Quantum kinetic theory with vector and axial gauge fields

Zhou Chen^{*} and Shu Lin[†]

School of Physics and Astronomy, Sun Yat-Sen University, Zhuhai 519082, China

(Received 27 September 2021; accepted 6 January 2022; published 18 January 2022)

In this paper we introduce the axial gauge field to the framework of the quantum kinetic theory with vector gauge field in the massless limit. Treating the axial gauge field on an equal footing with the vector gauge field, we construct a consistent solution to the kinetic equations up to the first order in gradient expansion, or equivalently the semiclassical expansion. The intuitive extension of quantum kinetic theory presented in this work provides a natural generalization, and turns out to give rise to the covariant anomaly and the covariant currents. The corresponding consistent currents can be obtained from the covariant ones by adding the Chern-Simons current. We use the consistent currents to calculate various correlation functions among currents and energy-momentum tensor in equilibrium state.

DOI: [10.1103/PhysRevD.105.014015](https://doi.org/10.1103/PhysRevD.105.014015)

I. INTRODUCTION

The quantum kinetic theories for spinning particles has received much attention over the past few years. The most prominent one is quantum kinetic theory for spin-1/2 particle, which has been widely used in the studies of spin-sensitive transports. Its massless limit, chiral kinetic theory (CKT) [1–18], has provided a novel description of the celebrated chiral magnetic effect [19–21] and chiral vortical effect [22–25]. More recently, generalization to the massive case, axial kinetic theory (AKT) [26–30], has revealed additional degrees of freedom not present in CKT. The inclusion of mass is a crucial step towards a realistic description of particle polarization. The axial kinetic theory has been applied to physics of spin polarization in heavy ion collisions (see [31] for a review). Collisional effects have been studied in [5,32–44]. Quantum kinetic theory for particles with other spins have been also constructed [45–47].

The quantum kinetic theory also offers a way to derive hydrodynamics from a microscopic theory. A key element of hydrodynamics is the response of spin-1/2 particles to external sources, including vector gauge field and torsionful metric. These allow for derivation of anomalous hydrodynamics [15,48–51], magnetohydrodynamics [17], and

spin hydrodynamics [37,52–54].¹ In fact, a more complete anomalous hydrodynamics also include response to axial gauge field [24,59]. It is desirable to extend the present framework of quantum kinetic theory to incorporate axial gauge fields as well. The purpose of introducing the axial gauge field is twofold. On one hand, while the axial gauge field is not a physical gauge field, it can be mimicked in system like Weyl semimetal leading to physical effects [60]. On the other hand, the axial gauge field can also serve as a convenient tool for deriving correlation functions of axial current. In the same spirit, vierbein and spin connection in torsionful metric allows for the derivation of correlation functions of the energy-momentum tensor and spin tensor [58]. Since the spin tensor and axial current are related, our approach can be an alternative to the theory with torsionful metric.

This study aims to integrate the axial gauge field in the framework of quantum kinetic theory. As we shall see, treating the vector and axial gauge fields on the same footing (equivalent to treating left-handed and right-handed fermions symmetrically) gives rise to the covariant current, for which neither the axial current nor the vector one is conserved. As is well known in quantum field theory, to guarantee the conservation of vector current, the consistent currents can be introduced by adding the Chern-Simons term to the covariant ones (see [61] for a review). Treating the axial gauge field as a source, we can utilize the consistent currents to calculate various correlation functions among currents and energy-momentum tensor. We will illustrate how to calculate

^{*}chenzh339@mail2.sysu.edu.cn

[†]linshu8@mail.sysu.edu.cn

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

¹The derivation based on responses works for more general systems where kinetic description might not apply (see [55–58] and references therein).

correlation functions with simple examples. A similar approach was taken in the physics of the Weyl semimetal, in which the axial gauge field can be induced and consistent current has been chosen as the physical current [62–65].

This paper is structured as follows. In Sec. II, we derive chiral kinetic theory with a vector/axial gauge field in the collisionless limit. In Sec. III, we present the solution of the kinetic equation up to first order in gradient. Section IV is devoted to the calculation of one-point functions, which give rise to anomalous transports and multipoint correlation functions. We summarize and provide an outlook in Sec. V. Details of calculations is left to the two appendixes.

II. QUANTUM KINETIC EQUATIONS WITH VECTOR/AXIAL GAUGE FIELDS

With both the vector and axial gauge fields, the chiral fermion Lagrangian with the gauge fields as backgrounds is given by

$$\mathcal{L} = i\bar{\psi}\not{D}\psi. \quad (1)$$

$D_\mu = \partial_\mu + iA_\mu + i\gamma^5 A_\mu^5$ is the extended covariant derivative, with A_μ and A_μ^5 being the vector and axial gauge potential, respectively. The coupling constants have been absorbed into the corresponding gauge potentials. In the current work, we restrict ourselves to the collisionless limit. The kinetic equation is formulated in terms of the Wigner function defined as

$$\begin{aligned} \bar{S}_{\alpha\beta}^<(x, y) &= \langle \bar{\psi}_\beta(y) \psi_\alpha(x) \rangle, \\ S_{\alpha\beta}^<(p, X) &= \int d^4s e^{ip \cdot s} \bar{S}_{\alpha\beta}^<(x, y), \end{aligned} \quad (2)$$

with $X = \frac{x+y}{2}$ and $s = x - y$. $\bar{S}_{\alpha\beta}^<(x, y)$ satisfies the Dirac equation

$$D_x \bar{S}_{\alpha\beta}^<(x, y) = 0. \quad (3)$$

Assuming constant field strength for the vector and axial fields, we can Fourier transform Eq. (3) to obtain the equation of motion of $S_{\alpha\beta}^<(p, X)$

$$\gamma_\mu \left(\mathcal{K}_\mu + \frac{i}{2} \mathcal{D}^\mu + \gamma^5 \Pi^\mu \right) S^<(p, X) = 0, \quad (4)$$

where $\mathcal{K}_\mu = p_\mu - A_\mu$, $\mathcal{D}_\mu = \partial_{X,\mu} + (\partial_{X,\nu} A_\mu) \partial_p^\nu$, and $\Pi_\mu = A_\mu^5 + \frac{i}{2} (\partial_{X,\nu} A_\mu^5) \partial_p^\nu$. $\mathcal{D}_\mu$ and Π_μ are organized as a gradient expansion in ∂_X ,² with $A_\mu, A_\mu^5 \sim O(\partial_X^0)$. To avoid the subtleties of the axial gauge symmetry, we proceed by

solving Eq. (4) directly without defining any gauge link. It turns out that the gauge-linked Wigner function can be obtained from the bare one by replacing the canonical momentum p_μ by the kinetic momentum $k_\mu = p_\mu - A_\mu \mp A_\mu^5$ corresponding to right-handed and left-handed components, respectively. It follows that the resulting solution is invariant under both vector and axial gauge symmetry, leading to covariant vector and axial currents.

The Wigner function $S^<(p, X)$, satisfying the “Hermitian” condition $S^<(p, X) = \gamma^0 S^<(p, X)^\dagger \gamma^0$, can be decomposed in terms of 16 independent generators of the Clifford algebra,

$$S^<(p, X) = \frac{1}{4} \left[\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{\sigma^{\mu\nu}}{2} \mathcal{S}_{\mu\nu} \right], \quad (5)$$

with the real coefficients $\mathcal{F}$, $\mathcal{P}$, $\mathcal{V}_\mu$, $\mathcal{A}_\mu$, and $\mathcal{S}_{\mu\nu}$ being its scalar, pseudoscalar, vector, axial-vector, and tensor components, respectively. To lighten the notation, the phase-space coordinates (p, X) of the Clifford-algebra coefficients have been omitted. We can further derive

$$\begin{aligned} \gamma^\mu S^<(p, X) &= \frac{1}{4} \left[\mathcal{V}^\mu - \gamma^5 \mathcal{A}^\mu + \gamma_\nu (g^{\mu\nu} \mathcal{F} + i\mathcal{S}^{\mu\nu}) \right. \\ &\quad \left. + \gamma^5 \gamma_\nu (\tilde{\mathcal{S}}^{\mu\nu} - i g^{\mu\nu} \mathcal{P}) \right. \\ &\quad \left. - \frac{\sigma_{\alpha\beta}}{2} (\epsilon^{\mu\nu\alpha\beta} \mathcal{A}_\nu + i(g^{\mu\alpha} \mathcal{V}^\beta - g^{\mu\beta} \mathcal{V}^\alpha)) \right], \end{aligned} \quad (6)$$

$$\begin{aligned} \gamma^5 \gamma^\mu S^<(p, X) &= \frac{1}{4} \left[-\mathcal{A}^\mu + \gamma^5 \mathcal{V}^\mu + \gamma_\nu (\tilde{\mathcal{S}}^{\mu\nu} - i g^{\mu\nu} \mathcal{P}) \right. \\ &\quad \left. + \gamma^5 \gamma_\nu (g^{\mu\nu} \mathcal{F} + i\mathcal{S}^{\mu\nu}) \right. \\ &\quad \left. + \frac{\sigma_{\alpha\beta}}{2} (\epsilon^{\mu\nu\alpha\beta} \mathcal{V}_\nu + i(g^{\mu\alpha} \mathcal{A}^\beta - g^{\mu\beta} \mathcal{A}^\alpha)) \right], \end{aligned} \quad (7)$$

where we have used the dual tensor $\tilde{\mathcal{S}}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \mathcal{S}_{\rho\sigma}$ and the following identities,

$$\begin{aligned} \gamma^\mu \gamma^\alpha \gamma^\beta &= g^{\mu\alpha} \gamma^\beta - g^{\mu\beta} \gamma^\alpha + g^{\alpha\beta} \gamma^\mu - i\epsilon^{\mu\alpha\beta\lambda} \gamma^5 \gamma_\lambda, \\ \gamma^5 \sigma_{\mu\nu} &= \frac{i}{2} \epsilon_{\mu\nu\kappa\lambda} \sigma^{\kappa\lambda}. \end{aligned} \quad (8)$$

Inserting Eqs. (6) and (7) into Eq. (4), and comparing the real and imaginary parts of the coefficients in the Clifford-algebra basis, we will obtain two sets of equations. We see that the equations for $\mathcal{A}_\mu$ and $\mathcal{V}_\mu$ are decoupled from other components,

$$\begin{aligned} \mathcal{K}_\mu \mathcal{V}^\mu - A_\mu^5 \mathcal{A}^\mu &= 0, \\ \mathcal{D}_\mu \mathcal{A}^\mu + (\partial_{X,\lambda} A_\mu^5) \partial_p^\lambda \mathcal{V}^\mu &= 0, \\ \mathcal{K}_\mu \mathcal{A}_\nu - \mathcal{K}_\nu \mathcal{A}_\mu - A_\mu^5 \mathcal{V}_\nu + A_\nu^5 \mathcal{V}_\mu &+ \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathcal{D}^\alpha \mathcal{V}^\beta + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial_\lambda A_\alpha^5) \partial_p^\lambda \mathcal{A}^\beta = 0, \end{aligned} \quad (9)$$

²The expansion in ∂_X is equivalent to expansion in $\hbar$.

$$\begin{aligned}
\mathcal{K}_\mu \mathcal{A}^\mu - A_\mu^5 \mathcal{V}^\mu &= 0, \\
\mathcal{D}_\mu \mathcal{V}^\mu + (\partial_{X,\lambda} A_\mu^5) \partial_p^\lambda \mathcal{A}^\mu &= 0, \\
\mathcal{K}_\mu \mathcal{V}_\nu - \mathcal{K}_\nu \mathcal{V}_\mu - A_\mu^5 \mathcal{A}_\nu + A_\nu^5 \mathcal{A}_\mu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathcal{D}^\alpha \mathcal{A}^\beta + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial_\lambda A_\alpha^5) \partial_p^\lambda \mathcal{V}^\beta &= 0.
\end{aligned} \tag{10}$$

The equations can be further decoupled in the chiral basis

$$k_\mu^s \mathcal{J}_s^\mu = 0, \tag{11}$$

$$\nabla_\mu^s \mathcal{J}_s^\mu = 0, \tag{12}$$

$$k_s^\mu \mathcal{J}_s^\nu - k_s^\nu \mathcal{J}_s^\mu + \frac{s}{2} \epsilon^{\mu\nu\alpha\beta} \nabla_\alpha^s \mathcal{J}_s^\beta = 0, \tag{13}$$

where the chiral components $\mathcal{J}_s^\mu$ of the Wigner function are defined as

$$\mathcal{J}_s^\mu = \frac{1}{2} (\mathcal{V}^\mu + s \mathcal{A}^\mu), \tag{14}$$

with $s = +$ and $s = -$ for right-handed and left-handed fermions, respectively. $k_\mu^s = p_\mu - A_\mu^s$ is the kinetic momentum with the gauge potential in the chiral basis being $A_\mu^s = A_\mu + s A_\mu^5$. $\nabla_\mu^s = \partial_\mu + (\partial_\nu A_\mu^s) \partial_p^\nu$ is the covariant derivative at $O(\partial_X)$. Note that the kinetic momenta and covariant derivatives differ for the right-handed and left-handed components in the presence of the axial gauge field.

III. SOLUTION TO THE KINETIC EQUATIONS

We solve $\mathcal{J}_\mu$ by gradient expansion up to first order,

$$\mathcal{J}_\mu^s = \mathcal{J}_\mu^{s(0)} + \mathcal{J}_\mu^{s(1)} + \dots, \tag{15}$$

where the superscripts (0), (1), ..., denote the orders in the expansion. Substituting this expansion into Eqs. (11)–(13) and requiring that the equations hold order by order, the equations for $\mathcal{J}_\mu^{s(n)}$ with $n = 1$ and $n = 0$ read

$$k_\mu^s \mathcal{J}_s^{s(n)\mu} = 0, \tag{16}$$

$$\nabla_\mu^s \mathcal{J}_s^{s(n)\mu} = 0, \tag{17}$$

$$k_\mu^s \mathcal{J}_s^{s(n)\nu} - k_\nu^s \mathcal{J}_s^{s(n)\mu} + \frac{s}{2} \epsilon_{\mu\nu\alpha\beta} \nabla_\alpha^s \mathcal{J}_s^{s(n-1)\beta} = 0, \tag{18}$$

where we have defined $\mathcal{J}_\mu^{s(-1)} = 0$. Contracting both sides of Eq. (18) with k_s^μ and using Eq. (16), we have

$$k_s^2 \mathcal{J}_\mu^{s(n)} = \frac{s}{2} \epsilon_{\mu\nu\alpha\beta} k_s^\nu \nabla_s^\alpha \mathcal{J}_s^{s(n-1)\beta}. \tag{19}$$

The solution of Eq. (19) implies a general form of $\mathcal{J}_\mu^{s(n)}$,

$$\mathcal{J}_\mu^{s(n)} = J_\mu^{s(n)} \delta(k_s^2) + \frac{s}{2k_s^2} \epsilon_{\mu\nu\alpha\beta} k_s^\nu \nabla_s^\alpha \mathcal{J}_s^{s(n-1)\beta}, \tag{20}$$

where $J_\mu^{s(n)}$ is nonsingular at $k_s^2 = 0$, and can be constrained by substituting Eq. (20) back into Eq. (18). From Eq. (16), we can also obtain a further constraint for $J_\mu^{s(n)}$, $k_s^\mu J_\mu^{s(n)} \delta(k_s^2) = 0$.

When $n = 0$, the equation of motion depend on kinetic momentum only. It is straightforward to write down the solution,

$$\mathcal{J}_\mu^{s(0)} = k_\mu^s \delta(k_s^2) f_s(k_s, X). \tag{21}$$

To be specific, in this paper we choose the zeroth-order distribution function to be a Fermi-Dirac distribution in the kinetic momentum

$$\begin{aligned}
f_s(k_s, X) &= \frac{2}{(2\pi)^3} [\theta(u \cdot k_s) f_{FD}(\beta \cdot k_s - \bar{\mu}_s) \\
&\quad + \theta(-u \cdot k_s) f_{FD}(-\beta \cdot k_s + \bar{\mu}_s)],
\end{aligned} \tag{22}$$

where the Fermi-Dirac distribution function $f_{FD}(z) \equiv [\exp(z) + 1]^{-1}$, $\beta^\mu \equiv \beta u^\mu$, with $\beta = \beta^\mu u_\mu \equiv 1/T$ being the inverse temperature and u^μ being the fluid velocity, and $\bar{\mu}_s = \beta \mu_s$ with $\mu_s = \mu + s \mu_5$ being the chemical potential for chirality $s = \pm 1$. The hydrodynamic quantities β^μ , β , and $\bar{\mu}_s$ can be dependent on X .

Using the definitions of k_μ^s and ∇_μ^s given in Sec. II, we can derive the following useful identities,

$$\begin{aligned}
\nabla_\mu^s k_\nu^s &= -F_{\mu\nu}^s, \\
\nabla_\mu^s \delta^{(n)}(k_s^2) &= -2F_{\mu\nu}^s k_s^\nu \delta^{(n+1)}(k_s^2),
\end{aligned} \tag{23}$$

where we have defined $F_{\mu\nu}^s \equiv \partial_\mu A_\nu^s - \partial_\nu A_\mu^s = F_{\mu\nu} + s F_{\mu\nu}^5$ and $\delta^{(n)}(k^2) = (\frac{d}{dk^2})^n \delta(k^2)$. Especially, it follows that $\nabla_\mu^s [k_s^\mu \delta(k_s^2)] = 0$. Using this identity, it is easy to show that

$$\begin{aligned}
\nabla_\mu^s \mathcal{J}_s^{(0)\mu} &= \nabla_\mu [k_s^\mu \delta(k_s^2) f_s(X, k_s)] \\
&= \delta(k_s^2) k_s^\mu \nabla_\mu f_s(X, k_s).
\end{aligned} \tag{24}$$

It implies the constraint equation $k_s^\mu \nabla_{s,\mu} f_s = 0$ in order for Eq. (17) to hold. With the distribution function given by Eq. (22), we can evaluate $k_s^\mu \nabla_{s,\mu} f_s = 0$ as

$$\begin{aligned}
k_s^\mu \nabla_\mu^s f_s &= \frac{\partial f_s}{\partial(\beta \cdot k_s)} k_s^\mu [\nabla_\mu(\beta \cdot k_s) - \nabla_\mu^s \bar{\mu}_s] \\
&= \frac{\partial f_s}{\partial(\beta \cdot k_s)} k_s^\mu [k_s^\nu \partial_\mu \beta_\nu - \partial_\mu \bar{\mu}_s - F_{\mu\nu}^s \beta^\nu]. \quad (25)
\end{aligned}$$

Accordingly, we deduce the conditions for the chiral systems to be in global equilibrium, that is

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad (26)$$

$$\partial_\mu \bar{\mu}_s = -F_{\mu\nu}^s \beta^\nu. \quad (27)$$

Equation (26) is the Killing condition for β_μ which leads to the solution $\beta_\mu = b_\mu - \Omega_{\mu\nu} x^\nu$ with b_μ and the thermal vorticity $\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$ being constants. Equation (27) takes a more familiar form in the original basis. With $\bar{\mu} = \frac{1}{2}(\bar{\mu}_+ + \bar{\mu}_-)$ and $\bar{\mu}_5 = \frac{1}{2}(\bar{\mu}_+ - \bar{\mu}_-)$, we have

$$\partial_\mu \bar{\mu} = -F_{\mu\nu} \beta^\nu, \quad (28)$$

$$\partial_\mu \bar{\mu}_5 = -F_{\mu\nu}^5 \beta^\nu. \quad (29)$$

Equation (28) implies the electric field is balanced by gradient of chemical potential. Equation (29) is just the axial counterpart of Eq. (28).

With the help of the conditions given by Eqs. (26) and (27), we can derive that

$$\nabla_\mu f_s = f_s' \Omega_{\mu\nu} k_s^\nu, \quad (30)$$

where $f_s' \equiv \frac{\partial f_s}{\partial(\beta \cdot k_s)}$. Substituting the zeroth-order solution Eq. (21) into Eq. (20) gives the first order solution ($n = 1$),

$$\begin{aligned}
\mathcal{J}_\mu^{s(1)} &= J_\mu^{s(1)} \delta(k_s^2) + \frac{s}{2k_s^2} \epsilon_{\mu\nu\alpha\beta} k_s^\nu \nabla_s^\alpha \mathcal{J}^{s(0)\beta} \\
&= J_\mu^{s(1)} \delta(k_s^2) + s \tilde{F}_{\mu\nu}^s k_s^\nu f_s \delta'(k_s^2), \quad (31)
\end{aligned}$$

where we have used $\tilde{F}_{\mu\nu}^s = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_s^{\alpha\beta}$ and $\delta'(k_s^2) = -\delta(k_s^2)/k_s^2$. Then, substituting Eqs. (21) and (31) into Eq. (18) we arrive at

$$\begin{aligned}
k_\mu^s \left(J_\nu^{s(1)} \delta(k_s^2) + s \tilde{F}_{\nu\rho}^s k_s^\rho f_s \delta'(k_s^2) \right) - [\mu \leftrightarrow \nu] \\
= -\frac{s}{2} \epsilon_{\mu\nu\alpha\rho} \nabla_s^\alpha [k_s^\rho f_s \delta(k_s^2)]. \quad (32)
\end{aligned}$$

It help us to determine $J_\mu^{s(1)} = -\frac{s}{2} \tilde{\Omega}_{\mu\nu} k_s^\nu f_s'$. Details of the determination are presented in Appendix A. We summarize the solution of the Wigner function up to the first order

$$\mathcal{J}_\mu^s = k_s^\mu \delta(k_s^2) f_s - \frac{s}{2} \tilde{\Omega}_{\mu\nu} k_s^\nu \delta(k_s^2) f_s' + s \tilde{F}_{\mu\nu}^s k_s^\nu \delta'(k_s^2) f_s. \quad (33)$$

Equation (33) generalizes the known solution to the case with axial gauge fields [15]. In fact, we can show they are equivalent upon proper identification. In the absence of axial gauge field, the vector gauge link is inserted as

$$\begin{aligned}
\bar{S}_{\text{link}}^<(x, y) &= \bar{S}^<(x, y) e^{-i \int_y^x dz \cdot A(z)} \\
&= \bar{S}^<(x, y) e^{-i s \cdot A(X)} + O(\partial_X^2). \quad (34)
\end{aligned}$$

We have used the subscript link to indicate that the corresponding quantities have explicit gauge link insertion. The vector gauge invariant $\bar{S}_{\text{link}}^<(x, y)$ is then Wigner transformed with kinetic momentum

$$S_{\text{link}}^<(k, X) = \int d^4 s e^{i k \cdot s} \bar{S}_{\text{link}}^<(x, y), \quad (35)$$

generating the gauge-invariant observables from components of the Wigner function. Importantly the momentum appearing in Eq. (35) should be the gauge-invariant kinetic momentum k . We choose to work with the bare Wigner function, which is then transformed with canonical momentum p . From Eq. (34), we can easily obtain

$$\begin{aligned}
e^{i k \cdot s} \bar{S}_{\text{link}}^<(x, y) &= e^{i p \cdot s} \bar{S}^<(x, y) \\
\Rightarrow S_{\text{link}}^<(k, X) &= S^<(p, X). \quad (36)
\end{aligned}$$

It shows that up to $O(\partial_X)$, our bare Wigner function is also vector gauge invariant; the gauge dependence in our bare Wigner function is canceled by the gauge dependence in the canonical momentum in the Wigner transform. Therefore, components of our $S^<(p, X)$ is also gauge invariant, whose momentum integration give rise to physical observables. When the axial gauge field is present, it is straightforward to deduce $S^<(p, X)$ is invariant under both vector and axial gauges. In fact, we may equivalently work with Wigner functions for right-handed and left-handed fermions with appropriate gauge link insertion

$$\begin{aligned}
\bar{S}_s(x, y) &= \langle \psi_s(x) \psi_s^\dagger(y) \rangle, \\
\bar{S}_{s,\text{link}}(x, y) &= \bar{S}_s(x, y) e^{-i \int_y^x dz \cdot A_s(z)}, \quad (37)
\end{aligned}$$

with $\psi_+ \equiv \psi_R$ and $\psi_- \equiv \psi_L$ standing for right-handed and left-handed fermions, respectively.

IV. ANOMALOUS TRANSPORTS AND CORRELATION FUNCTIONS

Integrating Eq. (33) over the kinetic momentum k_s , we obtain the left-handed and right-handed currents $j_s^\mu(X)$ up to the first order in ∂_X

$$j_s^\mu(X) = \int d^4 k_s \mathcal{J}_s^\mu(k_s, X). \quad (38)$$

Both the solution Eq. (33) and the integration measure are vector/axial gauge invariant, and it follows that the resulting currents are gauge invariant.

After the four-momentum integrations, we have the zeroth-order contribution given by $\mathcal{J}_s^{(0)\mu}$, and the first-order contribution given by $\mathcal{J}_s^{(1)\mu}$,

$$j_s^{(0)\mu} = n_s u^\mu, \quad (39)$$

$$j_s^{(1)\mu} = \xi_s \omega^\mu + \xi_{Bs} B_s^\mu, \quad (40)$$

with n_s being the fermion number density, ξ_s and ξ_{Bs} related to transport coefficients of chiral vortical effect (CVE), chiral magnetic effect (CME), and chiral separation effect (CSE). The vorticity and magnetic field are defined as $\omega^\mu = T\tilde{\Omega}^{\mu\nu}u_\nu$ and $B_s^\mu = \tilde{F}_s^{\mu\nu}u_\nu$ respectively, according to the decomposition

$$\begin{aligned} \tilde{F}_{\mu\nu}^s &= B_\mu^s u_\nu - B_\nu^s u_\mu - \epsilon_{\mu\nu\alpha\beta} u^\alpha E_\beta^s, \\ F_{\mu\nu}^s &= E_\mu^s u_\nu - E_\nu^s u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha B_\beta^s, \end{aligned} \quad (41)$$

$$\begin{aligned} T\tilde{\Omega}_{\mu\nu} &= \omega_\mu u_\nu - \omega_\nu u_\mu - \epsilon_{\mu\nu\alpha\beta} u^\alpha \varepsilon_\beta, \\ T\Omega_{\mu\nu} &= \varepsilon_\mu u_\nu - \varepsilon_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \omega_\beta. \end{aligned} \quad (42)$$

The coefficients n_s , ξ_s , and ξ_{Bs} are given by

$$n_s = \frac{\mu_s}{6\pi^2} \left(\frac{\pi^2}{\beta^2} + \mu_s^2 \right), \quad (43)$$

$$\xi_s = \frac{s}{12\pi^2} \left(\frac{\pi^2}{\beta^2} + 3\mu_s^2 \right), \quad (44)$$

$$\xi_{Bs} = \frac{s}{4\pi^2} \mu_s. \quad (45)$$

The vector and axial currents can be obtained from the linear combinations of $j_\pm^\mu$

$$j^\mu = j_+^\mu + j_-^\mu, \quad j_5^\mu = j_+^\mu - j_-^\mu. \quad (46)$$

Then the zeroth-order vector and axial currents read

$$j^{(0)\mu} = n u^\mu, \quad (47)$$

$$j_5^{(0)\mu} = n_5 u^\mu. \quad (48)$$

From the first-order currents, we can obtain the vector and axial currents in the CVE

$$j_\omega^\mu = \xi \omega^\mu, \quad (49)$$

$$j_{5,\omega}^\mu = \xi_5 \omega^\mu, \quad (50)$$

while the vector and axial currents in the CME are

$$j_B^\mu = \xi_B B^\mu + \xi_{B5} B_5^\mu, \quad (51)$$

$$j_{5,B}^\mu = \xi_{B5} B^\mu + \xi_B B_5^\mu, \quad (52)$$

where

$$\begin{aligned} n &= \frac{\mu}{3\pi^2} \left(\frac{\pi^2}{\beta^2} + \mu^2 + 3\mu_5^2 \right), \quad n_5 = \frac{\mu_5}{3\pi^2} \left(\frac{\pi^2}{\beta^2} + 3\mu^2 + \mu_5^2 \right), \\ \xi &= \frac{\mu\mu_5}{\pi^2}, \quad \xi_5 = \frac{1}{6\beta^2} + \frac{\mu^2}{2\pi^2} + \frac{\mu_5^2}{2\pi^2}, \quad \xi_B = \frac{\mu_5}{2\pi^2}, \\ \xi_{B5} &= \frac{\mu}{2\pi^2}. \end{aligned} \quad (53)$$

The gauge invariant canonical energy-momentum tensor can be obtained by

$$\begin{aligned} T^{\mu\nu} &= \int d^4 k_+ \mathcal{T}_+^\mu(k_+, X) k_+^\nu + \int d^4 k_- \mathcal{T}_-^\mu(k_-, X) k_-^\nu \\ &\equiv T_+^{\mu\nu} + T_-^{\mu\nu}. \end{aligned} \quad (54)$$

Then we can perform the four-momentum integrals to obtain

$$T_s^{(0)\mu\nu} = \left(u^\mu u^\nu - \frac{1}{3} \Delta^{\mu\nu} \right) \rho_s, \quad (55)$$

$$\begin{aligned} T_s^{(1)\mu\nu} &= s n_s (u^\mu \omega^\nu + u^\nu \omega^\mu) \\ &\quad + \frac{\xi_s}{2} (u^\mu B_s^\nu + u^\nu B_s^\mu - \epsilon^{\mu\nu\alpha\beta} u_\alpha E_\beta^s) \\ &\quad - \frac{s n_s}{2} (u^\mu \omega^\nu - u^\nu \omega^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha \varepsilon_\beta), \end{aligned} \quad (56)$$

with

$$\rho_s = \frac{7}{120} \frac{\pi^2}{\beta^4} + \frac{1}{4} \frac{\mu_s^2}{\beta^2} + \frac{1}{8\pi^2} \mu_s^4. \quad (57)$$

Equations (56) and (57) generalize the first-order results in [15] to the case with axial gauge field. We can further separate the symmetric and antisymmetric parts of the canonical energy-momentum tensor as³

$$T^{(0)\{\mu\nu\}} = \left(u^\mu u^\nu - \frac{1}{3} \Delta^{\mu\nu} \right) \rho, \quad (58)$$

$$\begin{aligned} T^{(1)\{\mu\nu\}} &= n_5 (u^\mu \omega^\nu + u^\nu \omega^\mu) + \frac{\xi}{2} (u^\mu B^\nu + u^\nu B^\mu) \\ &\quad + \frac{\xi_5}{2} (u^\mu B_5^\nu + u^\nu B_5^\mu), \end{aligned} \quad (59)$$

$$\begin{aligned} T^{(1)[\mu\nu]} &= -\frac{\xi}{2} \epsilon^{\mu\nu\alpha\beta} u_\alpha E_\beta - \frac{\xi_5}{2} \epsilon^{\mu\nu\alpha\beta} u_\alpha E_\beta^5 \\ &\quad - \frac{n_5}{2} (u^\mu \omega^\nu - u^\nu \omega^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha \varepsilon_\beta), \end{aligned} \quad (60)$$

³Implicitly we assume that the fluid velocity is not redefined in the presence of external sources including vector/axial gauge fields and vorticity. This chooses a particular hydrodynamic frame, termed thermodynamic frame in literature [66].

where $T^{\{\mu\nu\}} \equiv \frac{T^{\mu\nu} + T^{\nu\mu}}{2}$ and $T^{[\mu\nu]} \equiv \frac{T^{\mu\nu} - T^{\nu\mu}}{2}$. $\rho = \frac{7}{60} \frac{\pi^2}{\beta^4} + \frac{1}{2} \frac{\mu^2 + \mu_5^2}{\beta^2} + \frac{1}{4\pi^2} (\mu^4 + 6\mu^2\mu_5^2 + \mu_5^4)$ is the energy density. The right-hand side of Eq. (59) corresponds to heat flow along vorticity, magnetic, and axial magnetic fields respectively. The first two need chiral imbalance in the fluid to exist while the last one exists in a chirally-symmetric medium. As we shall see shortly, the last one can be related to the axial/chiral vortical effect by Onsager relation. Using the Killing conditions, we show in Appendix B the following conservation equations

$$\partial_\mu j_{\text{cov}}^\mu = \frac{1}{2\pi^2} (\vec{E}_5 \cdot \vec{B} + \vec{E} \cdot \vec{B}_5), \quad (61)$$

$$\partial_\mu j_{5,\text{cov}}^\mu = \frac{1}{2\pi^2} (\vec{E} \cdot \vec{B} + \vec{E}_5 \cdot \vec{B}_5), \quad (62)$$

$$\partial_\mu T^{\{\mu\nu\}} = F^{\nu\mu} j_{\text{cov},\mu} + F_5^{\mu\nu} j_{5,\text{cov},\mu}, \quad (63)$$

$$\partial_\mu T^{[\mu\nu]} = 0, \quad (64)$$

$$T^{[\mu\nu]} = \partial_\lambda S^{\lambda\mu\nu}. \quad (65)$$

Equations (61) and (62) correspond to the covariant anomaly, so we label the corresponding covariant current by the subscript “cov”, see also [62,63] for the derivation in the context of Weyl materials. Equations (63) and (64) are the energy-momentum conservation subject to external force by vector and axial gauge fields. Equation (65) corresponds to change rate of spin tensor $S^{\lambda\mu\nu}$.

We will be mainly interested in correlation functions among vector/axial currents, which are obtainable by functional derivatives with respect to vector/axial gauge potential. For this purpose, we may turn off the vorticity and set $\mu^\mu = (1, 0, 0, 0)$. We also assume constant temperature and vector/axial chemical potential for simplicity, which by the Killing condition, Eq. (27), implies $E_\mu = E_{\mu,5} = 0$. It follows that the antisymmetric part of the energy-momentum tensor vanishes identically.

Now we turn to the calculation of correlation functions. A convenient way to calculate the correlation function is to take functional derivatives of the effective action with respect to vector/axial gauge potential. Note that the effective action needs to satisfy the Wess-Zumino consistency condition, and the resulting current from functional derivative is the consistent one [61]

$$\begin{aligned} \frac{\delta}{\delta A_\mu(x)} e^{i\Gamma[A,A_5]} &= j_{\text{cons}}^\mu(x) e^{i\Gamma[A,A_5]}, \\ \frac{\delta}{\delta A_\mu^5(x)} e^{i\Gamma[A,A_5]} &= j_{5,\text{cons}}^\mu(x) e^{i\Gamma[A,A_5]}, \end{aligned} \quad (66)$$

with $\Gamma[A, A_5]$ being the effective action. Taking multiple derivatives give multipoint correlation functions. We

illustrate this with examples of two-point and three-point functions. Instead of finding the effective action, we start with the one-point function of consistent currents, which are related to the covariant currents in Eqs. (51) and (52) by [61]

$$\begin{aligned} j_{\text{cov}}^\mu &= j_{\text{cons}}^\mu + \frac{1}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} A_{5,\nu} F_{\rho\sigma}, \\ j_{5,\text{cov}}^\mu &= j_{5,\text{cons}}^\mu + \frac{1}{12\pi^2} \epsilon^{\mu\nu\rho\sigma} A_{5,\nu} F_{5\rho\sigma}. \end{aligned} \quad (67)$$

From Eqs. (51) and (52) we obtain

$$\begin{aligned} j_{\text{cons}}^i &= \frac{1}{2\pi^2} (\mu_5 - A_0^5) B_i + \frac{1}{2\pi^2} \mu B_5^i, \\ j_{5,\text{cons}}^i &= \frac{1}{2\pi^2} \left(\mu_5 - \frac{A_0^5}{3} \right) B_i^5 + \frac{1}{2\pi^2} \mu B^i. \end{aligned} \quad (68)$$

We will calculate correlation functions for equilibrium state without axial gauge field, i.e., $A_{5,\mu} = 0$. Equation (68) indicates the only nonvanishing correlation functions are two-point and three-point ones. The former comes from CME and CSE terms (and their analogs with axial magnetic fields). Fourier transforms of these terms give

$$\begin{aligned} \tilde{j}_{\text{cons}}^i(k) &= \frac{1}{2\pi^2} \mu_5 \tilde{B}_i(k) + \frac{1}{2\pi^2} \mu \tilde{B}_5^i(k), \\ \tilde{j}_{5,\text{cons}}^i(k) &= \frac{1}{2\pi^2} \mu_5 \tilde{B}_i^5(k) + \frac{1}{2\pi^2} \mu \tilde{B}^i(k), \end{aligned} \quad (69)$$

where we use tilde to indicate operators in Fourier space. The absence of electric fields requires momenta appearing in the Fourier transforms contain no temporal components. Taking the functional derivative once, and noting the coupling $\int d^4x A_\mu(x) j_{\text{cons}}^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \tilde{j}_{\text{cons}}^\mu(-k)$ and similarly for axial counterpart, we obtain

$$\begin{aligned} \langle \tilde{j}_{\text{cons}}^i(k) \tilde{j}_{\text{cons}}^j(-k) \rangle &= \frac{i\mu_5 \epsilon^{ijk} k^k}{2\pi^2}, \\ \langle \tilde{j}_{\text{cons}}^i(k) \tilde{j}_{5,\text{cons}}^j(-k) \rangle &= \frac{i\mu \epsilon^{ijk} k^k}{2\pi^2}, \\ \langle \tilde{j}_{5,\text{cons}}^i(k) \tilde{j}_{5,\text{cons}}^j(-k) \rangle &= \frac{i\mu_5 \epsilon^{ijk} k^k}{2\pi^2}, \\ \langle \tilde{j}_{5,\text{cons}}^i(k) \tilde{j}_{\text{cons}}^j(-k) \rangle &= \frac{i\mu \epsilon^{ijk} k^k}{2\pi^2}, \end{aligned} \quad (70)$$

where the left-hand side is defined by $\langle X(k)Y(p) \rangle = \langle X(k)Y(-k) \rangle (2\pi)^4 \delta^{(4)}(k+p)$. The terms quadratic in $A(A_5)$ give rise to the three-point correlation functions. Fourier transforms of these terms are given by

$$\begin{aligned}\tilde{j}_{\text{cons}}^i(q) &= \int \frac{d^4k}{(2\pi)^4} \left[-\frac{1}{2\pi^2} \tilde{A}_0^5(q-k) \tilde{B}_i(k) \right], \\ \tilde{j}_{5,\text{cons}}^i(q) &= \int \frac{d^4k}{(2\pi)^4} \left[-\frac{1}{6\pi^2} \tilde{A}_0^5(q-k) \tilde{B}_i^5(k) \right].\end{aligned}\quad (71)$$

Taking the functional derivatives twice, we obtain the following three-point correlation functions

$$\begin{aligned}\langle \tilde{j}_{\text{cons}}^i(q) \tilde{j}_{5,\text{cons}}^0(k-q) \tilde{j}_{\text{cons}}^j(-k) \rangle &= -i\epsilon^{ijk} \frac{k^k}{2\pi^2}, \\ \langle \tilde{j}_{5,\text{cons}}^i(q) \tilde{j}_{5,\text{cons}}^0(k-q) \tilde{j}_{5,\text{cons}}^j(-k) \rangle &= -i\epsilon^{ijk} \frac{k^k}{6\pi^2},\end{aligned}\quad (72)$$

where the left-hand side is defined by $\langle X(k)Y(q)Z(p) \rangle = \langle X(k)Y(q)Z(-k-q) \rangle (2\pi)^4 \delta^{(4)}(k+q+p)$. In the limit $q \rightarrow k$, Eq. (72) are in agreement with the results obtained with field theory and holography [25].

We can also calculate correlation functions between energy-momentum tensor and currents. Since the antisymmetric part of the energy-momentum tensor vanishes, the correlation functions involves only symmetric part of the energy-momentum tensor

$$\begin{aligned}\langle T^{\{0i\}}(k) j_{\text{cons}}^j(-k) \rangle &= \frac{\xi}{2} \epsilon^{ijk} i k^k, \\ \langle T^{\{0i\}}(k) j_{5,\text{cons}}^j(-k) \rangle &= \frac{\xi_5}{2} \epsilon^{ijk} i k^k,\end{aligned}\quad (73)$$

Indeed, they can be related to CVE in Eq. (50). For $u^\mu = (1, 0, 0, 0)$, ω can be induced by metric perturbation as

$$\omega^j = -\frac{1}{2} \epsilon^{ijk} \partial_k h_{0i}, \quad (74)$$

we find from Eqs. (49) and (50)

$$\begin{aligned}\langle j_{\text{cons}}^j(k) T^{\{0i\}}(-k) \rangle &= -\frac{\xi}{2} \epsilon^{ijk} i k^k, \\ \langle j_{5,\text{cons}}^j(k) T^{\{0i\}}(-k) \rangle &= -\frac{\xi_5}{2} \epsilon^{ijk} i k^k.\end{aligned}\quad (75)$$

Equations (75) and (73) are consistent with Onsager relation.⁴ The fact that Onsager relation holds is closely related to our choice of energy-momentum tensor. On one hand, in Eq. (75), the energy-momentum tensor follows from the functional derivative with respect to metric, which gives the symmetric Belinfante-Rosenfeld energy-momentum tensor. On the other hand, the energy-momentum tensor appearing in Eq. (73) corresponds to symmetric part of the canonical energy-momentum tensor. For free fermions, the two forms of tensor are known to agree, see [68] for a recent review. Had

we chosen other forms of the energy-momentum tensor, the Onsager relation might not be guaranteed.

V. SUMMARY AND OUTLOOK

We have derived chiral kinetic theory with both vector and axial gauge fields. We have also found a solution preserving both vector and axial gauge symmetry, leading to covariant vector and axial currents. The introduction of axial gauge field allows us to derive correlation functions of vector/axial current. This is done by first converting covariant currents to consistent current and then taking functional derivatives with respect to vector/axial gauge fields. We find the resulting correlation functions in agreement with known field theoretic results.

The present work can be extended in two aspects. Firstly, mass effect can be included. Since mass breaks axial symmetry explicitly, it requires a nontrivial modification to the solution presented in this work. It is known that mass introduces additional degrees of freedom. It would be interesting to see how the dynamics of these degrees of freedom are affected by the presence of an axial gauge field. More interestingly, collisional effect should be included for a complete description of dynamics, in particular for axial current (or spin density). This would provide a route to spin hydrodynamics for weakly-interacting fermion systems.

ACKNOWLEDGMENTS

This work is in part supported by NSFC under Grants No. 12075328, No. 11735007, and No. 11675274.

APPENDIX A: VERIFICATION OF EQUATIONS (17) AND DETERMINATION OF $J_\mu^{s(1)}$

We determine $J_\mu^{s(1)}$ from Eq. (32)

$$\begin{aligned}(k_\mu^s J_\nu^{s(1)} - k_\nu^s J_\mu^{s(1)}) \delta(k_s^2) &= -s(k_\mu^s \tilde{F}_{\nu\rho}^s - k_\nu^s \tilde{F}_{\mu\rho}^s) k_s^\rho f_s \delta'(k_s^2) \\ &\quad - \frac{s}{2} \epsilon_{\mu\nu\alpha\rho} \nabla_s^\alpha [k_s^\rho f_s \delta(k_s^2)] \\ &= -s(k_\mu^s \tilde{F}_{\nu\rho}^s - k_\nu^s \tilde{F}_{\mu\rho}^s + k_\rho^s \tilde{F}_{\mu\nu}^s) \\ &\quad \times k_s^\rho f_s \delta'(k_s^2) \\ &\quad + \frac{s}{2} \epsilon_{\mu\nu\rho\alpha} k_s^\rho \nabla_s^\alpha [f_s \delta(k_s^2)] \\ &= s \epsilon_{\mu\nu\rho\alpha} F_s^{\alpha\beta} k_\beta^s k_s^\rho f_s \delta'(k_s^2) \\ &\quad + \frac{s}{2} \epsilon_{\mu\nu\rho\alpha} k_s^\rho \nabla_s^\alpha [f_s \delta(k_s^2)] \\ &= \frac{s}{2} \epsilon_{\mu\nu\rho\alpha} k_s^\rho \Omega^{\alpha\beta} k_\beta^s f_s' \delta(k_s^2),\end{aligned}\quad (A1)$$

where we have used Eqs. (23), and (30), and the Schouten identity

$$g_\sigma^\rho \epsilon^{\mu\nu\alpha\beta} + g_\sigma^\beta \epsilon^{\rho\mu\nu\alpha} + g_\sigma^\alpha \epsilon^{\beta\rho\mu\nu} + g_\sigma^\nu \epsilon^{\alpha\beta\rho\mu} + g_\sigma^\mu \epsilon^{\nu\alpha\beta\rho} = 0, \quad (A2)$$

⁴See [67] for the case with a background magnetic field.

from which we can show that $k_s^\mu \tilde{F}_s^{\nu\rho} + k_s^\rho \tilde{F}_s^{\mu\nu} + k_s^\nu \tilde{F}_s^{\rho\mu} + \epsilon^{\mu\nu\rho\alpha} F_{\alpha\beta}^s k_s^\beta = 0$ by contracting both sides with $F_{\alpha\beta}^s k_s^\sigma$ and using the antisymmetric nature of the field strength tensors. At this stage, we can express the tensor in terms of the dual tensor $\Omega^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\kappa\lambda}\tilde{\Omega}_{\kappa\lambda}$. And the Levi-Civita tensors can be contracted as

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta} &= -2!\delta_{\alpha\beta}^{\rho\sigma} = -2(\delta_\alpha^\rho\delta_\beta^\sigma - \delta_\beta^\rho\delta_\alpha^\sigma), \\ \epsilon^{\mu\kappa\rho\sigma}\epsilon_{\mu\nu\alpha\beta} &= -\delta_{\nu\alpha\beta}^{\kappa\rho\sigma} = -\delta_\nu^\kappa(\delta_\alpha^\rho\delta_\beta^\sigma - \delta_\beta^\rho\delta_\alpha^\sigma) \\ &\quad - \delta_\alpha^\kappa(\delta_\beta^\rho\delta_\nu^\sigma - \delta_\nu^\rho\delta_\beta^\sigma) - \delta_\beta^\kappa(\delta_\nu^\rho\delta_\alpha^\sigma - \delta_\alpha^\rho\delta_\nu^\sigma). \end{aligned} \quad (\text{A3})$$

Using these contract relations, we can further derive for two arbitrary antisymmetric tensors, e.g., $\Omega_{\mu\nu}$ and $F_{\mu\nu}^s$, that

$$\frac{1}{2}\tilde{\Omega}^{\mu\nu}F_{\mu\nu}^s k_s^2 = \Omega^{\mu\alpha}\tilde{F}_{\mu\rho}^s k_\alpha^s k_s^\rho + \tilde{\Omega}^{\mu\alpha}F_{\mu\rho}^s k_\alpha^s k_s^\rho, \quad (\text{A4})$$

and especially,

$$\begin{aligned} \tilde{\Omega}^{\mu\nu}\Omega_{\mu\nu}k_s^2 &= 4\Omega^{\mu\alpha}\tilde{\Omega}_{\mu\rho}k_\alpha^s k_s^\rho, \\ \tilde{F}_s^{\mu\nu}F_{\mu\nu}^s k_s^2 &= 4F_{\mu\rho}^s\tilde{F}_s^{\mu\alpha}k_\alpha^s k_s^\rho. \end{aligned} \quad (\text{A5})$$

Then Eq. (A1) can be written as

$$\begin{aligned} (k_\mu^s J_\nu^{(1)} - k_\nu^s J_\mu^{(1)})\delta(k_s^2) &= \frac{s}{2}\epsilon_{\mu\nu\rho\alpha}k_s^\rho\Omega^{\alpha\beta}k_{\beta s}^s f'_s\delta(k_s^2) \\ &= -\frac{s}{4}\epsilon_{\mu\nu\rho\alpha}\epsilon^{\alpha\beta\kappa\lambda}\tilde{\Omega}_{\kappa\lambda}k_s^\rho k_{\beta s}^s f'_s\delta(k_s^2) \\ &= -\frac{s}{4}\delta_{\mu\nu\rho}^{\beta\kappa\lambda}k_s^\rho\tilde{\Omega}_{\kappa\lambda}k_{\beta s}^s f'_s\delta(k_s^2) \\ &= -\frac{s}{2}(k_\mu^s\tilde{\Omega}_{\nu\rho}k_s^\rho - k_\nu^s\tilde{\Omega}_{\mu\rho}k_s^\rho)f'_s\delta(k_s^2). \end{aligned} \quad (\text{A6})$$

We have thrown away the terms proportional to $k_s^2\delta(k_s^2) = 0$. In fact, Eq. (A6) cannot exclude the correction of the form $\delta J_\mu^{(1)} = k_\mu^s X^{(1)}$, which involves an unknown regular function $X^{(1)}$ that should be identified as of the first order in gradient. We may absorb this undetermined correction into the first order distribution function which we do not consider in this work. After that, we can set $J_\mu^{s(1)} = -\frac{s}{2}\tilde{\Omega}_{\mu\nu}k_s^\nu f'_s$ in line with Eq. (A6).

Now we check if the first-order solution given by Eq. (33) satisfies Eq. (17) with the global equilibrium conditions which have been embedded into Eq. (30). For convenience, we divide the Wigner function into two parts, $\mathcal{J}_\Omega^{s(1)\mu}$ and $\mathcal{J}_{EM}^{s(1)\mu}$, which are generated by vorticity and background fields respectively. We note that the vorticity $\Omega_{\mu\nu}$ is constant due to the Killing condition $\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$. Assuming the constant field strength tensors, i.e., $\partial_\rho F_{\mu\nu}^s = 0$, we can show that

$$\begin{aligned} \nabla_\mu^s \mathcal{J}_\Omega^{s(1)\mu} &= -\frac{s}{2}\tilde{\Omega}^{\mu\nu}\nabla_\mu^s[k_\nu^s f'_s\delta(k_s^2)] \\ &= \frac{s}{2}\tilde{\Omega}^{\mu\nu}F_{\mu\nu}^s f'_s\delta(k_s^2) + s\tilde{\Omega}^{\mu\nu}F_{\mu\rho}^s k_\nu^s k_s^\rho f'_s\delta'(k_s^2) \\ &\quad - \frac{s}{2}\tilde{\Omega}^{\mu\nu}\Omega_{\mu\rho}k_\nu^s k_s^\rho f'_s\delta(k_s^2) \\ &= -s\Omega^{\mu\nu}\tilde{F}_{\mu\rho}^s k_\nu^s k_s^\rho f'_s\delta'(k_s^2), \end{aligned} \quad (\text{A7})$$

and

$$\begin{aligned} \nabla_\mu^s \mathcal{J}_{EM}^{s(1)\mu} &= s\tilde{F}_s^{\mu\nu}\nabla_\mu^s[k_\nu^s f'_s\delta(k_s^2)] \\ &= -s\tilde{F}_s^{\mu\nu}F_{\mu\nu}^s f'_s\delta(k_s^2) - 2s\tilde{F}_s^{\mu\nu}F_{\mu\rho}^s k_\nu^s k_s^\rho f'_s\delta'(k_s^2) \\ &\quad + s\tilde{F}_s^{\mu\nu}\Omega_{\mu\rho}k_\nu^s k_s^\rho f'_s\delta(k_s^2) \\ &= s\tilde{F}_s^{\mu\nu}\Omega_{\mu\rho}k_\nu^s k_s^\rho f'_s\delta'(k_s^2), \end{aligned} \quad (\text{A8})$$

where we have used $\delta'(k^2) = -\delta(k^2)/k^2$, $\delta''(k^2) = -2\delta'(k^2)/k^2$ and Eq. (A5). By taking the sum of Eq. (A7) and Eq. (A8), we obtain

$$\nabla_\mu^s \mathcal{J}^{(1)\mu} = 0. \quad (\text{A9})$$

APPENDIX B: CONSERVATION EQUATIONS

Using Eqs. (26) and (42) with the definition $\beta_\mu \equiv u_\mu/T$, we can easily show that

$$\partial_\mu\beta^\mu \equiv \partial_\mu \frac{u^\mu}{T} = 0, \quad (\text{B1})$$

$$\partial_\mu\beta = u^\nu\partial_\mu\beta_\nu - \beta u^\nu\partial_\mu u_\nu = u^\nu\Omega_{\mu\nu} = \beta\epsilon_\mu, \quad (\text{B2})$$

$$u \cdot \partial\beta = 0, \quad \partial_\mu u^\mu = 0, \quad (\text{B3})$$

$$\partial_\mu u_\nu = T\Omega_{\mu\nu} - \epsilon_\mu u_\nu. \quad (\text{B4})$$

And we have used $u \cdot \epsilon = 0$, $u^2 = 1$ and $u^\nu\partial_\mu u_\nu = 0$. Then, from Eqs. (28) and (29) we have

$$\partial_\mu\mu = -\mu\epsilon_\mu - F_{\mu\nu}u^\nu = -\mu\epsilon_\mu - E_\mu, \quad (\text{B5})$$

$$\partial_\mu\mu_5 = -\mu_5\epsilon_\mu - F_{\mu\nu}^5 u^\nu = -\mu_5\epsilon_\mu - E_\mu^5, \quad (\text{B6})$$

and the integrability conditions for constant $F_{\mu\nu}(F_{\mu\nu}^5)$

$$F_{(5)\lambda}^\mu\Omega^{\nu\lambda} - F_{(5)\lambda}^\nu\Omega^{\mu\lambda} = 0, \quad (\text{B7})$$

or equivalently,

$$\begin{aligned} \epsilon_{\mu\nu\alpha\beta}(E_{(5)}^\alpha\omega^\beta - \epsilon_{(5)}^\alpha B_{(5)}^\beta) &= 0, \\ \epsilon_{\mu\nu\alpha\beta}(E_{(5)}^\alpha\epsilon_{(5)}^\beta + \omega_{(5)}^\alpha B_{(5)}^\beta) &= 0. \end{aligned} \quad (\text{B8})$$

Especially, we see that

$$u \cdot \partial \mu = u \cdot \partial \mu_5 = 0. \quad (\text{B9})$$

Finally we can derive the following useful identities

$$\partial_\mu n = -3n\epsilon_\mu - 2\xi_5 E_\mu - 2\xi E_\mu^5, \quad (\text{B10})$$

$$\partial_\mu n_5 = -3n_5\epsilon_\mu - 2\xi_5 E_\mu^5 - 2\xi E_\mu, \quad (\text{B11})$$

$$\partial_\mu \xi_5 = -2\xi_5 \epsilon_\mu - 2\xi_{B5} E_\mu - 2\xi_B E_\mu^5, \quad (\text{B12})$$

$$\partial_\mu \xi = -2\xi \epsilon_\mu - 2\xi_B E_\mu - 2\xi_{B5} E_\mu^5, \quad (\text{B13})$$

$$\partial_\mu \rho = -4\rho\epsilon_\mu - 3nE_\mu - 3n_5 E_\mu^5, \quad (\text{B14})$$

$$\begin{aligned} \partial_\mu \xi_B &= -\xi_B \epsilon_\mu - \frac{1}{2\pi^2} E_\mu^5, \\ \partial_\mu \xi_{B5} &= -\xi_{B5} \epsilon_\mu - \frac{1}{2\pi^2} E_\mu, \end{aligned} \quad (\text{B15})$$

which clearly gives

$$u \cdot \partial \{n, n_5, \xi, \xi_5, \xi_B, \xi_{B5}, \rho\} = 0. \quad (\text{B16})$$

On the other hand, we can also show

$$\partial_\mu \omega_\nu = \epsilon \cdot \omega g_{\mu\nu} - 2\epsilon_\mu \omega_\nu, \quad (\text{B17})$$

$$\begin{aligned} \partial_\mu \epsilon_\nu &= \omega_\mu \omega_\nu - \epsilon_\mu \epsilon_\nu + \epsilon^2 u_\mu u_\nu - \omega^2 \Delta_{\mu\nu} \\ &+ (u_\mu \epsilon_{\nu\lambda\rho\sigma} + u_\nu \epsilon_{\mu\lambda\rho\sigma}) u^\lambda \epsilon^\rho \omega^\sigma, \end{aligned} \quad (\text{B18})$$

$$\begin{aligned} \partial_\mu B_\nu &= -E_\mu \omega_\nu + \epsilon \cdot B u_\mu u_\nu + \omega \cdot E \Delta_{\mu\nu} \\ &- (u_\mu \epsilon_{\nu\lambda\rho\sigma} + u_\nu \epsilon_{\mu\lambda\rho\sigma}) u^\lambda \epsilon^\rho E^\sigma, \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \partial_\mu E_\nu &= B_\mu \omega_\nu + \epsilon \cdot E u_\mu u_\nu - \omega \cdot B \Delta_{\mu\nu} \\ &+ (u_\mu \epsilon_{\nu\lambda\rho\sigma} + u_\nu \epsilon_{\mu\lambda\rho\sigma}) u^\lambda E^\rho \omega^\sigma, \end{aligned} \quad (\text{B20})$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. For example,

$$\begin{aligned} \partial_\mu \omega_\nu &= \partial_\mu (T \tilde{\Omega}_{\nu\rho} u^\rho) = \tilde{\Omega}_{\nu\rho} (u^\rho \partial_\mu T + T \partial_\mu u^\rho) \\ &= \epsilon \cdot \omega g_{\mu\nu} - 2\epsilon_\mu \omega_\nu. \end{aligned} \quad (\text{B21})$$

These identities only hold with the constant $\Omega_{\mu\nu}$ and $F_{\mu\nu}^s$. Using the identities derived above, we can verify the following conservation laws,

$$\partial_\mu j^{(0)\mu} = 0, \quad \partial_\mu j_5^{(0)\mu} = 0, \quad (\text{B22})$$

$$\begin{aligned} \partial_\mu j^{(1)\mu} &= -\frac{1}{2\pi^2} (E \cdot B_5 + E_5 \cdot B) \\ &= \frac{1}{2\pi^2} (\vec{E} \cdot \vec{B}_5 + \vec{E}_5 \cdot \vec{B}), \end{aligned} \quad (\text{B23})$$

$$\begin{aligned} \partial_\mu j_5^{(1)\mu} &= -\frac{1}{2\pi^2} (E \cdot B + E_5 \cdot B_5) \\ &= \frac{1}{2\pi^2} (\vec{E} \cdot \vec{B} + \vec{E}_5 \cdot \vec{B}_5), \end{aligned} \quad (\text{B24})$$

$$\partial_\mu T^{\{\mu\nu\}} = F^{\nu\mu} j_\mu + F_5^{\mu\nu} j_{5,\mu}, \quad (\text{B25})$$

$$\partial_\mu T^{[\mu\nu]} = 0. \quad (\text{B26})$$

As an example, we evaluate the conservation law of the first-order symmetric energy current $T^{(1)\{\mu\nu\}}$ given by Eq. (59). For convenience, we divide the energy current into two parts, $T_\omega^{(1)\{\mu\nu\}}$ and $T_B^{(1)\{\mu\nu\}}$, which are generated by vorticity and background fields respectively. The derivative of $T_\omega^{(1)\{\mu\nu\}}$ can be evaluated as

$$\begin{aligned} \partial_\mu T_\omega^{(1)\{\mu\nu\}} &= \partial_\mu [n_5 (u^\mu \omega^\nu + u^\nu \omega^\mu)] \\ &= -2\xi_5 \omega \cdot E^5 u^\nu - 2\xi \omega \cdot E u^\nu, \end{aligned} \quad (\text{B27})$$

while the derivative of $T_B^{(1)\{\mu\nu\}}$ is

$$\begin{aligned} \partial_\mu T_B^{(1)\{\mu\nu\}} &= \partial_\mu \left[\frac{\xi}{2} (u^\mu B^\nu + u^\nu B^\mu) + \frac{\xi_5}{2} (u^\mu B_5^\nu + u^\nu B_5^\mu) \right] \\ &= \xi \omega \cdot E u^\nu + \xi_5 \omega \cdot E_5 u^\nu + \xi \epsilon_{\nu\mu\rho\sigma} \omega^\mu u^\rho B^\sigma \\ &+ \xi_5 \epsilon_{\nu\mu\rho\sigma} \omega^\mu u^\rho B_5^\sigma - u^\nu E \cdot (\xi_B B + \xi_{B5} B_5) \\ &- u^\nu E_5 \cdot (\xi_{B5} B + \xi_B B_5). \end{aligned} \quad (\text{B28})$$

Then we can verify the conservation law of the first-order symmetric energy-momentum tensor,

$$\begin{aligned} \partial_\mu T^{(1)\{\mu\nu\}} &= \partial_\mu T_\omega^{(1)\{\mu\nu\}} + \partial_\mu T_B^{(1)\{\mu\nu\}} \\ &= -u^\nu E \cdot j - u^\nu E_5 \cdot j_5 + \xi \epsilon_{\nu\mu\rho\sigma} \omega^\mu u^\rho B^\sigma \\ &+ \xi_5 \epsilon_{\nu\mu\rho\sigma} \omega^\mu u^\rho B_5^\sigma \\ &= F^{\nu\mu} j_\mu + F_5^{\mu\nu} j_{5,\mu}, \end{aligned} \quad (\text{B29})$$

where we have used the decomposition of Eq. (41) in the third line.

In addition to the conservation of currents and energy-momentum tensor, we can also obtain a relation between $T^{(1)[\mu\nu]}$ and spin tensor $S^{\lambda\mu\nu}$. The latter is determined by the axial current as

$$S^{\lambda\mu\nu} = \frac{1}{2} \epsilon^{\eta\lambda\mu\nu} j_{5\eta}. \quad (\text{B30})$$

Integrating Eq. (13) over k_s , we identify the first two terms as $T_s^{(1)[\mu\nu]}$. The last term gives $\frac{s}{2} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha j_\beta^s$, while the term dependent on A_μ^s inside ∇_μ^s simply drops out as a total derivative term. By taking the sum of right-handed and left-handed contributions, we arrive at

$$T^{(1)[\mu\nu]} = \partial_\lambda S^{\lambda\mu\nu}. \quad (\text{B31})$$

- [1] D. T. Son and N. Yamamoto, *Phys. Rev. D* **87**, 085016 (2013).
- [2] M. A. Stephanov and Y. Yin, *Phys. Rev. Lett.* **109**, 162001 (2012).
- [3] S. Pu, J. h. Gao, and Q. Wang, *Phys. Rev. D* **83**, 094017 (2011).
- [4] J. W. Chen, S. Pu, Q. Wang, and X. N. Wang, *Phys. Rev. Lett.* **110**, 262301 (2013).
- [5] Y. Hidaka, S. Pu, and D. L. Yang, *Phys. Rev. D* **95**, 091901 (2017).
- [6] C. Manuel and J. M. Torres-Rincon, *Phys. Rev. D* **89**, 096002 (2014).
- [7] C. Manuel and J. M. Torres-Rincon, *Phys. Rev. D* **90**, 076007 (2014).
- [8] A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, *Phys. Rev. D* **98**, 036010 (2018).
- [9] S. Carignano, C. Manuel, and J. M. Torres-Rincon, *Phys. Rev. D* **98**, 076005 (2018).
- [10] J. H. Gao, Z. T. Liang, Q. Wang, and X. N. Wang, *Phys. Rev. D* **98**, 036019 (2018).
- [11] Z. Wang, X. Guo, S. Shi, and P. Zhuang, *Phys. Rev. D* **100**, 014015 (2019).
- [12] S. Lin and A. Shukla, *J. High Energy Phys.* 06 (2019) 060.
- [13] J. H. Gao, Z. T. Liang, and Q. Wang, *Phys. Rev. D* **101**, 096015 (2020).
- [14] T. Hayata, Y. Hidaka, and K. Mameda, *J. High Energy Phys.* 05 (2021) 023.
- [15] S. Z. Yang, J. H. Gao, Z. T. Liang, and Q. Wang, *Phys. Rev. D* **102**, 116024 (2020).
- [16] S. Lin and L. Yang, *Phys. Rev. D* **101**, 034006 (2020).
- [17] S. Lin and L. Yang, *J. High Energy Phys.* 06 (2021) 054.
- [18] X. L. Luo and J. H. Gao, *J. High Energy Phys.* 11 (2021) 115.
- [19] D. Kharzeev, *Phys. Lett. B* **633**, 260 (2006).
- [20] D. Kharzeev and A. Zhitnitsky, *Nucl. Phys. A* **797**, 67 (2007).
- [21] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, *Phys. Rev. D* **78**, 074033 (2008).
- [22] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, *J. High Energy Phys.* 01 (2009) 055.
- [23] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, *J. High Energy Phys.* 01 (2011) 094.
- [24] Y. Neiman and Y. Oz, *J. High Energy Phys.* 03 (2011) 023.
- [25] K. Landsteiner, E. Megias, and F. Pena-Benitez, *Phys. Rev. Lett.* **107**, 021601 (2011).
- [26] K. Hattori, Y. Hidaka, and D. L. Yang, *Phys. Rev. D* **100**, 096011 (2019).
- [27] N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, *Phys. Rev. D* **100**, 056018 (2019).
- [28] J. H. Gao and Z. T. Liang, *Phys. Rev. D* **100**, 056021 (2019).
- [29] Y. C. Liu, K. Mameda, and X. G. Huang, *Chin. Phys. C* **44**, 094101 (2020); **45**, 089001(E) (2021).
- [30] X. Guo, *Chin. Phys. C* **44**, 104106 (2020).
- [31] J. H. Gao, G. L. Ma, S. Pu, and Q. Wang, *Nucl. Sci. Tech.* **31**, 90 (2020).
- [32] J. J. Zhang, R. H. Fang, Q. Wang, and X. N. Wang, *Phys. Rev. C* **100**, 064904 (2019).
- [33] S. Li and H. U. Yee, *Phys. Rev. D* **100**, 056022 (2019).
- [34] S. Carignano, C. Manuel, and J. M. Torres-Rincon, *Phys. Rev. D* **102**, 016003 (2020).
- [35] D. L. Yang, K. Hattori, and Y. Hidaka, *J. High Energy Phys.* 07 (2020) 070.
- [36] Z. Wang, X. Guo, and P. Zhuang, *Eur. Phys. J. C* **81**, 799 (2021).
- [37] S. Shi, C. Gale, and S. Jeon, *Phys. Rev. C* **103**, 044906 (2021).
- [38] N. Weickgenannt, E. Speranza, X. L. Sheng, Q. Wang, and D. H. Rischke, *Phys. Rev. Lett.* **127**, 052301 (2021).
- [39] D. Hou and S. Lin, *Phys. Lett. B* **818**, 136386 (2021).
- [40] N. Yamamoto and D. L. Yang, *Astrophys. J.* **895**, 56 (2020).
- [41] N. Weickgenannt, E. Speranza, X. L. Sheng, Q. Wang, and D. H. Rischke, *Phys. Rev. D* **104**, 016022 (2021).
- [42] X. L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, and Q. Wang, *Phys. Rev. D* **104**, 016029 (2021).
- [43] Z. Wang and P. Zhuang, *arXiv:2105.00915*.
- [44] S. Lin, *arXiv:2109.00184*.
- [45] X. G. Huang and A. V. Sadofyev, *J. High Energy Phys.* 03 (2019) 084.
- [46] X. G. Huang, P. Mitkin, A. V. Sadofyev, and E. Speranza, *J. High Energy Phys.* 10 (2020) 117.
- [47] K. Hattori, Y. Hidaka, N. Yamamoto, and D. L. Yang, *J. High Energy Phys.* 02 (2021) 001.
- [48] D. Satow, *Phys. Rev. D* **90**, 034018 (2014).
- [49] E. V. Gorbar, D. O. Rybalka, and I. A. Shovkovy, *Phys. Rev. D* **95**, 096010 (2017).
- [50] M. Buzzegoli, E. Grossi, and F. Becattini, *J. High Energy Phys.* 10 (2017) 091; 07 (2018) 119(E).
- [51] M. Buzzegoli and F. Becattini, *J. High Energy Phys.* 12 (2018) 002.
- [52] W. Florkowski, A. Kumar, and R. Ryblewski, *Prog. Part. Nucl. Phys.* **108**, 103709 (2019).
- [53] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, *Phys. Lett. B* **814**, 136096 (2021).
- [54] H. H. Peng, J. J. Zhang, X. L. Sheng, and Q. Wang, *Chin. Phys. Lett.* **38**, 116701 (2021).
- [55] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, *J. High Energy Phys.* 09 (2012) 046.
- [56] J. Hernandez and P. Kovtun, *J. High Energy Phys.* 05 (2017) 001.
- [57] A. D. Gallegos, U. Gürsoy, and A. Yarom, *SciPost Phys.* **11**, 041 (2021).
- [58] M. Hongo, X. G. Huang, M. Kaminski, M. Stephanov, and H. U. Yee, *J. High Energy Phys.* 11 (2021) 150.
- [59] D. T. Son and P. Surowka, *Phys. Rev. Lett.* **103**, 191601 (2009).
- [60] A. A. Zyuzin, S. Wu, and A. A. Burkov, *Phys. Rev. B* **85**, 165110 (2012).
- [61] K. Landsteiner, *Acta Phys. Pol. B* **47**, 2617 (2016).
- [62] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, *Phys. Rev. Lett.* **118**, 127601 (2017).
- [63] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, *Phys. Rev. B* **95**, 115202 (2017).
- [64] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, *Phys. Rev. B* **97**, 121105 (2018).
- [65] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, *Phys. Rev. B* **97**, 205119 (2018).
- [66] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, and A. Yarom, *Phys. Rev. Lett.* **109**, 101601 (2012).
- [67] Y. Bu and S. Lin, *Eur. Phys. J. C* **80**, 401 (2020).
- [68] E. Speranza and N. Weickgenannt, *Eur. Phys. J. A* **57**, 155 (2021).