Exact solutions for scalars and spinors on quantized Euclidean AdS_2 space and the correspondence principle

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We obtain the exact solutions to the field equations for massless scalar and massless spinor fields on quantized two-dimensional anti-de Sitter space. We then apply the AdS/CFT correspondence principle to get exact answers for the two-point correlation functions of the associated operators on the boundary. The results support the conclusion that conformal symmetry on the boundary is not spoiled by quantization of the bulk. Moreover, quantization of the bulk has no effect on the spinor two-point correlation function, while it induces an overall rescaling in the scalar two-point correlation function.

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I. INTRODUCTION

Research on the quantization of space-time has been carried out with the intention of probing the quasiclassical regime of quantum gravity. In this regard, arguments have been given supporting the idea that the quasiclassical regime can be described in terms of quantum field theory on a noncommutative, i.e., quantized, background [1]. Meanwhile, the AdS/CFT correspondence principle offers the possibility of a nonperturbative description of quantum gravity on asymptotically AdS spaces [2–5]. (Despite this, many applications of the correspondence principle rely on *classical* gravity calculations.) In an attempt to merge these different approaches, we have studied the AdS/CFT correspondence principle in a noncommutative setting [6–8]. By this we mean that the geometry on the gravity side of the correspondence was written on a quantized space-time, which we briefly review below.

In [6,7] we examined the example where the bulk geometry is the quantized version of two-dimensional anti-de Sitter space (AdS_2). In this regard, AdS_2 has a unique quantization upon imposing the requirement that all the isometries of the classical geometry are preserved under the quantization [9–14].¹ The AdS/CFT correspondence

principle posits a strong/weak duality between quantum gravity in some d -dimensional bulk and a field theory on its $d - 1$ boundary [2–5]. The bulk is traditionally taken to be an asymptotically anti-de Sitter space, with a corresponding conformal field theory (CFT) residing on its boundary, although many extensions of the correspondence having been proposed. We argued in [6,7] that the quantized version of AdS_2 is an asymptotically anti-de Sitter space. This is due to (i) the result that the star product used to realize the noncommutative algebra approaches the trivial commutative product in the boundary limit, and (ii) all quantum corrections to the Killing vectors vanish in this limit.² Therefore, according to the correspondence principle, a dual conformal quantum theory should reside on the boundary of quantized AdS_2 (barring known difficulties of the correspondence principle for two-dimensional anti-de Sitter space [15–18]). An initial study of the corresponding boundary theory was pursued in [6,7]. There we examined the two- and three-point correlation functions of operators on the boundary which were sourced by scalar fields on quantized AdS_2 . [More precisely, the analysis and quantization was carried out for the Euclidean version of AdS_2 (EAdS_2).] Due to rather involved calculations in the approach followed there, the best we could do was obtain the leading order quantum (noncommutative) corrections to the usual two- and three-point correlation functions, i.e., those associated with a “classical” space-time bulk.

In this article we reexamine the correspondence principle for quantized EAdS_2 , and obtain *exact* results for the two-point correlation functions of operators on the boundary. This is done for operators sourced by both scalar and spinor

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¹Although it may not be possible to make the same demand for arbitrary $d > 2$ dimensional anti-de Sitter spaces, higher-dimensional generalizations of such an isometry preserving quantization procedure do exist for other spaces, namely, indefinite complex projective spaces ($\mathbb{CP}^{p,q}$, $q \geq 1$), as was shown in [8].

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²Analogous results were shown for the quantization of $\mathbb{CP}^{p,q}$, $q \geq 1$, implying that these spaces are asymptotically commutative.

fields in the bulk. The exact results for the two-point correlators are possible due to the existence of exact solutions to both the scalar and spinor field equations on quantized EAdS₂. These equations, which involve the relevant Laplacian and Dirac operators, can be written purely algebraically. For simplicity, we shall specialize here to the case of massless scalar and spinor fields, although exact solutions should be obtainable for the massive case, as well. Our result shows that the conformal properties of the boundary two-point correlation functions are not spoiled by the quantization of the bulk. This is not unexpected since the corresponding isometries on the gravity side are preserved. Moreover, the boundary two-point correlation function sourced by spinors on quantized EAdS₂ is *identical* to that of its classical counterpart. Therefore, from such boundary correlation functions, one cannot distinguish whether the bulk theory is classical or quantum. On the other hand, the boundary two-point correlation function sourced by scalars on quantized EAdS₂ differs from its classical counterpart by a simple overall factor, which agrees with the leading order result of [6,7].

The outline for the rest of this article is as follows: In Sec. II we review the dynamics of massless scalar and spinor fields on EAdS₂. Two formulations of spinors are given, which we refer to as the noncovariant and covariant formulations, and a map between the two is given. While the noncovariant approach is simpler, the covariant approach has a natural generalization to quantum space-time. As a preliminary step towards quantization we introduce a Poisson structure to EAdS₂. The quantization of EAdS₂ is reviewed in Sec. III. There we write down the Laplacian and Dirac operators on quantized EAdS₂, and present exact solutions to the equations for both massless scalar and spinor fields on this space. From the asymptotic properties of these solutions, we derive exact results for the associated two-point correlation functions on the boundary in Sec. IV. Concluding remarks are presented in Sec. V. In Appendix A we show how to evaluate boundary terms on quantized EAdS₂. We find the quantum analog of the map between the covariant and noncovariant bases in Appendix B, and use it to construct the Dirac operator in the noncovariant basis. The solutions for massless spinors in that basis are trivially obtained. They can then be mapped back to the covariant basis, and the result agrees with the exact solutions of Sec. III.

II. SCALARS AND SPINORS ON EAdS₂

A. Preliminaries

We begin with a review of the Euclidean AdS₂ geometry. EAdS₂ is conventionally coordinatized using Fefferman-Graham coordinates (z, t) spanning a half-plane $\mathbb{R}_+^2$, $z > 0$, $-\infty < t < \infty$. The asymptotic boundary corresponds to $z \rightarrow 0$. The metric tensor in terms of these coordinates is given by

$$ds^2 = \frac{dz^2 + dt^2}{z^2}. \quad (2.1)$$

EAdS₂ is a maximally isometric space, with the isometries generated by the three Killing vectors

$$\begin{aligned} \mathcal{K}^- &= -\partial_t, & \mathcal{K}^1 &= -t\partial_t - z\partial_z, \\ \mathcal{K}^+ &= (z^2 - t^2)\partial_t - 2zt\partial_z, \end{aligned} \quad (2.2)$$

which form a basis for the $su(1,1)$ algebra. Defining $\mathcal{K}^0 = \frac{1}{2}(\mathcal{K}^+ + \mathcal{K}^-)$ and $\mathcal{K}^2 = \frac{1}{2}(\mathcal{K}^+ - \mathcal{K}^-)$, we get the commutator brackets

$$[\mathcal{K}^0, \mathcal{K}^1] = \mathcal{K}^2, \quad [\mathcal{K}^1, \mathcal{K}^2] = -\mathcal{K}^0, \quad [\mathcal{K}^2, \mathcal{K}^0] = \mathcal{K}^1. \quad (2.3)$$

The Laplacian on EAdS₂ is

$$\Delta = z^2(\partial_z^2 + \partial_t^2). \quad (2.4)$$

It can be written in terms of the three Killing vectors

$$\Delta = \mathcal{K}^a \mathcal{K}_a, \quad (2.5)$$

where the index $a = 0, 1, 2$ is raised and lowered using the metric tensor $[\eta_{ab}] = \text{diag}(-1, +1, +1)$. The equation for a free massless scalar Φ on EAdS₂,

$$\Delta\Phi = 0, \quad (2.6)$$

is easily solved. General solutions are expressed in terms of analytic and antianalytic functions of $\zeta = z + it$,

$$\Phi(\zeta, \bar{\zeta}) = f(\zeta) + g(\bar{\zeta}), \quad (2.7)$$

f and g being complex functions. Upon approaching the boundary $z = 0$ the solution behaves as

$$\begin{aligned} \Phi(\zeta, \bar{\zeta}) &= \phi_0(t) + z\phi_1(t) + \mathcal{O}(z^2), \\ \phi_0(t) &= f(it) + g(-it), \\ \phi_1(t) &= (f'(\zeta) + g'(\bar{\zeta}))|_{z=0}, \end{aligned} \quad (2.8)$$

the prime denoting a derivative.

The Dirac operator on an arbitrary d -dimensional commutative manifold is

$$D = \gamma^\mu(\partial_\mu + \omega_\mu), \quad (2.9)$$

where ω_μ , $\mu = 1, \dots, d$ are the components of the spin connection one-form and our convention is that the gamma matrices γ^μ satisfy the anticommutator

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\mathbb{1}. \quad (2.10)$$

Upon assuming that the manifold is locally Euclidean, the gamma matrices are transformed to a local Euclidean frame

using vielbein fields $e^i_\mu: \gamma^\mu \rightarrow \hat{\gamma}^i = e^i_\mu \gamma^\mu$. The transformed gamma matrices $\hat{\gamma}^i$ satisfy the anticommutation relations: $[\hat{\gamma}^i, \hat{\gamma}^j]_+ = 2\delta^{ij}\mathbb{1}$. We shall assume that the space has zero torsion. Denoting a basis of one-forms by $\{dx^\mu\}$, we write down the vielbein and spin connection one-forms $e = e_\mu dx^\mu$ and $\omega = \omega_\mu dx^\mu$, respectively. e_μ and ω_μ take values in the Clifford algebra generated by $\hat{\gamma}^i$, $e_\mu = e^i_\mu \hat{\gamma}_i$ and $\omega_\mu = \frac{1}{8} \omega_{ij\mu} [\hat{\gamma}^i, \hat{\gamma}^j]$. Then the zero torsion condition is

$$de + \omega \wedge e + e \wedge \omega = 0. \quad (2.11)$$

In the next two subsections we shall write down two different realizations of the Dirac operator, one in what we call the noncovariant basis and another in the covariant basis. While the former basis is more familiar, the latter is crucial for the isometry preserving quantization. We also write down the map between the two realizations. A relevant feature of this map is that it is singular at the asymptotic boundary. A preliminary step towards quantization is taken in the third subsection by introducing a natural Poisson structure on the manifold.

B. Dirac operator in a noncovariant basis

We denote gamma matrices, the Dirac operator and spinors in the noncovariant basis with a tilde. (We shall drop the tilde in the next subsection for the corresponding quantities in the covariant basis.) In writing down the Dirac operator on EAdS₂ we can take the two gamma matrices $\{\gamma^\mu\} = \{\tilde{\gamma}^z, \tilde{\gamma}^t\}$ to be

$$\tilde{\gamma}^z = z\sigma_1, \quad \tilde{\gamma}^t = z\sigma_2, \quad (2.12)$$

σ_i being Pauli matrices. This choice satisfies (2.10). The vielbein one-form is then

$$e = \sigma_i e^i_\mu dx^\mu = \frac{1}{z}(\sigma_1 dz + \sigma_2 dt). \quad (2.13)$$

The torsion free condition (2.11) is satisfied for

$$\omega = \frac{i}{2z} \sigma_3 dt. \quad (2.14)$$

The Dirac operator (2.9) then becomes

$$\begin{aligned} \tilde{D} &= z(\sigma_1 \partial_z + \sigma_2 \partial_t) - \frac{1}{2} \sigma_1 = z^{3/2}(\sigma_1 \partial_z + \sigma_2 \partial_t) \frac{1}{\sqrt{z}} \\ &= z^{3/2} \begin{pmatrix} 0 & \partial_z - i\partial_t \\ \partial_z + i\partial_t & 0 \end{pmatrix} \frac{1}{\sqrt{z}}. \end{aligned} \quad (2.15)$$

The corresponding chirality operator is σ_3 as it anticommutes with the Dirac operator $[\tilde{D}, \sigma_3]_+ = 0$, and squares to the identity. $\tilde{D}$ satisfies the Lichnerowicz formula:

$$\sqrt{z} \tilde{D} \frac{1}{z} \tilde{D} \sqrt{z} = \Delta. \quad (2.16)$$

The free massless spinor $\tilde{\Psi}$ on EAdS₂ satisfies $\tilde{D} \tilde{\Psi} = 0$, which is easily solved. General solutions are again expressed in terms of analytic and antianalytic functions of $\zeta = z + it$,

$$\tilde{\Psi}(\zeta, \bar{\zeta}) = \sqrt{z} \begin{pmatrix} f(\zeta) \\ g(\bar{\zeta}) \end{pmatrix}, \quad z = Re\zeta. \quad (2.17)$$

Here the complex functions f and g are associated with solutions of different chirality. In the boundary limit $z \rightarrow 0$,

$$\tilde{\Psi}(\zeta, \bar{\zeta}) \rightarrow \sqrt{z} \tilde{\psi}_0(t), \quad \tilde{\psi}_0(t) = \begin{pmatrix} f(it) \\ g(-it) \end{pmatrix}. \quad (2.18)$$

C. Dirac operator in the covariant basis

In formulating the covariant form of the Dirac operator, it is convenient to embed EAdS₂ in $\mathbb{R}^{1,2}$. Calling the embedding coordinates X^a , $a = 0, 1, 2$, EAdS₂ corresponds to a single sheeted hyperboloid

$$X^a X_a = -1, \quad (2.19)$$

where the metric on $\mathbb{R}^{1,2}$ is $[\eta_{ab}] = \text{diag}(-1, +1, +1)$, and, for convenience, the EAdS₂ scale is normalized to one.³ Expressed in terms of Fefferman-Graham coordinates the embedding is given by

$$X^0 = -\frac{z^2 + t^2 + 1}{2z}, \quad X^1 = -\frac{t}{z}, \quad X^2 = -\frac{z^2 + t^2 - 1}{2z}, \quad (2.20)$$

putting the single sheeted hyperboloid in the region $X^0 < 0$.

In the covariant formulation of the Dirac operator we choose the chirality operator to be

$$\gamma = X^a \tau_a, \quad (2.21)$$

where $\{\tau_a, a = 0, 1, 2\}$ is a basis for $su(1, 1)$ in the defining representation. Our choice is $(\tau_0, \tau_1, \tau_2) = (\sigma_3, i\sigma_1, i\sigma_2)$, σ_a denoting Pauli matrices, which satisfies

$$\tau_a \tau_b = -i\epsilon_{abc} \tau^c - \eta_{ab} \mathbb{1}, \quad (2.22)$$

ϵ_{abc} being the Levi-Civita symbol with $\epsilon_{012} = -1$. We note that γ is not Hermitian, and its square is the unit operator.

The chirality operator γ can be mapped from the chirality operator in the noncovariant basis, σ_3 , using the similarity transformation

³Our index convention here is different from that in [6,7]. In the latter, $a = 2$ corresponded to the timelike direction.

$$\gamma = U\sigma_3 U^{-1}, \quad (2.23)$$

where the 2×2 matrix U expressed in complex coordinates ζ and $\bar{\zeta}$ is given by

$$U = \frac{1}{2\sqrt{z}} \begin{pmatrix} -\zeta + 1 & -\bar{\zeta} - 1 \\ \zeta + 1 & \bar{\zeta} - 1 \end{pmatrix}. \quad (2.24)$$

U has determinant equal to one and satisfies

$$D = i \begin{pmatrix} tz\partial_z + \frac{1}{2}(1+t^2-z^2)\partial_t + i & z(t+i)\partial_z - \frac{1}{2}(z^2-(t+i)^2)\partial_t \\ -z(t-i)\partial_z + \frac{1}{2}(z^2-(t-i)^2)\partial_t & -tz\partial_z - \frac{1}{2}(1+t^2-z^2)\partial_t + i \end{pmatrix}. \quad (2.27)$$

In comparing with (2.9), it corresponds to making the following choice for the gamma matrices

$$\begin{aligned} \gamma^z &= iz \begin{pmatrix} t & t+i \\ -t+i & -t \end{pmatrix}, \\ \gamma^t &= \frac{i}{2} \begin{pmatrix} 1+t^2-z^2 & -z^2+(t+i)^2 \\ z^2-(t-i)^2 & -(1+t^2-z^2) \end{pmatrix}. \end{aligned} \quad (2.28)$$

It can be checked that they satisfy (2.10). The gamma matrices (2.28) are not obtained with a local Lorentz transformation from $\hat{\gamma}^i = \sigma_i$. The transformed Dirac operator D can be reexpressed in terms of the three Killing vectors (2.2),

$$\begin{aligned} D &= i \begin{pmatrix} -\frac{1}{2}(\mathcal{K}^- + \mathcal{K}^+) + i & \frac{1}{2}(\mathcal{K}^- - \mathcal{K}^+) - i\mathcal{K}^1 \\ -\frac{1}{2}(\mathcal{K}^- - \mathcal{K}^+) - i\mathcal{K}^1 & \frac{1}{2}(\mathcal{K}^- + \mathcal{K}^+) + i \end{pmatrix} \\ &= -i\tau_a \mathcal{K}^a - \mathbb{1}. \end{aligned} \quad (2.29)$$

This Dirac operator is preserved under simultaneous $2+1$ Lorentz transformations on τ_a and $\mathcal{K}^a$. By construction, D anticommutes with the chirality operator γ . This can also be independently verified using the properties (2.3), (2.22) and

$$[\mathcal{K}^a, X^b] = \epsilon^{abc} X_c, \quad X_a \mathcal{K}^a = \mathcal{K}^a X_a = 0. \quad (2.30)$$

The solution Ψ to $D\Psi = 0$ is obviously obtained by transforming the solution (2.17) by U

$$\Psi = U\tilde{\Psi} = \frac{1}{2} \begin{pmatrix} -\zeta + 1 & -\bar{\zeta} - 1 \\ \zeta + 1 & \bar{\zeta} - 1 \end{pmatrix} \begin{pmatrix} f(\zeta) \\ g(\bar{\zeta}) \end{pmatrix}. \quad (2.31)$$

The boundary behavior of Ψ differs from that of $\tilde{\Psi}$ in (2.18) due to the fact that U is singular at the boundary. Upon Taylor expanding (2.31) at $z = 0$ up to first order, we get

$$U^\dagger \sigma_3 U = -\sigma_3. \quad (2.25)$$

While U has unit determinant for all z , U is singular in the boundary limit $z \rightarrow 0$. More specifically, $\det \sqrt{z}U \rightarrow 0$ in the limit.

If we apply the same similarity transformation to the Dirac operator written in the noncovariant basis (2.15),

$$\tilde{D} \rightarrow D = U\tilde{D}U^{-1}, \quad (2.26)$$

we get

$$\begin{aligned} \Psi(\zeta, \bar{\zeta}) &= \frac{1}{2} \begin{pmatrix} 1-it \\ 1+it \end{pmatrix} (f(it) - g(-it)) \\ &\quad + \frac{z}{2} \begin{pmatrix} -f(it) - g(-it) + (1-it)(f'(it) - g'(-it)) \\ f(it) + g(-it) + (1+it)(f'(it) - g'(-it)) \end{pmatrix} \\ &\quad + \mathcal{O}(z^2). \end{aligned} \quad (2.32)$$

We can write the result as

$$\begin{aligned} \Psi(\zeta, \bar{\zeta}) &= (\sigma_3 - it\mathbb{1})\psi_0(t) - iz\partial_t((\sigma_3 - it\mathbb{1})\psi_1(t)) \\ &\quad + \mathcal{O}(z^2), \end{aligned} \quad (2.33)$$

where

$$\begin{aligned} \psi_0(t) &= \frac{1}{2}(f(it) - g(-it)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ \psi_1(t) &= \frac{1}{2}(f(it) + g(-it)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned} \quad (2.34)$$

D. Poisson structure

A preliminary step towards quantization is to attach a Poisson structure to the manifold. Poisson brackets $\{, \}$ can be uniquely defined on EAdS_2 upon demanding that they preserve the isometry. This choice corresponds to having the embedding coordinates X^a define an $su(1, 1)$ algebra

$$\{X^a, X^b\} = \epsilon^{abc} X_c. \quad (2.35)$$

These relations are consistent with writing down the following Poisson bracket for the Feffermann-Graham coordinates spanning $\mathbb{R}_+^2$ [6,7]

$$\{t, z\} = z^2, \quad (2.36)$$

which can be easily verified using (2.20). It follows that $z^{-1} = X^2 - X^0$ and t are canonically conjugate coordinates on $\mathbb{R}_+^2$,

$$\{z^{-1}, t\} = 1. \quad (2.37)$$

Alternatively, one can define a Darboux map to coordinates (x, y) spanning the entire plane, where

$$x = -\ln z, \quad y = t/z, \quad (2.38)$$

and consequently $\{x, y\} = 1$.

Using (2.2), the action of the Killing vectors on functions $\mathcal{F}$ on EAdS₂ is implemented from Poisson brackets with the embedding coordinates

$$\mathcal{K}^a \mathcal{F} = \{X^a, \mathcal{F}\}. \quad (2.39)$$

It then trivially follows that the Poisson bracket and the defining relation (2.19) are preserved under the action of the Killing vectors

$$\begin{aligned} \mathcal{K}^d \{X^a, X^b\} &= \{X^d, \{X^a, X^b\}\} = \epsilon^{abc} \{X^d, X_c\} = \epsilon^{abc} \mathcal{K}^d X_c, \\ \mathcal{K}^a (X^b X_b) &= \{X^a, X^b X_b\} = 2\{X^a, X^b\} X_b = 0. \end{aligned} \quad (2.40)$$

The field equations for the scalar and spinor can be expressed using Poisson brackets with the embedding coordinates. For the massless scalar field Φ ,

$$\Delta \Phi = \{X^a, \{X_a, \Phi\}\} = 0, \quad (2.41)$$

while for the massless spinor Ψ ,

$$D\Psi = -i\{\gamma, \Psi\} - \Psi = 0. \quad (2.42)$$

III. SCALARS AND SPINORS ON QUANTIZED EAdS₂

We begin with the natural isometry preserving quantization of EAdS₂ in the first subsection. There we show how the field equations for the scalar and spinor can be written in an algebraic manner. The exact solutions for massless fields are presented in the second subsection.

A. Quantization

The quantization of EAdS₂ is unique upon demanding that it preserves the isometry of EAdS₂ [9,12], for this one maps the embedding coordinates X^a to operators $\hat{X}^a$, satisfying the $su(1, 1)$ algebra

$$[\hat{X}^a, \hat{X}^b] = i\alpha \epsilon^{abc} \hat{X}_c, \quad (3.1)$$

where α is the quantization parameter. As with commutative EAdS₂, we fix the scale to be one

$$\hat{X}^a \hat{X}_a = -1, \quad (3.2)$$

in analogy with (2.19). (Any quantum corrections to the Casimir can be removed with a rescaling of $\hat{X}_a$ and α .)

$\hat{X}_a$ generate the algebra $\mathcal{A}$ of quantized EAdS₂. They form a basis for $su(1, 1)$ in a unitary irreducible representation. It was shown in [6] that both the commutative limit and boundary limit can be formulated in terms of these representations. In order for there to be a consistent commutative limit to EAdS₂, one has to restrict the allowed representations to the discrete series $D^\pm(k)$, where $k < 0$ labels the representation. Upon choosing $D^+(k)$, we can go to an eigenbasis of $\hat{X}^0$, which we denote by $|k, m\rangle$, $m = 0, 1, 2, \dots$. The action of $\hat{X}^a$ on these states is

$$\begin{aligned} \hat{X}_+ |k, m\rangle &= -\alpha c_m |k, m+1\rangle, \\ \hat{X}_- |k, m\rangle &= -\alpha c_{m-1} |k, m-1\rangle, \\ \hat{X}^0 |k, m\rangle &= -\alpha(m-k) |k, m\rangle, \end{aligned} \quad (3.3)$$

where $\hat{X}_\pm = \hat{X}^2 \pm i\hat{X}^1$ and $c_m = \sqrt{(m+1)(m-2k)}$. For the action of the quadratic Casimir one gets

$$\hat{X}^a \hat{X}_a |k, m\rangle = -\alpha^2 k(k+1) |k, m\rangle. \quad (3.4)$$

Upon comparing with (3.2), we get that k is related to the noncommutativity parameter,

$$\alpha^2 k(k+1) = 1, \quad (3.5)$$

and we should make the restriction $k < -1$. The commutative limit $\alpha \rightarrow 0$ then corresponds to $k \rightarrow -\infty$. In Sec. III B we shall argue that $\hat{X}^2 - \hat{X}^0$ is the quantum analog of z^{-1} . From (3.3), its expectation value goes to infinity when $m \rightarrow \infty$. So it follows that this is boundary limit.

It is straightforward to generalize the notion of Killing vectors to quantized EAdS₂. Denoting them by $\hat{\mathcal{K}}^a$, their action on functions $\hat{\mathcal{F}}$ on $\mathcal{A}$ is given by

$$\hat{\mathcal{K}}^a \hat{\mathcal{F}} = -\frac{i}{\alpha} [\hat{X}^a, \hat{\mathcal{F}}]. \quad (3.6)$$

Just as with the Killing vectors $\mathcal{K}^a$, $\hat{\mathcal{K}}^a$ satisfy the $su(1, 1)$ algebra

$$[\hat{\mathcal{K}}^a, \hat{\mathcal{K}}^b] = \epsilon^{abc} \hat{\mathcal{K}}_c, \quad (3.7)$$

and the action of $\hat{\mathcal{K}}^a$ leaves the defining relations (3.1) and (3.2) invariant.

In defining the Laplacian and Dirac operators on quantized EAdS₂ it is convenient to introduce left and right acting operators $\hat{X}^{La}$ and $\hat{X}^{Ra}$, respectively. Their action on functions $\hat{\mathcal{F}}$ on quantized EAdS₂ is defined to be

$$\hat{X}^{La} \hat{\mathcal{F}} = \hat{X}^a \hat{\mathcal{F}}, \quad \hat{X}^{Ra} \hat{\mathcal{F}} = \hat{\mathcal{F}} \hat{X}^a. \quad (3.8)$$

We note that the right action is an anti-involution, i.e., for two right acting operators $\hat{A}^R$ and $\hat{B}^R$, $(\hat{A}\hat{B})^R = \hat{B}^R\hat{A}^R$. As a result there will be a sign difference between commutators of right acting operators and the corresponding commutators of left acting operators. In addition, right acting operators commute with left acting operators. In other words, $\hat{A}^R$ and $\hat{B}^R$ are elements of the opposite algebra, $\mathcal{A}^{\text{op}}$. Thus $\hat{X}^{La}$ and $\hat{X}^{Ra}$ satisfy

$$\begin{aligned} [\hat{X}^{La}, \hat{X}^{Lb}] &= i\alpha\epsilon^{abc}\hat{X}_c^L, & [\hat{X}^{Ra}, \hat{X}^{Rb}] &= -i\alpha\epsilon^{abc}\hat{X}_c^R, \\ [\hat{X}^{La}, \hat{X}^{Rb}] &= 0, & \hat{X}^{La}\hat{X}_a^L &= \hat{X}^{Ra}\hat{X}_a^R = -\mathbb{1}. \end{aligned} \quad (3.9)$$

The quantum version of the Killing vectors can be expressed as

$$\hat{K}^a = -\frac{i}{\alpha}(\hat{X}^{La} - \hat{X}^{Ra}). \quad (3.10)$$

The obvious choice for the Laplacian operator on quantized EAdS₂ is $\hat{\Delta} = \hat{K}^a\hat{K}_a$. It can be rewritten as

$$\begin{aligned} \hat{\Delta} &= -\frac{1}{\alpha^2}(\hat{X}^{La} - \hat{X}^{Ra})(\hat{X}_a^L - \hat{X}_a^R) \\ &= \frac{2}{\alpha^2}(\hat{X}^{Ra}\hat{X}_a^L + \mathbb{1}). \end{aligned} \quad (3.11)$$

So the equation of motion for a massless scalar field $\hat{\Phi}$ on quantized EAdS₂ can be written in a purely algebraic way

$$\frac{\alpha^2}{2}\hat{\Delta}\hat{\Phi} = \hat{X}_a\hat{\Phi}\hat{X}^a + \hat{\Phi} = 0. \quad (3.12)$$

This equation reduces to its commutative analog (2.41) after requiring that the commutator of two functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ on quantized EAdS₂ goes to $i\alpha$ times the associated Poisson bracket in the limit $\alpha \rightarrow 0$

$$[\hat{\mathcal{F}}, \hat{\mathcal{G}}] \rightarrow i\alpha\{\hat{\mathcal{F}}, \hat{\mathcal{G}}\}, \quad \text{as } \alpha \rightarrow 0. \quad (3.13)$$

Before writing down the Dirac operator on quantized EAdS₂, we examine the chirality operator $\hat{\gamma}$. It is required to (i) commute with the algebra $\mathcal{A}$, (ii) square to unity, and (iii) it should reduce to γ in the commutative limit $\alpha \rightarrow 0$. For (i), we note that although $\hat{X}^{La}$ does not commute with $\mathcal{A}$, $\hat{X}^{Ra}$ does. Using associativity,

$$(\hat{X}^{Ra}\hat{\mathcal{F}})\hat{\mathcal{G}} = \hat{X}^{Ra}(\hat{\mathcal{F}}\hat{\mathcal{G}}) = \hat{\mathcal{F}}\hat{\mathcal{G}}\hat{X}^a,$$

for any two-function $\hat{\mathcal{F}}, \hat{\mathcal{G}} \in \mathcal{A}$, while we get the same result from

$$(\hat{\mathcal{F}}\hat{X}^{Ra})\hat{\mathcal{G}} = \hat{\mathcal{F}}(\hat{X}^{Ra}\hat{\mathcal{G}}) = \hat{\mathcal{F}}\hat{\mathcal{G}}\hat{X}^a.$$

Therefore $\hat{\gamma}$ should be constructed from $\hat{X}^{Ra}$, and since we want to recover (2.21) in the commutative limit, a natural

candidate would be $\hat{X}^{Ra}\tau_a$. However, this does not square to unity, and quantum corrections must be introduced to satisfy (ii). A chirality operator satisfying (i)–(iii) was obtained by Fakhri and Imaanpur [10]:

$$\hat{\gamma} = \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \left(\hat{X}^{Ra}\tau_a - \frac{\alpha}{2}\mathbb{1} \right). \quad (3.14)$$

Upon taking its square

$$\hat{\gamma}^2 = \frac{1}{1 + \frac{\alpha^2}{4}} \left(\hat{X}^{Ra}\hat{X}^{Rb}\tau_a\tau_b - \alpha\hat{X}^{Ra}\tau_a + \frac{\alpha^2}{4} \right) = \mathbb{1}. \quad (3.15)$$

In writing down the Dirac operator $\hat{D}$ on quantized AdS₂, we should require (a) that it anticommutes with the chirality operator $\hat{\gamma}$ and (b) that it reduces to the Dirac operator D on EAdS₂ (2.29) in the commutative limit. A natural candidate for $\hat{D}$ that is consistent with the commutative limit is $-i\tau_a\hat{K}^a - \mathbb{1}$; however, this does not satisfy (a). On the other hand, (a) is satisfied for $\hat{X}^{Ra}(\epsilon_{abc}\hat{K}^b\tau^c + \tau_a)$

$$\left[\hat{X}^{Ra}(\epsilon_{abc}\hat{K}^b\tau^c + \tau_a), \hat{X}^{Rd}\tau_d - \frac{\alpha}{2}\mathbb{1} \right]_+ = 0. \quad (3.16)$$

In order to satisfy (b), one can multiply by $-\hat{\gamma}$. The Dirac operator is then [10]

$$\hat{D} = -\hat{\gamma}\hat{X}^{Ra}(\epsilon_{abc}\hat{K}^b\tau^c + \tau_a) = \frac{i}{\alpha}\epsilon_{abc}\hat{\gamma}\hat{X}^{Ra}\hat{X}^{Lb}\tau^c. \quad (3.17)$$

Upon expanding out the chirality operator one gets

$$\begin{aligned} \hat{D} &= \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \left(-i\tau_a\hat{K}^a - \mathbb{1} + i\alpha\hat{X}_a^R\hat{K}^a \right. \\ &\quad \left. - \frac{\alpha}{2}(\epsilon_{abc}\hat{X}^{Ra}\hat{K}^b + \hat{X}_c^R)\tau^c - i\hat{X}_a^R\tau^a\hat{X}_b^R\hat{K}^b \right). \end{aligned} \quad (3.18)$$

We note that $\hat{X}_a^R\hat{K}^a$ acting on $\mathcal{A}$ is of order α , and so to leading order in α , $\hat{D} \sim -i\tau_a\hat{K}^a - \mathbb{1}$, which is consistent with the requirement (b).

Equation (3.17) is the quantization of the Dirac operator D (2.29) written in the covariant basis. In Appendix B we obtain the quantization of the Dirac operator D (2.15) written in the noncovariant basis. This is done by finding the quantum analog of the 2×2 matrix U in (2.24), and applying it to write down the inverse of the map (2.26).

Using (3.17), the Dirac equation for a massless particle on quantized EAdS₂ is

$$\hat{D}\hat{\Psi} = \frac{i}{\alpha}\epsilon_{abc}\hat{\gamma}\hat{X}^{Ra}\hat{X}^{Lb}\tau^c\hat{\Psi} = 0, \quad (3.19)$$

or simply

$$\epsilon_{abc}\tau^a\hat{X}^b\hat{\Psi}\hat{X}^c = 0. \quad (3.20)$$

This equation reduces to its commutative analog (2.42), after again requiring (3.13). Like the equation for the massless scalar field (3.12), the field equation for a massless spinor on quantized EAdS₂ has an algebraic form. Both of these equations have exact solutions which we show in the next subsection.

B. Exact solutions

Here we find the exact solutions to the quantized massless Klein-Gordon (3.12) and Dirac (3.20) equations. We will start with the scalar case and then we will show how to map the scalar solutions to the spinor ones. Because the exposition is rather technical, we will structure it as the sequence of several propositions.

As the first step we want to establish the quantum analog of the Fefferman-Graham coordinates and their relation to the quantum embedding coordinates. Classically, this relation is given in (2.20). Inverting this we have for the commutative Fefferman-Graham coordinates

$$z = \frac{1}{X^2 - X^0}, \quad t = -zX^1 \equiv -\frac{X^1}{X^2 - X^0}. \quad (3.21)$$

As a quantization of these relations we will take the symmetric ones

$$\begin{aligned} \hat{z} &= (\hat{X}^2 - \hat{X}^0)^{-1}, \\ \hat{t} &= -\frac{1}{2}(\hat{z}\hat{X}^1 + \hat{X}^1\hat{z}) \\ &\equiv -\frac{1}{2}((\hat{X}^2 - \hat{X}^0)^{-1}\hat{X}^1 + \hat{X}^1(\hat{X}^2 - \hat{X}^0)^{-1}) \end{aligned} \quad (3.22)$$

and show below that this is a consistent choice. Because $[\hat{z}, \hat{X}^a] = -\hat{z}[\hat{z}^{-1}, \hat{X}^a]\hat{z}$, it is easy to find all the commutators $[\hat{z}, \hat{X}^a]$ using $\hat{z}^{-1} = \hat{X}^2 - \hat{X}^0$ and (3.1)

$$\begin{aligned} [\hat{z}, \hat{X}^1] &= i\alpha\hat{z}, \\ [\hat{z}, \hat{X}^0] &= -i\alpha\hat{z}\hat{X}^1\hat{z} \equiv i\alpha\hat{z}\hat{t} - \frac{\alpha^2}{2}\hat{z}^2 \equiv i\alpha\hat{t}z + \frac{\alpha^2}{2}\hat{z}^2 \\ &\equiv \frac{i\alpha}{2}(\hat{z}\hat{t} + \hat{t}\hat{z}), \\ [\hat{z}, \hat{X}^2] &= [\hat{z}, \hat{X}^0]. \end{aligned} \quad (3.23)$$

From these commutators and the definition of $\hat{t}$ in (3.22) we immediately get

$$[\hat{t}, \hat{z}] = i\alpha\hat{z}^2, \quad [\hat{t}, \hat{z}^{-1}] = -i\alpha, \quad (3.24)$$

i.e. the correct quantization of (2.36) and (2.37) without any quantum corrections.

To find the commutators for $\hat{t}$ and $\hat{X}^a$, we first want to find the quantum analog of (2.20). It turns out that the quantization is nontrivial, compared to (3.22). While for $\hat{X}^1$ one trivially has from (3.22)

$$\hat{X}^1 = -\frac{1}{2}(\hat{z}^{-1}\hat{t} + \hat{t}\hat{z}^{-1}) \equiv -\hat{t}\hat{z}^{-1} - \frac{i\alpha}{2} \equiv -\hat{z}^{-1}\hat{t} + \frac{i\alpha}{2}, \quad (3.25)$$

the similar relations for $\hat{X}^{0,2}$ are less trivial. Using (3.2) we have

$$\begin{aligned} \hat{z}^{-1}(\hat{X}^2 + \hat{X}^0) &= (\hat{X}^2 - \hat{X}^0)(\hat{X}^2 + \hat{X}^0) \\ &= -1 - (\hat{X}^1)^2 + i\alpha\hat{X}^1. \end{aligned} \quad (3.26)$$

Then combining this with (3.25) we get

$$\hat{X}^2 + \hat{X}^0 = -\kappa(\alpha)^2\hat{z} - \hat{t}\hat{z}^{-1}\hat{t}, \quad (3.27)$$

where we define

$$\kappa(\alpha) := \sqrt{1 + \frac{\alpha^2}{4}}. \quad (3.28)$$

Notice that the same factor appears in the denominator of the quantum chirality operator (3.14) and Dirac operator (3.17). Furthermore, from (3.5), this factor can be related to the label k of the discrete series representation $D^+(k)$ in a simple way: $\frac{\kappa(\alpha)}{\alpha} = -k - \frac{1}{2}$. (Recall, $k < -1$.)

Combining (3.27) with the definition of $\hat{z}$ we finally get the full set of relations

$$\begin{aligned} \hat{X}^1 &= -\frac{1}{2}(\hat{z}^{-1}\hat{t} + \hat{t}\hat{z}^{-1}) \equiv -\hat{t}\hat{z}^{-1} - \frac{i\alpha}{2} \equiv -\hat{z}^{-1}\hat{t} + \frac{i\alpha}{2}, \\ \hat{X}^0 &= -\frac{1}{2}(\kappa(\alpha)^2\hat{z} + \hat{t}\hat{z}^{-1}\hat{t} + \hat{z}^{-1}), \\ \hat{X}^2 &= -\frac{1}{2}(\kappa(\alpha)^2\hat{z} + \hat{t}\hat{z}^{-1}\hat{t} - \hat{z}^{-1}). \end{aligned} \quad (3.29)$$

Note the appearance of a nontrivial deformation factor $\kappa(\alpha)^2$, which would be impossible to guess assuming some minimal quantization, as it was done in (3.22). It is easy to verify explicitly that this factor is essential to guarantee the correct algebraic relations (3.1) starting with the fundamental commutation relations (3.24).

Using the result (3.29), it is an easy exercise to find the commutators $[\hat{t}, \hat{X}^a]$

$$\begin{aligned}
[\hat{t}, \hat{X}^1] &= i\alpha\hat{t}, \\
[\hat{t}, \hat{X}^0] &= \frac{i\alpha}{2}(\hat{t}^2 - \kappa(\alpha)^2\hat{z}^2 + 1), \\
[\hat{t}, \hat{X}^2] &= \frac{i\alpha}{2}(\hat{t}^2 - \kappa(\alpha)^2\hat{z}^2 - 1). \quad (3.30)
\end{aligned}$$

Now we focus on solutions to the quantum scalar field equation of motion (3.12), and formulate our first proposition:

Proposition 1. $\hat{z}$ and $\hat{t}$, as defined in (3.22), are exact solutions of the quantum scalar field equation of motion (3.12).

This is the analog of the obvious fact that z and t solve the classical field equation (2.6). The proof of the proposition is as follows:

Proof.—Using (3.2) it easy to show that (3.12) can be equivalently rewritten as

$$\hat{X}_a\hat{\Phi}\hat{X}^a + \hat{\Phi} = 0 \Leftrightarrow \hat{X}_a[\hat{\Phi}, \hat{X}^a] = [\hat{X}_a, \hat{\Phi}]\hat{X}^a = 0. \quad (3.31)$$

Then $\hat{X}_a[\hat{z}, \hat{X}^a] = 0$ is an immediate consequence of (3.23), which shows that $\hat{z}$ is an exact solution.

For $\hat{t}$ we can proceed the same way, using (3.30), but let us prove it slightly differently by establishing some very useful criteria for a solution. Namely, if we know that some $\hat{A}$ is a solution, then $\hat{A}\hat{B}$ will be a solution for some $\hat{B}$ if and only if $\hat{X}_a\hat{A}[\hat{B}, \hat{X}^a] = 0$. This is trivially verified. Then one has that $\hat{t} = -\hat{z}\hat{X}^1 + \frac{i\alpha}{2}\hat{z}$ will be a solution if $\hat{X}_a\hat{z}[\hat{X}^1, \hat{X}^a] = 0$, which is trivially true after some simple algebra. ■

The classical solution (2.17) is given in terms of arbitrary functions of either ζ or $\tilde{\zeta}$. Motivated by this and by the possibility that the quantum $\hat{z}$ “coordinate” could be rescaled, as in (3.29), we will look for the general solutions of (3.12) in the form of an arbitrary polynomial in the quantum variable $\hat{\Xi} = C\hat{z} + i\hat{t}$, where C is a constant to be fixed later. First of all, in view of Proposition 1, $\hat{\Xi}$ is an exact solution for arbitrary C . The next proposition fixes this constant.

Proposition 2. $\hat{\Xi}^2$ is an exact solution of the quantum scalar equation of motion (3.12) if and only if $C = \pm\sqrt{1 + \frac{\alpha^2}{4}} \equiv \pm\kappa(\alpha)$.

Proof.—First of all, from the observation made in the proof of Proposition 1 and from the fact that $\hat{\Xi}$ is a solution, we know that $\hat{\Xi}^2$ will be a solution if and only if $\hat{X}_a\hat{\Xi}[\hat{\Xi}, \hat{X}^a] = [\hat{X}_a, \hat{\Xi}]\hat{\Xi} = 0$. Using $[\hat{X}^a, \hat{\Xi}]\hat{X}_a = 0$ (because $\hat{\Xi}$ is a solution), this is equivalent to

$$[\hat{X}_a, \hat{\Xi}][\hat{X}^a, \hat{\Xi}] = 0. \quad (3.32)$$

Each commutator in (3.32) can easily be calculated from (3.23) and (3.30) giving

$$\begin{aligned}
[\hat{X}^1, \hat{\Xi}] &= -i\alpha\hat{\Xi}, \\
[\hat{X}^0, \hat{\Xi}] &= -\frac{i\alpha}{2}C(\hat{z}\hat{t} + \hat{t}\hat{z}) + \frac{\alpha}{2}(\hat{t}^2 - \kappa(\alpha)^2\hat{z}^2 + 1), \\
[\hat{X}^2, \hat{\Xi}] &= -\frac{i\alpha}{2}C(\hat{z}\hat{t} + \hat{t}\hat{z}) + \frac{\alpha}{2}(\hat{t}^2 - \kappa(\alpha)^2\hat{z}^2 - 1). \quad (3.33)
\end{aligned}$$

Plugging this in (3.32) we get

$$0 = [\hat{X}_a, \hat{\Xi}][\hat{X}^a, \hat{\Xi}] = \alpha^2(\kappa(\alpha)^2 - C)\hat{z}^2 \quad (3.34)$$

from where the result follows. ■

So, we see that $\hat{\Xi}_{\pm} := \pm\kappa(\alpha)\hat{z} + i\hat{t}$ are the quantum analogs of the classical ζ and $-\tilde{\zeta}$. Using the found value for C , the relations (3.33) could be written in a more compact and closed form

$$\begin{aligned}
[\hat{X}^1, \hat{\Xi}] &= -i\alpha\hat{\Xi}, \\
[\hat{X}^0, \hat{\Xi}] &= -\frac{\alpha}{2}(\hat{\Xi}^2 - 1), \\
[\hat{X}^2, \hat{\Xi}] &= -\frac{\alpha}{2}(\hat{\Xi}^2 + 1). \quad (3.35)
\end{aligned}$$

Now this will allow us to prove the claimed result.

Proposition 3. An arbitrary polynomial in $\hat{\Xi}$ (either of $\hat{\Xi}_{\pm}$) is an exact solution of the quantum scalar equation of motion (3.12).

Proof.—The proof is by induction. We already know that $\hat{\Xi}$ and $\hat{\Xi}^2$ are solutions (we need to start with $\hat{\Xi}^2$, because only on this level do we get a nontrivial condition on C). Assume that $\hat{\Xi}^{n-1}$ is also a solution. Then we know that $\hat{\Xi}^n \equiv \hat{\Xi}^{n-1}\hat{\Xi}$ will be a solution if and only if $\hat{X}_a\hat{\Xi}^{n-1}[\hat{\Xi}, \hat{X}^a] = 0$. But from (3.35) we know that $\hat{\Xi}^{n-1}$ commutes with all $[\hat{\Xi}, \hat{X}^a]$, so we can write

$$\hat{X}_a\hat{\Xi}^{n-1}[\hat{\Xi}, \hat{X}^a] = \hat{X}_a[\hat{\Xi}, \hat{X}^a]\hat{\Xi}^{n-1} = 0, \quad (3.36)$$

where we used $\hat{X}_a[\hat{\Xi}, \hat{X}^a] = 0$ because $\hat{\Xi}$ is a solution. ■

So, we see that the general expression (2.17) for the classical solution has a nontrivial exact quantum counterpart. The general solution of the quantum scalar equation of motion (3.12) can be written in terms of $\hat{\Xi}_{+}$ and $\hat{\Xi}_{-}$, or equivalently, $\hat{\Xi} = \hat{\Xi}_{+}$ and $\hat{\Xi}^{\dagger}$. It is given by

$$\hat{\Phi} = \hat{F}(\hat{\Xi}) + \hat{G}(\hat{\Xi}^{\dagger}), \quad (3.37)$$

where $\hat{F}$ and $\hat{G}$ are arbitrary elements of the polynomial algebras (or of their closures) generated by $\hat{\Xi}$ and $\hat{\Xi}^{\dagger}$, respectively.

Now we want to ask whether the analogous exact solutions can be established for the quantum Dirac equation (3.20). The main motivation is the classical result (2.31), which could be interpreted as an arbitrary spinorial

solution written in terms of arbitrary scalar solutions. The natural quantization of (2.31) is

$$\hat{\Psi} = \frac{1}{2} \begin{pmatrix} 1 - \hat{\Xi} & -1 - \hat{\Xi}^\dagger \\ 1 + \hat{\Xi} & -1 + \hat{\Xi}^\dagger \end{pmatrix} \begin{pmatrix} \hat{F}(\hat{\Xi}) \\ \hat{G}(\hat{\Xi}^\dagger) \end{pmatrix}. \quad (3.38)$$

We want to see if it is a solution to (3.20). Due to the linearity of (3.20), it is enough to consider the case $\hat{F}(\hat{\Xi}) \neq 0$ and $\hat{G}(\hat{\Xi}^\dagger) = 0$. Then the general case will trivially follow. Also by the linearity, it suffices to work only with the scalar solutions given by monomials $\hat{F}_n := \hat{\Xi}^n$. So, we want to show that

$$\hat{\Psi}_n := \frac{1}{2} \begin{pmatrix} (1 - \hat{\Xi})\hat{F}_n \\ (1 + \hat{\Xi})\hat{F}_n \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \hat{F}_n - \hat{F}_{n+1} \\ \hat{F}_n + \hat{F}_{n+1} \end{pmatrix} \quad (3.39)$$

is an exact solution of (3.20). The following proposition will prove to be crucial in the demonstration.

Proposition 4. Let $\hat{\Psi}'_n := (\hat{F}'_n)$, then (3.39) is given by

$$2\alpha n \hat{\Psi}_n = \tau_a [\hat{X}^a, \hat{\Psi}'_n], \quad (3.40)$$

where τ_a are defined after (2.21).

Proof.—From the definition of τ_a we have

$$\tau_a [\hat{X}^a, \hat{\Psi}'_n] = \begin{pmatrix} [\hat{X}^0 + i\hat{X}^1 + \hat{X}^2, \hat{F}_n] \\ [-\hat{X}^0 + i\hat{X}^1 - \hat{X}^2, \hat{F}_n] \end{pmatrix}. \quad (3.41)$$

Now, the relations (3.35) can be easily generalized (e.g., by recursion) to

$$\begin{aligned} [\hat{X}^1, \hat{\Xi}^n] &= -in\alpha \hat{\Xi}^n, \\ [\hat{X}^0, \hat{\Xi}^n] &= -n\frac{\alpha}{2} (\hat{\Xi}^{n+1} - \hat{\Xi}^{n-1}), \\ [\hat{X}^2, \hat{\Xi}^n] &= -n\frac{\alpha}{2} (\hat{\Xi}^{n+1} + \hat{\Xi}^{n-1}). \end{aligned} \quad (3.42)$$

From here we have

$$[i\hat{X}^1 \pm (\hat{X}^0 + \hat{X}^2), \hat{\Xi}^n] = n\alpha (\hat{\Xi}^n \mp \hat{\Xi}^{n+1}), \quad (3.43)$$

which immediately proves the claim. ■

Now we finally can prove the main result for the spinorial case.

Proposition 5. The spinor (3.39) [and, as a trivial consequence, the general one given in (3.38)] is an exact solution of the quantum Dirac equation (3.20).

Proof.—Using the result of Proposition 4, we need to prove that

$$\epsilon_{abc} \tau^c \hat{X}^a \tau_k [\hat{X}^k, \hat{\Psi}'_n] \hat{X}^b = 0. \quad (3.44)$$

Using (2.22) and

$$\epsilon_{abc} \epsilon^c_{kl} = -(\eta_{ak} \eta_{bl} - \eta_{al} \eta_{bk}) \quad (3.45)$$

we have

$$\begin{aligned} \epsilon_{abc} \tau^c \hat{X}^a \tau_k [\hat{X}^k, \hat{\Psi}'_n] \hat{X}^b &= i\tau_b \hat{X}_a [\hat{X}^a, \hat{\Psi}'_n] \hat{X}^b \\ &\quad - i\tau_a \hat{X}^a [\hat{X}^b, \hat{\Psi}'_n] \hat{X}_b \\ &\quad - \epsilon_{abk} \hat{X}^a [\hat{X}^k, \hat{\Psi}'_n] \hat{X}^b. \end{aligned} \quad (3.46)$$

Because each component of $\hat{\Psi}'_n$ is an exact solution for the scalar case, cf. (3.31), the first two terms are equal to zero. The remaining term we can manipulate as follows:

$$\begin{aligned} \epsilon_{abk} \hat{X}^a [\hat{X}^k, \hat{\Psi}'_n] \hat{X}^b &= \epsilon_{abk} \hat{X}^a \hat{X}^k \hat{\Psi}'_n \hat{X}^b - \epsilon_{abk} \hat{X}^a \hat{\Psi}'_n \hat{X}^k \hat{X}^b \\ &= i\alpha X_b \hat{\Psi}'_n \hat{X}^b - i\alpha X^a \hat{\Psi}'_n \hat{X}_a = 0, \end{aligned} \quad (3.47)$$

where we used $\epsilon_{abc} \hat{X}^b \hat{X}^c = -i\alpha \hat{X}_a$. This proves (3.44). ■

Consequently (3.38) solves (3.20).⁴ Therefore, the main conclusion of this section is that the exact solutions to the quantized massless Klein-Gordon (3.12) and Dirac (3.20) equations are given by essentially the same expressions as in the classical case with the nontrivial quantization of the variable ζ given by $\hat{\Xi}$.

IV. EXACT BOUNDARY CORRELATION FUNCTIONS

Boundary correlation functions are generated from the on-shell bulk action. Before obtaining the boundary two-point correlation functions for operators sourced by massless scalars and spinors on EAdS₂ on quantized EAdS₂, we briefly recall how the correspondence works on the classical EAdS₂ bulk manifold in the first subsection. We give the quantized version for scalars and spinors in the following two subsections. As a result of the exact solutions of the previous section, our results are exact, and in fact, remarkably simple.

A. Two-point correlators on the boundary of EAdS₂

We begin by reviewing the calculation of the two-point correlation function for operators sourced by the massless scalar field on EAdS₂, and then for operators sourced by massless spinors. For the latter, we use the unconventional choice of the covariant description of the Dirac operator.

⁴An alternative proof is presented in Appendix B. There we construct the quantum version $\hat{U}$ of the matrix U in (2.24), and use it to map the Dirac operator to the analog of the noncovariant basis. The solutions for massless spinors in that basis are trivially obtained from the classical solutions (2.17) by replacing ζ by Ξ , with a particular choice of ordering for the $\sqrt{\hat{\Xi}}$ factor. Upon using $\hat{U}$ to map the solution back to the covariant basis, we recover (3.38).

1. Massless scalar field

For the case of the massless scalar field, one starts with the action

$$S[\Phi] = \frac{1}{2} \int_{\mathbb{R}_+^2} dt dz \{ (\partial_z \Phi)^2 + (\partial_t \Phi)^2 \}. \quad (4.1)$$

Variations $\delta\Phi$ of Φ in (4.1) give

$$\begin{aligned} \delta S[\Phi] = & - \int_{\mathbb{R}_+^2} dt dz \delta\Phi (\partial_z^2 + \partial_t^2) \Phi \\ & - \int_{\mathbb{R}} dt (\partial_z \Phi \delta\Phi) \Big|_{z=0}. \end{aligned} \quad (4.2)$$

Extremizing the action with Dirichlet boundary conditions yields the field equation (2.6). Since the equation is second order we should impose two boundary conditions to fix the solution (2.7). Solutions which are everywhere (and in particular at $z \rightarrow \infty$) regular can be expressed in terms of the boundary value of the field $\phi_0(t)$, using the boundary-to-bulk propagator [3,4]. Actually, in the free case, the full boundary-to-bulk propagator is not needed for the calculation of the boundary two-point function. The solution (2.7) can be expanded in powers of z , as was done in (2.8), and we only need to write the $\mathcal{O}(z)$ term, $\phi_1(t)$, in terms of $\phi_0(t)$. For this we can first write the functions $f(\zeta)$ and $g(\bar{\zeta})$ in (2.7) as⁵

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \frac{\phi_0(t')}{\zeta - it'}, \\ g(\bar{\zeta}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \frac{\phi_0(t')}{\bar{\zeta} + it'}. \end{aligned} \quad (4.3)$$

Then we can compute the z -derivative of $\Phi(\zeta, \bar{\zeta}) = f(\zeta) + g(\bar{\zeta})$, and take its boundary limit $z \rightarrow 0$ to get

$$\partial_z \Phi \rightarrow \phi_1(t) = \frac{1}{\pi} \int dt' \frac{\phi_0(t')}{(t-t')^2}, \quad \text{as } z \rightarrow 0. \quad (4.4)$$

We next substitute this back in the action (4.1), which can be rewritten as

⁵The boundary limit $\Phi(\zeta, \bar{\zeta}) \rightarrow \phi_0(t) = f(it) + g(-it)$ follows from the Sokhotski formula

$$\begin{aligned} \frac{1}{\zeta - it'} &\rightarrow \pi \delta(t-t') - i\mathcal{P}\left(\frac{1}{t-t'}\right) \\ \frac{1}{\bar{\zeta} + it'} &\rightarrow \pi \delta(t-t') + i\mathcal{P}\left(\frac{1}{t-t'}\right), \end{aligned} \quad \text{as } z \rightarrow 0,$$

where $\mathcal{P}$ denotes the principal value.

$$\begin{aligned} S[\Phi] = & -\frac{1}{2} \int_{\mathbb{R}_+^2} dt dz \Phi (\partial_z^2 + \partial_t^2) \Phi \\ & - \frac{1}{2} \int_{\mathbb{R}} dt (\Phi \partial_z \Phi) \Big|_{z=0}. \end{aligned} \quad (4.5)$$

Only the boundary term survives in the on-shell action. Using (4.4), the result is

$$S[\Phi]|_{\text{on-shell}} = -\frac{1}{2\pi} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \frac{\phi_0(t) \phi_0(t')}{(t-t')^2}. \quad (4.6)$$

In the AdS/CFT correspondence one identifies $S[\Phi]|_{\text{on-shell}}$ with the generating functional of the n -point connected correlation functions for the operator $\mathcal{O}$ associated with ϕ_0 . Here, both $\mathcal{O}$ and ϕ_0 are functions of only t ,

$$\langle \mathcal{O}(t_1) \cdots \mathcal{O}(t_n) \rangle = \frac{\delta^n S[\Phi]|_{\text{on-shell}}}{\delta \phi_0(t_1) \cdots \delta \phi_0(t_n)} \Big|_{\phi_0=0}. \quad (4.7)$$

So the two-point function in this example is

$$\langle \mathcal{O}(t) \mathcal{O}(t') \rangle = -\frac{1}{\pi} \frac{1}{(t-t')^2}. \quad (4.8)$$

2. Massless spinor field

Since we presented two different formulations for spinors on the classical EAdS₂ bulk manifold, we can proceed with two different derivations of the corresponding two-point correlator on the boundary. These two approaches cannot be easily mapped from one to another because the map (2.26) between the Dirac operators is singular at the $z \rightarrow 0$ boundary. Nevertheless, it can be checked that both approaches yield the same result for the boundary correlation function. Below we work with the Dirac operator D (2.29) of the covariant description.

The bulk action for massless spinors in the covariant description is given by

$$\begin{aligned} S_B[\bar{\Psi}, \Psi] &= - \int_{\mathbb{R}_+^2} \frac{dz dt}{z^2} \bar{\Psi} D \Psi \\ &= \int_{\mathbb{R}_+^2} \frac{dz dt}{z^2} \bar{\Psi} (i\tau_a \mathcal{K}^a + \mathbb{1}) \Psi, \end{aligned} \quad (4.9)$$

where the conjugate spinor is defined by $\bar{\Psi} = \Psi^\dagger \sigma_3$. With this definition one gets that $\bar{\Psi} \Psi$ is a scalar under $SU(1,1)$ transformations, $\Psi \rightarrow M \Psi$, where the 2×2 transformation matrix satisfies $M^\dagger \sigma_3 M = \sigma_3$.⁶ The action obviously leads to the field equation $D\Psi = 0$ and its Hermitian conjugate $\bar{\Psi} \bar{D} = 0$.

⁶Another standard choice is $M^\dagger \sigma_2 M = \sigma_2$. In this case we should define $\bar{\Psi} = \Psi^\dagger \sigma_2$. This choice would lead to the appearance of σ_2 instead of σ_3 in the formula for the two-point correlation function below, (4.21).

As is well known [19,20], the action for spinor fields should be supplemented with a boundary term. From [20], because the field equation is first order in derivatives, we cannot fix all components of the spinor at spatial infinity (corresponding to $z \rightarrow +0$), as well as demand that the fields are everywhere regular in the bulk. Therefore some spinor degrees of freedom at spatial infinity remain dynamical. The variations of the action with regards to these boundary degrees of freedom, as well as the bulk degrees of freedom, should be consistently made to vanish. This is not possible if the action consists only of (4.9). It is for this reason that boundary term should be included in the total action. Note, that this situation differs for the case of a scalar field, where $\delta\Phi$ could be consistently subjected to the Dirichlet boundary conditions in (4.2).

To obtain the boundary term consider an arbitrary variation in Ψ in the action (4.9)

$$\begin{aligned}\delta S_B[\bar{\Psi}, \Psi] &= - \int_{\mathbb{R}_+^2} \frac{dz dt}{z^2} \bar{\Psi} D \delta \Psi \\ &= \Sigma_B - \lim_{z \rightarrow 0} \int_{\mathbb{R}} dt \bar{\Psi} \Upsilon \delta \Psi,\end{aligned}\quad (4.10)$$

where Σ_B are the terms that vanish on shell, and Υ is defined by

$$\Upsilon = -\frac{i}{z} \begin{pmatrix} t & t+i \\ -t+i & -t \end{pmatrix}. \quad (4.11)$$

The boundary contribution in (4.10) resulted from ∂_z terms in (2.27). Υ can also be expressed in terms of the imbedding coordinates X^a and the chirality operator γ

$$\Upsilon = \epsilon_{abc} X^a \partial_t X^b \gamma \tau^c. \quad (4.12)$$

The asymptotic form of the solution for Ψ was given in (2.33), while for $\bar{\Psi}$ we have

$$\begin{aligned}\bar{\Psi}(\zeta, \bar{\zeta}) &= \bar{\psi}_0(t)(\sigma_3 + it\mathbb{1}) + iz\partial_t(\bar{\psi}_1(t)(\sigma_3 + it\mathbb{1})) \\ &+ \mathcal{O}(z^2),\end{aligned}\quad (4.13)$$

where ψ_0 and ψ_1 were defined in (2.34). We shall regard $\psi_0(t)$ [and $\bar{\psi}_0(t)$] as nondynamical, i.e., source fields, and so variations in (4.10) are to be carried out with respect to the remaining boundary fields $\psi_1(t)$. The boundary term in (4.10) is removed upon adding the following term to the bulk action (4.9):

$$S_{\partial B}[\bar{\Psi}, \Psi] = \lim_{z \rightarrow 0} \int_{\partial B} dt \bar{\psi}_0(t)(\sigma_3 + it\mathbb{1}) \Upsilon \Psi. \quad (4.14)$$

The total action is then $S_{\text{total}} = S_B + S_{\partial B}$.

Only the boundary term (4.14) survives in the on-shell action. To evaluate it we can rewrite the asymptotic expression for the solution (2.33) according to:

$$\begin{aligned}\Psi(\zeta, \bar{\zeta}) &= \frac{1}{2\pi} \int dt' \left(-\frac{i}{t-t'} (\sigma_3 - it\mathbb{1}) \psi_1(t') \right. \\ &\quad \left. + \frac{z}{(t-t')^2} (\sigma_3 - it'\mathbb{1}) \psi_0(t') + \mathcal{O}(z^2) \right).\end{aligned}\quad (4.15)$$

Here we have used the Cauchy formula. For this define a (clockwise) closed contour C in the complex plane (parametrized by $\zeta' = z' + it'$) which goes from $t' = -R$ to $t' = R$ along the t' axis and then returns to $t' = -R$ along the semi-circle $\zeta' = Re^{i\phi}$, where ϕ runs from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. Then

$$\begin{aligned}f(\zeta) &= -\frac{1}{2\pi i} \oint_C \frac{d\zeta' f(\zeta')}{\zeta' - \zeta}, \\ g(\bar{\zeta}) &= \frac{1}{2\pi i} \oint_C \frac{d\bar{\zeta}' g(\bar{\zeta}')}{\bar{\zeta}' - \bar{\zeta}}.\end{aligned}$$

Upon assuming that $f(Re^{i\phi})$ and $g(Re^{-i\phi})$ vanish in the limit $R \rightarrow \infty$, we get that

$$\begin{aligned}f(\zeta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \frac{f(it')}{\zeta - it'}, \\ g(\bar{\zeta}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \frac{g(-it')}{\bar{\zeta} + it'}.\end{aligned}$$

Upon setting $z = 0$ we get

$$\begin{aligned}f(it) &= \frac{1}{2\pi i} \int dt' \frac{f(it')}{t-t'}, \\ g(-it) &= -\frac{1}{2\pi i} \int dt' \frac{g(-it')}{t-t'},\end{aligned}\quad (4.16)$$

and consequently

$$\begin{aligned}f'(it) &= \frac{1}{2\pi} \int dt' \frac{f(it')}{(t-t')^2}, \\ g'(-it) &= \frac{1}{2\pi} \int dt' \frac{g(-it')}{(t-t')^2}.\end{aligned}\quad (4.17)$$

Substituting (4.16) and (4.17) in (2.33) gives (4.15).

So evaluating S_{total} on shell gives

$$\begin{aligned}S_{\text{total}}[\bar{\Psi}, \Psi]|_{\text{on-shell}} &= \int dt dt' \frac{z}{2\pi(t-t')^2} \bar{\psi}_0(t)(\sigma_3 + it\mathbb{1}) \\ &\quad \times \Upsilon(\sigma_3 - it'\mathbb{1}) \psi_0(t').\end{aligned}\quad (4.18)$$

The z independent term in the integrand of (4.15) does not contribute to the on-shell action due to the result that $\bar{\psi}_0(t)(\sigma_3 + it\mathbb{1})\Upsilon(\sigma_3 - it\mathbb{1})\psi_1(t')$ is identically zero. Notice also that the factor z in (4.18) cancels with the $1/z$ appearing in Υ (4.12). Finally, use the identity

$$\begin{aligned} & \bar{\psi}_0(t)(\sigma_3 + it\mathbb{1})\Upsilon(\sigma_3 - it'\mathbb{1})\psi_0(t') \\ &= -\frac{i}{z}(t - t')\bar{\psi}_0(t)\sigma_3\psi_0(t') \end{aligned} \quad (4.19)$$

to simplify the result to

$$\begin{aligned} S_{\text{total}}[\bar{\Psi}, \Psi]|_{\text{on-shell}} \\ = -\frac{i}{2\pi} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \bar{\psi}_0(t) \frac{\sigma_3}{t - t'} \psi_0(t'). \end{aligned} \quad (4.20)$$

The resulting two-point correlation function on the boundary is

$$\begin{aligned} \langle \mathcal{O}_{\bar{\psi}_0}(t) \mathcal{O}_{\psi_0}(t') \rangle &= \frac{\delta^2 S_{\text{total}}[\bar{\Psi}, \Psi]|_{\text{on-shell}}}{\delta \bar{\psi}_0(t) \delta \psi_0(t')} \Big|_{\bar{\psi}_0=\psi_0=0} \\ &= \frac{1}{2\pi i} \frac{\sigma_3}{t - t'}. \end{aligned} \quad (4.21)$$

B. Scalar two-point correlator on the boundary of quantized EAdS₂

The noncommutative generalization of the action (4.1) is

$$\hat{S}[\hat{\Phi}] = -\frac{1}{2} \text{Tr}\{[\hat{X}^\mu, \hat{\Phi}][\hat{X}_\mu, \hat{\Phi}]\}, \quad (4.22)$$

where Tr denotes a trace. The field equation (3.12) follows from extremizing the action with respect to variations in $\hat{\Phi}$.

Here it is convenient to introduce the star product realization of the operator product. The Moyal-Weyl star product can be employed for this purpose, provided one transforms to an appropriate (canonical) pair of coordinates (x, y) , as was discussed in [6,7]. From (2.37), that canonical pair can be taken to be z^{-1} and t . As (z^{-1}, t) span $\mathbb{R}_+^2$, appropriate boundary conditions should be imposed at $z^{-1} = 0$ (which recall is *not* the asymptotic boundary of EAdS₂), i.e., all functions should vanish sufficiently rapidly as $z \rightarrow \infty$. Alternatively, one can employ a pair of canonical coordinates, denoted by (x, y) , that span the entire plane, as was done in [6,7]. Then the map back to Fefferman-Graham coordinates is

$$t = ye^{-x}, \quad z = e^{-x}. \quad (4.23)$$

In the quantum theory x and y get promoted to operators $\hat{x}$ and $\hat{y}$, satisfying the standard canonical commutation relation

$$[\hat{x}, \hat{y}] = i\alpha\mathbb{1}. \quad (4.24)$$

The quantization of the map (4.23) is obtained using symmetric ordering:

$$\hat{t} = \frac{1}{2}(\hat{y}e^{-\hat{x}} + e^{-\hat{x}}\hat{y}), \quad \hat{z} = e^{-\hat{x}}. \quad (4.25)$$

Then from (4.24) and (4.23) we recover the fundamental commutation relations for $\hat{z}$ and $\hat{t}$ (3.24).

The product of operators $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ can be expressed in terms of the Moyal-Weyl star product of their symbols (which we also denote by $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$, respectively), which, when expressed in terms of canonical variables (x, y) [or (z^{-1}, t)], is defined in the standard way

$$[\hat{\mathcal{F}} \star \hat{\mathcal{G}}](x, y) = \hat{\mathcal{F}}(x, y) \exp\left\{\frac{i\alpha}{2}(\vec{\partial}_x \vec{\partial}_y - \vec{\partial}_y \vec{\partial}_x)\right\} \hat{\mathcal{G}}(x, y). \quad (4.26)$$

Furthermore, the trace on the algebra, Tr , becomes $\frac{1}{\alpha^2} \int_{\mathbb{R}^2} dx dy$. Then the operator equations (4.25) are replaced by the corresponding equations for the symbols $\hat{t}$ and $\hat{z}$

$$\hat{t} = \frac{1}{2}(\hat{y} \star e^{-\hat{x}} + e^{-\hat{x}} \star \hat{y}), \quad \hat{z} = e^{-\hat{x}}, \quad (4.27)$$

and we recover (3.24), where $[\hat{\mathcal{F}}, \hat{\mathcal{G}}]$ now denotes a star commutator $\hat{\mathcal{F}} \star \hat{\mathcal{G}} - \hat{\mathcal{G}} \star \hat{\mathcal{F}}$. The star commutator goes to $i\alpha$ times the corresponding Poisson bracket in the commutative limit $\alpha \rightarrow 0$, (3.13). From (3.24) it follows that given any two functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ that are well behaved when $z \rightarrow 0$, their star commutator must vanish in the boundary limit, with the leading order term in z agreeing with the commutative limit,

$$[\hat{\mathcal{F}}, \hat{\mathcal{G}}] \rightarrow i\alpha z^2(\partial_t \hat{\mathcal{F}} \partial_z \hat{\mathcal{G}} - \partial_t \hat{\mathcal{G}} \partial_z \hat{\mathcal{F}}) = i\alpha\{\hat{\mathcal{F}}, \hat{\mathcal{G}}\} \quad \text{as } z \rightarrow 0. \quad (4.28)$$

Moreover, the star product of any two functions with a well-behaved $z \rightarrow 0$ limit reduces to the pointwise product in the boundary limit.

The expressions for the noncommutative analogs $\hat{X}^a$ of the embedding coordinates in terms of $\hat{t}$ and $\hat{z}$, given in (3.29), can be realized in terms of the star product. From them we recover (3.1) and (3.2), the latter now written as $\hat{X}^a \star \hat{X}_a = -\mathbb{1}$. Upon expanding (3.29) in α , it can be shown that embedding coordinates X^0 and X^2 only pick up a second order correction in α upon quantization (in fact, the corrections to X^0 and X^2 are identical), while X^1 is undeformed

$$\hat{X}^0 = X^0 - \frac{\alpha^2}{8}z, \quad \hat{X}^1 = X^1, \quad \hat{X}^2 = X^2 - \frac{\alpha^2}{8}z. \quad (4.29)$$

We see that the quantum corrections vanish in the boundary limit $z \rightarrow 0$. Recall that the quantum analogs $\hat{\mathcal{K}}^a$ of Killing vectors $\mathcal{K}^a$ were constructed from commutators (or star commutators) with $\hat{X}_a$ (3.6). Since $\hat{X}^a$ reduces to embedding coordinates X^a in the boundary limit, and the star commutator goes to $i\alpha$ times the corresponding Poisson bracket in the limit (4.28), it follows that $\hat{\mathcal{K}}^a$ must reduce to the classical Killing vectors $\mathcal{K}^a$ (2.39) on the boundary. This led to the conclusion in [6,7] that quantized EAdS₂ is an asymptotically anti-de Sitter space.

From the above we can write (4.22) according to

$$\hat{S}[\hat{\Phi}] = -\frac{1}{2\alpha^2} \int_{\mathbb{R}_+^2} \frac{dt dz}{z^2} \{[\hat{X}^\mu, \hat{\Phi}] \star [\hat{X}_\mu, \hat{\Phi}]\}, \quad (4.30)$$

where again $[\cdot, \cdot]$ now denotes a star commutator. The action can be expressed as a sum of two terms, where one term vanishes on shell, and the other term, which we denote by $\hat{S}_{\partial D}[\hat{\Phi}]$, is only defined on the asymptotic boundary. As was argued in [6,7], the expression for the boundary action is identical in form to that appearing in the commutative theory (4.5),

$$\hat{S}_{\partial D}[\hat{\Phi}] = -\frac{1}{2} \lim_{z \rightarrow 0} \int dt \hat{\Phi} \partial_z \hat{\Phi}. \quad (4.31)$$

The exact solution to the quantized scalar field equation is given by (3.37). Near the boundary $z = 0$ the solution behaves as

$$\begin{aligned} \hat{\Phi}(\hat{\Xi}, \hat{\Xi}^\dagger) &= \phi_0(t) + \kappa(\alpha) z \phi_1(t) + \mathcal{O}(z^2), \\ \phi_0(t) &= \hat{F}(it) + \hat{G}(-it), \\ \phi_1(t) &= (\hat{F}'(\hat{\Xi}) + \hat{G}'(\hat{\Xi}^\dagger))|_{z=0}, \end{aligned} \quad (4.32)$$

the prime denoting a derivative. Here we used $z\phi_1(t) = z\star\phi_1(t) + \mathcal{O}(z^2) = \phi_1(t)\star z + \mathcal{O}(z^2)$. So, like with the classical solution, $\hat{\Phi}$ tends to $\phi_0(t)$ as $z \rightarrow 0$, while $\partial_z \hat{\Phi}(z, t)$ picks up a factor of $\kappa(\alpha)$:

$$\begin{aligned} \partial_z \hat{\Phi}(z, t) &\rightarrow \kappa(\alpha) \partial_z \Phi(z, t)|_{z \rightarrow 0} \\ &\rightarrow \frac{\kappa(\alpha)}{\pi} \int dt' \frac{\phi_0(t')}{(t-t')^2}, \end{aligned} \quad (4.33)$$

where we used (4.4). Upon substituting into (4.31), the result for the on-shell action is

$$S[\Phi]|_{\text{on-shell}} = -\frac{\kappa(\alpha)}{2\pi} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \frac{\phi_0(t)\phi_0(t')}{(t-t')^2}. \quad (4.34)$$

So the boundary two-point function for massless scalar fields on quantized EAdS₂ picks up an overall factor of $\kappa(\alpha)$

$$\langle \mathcal{O}(t) \mathcal{O}(t') \rangle = -\frac{\kappa(\alpha)}{\pi} \frac{1}{(t-t')^2}. \quad (4.35)$$

This agrees up to $\mathcal{O}(\alpha^4)$ with the leading order perturbative result found in [6,7].

C. Spinor two-point correlator on the boundary of quantized EAdS₂

As in the previous subsection we can utilize the star product realization of the operator product to write down the equations of motion and the action. Concerning the latter, the bulk action for a massless spinor on quantized EAdS₂ can be taken to be

$$\begin{aligned} \hat{S}_B[\tilde{\Psi}, \hat{\Psi}] &= - \int_B \frac{dz dt}{z^2} \tilde{\Psi} \star \hat{D} \hat{\Psi} \\ &= -\frac{i}{\alpha} \int_B \frac{dz dt}{z^2} \epsilon_{abc} \tilde{\Psi} \star \hat{\gamma}(\tau^c \hat{X}^b \star \hat{\Psi} \star \hat{X}^a) \\ &= -\frac{i}{\alpha \sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dz dt}{z^2} \epsilon_{abc} \\ &\quad \times \tilde{\Psi} \star (\tau_d \tau^c \hat{X}^b \star \hat{\Psi} \star \hat{X}^a \star \hat{X}^d - \frac{\alpha}{2} \tau^c \hat{X}^b \star \hat{\Psi} \star \hat{X}^a). \end{aligned} \quad (4.36)$$

Like in the commutative case, the conjugate spinor is defined by $\tilde{\Psi} = \hat{\Psi}^\dagger \sigma_3$. Variations of $\tilde{\Psi}$ in the action lead to the equations of motion in the bulk (3.20). Using the cyclic property of the Moyal-Weyl star product, $\int dx dy \mathcal{F} \star \mathcal{G} \star \mathcal{H} = \int dx dy \mathcal{H} \star \hat{\mathcal{F}} \star \hat{\mathcal{G}} + \text{boundary terms}$, the action (3.20) can also be written as

$$\begin{aligned} \hat{S}_B[\tilde{\Psi}, \hat{\Psi}] &= -\frac{i}{\alpha \sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dz dt}{z^2} \epsilon_{abc} \hat{X}^a \star \left(\hat{X}^d \star \tilde{\Psi} \tau_d - \frac{\alpha}{2} \tilde{\Psi} \right) \star \hat{X}^b \tau^c \star \hat{\Psi} + B.T. \\ &= \frac{i}{\alpha \sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dz dt}{z^2} \epsilon_{abc} \left(\hat{X}^d \star \hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c \tau_d - \frac{\alpha}{2} \hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c \right) \star \hat{\Psi} + B.T. \\ &= \frac{i}{\alpha} \int_B \frac{dz dt}{z^2} \epsilon_{abc} \hat{\gamma}(\hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c) \star \hat{\Psi} + B.T., \end{aligned} \quad (4.37)$$

where $B.T.$ denotes boundary terms, and we used the identity

$$\epsilon_{abc} \hat{X}^a \star \hat{X}^d \star \tilde{\Psi} \star \hat{X}^b \tau_d \tau^c = -\epsilon_{abc} \hat{X}^d \star \hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c \tau_d + \alpha \epsilon_{abc} \hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c. \quad (4.38)$$

Then variations of $\hat{\Psi}$ in the action lead to the equations of motion

$$\epsilon_{abc} \hat{X}^a \star \tilde{\Psi} \star \hat{X}^b \tau^c = 0. \quad (4.39)$$

This is the Hermitian conjugate of the Dirac equation (3.20). To show this we can use the identity $\tau_a^\dagger = \sigma_3 \tau_a \sigma_3$.

The exact solution to the Dirac equation (3.20) is given by (3.38). As before, we only need the leading and next to leading behavior as $z \rightarrow 0$ to obtain the two-point correlator on the boundary. The expansion of (3.38) near $z = 0$ is

$$\hat{\Psi}(\hat{\Xi}, \hat{\Xi}^\dagger) = \frac{1}{2} \begin{pmatrix} 1 - it \\ 1 + it \end{pmatrix} (\hat{F}(it) - \hat{G}(-it)) + \frac{\kappa(\alpha)z}{2} \begin{pmatrix} -\hat{F}(it) - \hat{G}(-it) + (1 - it)(\hat{F}'(it) - \hat{G}'(-it)) \\ \hat{F}(it) + \hat{G}(-it) + (1 + it)(\hat{F}'(it) - \hat{G}'(-it)) \end{pmatrix} + \mathcal{O}(z^2). \quad (4.40)$$

It, along with its Hermitian conjugate, can be reexpressed as

$$\begin{aligned} \hat{\Psi}(\hat{\Xi}, \hat{\Xi}^\dagger) &= (\sigma_3 - it\mathbb{1})\psi_0(t) - iz\kappa(\alpha)\partial_t((\sigma_3 - it\mathbb{1})\psi_1(t)) + \mathcal{O}(z^2), \\ \tilde{\Psi}(\hat{\Xi}, \hat{\Xi}^\dagger) &= \bar{\psi}_0(t)(\sigma_3 + it\mathbb{1}) + iz\kappa(\alpha)\partial_t(\bar{\psi}_1(t)(\sigma_3 + it\mathbb{1})) + \mathcal{O}(z^2), \end{aligned} \quad (4.41)$$

where ψ_0 and ψ_1 are the quantum analogs of (2.34).

As was true in the commutative case, the action should be supplemented with a boundary term so that variations of the total action can be consistently made to vanish. A general variation in $\hat{\Psi}$ induces the following change in the bulk action (4.36):

$$\begin{aligned} \delta \hat{S}_B &= -\frac{i}{\alpha} \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dzdt}{z^2} \epsilon_{abc} \tilde{\Psi} \star \left(\tau_d \tau^c \hat{X}^b \star \delta \hat{\Psi} \star \hat{X}^a \star \hat{X}^d - \frac{\alpha}{2} \tau^c \hat{X}^b \star \delta \hat{\Psi} \star \hat{X}^a \right) \\ &= -\frac{i}{\alpha} \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dzdt}{z^2} \epsilon_{abc} \hat{X}^a \star \left(\hat{X}^d \star \tilde{\Psi} \tau_d - \frac{\alpha}{2} \tilde{\Psi} \right) \star \hat{X}^b \tau^c \star \delta \hat{\Psi} + b.v., \end{aligned} \quad (4.42)$$

where the boundary variation is

$$b.v. = -\frac{i}{\alpha} \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \int_B \frac{dzdt}{z^2} \epsilon_{abc} \left([\tilde{\Psi} \star \tau_d \tau^c \hat{X}^b \star \delta \hat{\Psi}, \hat{X}^a \star \hat{X}^d] - \frac{\alpha}{2} [\tilde{\Psi} \star \tau^c \hat{X}^b \star \delta \hat{\Psi}, \hat{X}^a] \right). \quad (4.43)$$

The boundary term results from the integral of a star commutator, and as shown in Appendix A, such an integral is given explicitly by

$$\int_D \frac{dzdt}{z^2} [\hat{\mathcal{F}}, \hat{\mathcal{G}}](z, t) = -i\alpha \int dt (\partial_t \hat{\mathcal{F}} \hat{\mathcal{G}}) \Big|_{z=0}, \quad (4.44)$$

for any two functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ on quantized EAdS₂. Applying this to (4.43) gives

$$b.v. = -\frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \int dt \epsilon_{abc} \left((\hat{X}^a \star \hat{X}^d) \partial_t (\tilde{\Psi} \star \hat{X}^b \star \tau_d \tau^c \delta \hat{\Psi}) - \frac{\alpha}{2} \hat{X}^a \partial_t (\tilde{\Psi} \star \hat{X}^b \star \tau^c \delta \hat{\Psi}) \right) \Big|_{z=0}. \quad (4.45)$$

Using the result that the star product reduces to the pointwise product in the boundary limit, and that $\hat{X}^a$ go to the embedding coordinates X^a in the boundary limit, this simplifies to

$$\begin{aligned}
b.v. &= -\frac{1}{\sqrt{1+\frac{\alpha^2}{4}}} \int dt \epsilon_{abc} X^a \partial_t X^b \tilde{\Psi} \left(X^d \tau_d - \frac{\alpha}{2} \right) \tau^c \delta \hat{\Psi} \Big|_{z=0} \\
&= -\frac{1}{\sqrt{1+\frac{\alpha^2}{4}}} \int dt \tilde{\Psi} \left(\mathbb{1} - \frac{\alpha}{2} \gamma \right) \Upsilon \delta \hat{\Psi} \Big|_{z=0},
\end{aligned} \tag{4.46}$$

where γ is the chirality operator on the classical manifold (2.21), and Υ was defined in (4.12). We see that the boundary term reduces to its commutative counterpart (4.10) in the limit $\alpha \rightarrow 0$.

As in Sec. IV A b, we need to add a term to the action that will cancel the boundary variation (4.46). For this we shall again assume that ψ_0 is the source field. Then (4.46) is eliminated upon adding

$$\hat{S}_{\partial B}[\tilde{\Psi}, \hat{\Psi}] = \frac{1}{\sqrt{1+\frac{\alpha^2}{4}}} \int dt \bar{\psi}_0(t) (\sigma_3 + it\mathbb{1}) \left(\mathbb{1} - \frac{\alpha}{2} \gamma \right) \Upsilon \hat{\Psi} \Big|_{z=0} \tag{4.47}$$

to the bulk action (4.36). Upon using the identity,

$$\bar{\psi}_0(t) (\sigma_3 + it\mathbb{1}) \gamma \Upsilon = -\bar{\psi}_0(t), \tag{4.48}$$

(4.47) simplifies to

$$\hat{S}_{\partial B}[\tilde{\Psi}, \hat{\Psi}] = \frac{1}{\sqrt{1+\frac{\alpha^2}{4}}} \int dt \bar{\psi}_0(t) \left((\sigma_3 + it\mathbb{1}) \Upsilon + \frac{\alpha}{2} \right) \hat{\Psi} \Big|_{z=0}. \tag{4.49}$$

Once again only the boundary term survives in the total action $\hat{S}_{\text{total}} = \hat{S}_B + \hat{S}_{\partial B}$ when evaluating it on shell. To evaluate it we can rewrite the asymptotic expression for the solution for $\tilde{\Psi}$ (4.41) using (4.3) and (4.4):

$$\hat{\Psi}(\hat{\Xi}, \hat{\Xi}^\dagger) = \frac{1}{2\pi} \int dt' \left(-\frac{i}{t-t'} (\sigma_3 - it'\mathbb{1}) \psi_1(t') + \frac{\kappa(\alpha)z}{(t-t')^2} (\sigma_3 - it'\mathbb{1}) \psi_0(t') + \mathcal{O}(z^2) \right). \tag{4.50}$$

Substituting this result into $\hat{S}_{\partial B}$, and regarding ψ_0 as the source field, we get

$$\hat{S}_{\text{total}}[\tilde{\Psi}, \hat{\Psi}]|_{\text{on-shell}} = \frac{\kappa(\alpha)}{\sqrt{1+\frac{\alpha^2}{4}}} \int dt dt' \frac{z}{2\pi(t-t')^2} \bar{\psi}_0(t) (\sigma_3 + it\mathbb{1}) \Upsilon (\sigma_3 - it'\mathbb{1}) \psi_0(t'). \tag{4.51}$$

Like in the commutative case, this term survives after taking the $z \rightarrow 0$ limit because Υ (4.12) contains a factor of $1/z$, while the term that is linear in α in the integrand of (4.49) does not survive. Using the identity (4.19), we then get

$$\hat{S}_{\text{total}}[\tilde{\Psi}, \hat{\Psi}]|_{\text{on-shell}} = -\frac{i}{2\pi} \frac{\kappa(\alpha)}{\sqrt{1+\frac{\alpha^2}{4}}} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \bar{\psi}_0(t) \frac{\sigma_3}{t-t'} \psi_0(t'). \tag{4.52}$$

But we saw in (3.28) that $\kappa(\alpha) = \sqrt{1+\frac{\alpha^2}{4}}$, so the on-shell action and resulting two-point function on the boundary are identical to the commutative case to all orders in α .

V. CONCLUDING REMARKS

In this paper we have continued our examination of the noncommutative AdS/CFT correspondence. Though here we restricted our attention to the case of massless fields, we

have extended previous results in two directions. First of all, by considering spinors, we have gone beyond the case of scalar fields examined in [6–8]. An essential tool for the study of spinor fields on the quantized EAdS₂ was the map (2.26) to the covariant formulation of the Dirac operator, which we were able to extend to the quantized space [cf. Appendix B]. The second direction of this paper makes what we regard as a notable advance in the study of fields on quantized spaces-times. In our previous works we obtained results only to leading nontrivial order in the

noncommutative parameter α , while here we were able to solve the free scalar and spinor equations exactly, i.e., nonperturbatively in α . The results suggest more generally that there may be a simple prescription for finding exact solutions on this quantum space. It is to replace functions of the complex variables $\zeta = z + it$ and $\bar{\zeta} = z - it$ with functions of the quantum operators $\hat{\Xi} = \kappa(\alpha)\hat{z} + i\hat{t}$ and $\hat{\Xi}^\dagger = \kappa(\alpha)\hat{z} - i\hat{t}$, respectively, where $\hat{z}$ and $\hat{t}$ are the noncommutative analogs of the Fefferman-Graham coordinates z and t . Thus it appears that the z coordinate (which is the inverse of the radial coordinate) gets rescaled by $\kappa(\alpha)$ upon quantization. The z coordinate is said to be associated with the energy scale of the corresponding CFT. The implication then is that the CFT energy scale gets rescaled by the quantization of the bulk.

We conjectured in our previous works that boundary conformal symmetry survives the quantization of the AdS bulk space-time. By having exact results we have now been able to prove the conjecture for free massless fields. Furthermore, we have shown that the boundary two-point function is undeformed for the case of a spinor source, while it gets rescaled by $\kappa(\alpha)$ for the case of a scalar source.

There are a number of extensions of this work that can be considered, some of them being crucial (in particular, item 2 below) to the study of the noncommutative correspondence. Among them are the following:

- (1) One obvious extension of our work is to obtain exact solutions for the case of massive scalars and spinors on quantized EAdS₂. Exact results are possible and will likely involve special functions, with the domain spanned by operators $\hat{\Xi}$ and $\hat{\Xi}^\dagger$. Exact results for the corresponding boundary two-point functions can then be deduced from the asymptotic properties of these solutions, as was done here.
- (2) Another obvious extension is to include interactions in scalar and spinor field theory on quantized

EAdS₂, such as $\lambda\Phi^3$ or a Yukawa coupling. Partial results for the $\lambda\Phi^3$ interaction were obtained in [7], but because we did not have exact solutions they required a rather involved expansion in two parameters, λ and the noncommutativity parameter α . A simpler one parameter expansion can now be pursued by exploiting the exact solution we have found here. It should admit leading order expressions of $n(> 2)$ point correlators on the boundary. A question raised previously was whether or not the rescaling of the boundary two-point function (in the case of scalars) could be due to a trivial field renormalization. The study of interactions using the exact results found here should resolve this issue.

- (3) Natural generalizations of quantized EAdS₂ were obtained in [8]. They correspond to the quantization of $\mathbb{CP}^{p,q}$. As with EAdS₂, the quantization is uniquely given upon demanding that it preserves the isotropies. A construction of the Laplacian and Dirac operators on these spaces should be possible, from which one can search for exact solutions to the equations for massless scalar and spinor fields analogous to the ones found in this article.

We plan to address these and related topics in forthcoming articles.

APPENDIX A: BOUNDARY TERMS

The integral of the commutator of the Moyal-Weyl star product of any two functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ on the plane vanishes provided that the functions vanish sufficiently rapidly at infinity. The integral does not vanish, however, if the domain D (which say is two dimensional) has a boundary ∂D and the functions are unconstrained on the boundary. Below we show that if D is the bulk region of EAdS₂, we get the result (4.44).

Say that D is some two-dimensional region with a boundary ∂D . From the definition of the Moyal-Weyl star product (4.26), one gets

$$\int_D dx dy [\hat{\mathcal{F}}, \hat{\mathcal{G}}](x, y) = \int_D dx dy [\partial_x \mathcal{V}_y - \partial_y \mathcal{V}_x](x, y) = \int_{\partial D} (\mathcal{V}_x dx + \mathcal{V}_y dy). \quad (\text{A1})$$

Upon expanding $\mathcal{V}_x$ and $\mathcal{V}_y$ to leading and next to leading orders in α one has

$$\begin{aligned} \mathcal{V}_x &= i\alpha \left(-\partial_x \hat{\mathcal{F}} \hat{\mathcal{G}} + \frac{\alpha^2}{24} (\partial_x^3 \hat{\mathcal{F}} \partial_y^2 \hat{\mathcal{G}} + \partial_x \partial_y^2 \hat{\mathcal{F}} \partial_x^2 \hat{\mathcal{G}} - 2\partial_x^2 \partial_y \hat{\mathcal{F}} \partial_x \partial_y \hat{\mathcal{G}}) + \mathcal{O}(\alpha^4) \right), \\ \mathcal{V}_y &= i\alpha \left(-\partial_y \hat{\mathcal{F}} \hat{\mathcal{G}} + \frac{\alpha^2}{24} (\partial_y^3 \hat{\mathcal{F}} \partial_x^2 \hat{\mathcal{G}} + \partial_x^2 \partial_y \hat{\mathcal{F}} \partial_y^2 \hat{\mathcal{G}} - 2\partial_x \partial_y^2 \hat{\mathcal{F}} \partial_x \partial_y \hat{\mathcal{G}}) + \mathcal{O}(\alpha^4) \right). \end{aligned} \quad (\text{A2})$$

Now let D correspond to the bulk region of EAdS₂ with ∂D corresponding to $z = 0$, where the map between the canonical coordinates (x, y) to Fefferman-Graham coordinates can be taken to be (2.38). A constant z slice corresponds to $x = \text{const}$. So using $dy = \frac{dt}{z}$, we get the following integral along the asymptotic boundary:

$$\int_D dx dy [\hat{\mathcal{F}}, \hat{\mathcal{G}}](x, y) = \int dt \frac{\mathcal{V}_y}{z} \Big|_{z=0}. \quad (\text{A3})$$

$\mathcal{V}_y$ is reexpressed in terms for Fefferman-Graham coordinates, using

$$\partial_x = -z\partial_z - t\partial_t, \quad \partial_y = z\partial_t. \quad (\text{A4})$$

Assuming that $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$, along with their derivatives, go to well-defined functions of t in the limit $z \rightarrow 0$, we get that the order α^3 correction terms in $\mathcal{V}_y$ go like z^3 , and so do not contribute in the limit. Thus only the $\mathcal{O}(\alpha)$ term survives in the limit, giving the result (4.44).

APPENDIX B: QUANTUM MAP TO NONCOVARIANT BASIS

In Sec. II we gave the map (2.26) between the covariant and noncovariant formulations of the Dirac operator on EAdS₂. Here we give the quantum analog of the map. This will allow us to construct the Dirac operator on quantized EAdS₂ in the noncovariant basis. We shall see that solutions to the Dirac equation for massless spinors are trivial in this basis. As a check, we can then map these solutions back to the covariant basis. The result agrees with the exact solutions obtained in Sec. III, (3.38).

To obtain the quantum map we will need the quantum version of the 2×2 matrix U (2.24). For this we first introduce the left and right acting operators $\hat{z}^A$ and $\hat{t}^A$, $A = L, R$, which act on functions $\hat{\mathcal{F}}$ on quantized EAdS₂ according to

$$\begin{aligned} \hat{z}^L \hat{\mathcal{F}} &= \hat{z} \hat{\mathcal{F}}, & \hat{t}^L \hat{\mathcal{F}} &= \hat{t} \hat{\mathcal{F}}, \\ \hat{z}^R \hat{\mathcal{F}} &= \hat{\mathcal{F}} \hat{z}, & \hat{t}^R \hat{\mathcal{F}} &= \hat{\mathcal{F}} \hat{t}. \end{aligned} \quad (\text{B1})$$

From the commutation relations (3.24), the left acting operators $\hat{t}^L$ and $\hat{z}^L$ satisfy

$$[\hat{t}^L, \hat{z}^L] = i\alpha(\hat{z}^L)^2, \quad (\text{B2})$$

while the opposite sign appears in the commutator of $\hat{t}^R$ and $\hat{z}^R$

$$[\hat{t}^R, \hat{z}^R] = -i\alpha(\hat{z}^R)^2. \quad (\text{B3})$$

All other commutators between $\hat{z}^A$ and $\hat{t}^A$ vanish. From (B3) it follows that

$$[\hat{t}^R, (\hat{z}^R)^{-1}] = i\alpha \mathbb{1}. \quad (\text{B4})$$

We can further introduce the right-acting version of the operator $\hat{\Xi}$ defined in Sec. III B, along with its Hermitian conjugate:

$$\hat{\Xi}^R = \kappa(\alpha)\hat{z}^R + i\hat{t}^R, \quad \hat{\Xi}^{\dagger R} = \kappa(\alpha)\hat{z}^R - i\hat{t}^R. \quad (\text{B5})$$

We now claim that the quantization of the 2×2 matrix (2.24) is

$$\hat{U}^R = \begin{pmatrix} -\hat{\Xi}^R + 1 & -\hat{\Xi}^{\dagger R} - 1 \\ \hat{\Xi}^R + 1 & \hat{\Xi}^{\dagger R} - 1 \end{pmatrix} \frac{1}{2\sqrt{\kappa(\alpha)\hat{z}^R}}. \quad (\text{B6})$$

It satisfies the analog of the identity (2.25), i.e.,

$$\hat{\Xi}^{\dagger R} \sigma_3 \hat{U}^R = -\sigma_3, \quad (\text{B7})$$

where⁷

$$\hat{U}^{\dagger R} = \frac{1}{2\sqrt{\kappa(\alpha)\hat{z}^R}} \begin{pmatrix} -\hat{\Xi}^{\dagger R} + 1 & \hat{\Xi}^{\dagger R} + 1 \\ -\hat{\Xi}^R - 1 & \hat{\Xi}^R - 1 \end{pmatrix}. \quad (\text{B8})$$

The inverse of $\hat{U}^R$ is

$$(\hat{U}^R)^{-1} = \frac{1}{2\sqrt{\kappa(\alpha)\hat{z}^R}} \begin{pmatrix} \hat{\Xi}^{\dagger R} - 1 & \hat{\Xi}^{\dagger R} + 1 \\ -\hat{\Xi}^R - 1 & -\hat{\Xi}^R + 1 \end{pmatrix}. \quad (\text{B9})$$

It is easy to check that it is the left inverse of $\hat{U}^R$, $(\hat{U}^R)^{-1} \hat{U}^R = \mathbb{1}$. To check that it is the right inverse, $\hat{U}^R (\hat{U}^R)^{-1} = \mathbb{1}$, use

$$[\hat{\Xi}^R, (\hat{z}^R)^{-1}] = -[\hat{\Xi}^{\dagger R}, (\hat{z}^R)^{-1}] = -\alpha, \quad (\text{B10})$$

which follows from (B4).

From (2.25), we saw that U maps the classical chirality operator σ_3 of the noncovariant basis to the chirality operator γ of the covariant basis. The analogous property holds for $\hat{U}$ in the quantized system. Upon computing $\hat{U}^R \sigma_3 (\hat{U}^R)^{-1}$, we get

$$\hat{U}^R \sigma_3 (\hat{U}^R)^{-1} = \frac{1}{\sqrt{1 + \frac{\alpha^2}{4}}} \left(\hat{X}^{Ra} \tau_a - \frac{\alpha}{2} \mathbb{1} \right), \quad (\text{B11})$$

which is the chirality operator $\hat{\gamma}$ on quantized EAdS₂. The inverse map takes $\hat{\gamma}$ back to σ_3 ,

$$(\hat{U}^R)^{-1} \hat{\gamma} \hat{U}^R = \sigma_3. \quad (\text{B12})$$

Now we apply the map to the Dirac operator $\hat{D}$, which in Sec. III was formulated in the covariant basis. We write this Dirac operator as

⁷We can assume that Hermitian conjugation of right acting operators $\hat{A}^R$ is an anti-involution. For this define the action of $(\hat{A}^R)^\dagger$ on functions $\hat{\mathcal{F}}$ on quantized EAdS₂ by

$$\hat{\mathcal{F}}(\hat{A}^R)^\dagger = (\hat{A}^R \hat{\mathcal{F}}^\dagger)^\dagger = (\hat{\mathcal{F}}^\dagger \hat{A})^\dagger = \hat{A}^\dagger \hat{\mathcal{F}}.$$

Then for two right acting operators $\hat{A}^R$ and $\hat{B}^R$

$$\begin{aligned} \hat{\mathcal{F}}(\hat{A}^R)^\dagger (\hat{B}^R)^\dagger &= (\hat{A}^\dagger \hat{\mathcal{F}}) (\hat{B}^R)^\dagger = \hat{B}^\dagger \hat{A}^\dagger \hat{\mathcal{F}} = (\hat{A} \hat{B})^\dagger \hat{\mathcal{F}} \\ &= \hat{\mathcal{F}}((\hat{A} \hat{B})^R)^\dagger = \hat{\mathcal{F}}((\hat{B})^R (\hat{A})^R)^\dagger. \end{aligned}$$

$$\hat{D} = \frac{i}{\alpha} \hat{\gamma} \Delta_c \tau^c, \quad \Delta_c = \epsilon_{abc} \hat{X}^{Ra} \hat{X}^{Lb}, \quad (\text{B13})$$

where $\hat{X}^{Aa}$, $A = L, R$, give the left and right action of $\hat{X}^a$, (3.8). Using (3.29), we can express them in terms of $\hat{z}^A$ and $\hat{\tau}^A$ according to

$$\begin{aligned} \hat{X}^{A0} &= -\frac{1}{2} (\hat{\tau}^A (\hat{z}^A)^{-1} \hat{\tau}^A + \kappa(\alpha)^2 \hat{z}^A + (\hat{z}^A)^{-1}), \\ \hat{X}^{A1} &= -\frac{1}{2} (\hat{\tau}^A (\hat{z}^A)^{-1} + \hat{\tau}^A (\hat{z}^A)^{-1}), \\ \hat{X}^{A2} &= -\frac{1}{2} (\hat{\tau}^A (\hat{z}^A)^{-1} \hat{\tau}^A + \kappa(\alpha)^2 \hat{z}^A - (\hat{z}^A)^{-1}). \end{aligned} \quad (\text{B14})$$

To write down the Dirac operator in the noncovariant basis we apply a similarity transformation to $\hat{D}$. We get

$$\hat{D}^{(\hat{U}^R)^{-1}} = (\hat{U}^R)^{-1} \hat{D} \hat{U}^R = \frac{i}{\alpha} \sigma_3 (\hat{U}^R)^{-1} \Delta_c \tau^c \hat{U}^R, \quad (\text{B15})$$

where we used (B12). Substituting (B6) and (B9) into (B15) gives

$$\hat{D}^{(\hat{U}^R)^{-1}} = \frac{i}{2\alpha\kappa(\alpha)} \sigma_3 \frac{1}{\sqrt{\hat{z}^R}} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \frac{1}{\sqrt{\hat{z}^R}}, \quad (\text{B16})$$

where the matrix elements M_{ab} are

$$\begin{aligned} M_{11} &= \hat{\Xi}^{\dagger R} \Delta_+ \hat{\Xi}^R + \Delta_- + i(\hat{\Xi}^{\dagger R} \Delta_1 - \Delta_1 \hat{\Xi}^R), \\ M_{12} &= \hat{\Xi}^{\dagger R} \Delta_+ \hat{\Xi}^{\dagger R} - \Delta_- - i(\hat{\Xi}^{\dagger R} \Delta_1 + \Delta_1 \hat{\Xi}^{\dagger R}), \\ M_{21} &= -\hat{\Xi}^R \Delta_+ \hat{\Xi}^R + \Delta_- - i(\hat{\Xi}^R \Delta_1 + \Delta_1 \hat{\Xi}^R), \\ M_{22} &= -\hat{\Xi}^R \Delta_+ \hat{\Xi}^{\dagger R} - \Delta_- + i(\hat{\Xi}^R \Delta_1 - \Delta_1 \hat{\Xi}^{\dagger R}), \end{aligned} \quad (\text{B17})$$

and $\Delta_{\pm}$ and Δ_1 are defined according to

$$\begin{aligned} \Delta_{\pm} &= \Delta_0 \pm \Delta_2 = \epsilon_{AB} \hat{X}^{A\mp} \hat{X}^{B1}, \\ \Delta_1 &= \frac{1}{2} \epsilon_{AB} \hat{X}^{A+} \hat{X}^{B-}. \end{aligned} \quad (\text{B18})$$

Here $\hat{X}^{A\pm} = \hat{X}^{A2} \pm \hat{X}^{A0}$, and ϵ_{AB} is antisymmetric, with $\epsilon_{LR} = 1$.

The rather involved expressions for the matrix elements can be simplified. Starting with M_{11} , we can write

$$\begin{aligned} M_{11} &= \hat{X}^{L+} \left(\hat{X}^{R1} + \frac{i}{2} (\hat{\Xi}^{\dagger R} \hat{X}^{R-} - \hat{X}^{R-} \hat{\Xi}^{\dagger R}) \right) \\ &\quad + \hat{X}^{L-} \left(\hat{\Xi}^{\dagger R} \hat{X}^{R1} \hat{\Xi}^R - \frac{i}{2} (\hat{\Xi}^{\dagger R} \hat{X}^{R+} - \hat{X}^{R+} \hat{\Xi}^{\dagger R}) \right) \\ &\quad - \hat{X}^{L1} (\hat{\Xi}^{\dagger R} \hat{X}^{R-} \hat{\Xi}^R + \hat{X}^{R+}). \end{aligned} \quad (\text{B19})$$

Some algebra shows that this is zero. [In fact, all three lines separately vanish. For this we need to use (B5) and (B14).] The same result follows for M_{22} , which is expanded as

$$\begin{aligned} M_{22} &= \hat{X}^{L+} \left(-\hat{X}^{R1} + \frac{i}{2} (\hat{\Xi}^R \hat{X}^{R-} - \hat{X}^{R-} \hat{\Xi}^{\dagger R}) \right) \\ &\quad - \hat{X}^{L-} \left(\hat{\Xi}^R \hat{X}^{R1} \hat{\Xi}^{\dagger R} + \frac{i}{2} (\hat{\Xi}^R \hat{X}^{R+} - \hat{X}^{R+} \hat{\Xi}^{\dagger R}) \right) \\ &\quad + \hat{X}^{L1} (\hat{\Xi}^R \hat{X}^{R-} \hat{\Xi}^{\dagger R} + \hat{X}^{R+}). \end{aligned} \quad (\text{B20})$$

Again, all three lines separately vanish. So as with the commutative Dirac operator in the noncovariant basis (2.15), the diagonal matrix elements vanish, $M_{11} = M_{22} = 0$. Concerning the off-diagonal matrix elements, we first expand M_{21} :

$$\begin{aligned} M_{21} &= \hat{X}^{L+} \left(\hat{X}^{R1} - \frac{i}{2} (\hat{\Xi}^R \hat{X}^{R-} + \hat{X}^{R-} \hat{\Xi}^R) \right) \\ &\quad + \hat{X}^{L-} \left(-\hat{\Xi}^R \hat{X}^{R1} \hat{\Xi}^R + \frac{i}{2} (\hat{\Xi}^R \hat{X}^{R+} + \hat{X}^{R+} \hat{\Xi}^R) \right) \\ &\quad + \hat{X}^{L1} (\hat{\Xi}^R \hat{X}^{R-} \hat{\Xi}^R - \hat{X}^{R+}). \end{aligned} \quad (\text{B21})$$

With some work this can be simplified to

$$\begin{aligned} M_{21} &= \kappa(\alpha) (-i\hat{X}^{L+} - i\hat{X}^{L-} (\hat{\Xi}^R)^2 + 2\hat{X}^{L1} \hat{\Xi}^R) \\ &= i\kappa(\alpha) (\kappa(\alpha)^2 \hat{z}^L + \hat{\tau}^L (\hat{z}^L)^{-1} \hat{\tau}^L - (\hat{z}^L)^{-1} (\hat{\Xi}^R)^2 \\ &\quad + i((\hat{z}^L)^{-1} \hat{\tau}^L + \hat{\tau}^L (\hat{z}^L)^{-1}) \hat{\Xi}^R) \\ &= i\kappa(\alpha) (\hat{z}^L)^{-1} (\hat{\Xi}^{\dagger L} + \hat{\Xi}^R - \alpha \hat{z}^L) (\hat{\Xi}^L - \hat{\Xi}^R). \end{aligned} \quad (\text{B22})$$

Finally for M_{12} we get

$$\begin{aligned} M_{12} &= -\hat{X}^{L+} \left(\hat{X}^{R1} + \frac{i}{2} (\hat{\Xi}^{\dagger R} \hat{X}^{R-} + \hat{X}^{R-} \hat{\Xi}^{\dagger R}) \right) \\ &\quad + \hat{X}^{L-} \left(\hat{\Xi}^{\dagger R} \hat{X}^{R1} \hat{\Xi}^{\dagger R} + \frac{i}{2} (\hat{\Xi}^{\dagger R} \hat{X}^{R+} + \hat{X}^{R+} \hat{\Xi}^{\dagger R}) \right) \\ &\quad + \hat{X}^{L1} (-\hat{\Xi}^{\dagger R} \hat{X}^{R-} \hat{\Xi}^{\dagger R} + \hat{X}^{R+}), \end{aligned} \quad (\text{B23})$$

which simplifies to

$$\begin{aligned} M_{12} &= \kappa(\alpha) (-i\hat{X}^{L+} - i\hat{X}^{L-} (\hat{\Xi}^{\dagger R})^2 - 2\hat{X}^{L1} \hat{\Xi}^{\dagger R}) \\ &= i\kappa(\alpha) (\kappa(\alpha)^2 \hat{z}^L + \hat{\tau}^L (\hat{z}^L)^{-1} \hat{\tau}^L - (\hat{z}^L)^{-1} (\hat{\Xi}^{\dagger R})^2 \\ &\quad - i((\hat{z}^L)^{-1} \hat{\tau}^L + \hat{\tau}^L (\hat{z}^L)^{-1}) \hat{\Xi}^{\dagger R}) \\ &= i\kappa(\alpha) (\hat{z}^L)^{-1} (\hat{\Xi}^{\dagger R} + \hat{\Xi}^L + \alpha \hat{z}^L) (\hat{\Xi}^{\dagger L} - \hat{\Xi}^{\dagger R}). \end{aligned} \quad (\text{B24})$$

So the transformed Dirac operator $\hat{D}^{(\hat{U}^R)^{-1}}$ is

$$\frac{1}{2\alpha\hat{z}^L\sqrt{\hat{z}^R}} \begin{pmatrix} 0 & -(\hat{\Xi}^{\dagger R} + \hat{\Xi}^L + \alpha\hat{z}^L)(\hat{\Xi}^{\dagger L} - \hat{\Xi}^{\dagger R}) \\ (\hat{\Xi}^{\dagger L} + \hat{\Xi}^R - \alpha\hat{z}^L)(\hat{\Xi}^L - \hat{\Xi}^R) & 0 \end{pmatrix} \frac{1}{\sqrt{\hat{z}^R}}. \quad (\text{B25})$$

It is the quantization of the standard Dirac operator on EAdS₂ (2.15).

When the Dirac operator (B25) acts on a spinor $\hat{\Psi} = \begin{pmatrix} \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{pmatrix}$ it gives

$$\hat{D}^{(\hat{U}^R)^{-1}} \hat{\Psi} = \frac{1}{2\alpha\hat{z}} \begin{pmatrix} -[\hat{\Xi}^{\dagger}, \hat{\Psi}_2 \frac{1}{\sqrt{\hat{z}}}] \hat{\Xi}^{\dagger} - (\hat{\Xi} + \alpha\hat{z})[\hat{\Xi}^{\dagger}, \hat{\Psi}_2 \frac{1}{\sqrt{\hat{z}}}] \\ [\hat{\Xi}, \hat{\Psi}_1 \frac{1}{\sqrt{\hat{z}}}] \hat{\Xi} + (\hat{\Xi}^{\dagger} - \alpha\hat{z})[\hat{\Xi}, \hat{\Psi}_1 \frac{1}{\sqrt{\hat{z}}}] \end{pmatrix} \frac{1}{\sqrt{\hat{z}}}. \quad (\text{B26})$$

For the case of massless spinors, the Dirac equation implies that $[\hat{\Xi}, \hat{\Psi}_1 \frac{1}{\sqrt{\hat{z}}}] = [\hat{\Xi}^{\dagger}, \hat{\Psi}_2 \frac{1}{\sqrt{\hat{z}}}] = 0$, and therefore we get the simple solution

$$\hat{\Psi} = \begin{pmatrix} F(\hat{\Xi}) \\ G(\hat{\Xi}^{\dagger}) \end{pmatrix} \sqrt{\hat{z}}, \quad (\text{B27})$$

yielding the quantum version of (2.17). Once again, $\hat{F}$ and $\hat{G}$ are arbitrary elements of the polynomial algebras

generated by $\hat{\Xi}$ and $\hat{\Xi}^{\dagger}$, respectively. Finally, if we act on the solution (B27) with $\hat{U}$, we get

$$\begin{aligned} \hat{\Psi} &= \hat{U}^R \hat{\Psi} = \frac{1}{2\sqrt{\kappa(\alpha)}} \begin{pmatrix} -\hat{\Xi}^R + 1 & -\hat{\Xi}^{\dagger R} - 1 \\ \hat{\Xi}^R + 1 & \hat{\Xi}^{\dagger R} - 1 \end{pmatrix} \begin{pmatrix} F(\hat{\Xi}) \\ G(\hat{\Xi}^{\dagger}) \end{pmatrix} \\ &= \frac{1}{2\sqrt{\kappa(\alpha)}} \begin{pmatrix} (1 - \hat{\Xi})F(\hat{\Xi}) - (1 + \hat{\Xi}^{\dagger})G(\hat{\Xi}^{\dagger}) \\ (1 + \hat{\Xi})F(\hat{\Xi}) + (1 - \hat{\Xi}^{\dagger})G(\hat{\Xi}^{\dagger}) \end{pmatrix}, \end{aligned} \quad (\text{B28})$$

which agrees with the exact solution (3.38).

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