From AdS to dS exchanges: Spectral representation, Mellin amplitudes, and crossing

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In this paper we take the first steps towards bridging the gap between the conformal and cosmological bootstrap programs. We present a simple general relation between tree-level exchanges in anti-de Sitter (AdS) and de Sitter (dS) space with Bunch-Davies initial conditions, providing a complete AdS to dS dictionary at tree-level. This allows us to directly import techniques and results for conformal field theory correlators dual to AdS-Witten diagrams to boundary correlation functions in dS. We apply this relation to extend various indispensable tools from the conformal bootstrap to boundary correlation functions in dS, including Mellin amplitudes and the spectral representation for dS exchanges. We also derive their conformal block decomposition, both in the direct and crossed channels, from their AdS counterparts. The relation between AdS and dS exchanges itself is derived using a recently introduced Mellin-Barnes representation for boundary correlators in momentum space, where (A)dS exchanges are straightforwardly fixed by a combination of factorization, conformal symmetry and boundary conditions.

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I. INTRODUCTION

The last decade has seen significant progress in our understanding of scattering in anti-de Sitter (AdS) space. Through the AdS/CFT correspondence [1] we can reformulate scattering processes in AdS in terms of correlation functions in conformal field theory (CFT), which are sharply defined by the requirements of conformal symmetry, unitarity and a consistent operator product expansion (OPE). Accordingly, numerous highly effective techniques for the study of scattering in AdS have been developed within the conformal bootstrap program, including Mellin amplitudes [2,3], spectral representation [4,5], large spin expansion [6–8], dispersion relations, and inversion formulas [9–11].

With this paper we take some simple steps towards extending some of these techniques to scattering in de Sitter (dS) space. Conformal symmetry has indeed been effective in constraining boundary observables in dS in numerous previous works [12–20], and most recently within the cosmological bootstrap program [21–28]. Our understanding of the properties required of consistent correlation functions in Euclidean CFTs dual to physics in dS is, however, very primordial compared to that which we enjoy in AdS. In this work we take some first steps towards closing the gap between the conformal bootstrap and cosmological bootstrap programs. Using the Mellin-Barnes representation for the momentum space-boundary correlators introduced in [24,25], we strengthen the connection between in-in correlators on the late-time boundary of dS and Witten diagrams in AdS. For Bunch-Davies initial conditions, we present a simple general relation (28) between exchanges in dS and AdS at tree level, which allows us to directly import AdS techniques (both in position and momentum space [29]) to in-in correlators in dS, which we briefly explore in the final section of this paper. This includes spectral representation in

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Sec. IV A and Mellin amplitudes in Sec. IV B for dS exchanges, as well as their conformal block decomposition both in the direct and crossed channels in Sec. IV C.

The Mellin-Barnes representation [24,25] itself provides us with a new systematic understanding of boundary correlators in AdS and dS. As we shall see, it allows us to straightforwardly reconstruct an exchange diagram from its on-shell part. The latter is fixed by a combination of factorization, conformal symmetry, and boundary conditions which, as we shall see, can be seamlessly implemented using the Mellin-Barnes representation.

Notation and Conventions. Whenever referring to the bulk of Euclidean anti-de Sitter space (EAdS) and dS we use Poincaré coordinates

$$ds_{\text{EAdS}}^2 = \frac{dz^2 + d\mathbf{x}^2}{z^2}, \quad ds_{\text{dS}}^2 = \frac{-d\eta^2 + d\mathbf{x}^2}{\eta^2}, \quad (1)$$

where $z \in [0, \infty)$ and $\eta \in (-\infty, 0]$. The three-dimensional vector $\mathbf{x}$ parametrizes the boundary directions and $\mathbf{k}$ will denote the boundary momentum. Momentum-conserving delta functions will always be left implicit. Often we will parametrize scaling dimensions as $\Delta_{\pm} = \frac{d}{2} \pm i\mu$ and $\Delta_j = \frac{d}{2} + i\mu_j$. Unless specified, the latter scaling dimensions will be kept generic throughout.

II. (A)dS PROPAGATORS

A. Bulk-Boundary Propagators

The in-in bulk-boundary propagators in dS for the $\Delta = \Delta_{\pm}$ modes in Poincaré coordinates (1) can be obtained from their counterparts in EAdS through the analytic continuation $z = -\eta e^{\pm \frac{i\pi}{2}}$ [24,25]

$$K_{\Delta,J}^{\pm}(\eta, \mathbf{x}; \mathbf{x}') = c_{\Delta}^{\text{dS-AdS}} e^{\mp \Delta \frac{i\pi}{2}} K_{\Delta,J}^{\text{AdS}}(-\eta e^{\pm \frac{i\pi}{2}}, \mathbf{x}; \mathbf{x}'), \quad (2)$$

where the factor

$$c_{\Delta}^{\text{dS-AdS}} = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\sqrt{\pi}}, \quad (3)$$

accounts for the change in normalization as we move from AdS to dS. Bulk-boundary propagators in EAdS for scalar fields are given by a modified Bessel function of the second kind in Poincaré coordinates [30]. Bulk-boundary propagators for spinning fields (see e.g., [31]) with the same Δ can be obtained from the latter by acting with a differential operator in the boundary momentum $\mathbf{k}$ [43].

In the Mellin-Barnes representation [24,25] the analytic continuation (2) reduces to multiplying the AdS bulk-boundary propagator by a simple phase. For scalar fields it reads [24,25]

$$K_{\Delta,J}^{\text{AdS}}(z, \mathbf{k}) = \int_{-i\infty}^{+i\infty} [ds] K_{\Delta,J}^{\text{AdS}}(z, \mathbf{k}, s), \quad (4a)$$

$$K_{\Delta,0}^{\text{AdS}}(z, \mathbf{k}, s) = \frac{z^{d-\Delta}}{2\sqrt{\pi}\Gamma(\Delta - \frac{d}{2})} \Gamma\left(s \pm \frac{1}{2}\left(\Delta - \frac{d}{2}\right)\right) \times \left(\frac{z|\mathbf{k}|}{2}\right)^{2s+\Delta-\frac{d}{2}}, \quad (4b)$$

where $[ds] = \frac{ds}{2\pi i}$ and $\Gamma(a \pm b) \equiv \Gamma(a+b)\Gamma(a-b)$. We refer to variables s as external Mellin variables, associated to each external leg. Combined with (2), we have

$$K_{\Delta,J}^{\text{dS},\pm}(\eta, \mathbf{k}, s) = c_{\Delta}^{\text{dS-AdS}} e^{\mp(s+\frac{1}{2}(\Delta-\frac{d}{2}))\pi i} K_{\Delta,J}^{\text{AdS}}(-\eta, \mathbf{k}, s). \quad (5)$$

This also holds for fields of spin- J , which can be understood from the relation between spinning and scalar bulk-boundary propagators given in [43].

B. Bulk-Bulk propagators

For bulk-bulk propagators it is useful to consider a special class of bilocal eigenfunctions of the Laplacian

$$(\nabla_1^2 - m^2)\Omega_{\mu,J}^{\text{AdS}}(x_1; x_2) = 0, \quad (6)$$

where $m^2 = -(\Delta_+ \Delta_- + J)$, which are nonsingular in the short distance limit. These are given by the following product of bulk-boundary propagators [5,44]

$$\Omega_{\mu,J}^{\text{AdS}}(z_1, \mathbf{k}; z_2, -\mathbf{k}) = \frac{\mu^2}{\pi} K_{\Delta_+,J}^{\text{AdS}}(z_1, \mathbf{k}) K_{\Delta_-,J}^{\text{AdS}}(z_2, -\mathbf{k}). \quad (7)$$

Such harmonic functions are a complete basis for bilocal functions in AdS [5]. The bulk-bulk propagator for the Δ_+ boundary condition admits the following spectral representation [5,45]

$$\Pi_{\Delta_+,J}^{\text{AdS}} = \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + (\Delta_+ - \frac{d}{2})^2} \Omega_{\nu,J}^{\text{AdS}} + \dots, \quad (8)$$

where the ... denote lower-spin contact terms. For the boundary condition Δ_- the spectral integral is supplemented by a single harmonic function (7), [46]

$$\Pi_{\Delta_-,J}^{\text{AdS}} = \frac{2\pi i}{\mu} \Omega_{\mu,J}^{\text{AdS}} + \Pi_{\Delta_+,J}^{\text{AdS}}. \quad (9)$$

The propagators (8) and (9) are placed on an equal footing by adopting the Mellin-Barnes representation [24,25],

$$\Pi_{\Delta_{\pm},J}^{\text{AdS}} = \int_{-i\infty}^{+i\infty} \frac{dud\bar{u}}{(2\pi i)^2} \Pi_{\Delta_{\pm},J}^{\text{AdS}}(u; \bar{u}), \quad (10)$$

where the spectral integral can be lifted in momentum space [25]

$$\begin{aligned} \Pi_{\Delta_{\pm},J}^{\text{AdS}}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u}) &= \csc(\pi(u + \bar{u})) \omega_{\Delta_{\pm}}(u, \bar{u}) \\ &\times \Gamma(i\mu) \Gamma(-i\mu) \Omega_{\mu,J}^{\text{AdS}} \\ &\times (z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u}), \end{aligned} \quad (11)$$

with

$$\omega_{\Delta_{\pm}}(u, \bar{u}) = 2 \sin\left(u \mp \frac{i\mu}{2}\right) \sin\left(\bar{u} \mp \frac{i\mu}{2}\right). \quad (12)$$

We often refer to u and $\bar{u}$ as internal Mellin variables, associated to an internal leg. The Mellin-Barnes representation $\Omega_{\mu,J}^{\text{AdS}}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u})$ of the harmonic function (7) is inherited from that of its constituent bulk-boundary propagators.

The discontinuity of (11) with respect to $\mathbf{s} = |\mathbf{k}|^2$, defined as

$$-2i \text{Disc}_{\mathbf{s}}[f(\mathbf{s})] = f(e^{i\pi}\mathbf{s}) - f(e^{-i\pi}\mathbf{s}), \quad (13)$$

gives the corresponding on-shell propagator. This reads

$$\begin{aligned} \text{Disc}_{\mathbf{s}}[\Pi_{\Delta_{\pm},J}^{\text{AdS}}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u})] \\ = \omega_{\Delta_{\pm}}(u, \bar{u}) \Gamma(i\mu) \Gamma(-i\mu) \Omega_{\mu,J}^{\text{AdS}}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u}). \end{aligned} \quad (14)$$

These encode the physical exchanged single-particle state and are also solutions to the homogeneous equation (6). The bulk-contact terms in the exchange are instead encoded by the factor $\csc(\pi(u + \bar{u}))$, which is killed by the discontinuity through the identity

$$\text{Disc}_{\mathbf{s}}[|\mathbf{k}|^{2(u+\bar{u})}] = \sin(\pi(u + \bar{u})) |\mathbf{k}|^{2(u+\bar{u})}. \quad (15)$$

The functions $\omega_{\Delta_{\pm}}(u, \bar{u})$ can be regarded as projectors onto the $\Delta_{\pm}$ boundary behaviors, with the most general propagator in AdS being a linear combination of the two

$$\Pi_{\alpha\Delta_+ + \beta\Delta_-}^{\bullet}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u}) = \underbrace{\csc(\pi(u + \bar{u}))}_{\text{contact terms}} \underbrace{[\alpha\omega_{\Delta_+}(u, \bar{u}) + \beta\omega_{\Delta_-}(u, \bar{u})]}_{\text{boundary conditions}} \times \underbrace{\Gamma(i\mu)\Gamma(-i\mu)\Omega_{\mu,J}^{\bullet}(z_1, \mathbf{k}, u; z_2, -\mathbf{k}, \bar{u})}_{\text{harmonic function}}, \quad (16)$$

where $\bullet = \text{AdS}$. In-in propagators in dS take the same form, where for the $\pm\pm$ branch we have $\bullet = \text{dS}, \pm, \pm$. The $\pm\pm$ harmonic function is the following analytic continuation of the AdS harmonic function (7)

$$\Omega_{\mu,J}^{\text{dS},\pm,\pm}(\eta_1, \mathbf{k}; \eta_2, -\mathbf{k}) \quad (17)$$

$$\begin{aligned} &= c_{\frac{d}{2}+i\mu}^{\text{dS-AdS}} c_{\frac{d}{2}-i\mu}^{\text{dS-AdS}} e^{\mp\frac{i\mu}{2}(\frac{d}{2}+i\mu)} e^{\mp\frac{i\mu}{2}(\frac{d}{2}-i\mu)} \\ &\times \Omega_{\mu,J}^{\text{AdS}}(-\eta_1 e^{\pm\frac{i\mu}{2}}, \mathbf{k}; -\eta_2 e^{\mp\frac{i\mu}{2}}, -\mathbf{k}) \end{aligned} \quad (18)$$

$$= \frac{\mu^2}{\pi} K_{\Delta_+,J}^{\pm}(\eta_1, \mathbf{k}) K_{\Delta_-,J}^{\pm}(\eta_2, -\mathbf{k}). \quad (19)$$

The precise linear combination of $\Delta_{\pm}$ modes appearing in the dS in-in propagators (16) is fixed by

the vacuum at early times. The Bunch-Davies vacuum corresponds to

$$\alpha_{\pm\pm}^{\text{B.D.}} = -\frac{i}{8\mu} e^{\pm\pi\mu}, \quad \beta_{\pm\pm}^{\text{B.D.}} = \frac{i}{8\mu} e^{\mp\pi\mu}, \quad (20a)$$

$$\alpha_{\pm\mp}^{\text{B.D.}} = \frac{i}{2\mu} e^{\mp\pi\mu}, \quad \beta_{\pm\mp}^{\text{B.D.}} = -\frac{i}{2\mu} e^{\mp\pi\mu}. \quad (20b)$$

The fact that these coefficients are different for each of the in-in propagators tells us that they cannot be regarded as analytic continuations of one and the same AdS bulk-bulk propagator (16). In particular, from (16) we have

$$\begin{aligned} \Pi_{\mu,J}^{\pm,\pm}(\eta_1, \mathbf{k}; \eta_2, -\mathbf{k}) &= c_{\frac{d}{2}+i\mu}^{\text{dS-AdS}} c_{\frac{d}{2}-i\mu}^{\text{dS-AdS}} e^{\mp\frac{i\mu}{2}(\frac{d}{2}+i\mu)} e^{\mp\frac{i\mu}{2}(\frac{d}{2}-i\mu)} \times \left[\alpha^{\pm,\pm} \Pi_{\frac{d}{2}+i\mu,J}^{\text{AdS}}(-\eta_1 e^{\pm\frac{i\mu}{2}}, \mathbf{k}; -\eta_2 e^{\mp\frac{i\mu}{2}}, -\mathbf{k}) \right. \\ &\quad \left. + \beta^{\pm,\pm} \Pi_{\frac{d}{2}-i\mu,J}^{\text{AdS}}(-\eta_1 e^{\pm\frac{i\mu}{2}}, \mathbf{k}; -\eta_2 e^{\mp\frac{i\mu}{2}}, -\mathbf{k}) \right], \end{aligned} \quad (21)$$

which, combined with (20), makes it clear that in-in propagators are each analytic continuations of different linear combinations of $\Delta^{\pm}$ AdS propagators.

III. FOUR-POINT (A)dS EXCHANGES

The on-shell exchange of a particle is represented in four-point functions on the boundary of (A)dS space by Casimir invariants of the conformal group. In particular, the source-free equation of motion (6) satisfied by the corresponding on-shell propagator is equivalent to the quadratic Casimir equation

$$\{\hat{\mathcal{C}}_2 - [J(d+J-2) - \Delta(d-\Delta)]\} f_{\Delta,J}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = 0, \quad (22)$$

where $\hat{\mathcal{C}}_2$ is the quadratic Casimir of the conformal group [47]. An on-shell exchange in (A)dS with general boundary conditions is given by the general solution to this equation with a normalization that is fixed by on-shell factorization. In the following, for ease of presentation we focus on exchanges between generic external scalar fields.

Since boundary correlators dual to scattering processes in AdS are single valued functions of the cross ratios in the Euclidean regime [48] it is natural to consider single-valued solutions $\mathcal{F}_{\mu,J}$ to the quadratic Casimir equation (22), which we refer to as conformal partial waves [4]. These are the boundary counter parts of the bulk harmonic function $\Omega_{\mu,J}$. This duality is made manifest by the following representation of conformal partial waves,

$$\mathcal{F}_{\mu,J}^*(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = \mathcal{A}_{\mu_1, \mu_2, \mu, J}^*(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}) \times \mathcal{A}_{-\mu, J; \mu_3, \mu_4}^*(-\mathbf{k}; \mathbf{k}_3, \mathbf{k}_4), \quad (23)$$

$$\text{Disc}_{|\mathbf{k}|^2} [\mathcal{A}_{\alpha\Delta_+ + \beta\Delta_-, J}^*(\mathbf{k}_1, s_1, \mathbf{k}_2, s_2; \mathbf{k}_3, s_3, \mathbf{k}_4, s_4)] = \underbrace{[\alpha\omega_{\Delta_+}(u, \bar{u}) + \beta\omega_{\Delta_-}(u, \bar{u})]}_{\text{boundary conditions}} \times \Gamma(i\mu)\Gamma(-i\mu) \underbrace{\mathcal{F}_{\mu, J}^*(\mathbf{k}_1, s_1, \mathbf{k}_2, s_2; \mathbf{k}_3, s_3, \mathbf{k}_4, s_4)}_{\text{Conformal Partial Wave}}, \quad (24)$$

where the Mellin-Barnes representation of the conformal partial wave (23) is simply inherited from the Mellin-Barnes representation of the three-point functions, which is given in the supplementary material [49].

Above we fixed the form of an on-shell exchange in (A) dS by solving the Casimir equation (22) and fixing the normalization from the on-shell factorization. The full, off-shell exchange satisfies the same equation but with a

which is the product of the three-point functions generated by the cubic vertices that mediate the exchange. As before, $\bullet = \text{AdS or dS}, \pm, \hat{\pm}$, which fixes the normalization by the requirement that it is consistent with on-shell factorization of the AdS exchange or the $\pm\hat{\pm}$ contribution to the dS exchange. To obtain the general solution to the conformal invariance equation (22) we employ the duality between $\mathcal{F}_{\mu, J}$ and $\Omega_{\mu, J}$, where at the level of the Mellin-Barnes representation one simply multiplies with a general linear combination of projectors (12)

$$\mathcal{A}_{\alpha\Delta_+ + \beta\Delta_-, J}^*(\mathbf{k}_{1,2}, s_{1,2}; \mathbf{k}_{3,4}, s_{3,4}) = \underbrace{\csc(\pi(u + \bar{u}))}_{\text{contact terms}} \underbrace{[\alpha\omega_{\Delta_+}(u, \bar{u}) + \beta\omega_{\Delta_-}(u, \bar{u})]}_{\text{boundary conditions}} \times \Gamma(i\mu)\Gamma(-i\mu) \underbrace{\mathcal{F}_{\mu, J}^*(\mathbf{k}_{1,2}, s_{1,2}; \mathbf{k}_{3,4}, s_{3,4})}_{\text{Conformal Partial Wave}}. \quad (25)$$

where for dS the full exchange is the sum of the in-in contributions $\bullet = \text{dS}, \pm, \hat{\pm}$,

$$\mathcal{A}_{\mu, J}^{\text{dS}} = \sum_{\pm, \hat{\pm}} (\pm i)(\hat{\pm} i) \mathcal{A}_{\alpha_{\pm, \hat{\pm}} \Delta_+ + \beta_{\pm, \hat{\pm}} \Delta_-, J}^{\text{dS}, \pm, \hat{\pm}}. \quad (26)$$

Recall that $\alpha_{\pm, \hat{\pm}}$ and $\beta_{\pm, \hat{\pm}}$ parametrize the vacuum at early times and for the Bunch-Davies vacuum are given by (20).

Note that the conformal partial wave (23) for dS exchanges, $\bullet = \text{dS}, \pm, \hat{\pm}$, is proportional to that for exchanges in AdS, $\bullet = \text{AdS}$, since it amounts to a choice of normalization determined by the requirement of on-shell

contact term source on the right-hand side. The solution to the latter can be reconstructed from the knowledge of the solution to its homogeneous counterpart—i.e., reconstructing the full exchange from its on-shell part (24). As we saw in Sec. II this is straightforward at the level of the Mellin-Barnes representation, where the bulk-bulk propagator is obtained from its on-shell part by multiplying it with $\csc(\pi(u + \bar{u}))$. The full exchange is therefore

factorization. The precise proportionality constant is given by the proportionality constants between the $\pm$ and $\hat{\pm}$ three-point functions in dS and their AdS counterparts, which are given in the supplementary material [49]. In particular,

$$\mathcal{F}_{\mu, J}^{\text{dS}, \pm, \hat{\pm}}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = e^{(\mp\frac{\pi}{2}\hat{\pm}\frac{\pi}{2} + i(\mp\mu_1 \mp\mu_2 \mp\mu \hat{\pm}\mu \hat{\pm}\mu_3 \hat{\pm}\mu_4))\frac{\pi i}{2}} \times \mathcal{F}_{\mu, J}^{\text{AdS}}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4). \quad (27)$$

This means that we can express the in-in exchange in dS as a sum of AdS exchanges $\mathcal{A}_{\pm, J}^{\text{AdS}}$. For the Bunch-Davies initial condition we have [50]:

$$\mathcal{A}_{\mu, J}^{\text{dS, B.D.}} = -\frac{1}{2\mu i} \sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_- + J - d)\right) \sin\left(\frac{\pi}{2}(\Delta_3 + \Delta_4 + \Delta_- + J - d)\right) \mathcal{A}_{\Delta_-, J}^{\text{AdS}} + \frac{1}{2\mu i} \sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_+ + J - d)\right) \sin\left(\frac{\pi}{2}(\Delta_3 + \Delta_4 + \Delta_+ + J - d)\right) \mathcal{A}_{\Delta_+, J}^{\text{AdS}}. \quad (28)$$

This equation is the main result. Using this relation, techniques and results enjoyed by both the position and momentum space exchanges in AdS can be directly imported to dS, some of which we shall explore in the following section.

Note that the above relation between dS and AdS exchanges reproduces the OPE limit of the dS exchange derived in [24,25]. The factors multiplying each AdS exchange are the products of sinusoidal factors from the corresponding three-point contact diagrams. *A posteriori*, one could have recovered the full expression for the dS exchange from its OPE limit by completing each single-trace conformal block to its corresponding AdS exchange. Assuming single valuedness of the dS exchange, the result (28) would have then been the unique possible solution. The way we arrived to the expression (28) indeed confirms that single valuedness of dS exchanges would have been a valid assumption.

IV. SPECTRAL REPRESENTATION, MELLIN AMPLITUDES AND CROSSING IN DE SITTER

A. Spectral representation

The spectral representation of a conformal four-point function reads [9,10,51]

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = (\text{non-norm}) + \sum_J \int_{-\infty}^{\infty} d\nu a(\nu) \mathcal{F}_{\nu,J}, \quad (29)$$

where the spectral integral captures only the normalizable part of the four-point function and $a(\nu)$ has poles at the scaling dimensions of physical exchanged primary operators. The spectral representation of spinning AdS exchanges was given in [5,52], which follows from the harmonic function decomposition (8) and (9) of the bulk-bulk propagators. From the relation (28) we can immediately write down the spectral representation for exchanges in dS,

$$\begin{aligned} \mathcal{A}_{\mu,J}^{\text{dS,B.D.}} = & \underbrace{-\sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_- + J - d)\right) \sin\left(\frac{\pi}{2}(\Delta_3 + \Delta_4 + \Delta_- + J - d)\right) \mathcal{F}_{\mu,J}^{\text{AdS}}}_{\text{non-normalizable}} \\ & + \frac{1}{2} \frac{\pi}{\mu^2} \frac{1}{\Gamma(i\mu)\Gamma(-i\mu)} \sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + 2J - d)\right) \int_{-\infty}^{\infty} d\nu \frac{1}{\nu^2 + (\Delta_+ - \frac{d}{2})^2} \frac{\nu^2}{\pi} \mathcal{F}_{\nu,J}^{\text{AdS}}. \end{aligned} \quad (30)$$

This also formally extends to Δ_+ on the principal series using the appropriate ϵ -prescription, as was shown in Sec. 4.7 of [25].

Note that the conformal partial wave on the first line of (30) does not contain bulk contact terms since it is a solution to the quadratic Casimir equation. The relation (28) tells us that the four-point contact terms are given by the AdS counterparts multiplied by the factor

$\sin(\frac{\pi}{2}(2J - d + \sum_{j=1}^4 \Delta_j))$. This was also observed in [24,25] and extends to four points the relation between three-point AdS and dS contact diagrams.

B. Mellin amplitudes

Similarly we can write down Mellin amplitudes for dS exchanges,

$$\begin{aligned} \mathcal{M}_{\mu,J}^{\text{dS,B.D.}} = & -\frac{1}{2\mu i} \sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_- + J - d)\right) \sin\left(\frac{\pi}{2}(\Delta_3 + \Delta_4 + \Delta_- + J - d)\right) \mathcal{M}_{\Delta_-,J}^{\text{AdS}} \\ & + \frac{1}{2\mu i} \sin\left(\frac{\pi}{2}(\Delta_1 + \Delta_2 + \Delta_+ + J - d)\right) \sin\left(\frac{\pi}{2}(\Delta_3 + \Delta_4 + \Delta_+ + J - d)\right) \mathcal{M}_{\Delta_+,J}^{\text{AdS}}, \end{aligned} \quad (31)$$

in terms of their AdS counterparts [3,51]

$$\begin{aligned} \mathcal{M}_{\Delta_{\pm},J}^{\text{AdS}}(s,t) = & C_{\Delta_1,\Delta_2,\Delta_{\pm};J} C_{\Delta_3,\Delta_4,\Delta_{\pm};J} \\ & \times \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{\tau_{\pm},J|m}(s)}{t - \tau_{\pm} - 2m} + \text{contact}, \end{aligned} \quad (32)$$

where $\tau_{\pm} = \Delta_{\pm} - J$ is the twist and $C_{\Delta_i,\Delta_j,\Delta_{\pm};J}$ are the OPE coefficients [53]. The $\mathcal{Q}_{\tau_{\pm},J|m}(s)$ are kinematic polynomials. We employed the conventions and notations of [51], to which we refer the reader for details.

Note that this result unlocks another tool for studying the flat limit of dS correlators. In particular, one might try to make use of Penedones' formula [3] relating a Mellin amplitude in a large- N CFT and the corresponding flat space S -matrix. For dS, naively one would write

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-i\infty}^{+i\infty} d\alpha e^{\alpha} \alpha^{\frac{d-\Delta_1-\Delta_2-\Delta_3-\Delta_4}{2}} \\ \times \mathcal{M}_{\mu,J}^{\text{dS,B.D.}}\left(\frac{R^2 s}{4\alpha}, \frac{R^2 t}{4\alpha}; \Delta_j = iRm_j\right), \end{aligned} \quad (33)$$

up to an overall normalization following from that of the bulk-boundary propagators.

C. Conformal block decomposition and crossing

Using (28) the conformal block decomposition of dS exchanges can be written down from the knowledge of the decomposition of their AdS counterparts $\mathcal{A}_{\Delta_{\pm},J}^{\text{AdS}}$. The decomposition of the s -channel dS exchange (28) in terms of direct channel conformal blocks reads

$$\begin{aligned} {}^{(s)}\mathcal{A}_{\mu,J}^{\text{dS,B.D.}}(u,v) &= a_{\Delta_+,J} g_{\Delta_+,J}(u,v) \\ &+ \sum_{n,J'=0}^J [{}^{(s)}a_{n,J'|\Delta_+,J} g_{\Delta_1+\Delta_2+2n,J'}(u,v) \\ &+ {}^{(s)}b_{n,J'|\Delta_+,J} g_{\Delta_3+\Delta_4+2n,J'}(u,v)] + (\Delta_+ \rightarrow \Delta_-), \end{aligned} \quad (34)$$

where u, v are the usual cross ratios and the OPE coefficients are given in terms of their AdS counterparts via

$$\begin{aligned} \frac{a_{\Delta_{\pm},J}}{a_{\Delta_{\pm},J}^{\text{AdS}}} &= \frac{{}^{(s)}a_{n,J'|\Delta_{\pm},J}}{{}^{(s)}a_{n,J'|\Delta_{\pm},J}^{\text{AdS}}} = \frac{{}^{(s)}b_{n,J'|\Delta_{\pm},J}}{{}^{(s)}b_{n,J'|\Delta_{\pm},J}^{\text{AdS}}} \\ &= \frac{1}{2\Delta_{\pm} - d} \sin\left(\frac{\pi}{2}(\Delta_{1+2} + \Delta_{\pm} + J - d)\right) \\ &\times \sin\left(\frac{\pi}{2}(\Delta_{3+4} + \Delta_{\pm} + J - d)\right). \end{aligned} \quad (35)$$

For the single-trace exchanges $a_{\Delta_{\pm},J}$ on the first line of (34), the AdS OPE coefficients are simply

$$a_{\Delta_{\pm},J}^{\text{AdS}} = \frac{B_{\Delta_1,\Delta_2,\Delta_{\pm},J} B_{\Delta_3,\Delta_4,\Delta_{\pm},J}}{C_{\Delta_{\pm},J}}. \quad (36)$$

The AdS OPE coefficients $a_{n,J'|\Delta_{\pm},J}^{\text{AdS}}$ and $b_{n,J'|\Delta_{\pm},J}^{\text{AdS}}$ for exchanges of double-trace operators on the second and third lines of (34) are nonanalytic in spin J' . These have been computed e.g., in [54–58].

We can also determine the conformal block decomposition of exchanges under crossing. The decomposition of a

t - or u -channel dS exchange in terms of direct channel conformal blocks reads

$$\begin{aligned} {}^{(t,u)}\mathcal{A}_{\mu,J}^{\text{dS,B.D.}}(u,v) &= \sum_{n,J'=0}^{\infty} [{}^{(t,u)}a_{n,J'|\Delta_+,J}^{\text{dS}} g_{\Delta_1+\Delta_2+2n,J'}(u,v) \\ &+ {}^{(t,u)}b_{n,J'|\Delta_+,J}^{\text{dS}} g_{\Delta_3+\Delta_4+2n,J'}(u,v)] \\ &+ (\Delta_+ \rightarrow \Delta_-), \end{aligned} \quad (37)$$

where now ${}^{(t,u)}a_{n,J'|\Delta_{\pm},J}^{\text{dS}}$ and ${}^{(t,u)}b_{n,J'|\Delta_{\pm},J}^{\text{dS}}$ are given by (35) with $\Delta_2 \leftrightarrow \Delta_3$ and $\Delta_2 \leftrightarrow \Delta_4$ respectively in the t and u channel. The AdS OPE coefficients ${}^{(t,u)}a_{n,J'|\Delta_{\pm},J}^{\text{AdS}}$ and ${}^{(t,u)}b_{n,J'|\Delta_{\pm},J}^{\text{AdS}}$ of double-trace operators induced by crossed channel exchanges have a universal component that is analytic in spin J' , which have been computed in various works [54–57,59–61]. In addition to the analytic in-spin part there are also nonanalytic in spin contributions coming from bulk contact terms, which have been computed e.g., in [54–58,62].

Regarding bulk exchanges as a basis to decompose CFT four-point functions in the spirit of the Polyakov bootstrap [54,63], the freedom to add bulk contact terms is a source of ambiguity in the OPE data. A prescription to fix this contact-term ambiguity for CFT correlators dual to physics in AdS was proposed in [58,64] (see also the related [65]). Given our result (28), we expect this prescription and the corresponding (cyclic) Polyakov blocks to carry over to correlators in Euclidean CFTs dual to physics in dS, although their relevance at the nonperturbative level in dS should be clarified.

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- [48] Below we shall see that this is also true for dS exchanges with Bunch Davies initial conditions.
- [49] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevD.104.L081902> From AdS to dS Exchanges: Spectral Representation, Mellin Amplitudes and Crossing.

- [50] Here for concision we have omitted the factors (3). Note that for each external leg one should multiply by the factor

$$\mathcal{N}_{\frac{d}{2}+i\mu_j, J}^{\pm}(\eta_0) = (-\eta_0 e^{\mp \frac{i\pi}{2}})^{\frac{d}{2}+i\mu_j-J} \frac{\Gamma(-i\mu_j)}{2\sqrt{\pi}}, \quad (38)$$

See [25] for the details.

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