r)|$, and so by definition $|t_f - r_p| \leq \Delta r$. So, the Taylor expansion of $f(r)$ at r_p valued for $t = t_f$ becomes $f(t_f) = f(r_p) + f'(r_p)(t_f - r_p) + \mathcal{O}((\Delta r)^2)$ and then

$$t_f = r_p + \frac{f(0)/2 - f(r_p)}{f'(r_p)} + \mathcal{O}((\Delta r)^2). \tag{C4}$$

We use (C4) to compute t_f in first order of Δr, and the same follows for getting the thickness of the φ_a fields. In Table II of Sec. IV C we present the values of the thickness for each of the fields and for each of the values of σ_1 and σ_2 of Table I.

[3]Equation (4.37) can be very sensible at $k = 1$, where the derivatives are calculated by the second-order central formula, in the sense that small variations in f can lead to large variations of ΔE_f, which in turn may be at odds with its neighboring values in the grid. For example, consider the particular case $\sigma_2 = 0$ and take a self-dual field configuration that satisfies (4.37) as a seed configuration, such as (C1), i.e., for which we have exactly $\Delta E_f = 0$. The numerical value of $\Delta E_{f,1}$, obtained without replacing (C1) in (4.37), can become non-negligible, differing significantly from $\Delta E_{f,2}$. So, it may be preferable to use the expression (C3) to calculate f_1 instead the central formula. In addition, since ΔE_f can be very sensitive by small variations of f, a very small value of the step size $\Delta\alpha_f$ is used. Finally, the adaptive step size $\Delta\alpha_f$ decreases and the field configuration update is not accepted when the energy (4.41) grows or $\Delta_f E$ deforms drastically, which can avoid discontinuity problems in the function ΔE_f.

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