Conformal block expansion in celestial CFT

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The 4D four-point scattering amplitude of massless scalars via a massive exchange is expressed in a basis of conformal primary particle wave functions. This celestial amplitude is expanded in a basis of 2D conformal partial waves on the unitary principal series, and then rewritten as a sum over 2D conformal blocks via contour deformation. The conformal blocks include intermediate exchanges of spinning light-ray states, as well as scalar states with positive integer conformal weights. The conformal block prefactors are found as expected to be quadratic in the celestial operator product expansion coefficients.

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I. INTRODUCTION

Celestial conformal field theories (CCFTs) provide potential holographic duals for 4D quantum theories of gravity in asymptotically flat spacetimes [1–10]. Correlators on the 2D celestial sphere are defined as spacetime scattering amplitudes of particles with $SO(3, 1)$ conformal primary wave functions. These celestial correlators have some, but not all, of the properties of standard CFT_2 correlators. In order to better understand their properties, in this paper we derive a conformal block expansion of the massless scalar four-point correlators arising from massive scalar exchange. Our formula reveals a number of interesting features and provides insight into the structure of CCFT. The expansion is a new decomposition of spacetime scattering amplitudes as sums over s , t , or u channel exchanges, unlike the Feynman diagrammatic decomposition involving s , t , and u exchanges. This is one of many instances in which stringy behavior emerges in celestial holography.

One motivation for our analysis is to find a suitable basis of primary operators in CCFT. Depending on its presentation, the problem of finding a complete basis of primaries of any conformal field theory can be quite nontrivial. For example in Liouville theory, presented as an action for a scalar field, exponentials of the Liouville field formally describe primaries with arbitrary conformal weight Δ . However a complete basis includes only primaries with $\Delta > \frac{c-1}{12}$ on the real axis [11,12]. This conclusion requires analysis of modular properties, OPE locality, and closure

and crossing symmetry of the four-point function. In sine-Liouville theory the situation is even more subtle and some discrete states are also required [13].

The presentations of CCFTs given so far¹ are novel: 2D correlators are defined as Mellin transforms² of 4D momentum-space scattering amplitudes. Soft theorems [14–16] imply that these transform as conformal primaries under the Virasoro action on the celestial sphere.³ The Mellin transform formally produces primaries of any complex conformal weight. Yet further primaries are obtained from these via shadow or light-ray transforms. All of these together are clearly an overcomplete basis. An important constraint on a complete basis is that it should supply all the intermediate exchange channels represented in the conformal block expansion of the four-point correlators. Hence the present work sheds some light on this open problem.

Pasterski and Shao [2] showed—up to zero mode subtleties [17]—that states on the unitary principal series with $\Delta = 1 + i\lambda$, $\lambda \in \mathbb{R}$ form a complete basis for normalizable (massless) particle scattering states. However such states are not in representations of the translation group, which shift $\Delta \rightarrow \Delta + 1$, and in gauge or gravity theory they do not include the states at nonpositive integer Δ which generate the soft symmetries [3,5,10,18] or their Goldstone boson partners at positive integer Δ . States with real integral Δ do include these states and provide

¹An intrinsic construction of any CCFT starting from a microscopic theory of gravity such as string theory would be major progress.

²This paper considers massless scattering only. In the massive case a hyperbolic transform replaces the Mellin transform.

³This only holds when gravity is included, but we nevertheless expect the pure field theory correlators discussed here to capture some aspects of conformal symmetry, particularly the global Lorentz subgroup of Virasoro.

representations of the symmetry groups. Moreover the poles in their scattering amplitudes encode all information about the Wilsonian effective action [19]. However it is not known if the integer-weight states are in any sense complete.

Our analysis is simplified by working in (2,2) signature Klein space which means the CCFT lives in (1,1) signature [20], and by defining correlators with general complex conformal weights via analytic continuation. Our explicit formulas decompose four-point scattering of massless scalars via a contact interaction or massive t -channel scalar exchange. All of the intermediate conformal blocks exchanged can be identified as one of two types of conformal primary states of the massive scalar. The first type are spin $J = 0$ blocks with real, positive, integral conformal weights $\Delta = 2, 3, \dots$ ⁴ The second are the so-called light-ray states which have appeared in a number of recent conformal block expansions [21–23]. They have imaginary spin [as allowed in (1,1)] and $\Delta = 1$. They are related by the light-ray transform, which is a shadow in one null direction, to ordinary primaries on the unitary principal series. New conformal primary solutions to the bulk scalar wave equation are herein constructed which describe the wave functions of the associated spacetime scalar particles.

Conformal symmetry predicts that the coefficients of the conformal blocks are quadratic in the OPE coefficients. The OPE coefficients in CCFT can be extracted from the three-point amplitude and we indeed find this to be the case. For the integer blocks the relevant OPEs are Euler beta functions with arguments related to the conformal weights, while for the light-ray transforms they are just the scalar coupling in an appropriate basis.

The match is impressive in some details but not perfect. There is an anomalous overall coefficient of $\frac{1}{2} \csc \frac{\pi\beta}{2}$, with β a sum of external conformal weights as well as a product of cosines of the weights appearing in the integral blocks which we do not understand. Speculations on the origin of these anomalous factors are given in Sec. VA.

A fascinating feature of this expansion is that the conformal block expansion in massive scalar exchanges pertains even to the pure $\lambda\phi^4$ scattering in which there are no massive scalars, or indeed any intermediate states. This can be traced back to the fact that the Mellin transform from momentum space to celestial amplitudes is well defined only in theories with soft UV behavior. In scale invariant theories like $\lambda\phi^4$ or Yang-Mills (YM), theory the transforms are only marginally convergent and a regulator is needed to define them. In $\lambda\phi^4$ the regulation effectively amounts to replacing the contact interaction with a massive scalar and taking the infinite mass limit for fixed g/m . But

since celestial amplitudes have no mass scale the massive scalar does not decouple in this limit. This resonates both with string theory lore and with observations in [19] that pointlike interactions in quantum gravity should be softened by massive particle exchanges. Of course scalar scattering is not in itself quantum gravity, but the argument that it should be embedded in a well-behaved CCFT requires quantum gravity [14].

Another interesting feature is the CFT_2 origin of the $\delta(z - \bar{z})$ singularity in the celestial amplitude, where z is the 2D conformal cross ratio. In momentum space, one starts with a product of four delta functions enforcing momentum conservation. Mellin transforms eliminate three of the four delta functions, with $\delta(z - \bar{z})$ remaining as the constraint that all four insertions lie on a celestial equator. In a unitary, rational, local CFT_2 , the four-point function can never be singular at noncoincident points. However here translation invariance (i.e., momentum conservation) shifts $\Delta \rightarrow \Delta + 1$ and thereby requires an infinite number of primaries. We see explicitly herein that summing over these primaries cancels the finite contribution from the light-ray continuum, while the remaining integral over light-ray blocks can be shown to produce the delta function.

Our work builds on several related previous analyses. Lam and Shao [24] performed a partial wave expansion for scalar four-point scattering in three dimensions and verified a factorization formula implied by the optical theorem. A partial wave expansion of a scalar 4-point in a different channel and a YM 4-point was given by Nandan, Schreiber, Volovich, and Zlotnikov in [25]. Fan, Fotopoulos, Stieberger, Taylor, and Zhu [26] gave a conformal block decomposition for this 4 point after shadowing an external leg. They also found light-ray-like states (with integral weights) but did not match OPE coefficients.

This paper is organized as follows. Section II reviews some basic celestial formulas and establishes conventions. Section III presents the massless scalar 4-point amplitude mediated by massive scalar exchange in the conformal primary basis. In Sec. IV we review completeness properties of partial waves on the unitary principal series. In Sec. V we transform the celestial amplitude to that basis, deform the contour integral over the unitary principal series, and derive the conformal block expansion from pole contributions. Properties of the light transform are discussed and the explicit bulk wave function of exchanged states is derived. Section VI describes the infinite-mass limit and the resulting contact amplitude. Appendix A presents two nontrivial checks of our main formula (5.4). The normalization of the massive celestial two-point function is computed in Appendix B. Appendix C provides an interpretation of the overall $\csc \frac{\pi\beta}{2}$ as arising from a sum over off-shell exchanges. The conformal block decomposition of the full 4-point celestial amplitude of massless

⁴The full four-point scattering mediated by massive exchange also includes spinning exchanges of dimensions $\beta/2$ shifted by integers.

scalars mediated by massive exchange is worked out in Appendix D.

II. PRELIMINARIES

Momentum-space scattering amplitudes can be mapped to correlation functions on the celestial sphere transforming covariantly under the $SL(2, \mathbb{C})$ Lorentz group [1,2]. In this section, we review the simplest case of celestial amplitudes associated with the scattering of 4 massless scalars. We will treat z and $\bar{z}$ as independent real variables. This amounts to considering scattering in (2,2) signature Klein space where massless momenta can be parametrized by their energy ω_i and celestial coordinates $(z_i, \bar{z}_i)$ as⁵

$$p_i^\mu = \epsilon_i \omega_i q^\mu(z_i, \bar{z}_i), \quad q^\mu(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, z - \bar{z}, 1 - z\bar{z}). \quad (2.1)$$

Four-point celestial amplitudes of massless scalars were shown in [19,27] to be related to their momentum-space counterparts by a Mellin transform,

$$\mathcal{A}(z_i, \bar{z}_i; \Delta_i) = K(z_i, \bar{z}_i) X(z, \beta) \int_0^\infty d\omega \omega^{\beta-1} \mathcal{M}(\omega^2, -z\omega^2). \quad (2.2)$$

Here

$$K(z_i, \bar{z}_i) \equiv \prod_{i < j} z_{ij}^{\frac{h_i}{2} - h_j - h_i} \bar{z}_{ij}^{\frac{\bar{h}_i}{2} - \bar{h}_j - \bar{h}_i} \quad (2.3)$$

is a (symmetric) conformally covariant factor and

$$X(z, \beta) \equiv 2^{-\beta-2} |z(1-z)|^{\frac{1}{2}(\beta+4)} \delta(z - \bar{z}). \quad (2.4)$$

$\mathcal{M}(s, t)$ is the four-point momentum-space scattering amplitude of scalars upon stripping off the momentum conserving delta function, $h_i = \bar{h}_i = \frac{\Delta_i}{2}$ are conformal weights,

$$\beta = \sum_{i=1}^4 \Delta_i - 4, \quad (2.5)$$

and z is the standard cross ratio

$$z = \frac{z_{13} z_{24}}{z_{12} z_{34}}. \quad (2.6)$$

⁵These are related to the Klein space metric $ds^2 = -(dX^0)^2 + (dX^1)^2 - (dX^2)^2 + (dX^3)^2$ by $X^\mu = \frac{1}{2}(un^\mu + r q^\mu)$, where $n^\mu = (1, 0, 0, -1)$.

FIG. 1. Four-point kinematics with particles 1,2 incoming and particles 3,4 outgoing. We will study the 13–24 OPE limit ($z \rightarrow 0$) in the CCFT which corresponds to the collinear limit of 1 and 3 in the bulk.

In terms of the Mandelstam invariants

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_3)^2, \quad (2.7)$$

one finds the cross ratio is

$$z \equiv -\frac{t}{s}. \quad (2.8)$$

In this paper we will study the physical configuration $s \geq 0, t \in [-s, 0] \Rightarrow z \in [0, 1]$, corresponding to particles 1 and 2 incoming ($\epsilon_1 = \epsilon_2 = -1$) and particles 3 and 4 outgoing ($\epsilon_3 = \epsilon_4 = 1$). We illustrate this in Fig. 1.

While (2.2) resembles a 2D CFT four-point function constrained by global conformal symmetry, it enjoys two additional properties: upon stripping off the conformally covariant prefactor (2.3), it depends on the external weights only through their sum β and it is multiplied by a delta function enforcing that the four points lie on a circle. Both of these properties follow more generally from translation invariance [28].

In this paper we will be interested in the expansion of celestial amplitudes (2.2) around the $z \rightarrow 0$ limit which should encode information about the operator product expansions (OPE) of particles 1, 3 and 2, 4. To this end, it will be convenient to relate $K(z_i, \bar{z}_i)$ to the conformally covariant structure symmetric under 1–3 and 2–4 exchanges,

$$I_{13-24}(z_i, \bar{z}_i) \equiv \frac{\left(\frac{z_{34}}{z_{14}}\right)^{h_{13}} \left(\frac{z_{14}}{z_{12}}\right)^{h_{24}} \left(\frac{\bar{z}_{34}}{\bar{z}_{14}}\right)^{\bar{h}_{13}} \left(\frac{\bar{z}_{14}}{\bar{z}_{12}}\right)^{\bar{h}_{24}}}{z_{13}^{h_1+h_3} z_{24}^{h_2+h_4} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3} \bar{z}_{24}^{\bar{h}_2+\bar{h}_4}}. \quad (2.9)$$

We find

$$\begin{aligned} K(z_i, \bar{z}_i)X(z, \beta) &= \prod_{i < j} z_{ij}^{\frac{h}{3}-h_i-h_j} \bar{z}_{ij}^{\frac{h}{3}-\bar{h}_i-\bar{h}_j} 2^{-\beta-2} |z(1-z)|^{\frac{1}{6}(\beta+4)} \delta(z-\bar{z}) \\ &= I_{13-24}(z_i, \bar{z}_i) 2^{-\beta-2} |z|^{2+\beta/2} |1-z|^{h_{13}-h_{24}} \delta(z-\bar{z}). \end{aligned} \quad (2.10)$$

This allows us to put generic celestial four-point amplitudes into the form

$$\mathcal{A}(z_i, \bar{z}_i; \Delta_i) = I_{13-24}(z_i, \bar{z}_i) f(z, \bar{z}), \quad (2.11)$$

with $f(z, \bar{z})$ obtained from (2.10) and (2.2).

In CCFT we are *a priori* interested in arbitrary complex dimensions $(h_i, \bar{h}_i)$ for which Mellin integrals or other quantities may be ill defined. Wherever such ambiguities arise we invoke analytic continuation from real values of $(h_i, \bar{h}_i)$. A discussion of the analytic properties of four-point scattering can be found in [19].

III. SCALAR SCATTERING

A. Massive exchange

Here we consider the scattering of four massless scalars ϕ mediated by a scalar Ψ of mass m . The stripped momentum-space amplitude is

$$\mathcal{M}(s, t) = -g^2 \left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{-t-s-m^2} \right). \quad (3.1)$$

Plugging (3.1) into (2.2) and evaluating the integral using contours, we find the celestial amplitude takes the form (2.11) with

$$\begin{aligned} f_{\text{stu}}(z, \bar{z}) &= N_{gm}(\beta) \delta(z-\bar{z}) |z|^2 |z-1|^{h_{13}-h_{24}} \\ &\times \left[e^{i\pi\beta/2} |z|^{\beta/2} + 1 + \left| \frac{z}{z-1} \right|^{\beta/2} \right]. \end{aligned} \quad (3.2)$$

The z -independent prefactor is

$$N_{gm}(\beta) = \frac{g^2 \pi}{8m^2} \frac{(m/2)^\beta}{\sin \pi\beta/2}, \quad (3.3)$$

with g the strength of the $\phi^2\Psi$ interaction and $h_{ij} \equiv h_i - h_j$. This agrees with [25] upon redefinition of the cross ratio.

It is possible to consistently project out the s , t , or u channel exchanges. In a theory with two massless scalars ϕ_1, ϕ_2 which couple to Ψ via equal strength interactions $g\phi_i^2\Psi$, the $\phi_1\phi_2 \rightarrow \phi_1\phi_2$ amplitude contains only the second t -channel term and $f_{\text{stu}}(z, \bar{z})$ simply reduces to

$$f_t(z, \bar{z}) = N_{gm}(\beta) |z|^2 |z-1|^{h_{13}-h_{24}} \delta(z-\bar{z}). \quad (3.4)$$

For simplicity we concentrate on this four-point scattering in the following. However we note that all three terms in

(3.2) have an albeit more complicated conformal block decomposition in the 13–24 channel (this case is deferred to the Appendix D). While discrete and continuous exchanges are still present, with discrete states of dimensions differing by integers, these two kinds of contributions are in this case not obviously related by a light-ray transform; their dimensions are in general β dependent and $J \neq 0$ contributions are allowed. Moreover, the coefficients of the decompositions do not factorize, although simplifications may occur at special values of the external dimensions. We leave a full analysis to future work.

B. Contact interaction

It is not difficult to see from a change of variables in (2.2) that any s -independent amplitude $\mathcal{M}$ will take the form (2.11) with $f(z, \bar{z})$ having the same z dependence as (3.4). For example tree-level ϕ^4 four-point contact scattering has

$$\mathcal{M}(s, t) = \lambda. \quad (3.5)$$

Plugging this into (2.2) one immediately finds

$$f_\lambda(z, \bar{z}) = \frac{\pi\lambda}{2} \delta(\beta) |z|^2 |z-1|^{h_{13}-h_{24}} \delta(z-\bar{z}). \quad (3.6)$$

In fact one can show that (3.2) reduces to the four-point contact interaction as $m \rightarrow \infty$ for fixed g/m . The following identity which holds for purely imaginary β or equivalently external states on the principal series will be useful⁶

$$\begin{aligned} \lim_{z \rightarrow 0} z^{-\beta} &= \frac{1}{\Gamma(\beta)} \lim_{z \rightarrow 0} \int_0^\infty d\omega \omega^{\beta-1} e^{-\omega z} \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty d\omega \omega^{\beta-1} = 2\pi\beta\delta(\beta). \end{aligned} \quad (3.7)$$

Looking now at the prefactor of (3.2), we see

$$\lim_{m \rightarrow \infty} N_{gm}(\beta) = \frac{g^2}{8m^2} \frac{\pi\beta}{\sin \pi\beta/2} 2\pi\delta(\beta) = \frac{g^2\pi}{2m^2} \delta(\beta). \quad (3.8)$$

Since β is set to 0, we deduce that the $|z|^{\beta/2}$ and $|\frac{z}{z-1}|^{\beta/2}$ terms are set to 1 and the celestial amplitude collapses to

⁶A prescription for analytically continuing this result to $\beta \in \mathbb{C}$ was proposed in [29].

$$\lim_{m \rightarrow \infty} f_{\text{stu}}(z, \bar{z}) = \frac{3g^2\pi}{2m^2} \delta(\beta) |z|^2 |1 - z|^{h_{13}-h_{24}} \delta(z - \bar{z}). \quad (3.9)$$

This is precisely the amplitude of the four-point contact interaction (3.6) subject to the identification

$$\frac{3g^2}{m^2} = \lambda. \quad (3.10)$$

IV. PARTIAL WAVE BASIS

Any conformally covariant four-point function admits a conformal partial wave expansion in a complete basis of orthonormal solutions to the two-particle conformal Casimir equation. One then hopes to reexpress this decomposition as a sum or integral over conformal blocks associated with intermediate conformal primary states. This representation of the four-point function gives information both about the complete basis of primary states and their OPEs. In particular when a complete basis is identified, their OPE should close and no other primaries should appear in the conformal block expansion. In CCFT one typically discusses conformal primary wave functions as well as their shadows and light transforms with arbitrary complex weight, with soft currents and Goldstone bosons arising at real integer weights. All of these together are clearly an overcomplete basis. Which if any subset comprise a complete basis on which all the symmetries act is an open question which we hope the current work will help address.

In a standard 2D CFT with a nondegenerate spectrum, the OPE

$$\begin{aligned} \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_3(z_3, \bar{z}_3) \sim \sum_{\mathcal{O}_k \text{ primary}} z_{13}^{h_k-h_1-h_3} \bar{z}_{13}^{\bar{h}_k-\bar{h}_1-\bar{h}_3} \\ \times C_{13}^k \mathcal{O}_k(z_3, \bar{z}_3) + \dots, \end{aligned} \quad (4.1)$$

where ... includes a sum over descendants, allows CFT four-point correlation functions to be written as [30]

$$\begin{aligned} \left\langle \prod_{i=1}^4 \mathcal{O}_i^{h_i, \bar{h}_i}(z_i, \bar{z}_i) \right\rangle = I_{13-24}(z_i, \bar{z}_i) \\ \times \sum_{\mathcal{O}_k \text{ primary}} C_{13k} C_{24k} k_{h_k, \bar{h}_k}^{\{h_i, \bar{h}_i\}}(z, \bar{z}). \end{aligned} \quad (4.2)$$

Here $k_{h, \bar{h}}^{\{h_i, \bar{h}_i\}}$ are $\text{SL}(2, \mathbb{C})$ conformal blocks which resum the contribution from a primary $\mathcal{O}_{h, \bar{h}}$ and its $\text{SL}(2, \mathbb{C})$ descendants appearing in the $\mathcal{O}_1, \mathcal{O}_3$ OPE, and $z = \frac{z_{13}z_{24}}{z_{12}z_{34}}$ as in (2.6). From now on, we will suppress any labels indicating the dependence of the conformal blocks on external dimensions. C_{ijk} are coefficients of three-point functions

$$\begin{aligned} \langle \mathcal{O}_i(z_i, \bar{z}_i) \mathcal{O}_j(z_j, \bar{z}_j) \mathcal{O}_k(z_k, \bar{z}_k) \rangle \\ = \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{ik}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}} \end{aligned} \quad (4.3)$$

and $C_{ij}^k = C_{ijl} D^{lk}$ where

$$\langle \mathcal{O}_i(z_i, \bar{z}_i) \mathcal{O}_j(z_j, \bar{z}_j) \rangle = \frac{D_{ij}}{z_{ij}^{h_i+h_j} \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j}}. \quad (4.4)$$

Conformal invariance requires $D_{ij} = 0$ for operators of different scaling dimension. Equation (4.2) can be derived by replacing $\mathcal{O}_1 \mathcal{O}_3$ on the left-hand side by the OPE (4.1) (including descendants) and resumming the three-point functions (4.3) and their analogs involving descendants of one of the operators into $k_{h, \bar{h}}(z, \bar{z})$.

Alternatively, conformal symmetry implies that the blocks are eigenfunctions of the conformal Casimir acting on operators 1 and 3. This reduces to the following differential equation

$$(\mathcal{D}_z + \mathcal{D}_{\bar{z}}) k_{h, \bar{h}}(z, \bar{z}) = [h(h-1) + \bar{h}(\bar{h}-1)] k_{h, \bar{h}}(z, \bar{z}), \quad (4.5)$$

where

$$\begin{aligned} \mathcal{D}_z = z^2(1-z) \frac{\partial^2}{\partial z^2} - (1-h_{13}+h_{24})z^2 \frac{\partial}{\partial z} + h_{13}h_{24}z, \\ h_{ij} \equiv h_i - h_j. \end{aligned} \quad (4.6)$$

It turns out that (4.5) admits a complete, orthogonal basis of solutions [31–33]. We treat $z, \bar{z}$ as real independent variables, or equivalently we work in (2,2) Klein space. In this case

$$\Psi_{h, \bar{h}}(z, \bar{z}) = \Psi_h(z) \Psi_{\bar{h}}(\bar{z}) \quad (4.7)$$

forms such a set provided Ψ_h and $\Psi_{\bar{h}}$ are both complete and orthogonal solutions to the $\text{SL}(2, \mathbb{R})$ conformal Casimir equations

$$\mathcal{D}_z \Psi_h(z) = h(h-1) \Psi_h(z), \quad \mathcal{D}_{\bar{z}} \Psi_{\bar{h}}(\bar{z}) = \bar{h}(\bar{h}-1) \Psi_{\bar{h}}(\bar{z}). \quad (4.8)$$

Symmetry of (4.8) under $h \rightarrow 1-h$ implies Ψ_h is a linear combination of the $\text{SL}(2, \mathbb{R})$ conformal blocks

$$k_h(z) = z^h {}_2F_1 \left[\begin{matrix} h-h_{13}, h+h_{24} \\ 2h \end{matrix}; z \right] \quad (4.9)$$

and their shadows $k_{1-h}(z)$. For real external dimensions, it was shown in [34,35] that $\mathcal{D}_z$ is self-adjoint with respect to the inner product

$$\langle F, G \rangle = \int_0^1 \frac{dz}{z^2} (1-z)^{-h_{13}+h_{24}} F^*(z) G(z). \quad (4.10)$$

A generic set of $\mathcal{D}_z$ eigenfunctions with real eigenvalues $\alpha^2 - \frac{1}{4}$ and $h = \frac{1}{2} + \alpha$, $\alpha \in i\mathbb{R}$ is

$$\Psi_\alpha(z) = Ak_{1/2+\alpha}(z) + Bk_{1/2-\alpha}(z), \quad \alpha \in i\mathbb{R}. \quad (4.11)$$

Imposing the boundary condition $\Psi(1) = 1$ fixes A and B and yields a basis of solutions [35]

$$\begin{aligned} \Psi_\alpha &= \frac{1}{2} (Q(\alpha)k_{1/2+\alpha}(z) + Q(-\alpha)k_{1/2-\alpha}(z)), \\ Q(\alpha) &= \frac{2\Gamma(-2\alpha)\Gamma(1-h_{13}+h_{24})}{\Gamma(\frac{1}{2}-\alpha-h_{13})\Gamma(\frac{1}{2}-\alpha+h_{24})}. \end{aligned} \quad (4.12)$$

The conformal partial waves (4.12) are real-valued ($\Psi_\alpha^* = \Psi_{-\alpha} = \Psi_\alpha$) for real h_{13} , h_{24} , orthogonal with respect to the inner product (4.10)

$$\begin{aligned} \langle \Psi_\alpha, \Psi_{\alpha'} \rangle &= \int_0^1 \frac{dz}{z^2} (1-z)^{-h_{13}+h_{24}} \Psi_\alpha(z) \Psi_{\alpha'}(z) \\ &= 2\pi i \frac{N(\alpha)}{2} [\delta(\alpha - \alpha') + \delta(\alpha + \alpha')], \end{aligned} \quad (4.13)$$

with

$$N(\alpha) = \frac{Q(\alpha)Q(-\alpha)}{2} \quad (4.14)$$

and obey the completeness relation

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\alpha}{N(\alpha)} \Psi_\alpha(z) \Psi_\alpha(w) = z^2 (1-z)^{h_{13}-h_{24}} \delta(z-w). \quad (4.15)$$

Using a hypergeometric identity Ψ_α can be compactly written as

$$\Psi_\alpha(z) = z^{h_{13}} {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \alpha - h_{13}, \frac{1}{2} - \alpha - h_{13} \\ 1 - h_{13} + h_{24} \end{matrix}; -\frac{1-z}{z} \right]. \quad (4.16)$$

This argument can be easily generalized [33] to show that

$$\Psi_{\alpha, \bar{\alpha}}(z, \bar{z}) = \Psi_\alpha(z) \Psi_{\bar{\alpha}}(\bar{z}), \quad \alpha, \bar{\alpha} \in i\mathbb{R} \quad (4.17)$$

form a basis of the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ conformal Casimir equation on the Lorentzian square $z, \bar{z} \in [0, 1]$. It follows from (4.13) and (4.15) that any conformally invariant function $f(z, \bar{z})$ can be decomposed as

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\alpha}{N(\alpha)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\bar{\alpha}}{N(\bar{\alpha})} \hat{f}(\alpha, \bar{\alpha}) \Psi_{\alpha, \bar{\alpha}}(z, \bar{z}), \quad (4.18)$$

where

$$\begin{aligned} \hat{f}(\alpha, \bar{\alpha}) &= \int_0^1 \frac{dz}{z^2} (1-z)^{-h_{13}+h_{24}} \\ &\times \int_0^1 \frac{d\bar{z}}{\bar{z}^2} (1-\bar{z})^{-h_{13}+h_{24}} f(z, \bar{z}) \Psi_{\alpha, \bar{\alpha}}(z, \bar{z}). \end{aligned} \quad (4.19)$$

We emphasize that (4.17) form a basis on the Lorentzian square $z, \bar{z} \in [0, 1]$ while standard $SL(2, \mathbb{C})$ conformal partial waves such as in [31] form a basis on the Euclidean plane. It is this property that simplifies the analysis of celestial four-point amplitudes with nontrivial support for $z \in [0, 1]$ and $z = \bar{z}$ which we turn to next.

V. CONFORMAL BLOCK DECOMPOSITION

In this section we decompose the tree-level four-scalar celestial amplitude in the partial wave basis (4.7), and then use contour deformation to reexpress it as a sum plus integral over exchanges of conformal blocks associated to primary operators multiplied by the appropriate squared OPE coefficients.

We take $f(z, \bar{z})$ to be the t -channel amplitude $f_t(z, \bar{z})$ in (3.4). We will first take the external dimensions to be real, and then obtain the final result for external dimensions on the principal series by analytic continuation. We begin by evaluating (4.19) for $f = f_t$

$$\begin{aligned} \hat{f}_t(\alpha, \bar{\alpha}) &= N_{gm}(\beta) \int_0^1 \frac{dz}{z^2} (1-z)^{-h_{13}+h_{24}} \int_0^1 \frac{d\bar{z}}{\bar{z}^2} (1-\bar{z})^{-\bar{h}_{13}+\bar{h}_{24}} \\ &\times z^2 (1-z)^{h_{13}-h_{24}} \delta(z-\bar{z}) \Psi_\alpha(z) \Psi_{\bar{\alpha}}(\bar{z}) \\ &= N_{gm}(\beta) \int_0^1 \frac{dz}{z^2} (1-z)^{-h_{13}+h_{24}} \Psi_\alpha(z) \Psi_{\bar{\alpha}}(z), \end{aligned} \quad (5.1)$$

where we used $h_{13} = \bar{h}_{13}$, $h_{24} = \bar{h}_{24}$ for the four-point scalar amplitude. We recognize the last line of (5.1) as the orthogonality relation (4.13), yielding

$$\hat{f}_t(\alpha, \bar{\alpha}) = N_{gm}(\beta) \pi i \frac{Q(\alpha)Q(-\alpha)}{2} [\delta(\alpha - \bar{\alpha}) + \delta(\alpha + \bar{\alpha})]. \quad (5.2)$$

Plugging (5.2) back into (4.18) results in the partial wave decomposition

$$z^2 (1-z)^{-h_{13}+h_{24}} \delta(z-\bar{z}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\alpha}{N(\alpha)} \Psi_\alpha(z) \Psi_\alpha(\bar{z}). \quad (5.3)$$

Next we use (4.12) and deform the contour in the right-hand complex α plane for the term proportional to

$k_{\frac{1}{2}+\alpha}(z)k_{\frac{1}{2}+\alpha}(\bar{z})$. This can be evaluated at the poles of the $Q(\alpha)$ coefficients in (4.12), yielding

$$f_t(z, \bar{z}) = \frac{1}{2} \csc \frac{\pi\beta}{2} \left[\sum_{n=1}^{\infty} D^{nn} C_{13n} C_{24n} \times \cos \pi \left(\frac{n}{2} + h_{13} \right) \cos \pi \left(\frac{n}{2} + h_{24} \right) k_{\frac{1+n}{2}}(z) k_{\frac{1+n}{2}}(\bar{z}) + \frac{1}{2} \int_{-i\infty}^{i\infty} d\alpha C_{13\alpha}^L C_{24\alpha}^L D^{L,\alpha\alpha} k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) \right], \quad (5.4)$$

$$C_{ijn} = \frac{2g}{m^4} \left(\frac{m}{2} \right)^{2h_i+2h_j} B \left(\frac{1+n}{2} + h_{ij}, \frac{1+n}{2} - h_{ij} \right),$$

$$C_{ij\alpha}^L = -\pi i \frac{2g}{m^4} \left(\frac{m}{2} \right)^{2h_i+2h_j} \frac{1}{\alpha} \quad (5.5)$$

and

$$D^{nn} = \frac{nm^2}{2\pi}, \quad D^{L,\alpha\alpha} = -\frac{m^2\alpha^2}{\pi^2 i}. \quad (5.6)$$

We demonstrate explicitly in Appendix A that the right-hand-side of (5.4) vanishes for $z \neq \bar{z}$ due to a cancellation between the two terms. Here D^{nn} and $D^{L,\alpha\alpha}$ are the (inverse of the) coefficient of the celestial two-point function of massive scalars and its light transform respectively. These are derived in Appendix B and Sec. VB 3.

A. Integer mode exchanges

Consider the first term in (5.4). The coefficients (5.5) are precisely the previously known OPEs of two massless and one massive scalar, as computed from the three-point celestial amplitude of two massless and one massive particle [24].⁷ Hence this term corresponds to a tower of massive exchanges of positive integer dimensions $\Delta = n + 1$ and spin $J = 0$. The factor of n in (5.4) arises from the normalization of the celestial two-point function of massive scalars of dimension $\Delta = 1 + n$ as detailed in Appendix B.

This remarkable match still leaves unexplained the prefactor of $\frac{1}{2} \csc \frac{\pi\beta}{2}$ as well as the product of $\cos \pi(\frac{n}{2} + h_{ij})$. The first factor, which depends only on β arises in a universal manner (see Appendix C), and has analogs present in all previous discussions of conformal factorization [24–26]. It appears to be a signature of the fact that, from a bulk point of view, the exchanges are off-shell. Interestingly it drops out of the residues of the poles of the amplitude (indeed it is the source of them), which were argued in [19] to contain all the information about the low

energy effective action. The form of the second factor suggests it might be absorbed in the OPE coefficients. Moreover, it disappears when restricted to integer weights. Finally it is interesting to note that similar factors are generated in Lorentzian CFT correlators involving terms with different time orderings (see for example Sec. 2.5 of [22]). For now we simply leave these discrepancies to hopefully be resolved in future work.

The second term in (5.4) corresponds to a continuum exchange of operators of $\Delta = 1$ and $J = -i\lambda$ which coincide with dimensions of operators obtained from scalars on the principal series (ie. with $\Delta = 1 + i\lambda$) via the so-called light-ray transform [22] which takes $(\Delta, J) \rightarrow (1 - J, 1 - \Delta)$. The next section considers these in detail.

B. Light transform exchanges

The conformal block decomposition (5.4) contains terms involving exchanges of dimension $\Delta = 1$ and imaginary spin $J = h - \bar{h}$. Such contributions are not allowed in Euclidean CFT, where the solutions to (4.5) are constrained to be single-valued and J is accordingly quantized. However we are working in (2,2) Klein space and the boundary supports a Lorentzian (1,1) CFT. There is no quantization of spin in (1,1) signature and indeed imaginary spin values are regularly encountered in conformal block decompositions [21–23]. In this section we show that the imaginary spin exchanges appearing in the decomposition of Lorentzian celestial four-point amplitudes correspond to new Klein space solutions of the massless and massive wave equations which have not previously been discussed.⁸ The light transform relates these to scalar solutions on the principal series $\Delta = 1 + i\lambda, J = 0$. Moreover, we demonstrate that the coefficient of the imaginary spin block is proportional to the squared OPE coefficients of two massless scalars and a light-transformed massive scalar.

1. Boundary

In a generic conformal field theory, the light transform is an integral of a conformal primary operator $\mathcal{O}$ along a light ray. In 1 + 1 dimensions, there are two null directions, corresponding to $z = 0$ or $\bar{z} = 0$. Therefore, light transforms of spin 0 operators correspond to the holomorphic or antiholomorphic integrals given by⁹

$$\begin{aligned} \mathbf{L}[\Phi_\Delta](z, \bar{z}, +) &= \int_{-\infty}^{\infty} dz' \frac{1}{(z' - z)^{2-\Delta}} \Phi_\Delta(z', \bar{z}), \\ \mathbf{L}[\Phi_\Delta](z, \bar{z}, -) &= \int_{-\infty}^{\infty} d\bar{z}' \frac{1}{(\bar{z}' - \bar{z})^{2-\Delta}} \Phi_\Delta(z, \bar{z}'). \end{aligned} \quad (5.7)$$

⁷In (3,1) signature the three-point functions of one massless incoming, one massless outgoing, and one massive incoming or outgoing particle vanishes. On the other hand, in the (2,2) case considered here, this kinematic configuration is allowed.

⁸Light transforms also exist in (3,1) for integer J .

⁹Equation (5.7) is related to the formula given in [22] by a change of variables.

Notice that the product of these two light-ray transforms is the more commonly encountered shadow transform in Euclidean signature [36].

2. Bulk

In this subsection we construct the holomorphic light transforms of bulk scalar primary wave functions as solutions to the scalar wave equation of weights $h' = 1 - h$ and $\bar{h}' = \bar{h}$ respectively. This corresponds to dimension $\Delta' = 1$ and spin $J' = 1 - \Delta$. In particular, light-transformed primaries of operators with dimension $\Delta = 1 + i\lambda$ on the principal series have imaginary spin $J' = -i\lambda$.

Massless scalar particles in Klein space satisfy the wave equation

$$\square\phi = 0, \quad ds^2 = -(dX^0)^2 + (dX^1)^2 - (dX^2)^2 + (dX^3)^2. \quad (5.8)$$

With the parameterization of q in (2.1),

$$q \cdot X = -(X^0 - X^3) + z(X^1 - X^2) + \bar{z}(X^1 + X^2) - z\bar{z}(X^0 + X^3). \quad (5.9)$$

Massless scalar primaries take the form¹⁰[20]

$$\phi_\Delta(z, \bar{z}) = \frac{i^\Delta \Gamma(\Delta)}{(-q \cdot X)^\Delta}. \quad (5.10)$$

The holomorphic light transform of (5.10) is

$$\int_{-\infty}^{\infty} \frac{dz'}{(z' - z)^{2-\Delta}} \frac{e^{i\pi\Delta/2} \Gamma(\Delta)}{(-q(z', \bar{z}) \cdot X)^\Delta} = \int_{-\infty}^{\infty} \frac{dz'}{(z' - z)^{2-\Delta}} \frac{e^{i\pi\Delta/2} \Gamma(\Delta)}{((X^0 - X^3) - \bar{z}(X^1 + X^2) - z'(X^1 - X^2 - \bar{z}(X^0 + X^3)))^\Delta}. \quad (5.11)$$

Upon changing variables, this integral becomes proportional to

$$\int_{-\infty}^{\infty} \frac{dz'}{(z' - z)^{2-\Delta}} \frac{1}{(-q(z', \bar{z}) \cdot X)^\Delta} = \int_{-\infty}^{\infty} dz' z'^{\Delta-2} (-q \cdot X - z' \partial_z q \cdot X)^{-\Delta} = \frac{(\partial_z q \cdot X)^{1-\Delta}}{-q \cdot X} \int_{-\infty}^{\infty} dz' z'^{\Delta-2} (1 - z')^{-\Delta}. \quad (5.12)$$

The remaining integral is directly related to the Euler beta function¹¹

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dz z^{\Delta-2+\epsilon} (1 - z)^{-\Delta} = \lim_{\epsilon \rightarrow 0} \frac{-2\pi^2 i}{\sin \pi \epsilon \Gamma(-\Delta + 2 + \epsilon) \Gamma(\Delta) \Gamma(\epsilon)} = -2\pi i \frac{1}{\Gamma(\Delta) \Gamma(-\Delta + 2)}. \quad (5.13)$$

Putting everything together, we get the light-transformed field

$$\tilde{\phi}(z, \bar{z}; X) = \frac{2\pi i e^{i\pi\Delta/2}}{\Gamma(2 - \Delta)} \frac{(\partial_z q \cdot X)^{1-\Delta}}{q \cdot X}. \quad (5.14)$$

Equation (5.14) has the correct scaling properties, namely under $\bar{z} \rightarrow \alpha \bar{z}$, $q \cdot X \rightarrow \alpha^{1/2} q \cdot X$ and (5.14) is rescaled by $\alpha^{-1/2 - i\lambda/2}$. Similarly, $z \rightarrow \alpha z$ gives a net factor of $\alpha^{-1/2 + i\lambda/2}$. It follows that (5.14) has dimension and spin $\Delta = 1, J = -i\lambda$ consistent with the light-transform of a conformal scalar primary of dimension $\Delta = 1 + i\lambda$.

¹⁰We suppress the $\pm i\epsilon$ prescription. In Klein space the $\pm$ sign can be absorbed by a rotation of q up to normalization.

¹¹This requires a careful choice of branch for the indefinite integral. There exists a second solution $\tilde{\phi} \propto \frac{(\partial_z q \cdot X)^{2-\Delta}}{q \cdot X} \delta(\partial_z q \cdot X)$ with the correct transformation properties whose relevance we leave to future work.

This analysis is easily generalized to massive primaries [1]

$$\frac{4\pi}{im} \frac{(\sqrt{-X^2})^{\Delta-1}}{(-q \cdot X)^\Delta} K_{\Delta-1}(m\sqrt{X^2}) \quad (5.15)$$

since the light transform only affects the $(-q \cdot X)^{-\Delta}$ piece. Using (5.14), we immediately find the light-transformed massive conformal primary wave functions

$$\tilde{\Phi}(z, \bar{z}; X) = \frac{8\pi^2}{m\Gamma(\Delta)\Gamma(-\Delta+2)} \left(\sqrt{-X^2} \right)^{\Delta-1} \times \frac{(\partial_z q \cdot X)^{1-\Delta}}{q \cdot X} K_{\Delta-1}(m\sqrt{X^2}). \quad (5.16)$$

The antiholomorphic light transforms can be found using similar methods.

3. Light-ray OPEs

In this section we show that the light-ray-like exchanges

$$\frac{1}{2} \int_{-i\infty}^{i\infty} d\alpha C_{13\alpha}^L C_{24\alpha}^L D^{L,\alpha\alpha} k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) \quad (5.17)$$

correspond to light transforms of massive scalars (5.7) by matching the OPE coefficients (5.5) to OPE coefficients of celestial three-point functions involving light transforms of massive scalars on the principal series.

Following [32] we write a conformal block associated with an exchange of dimension Δ and spin J as

$$k_{\Delta,J}^{\Delta_i}(z, \bar{z}) = \frac{\langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_{\Delta,J} \rangle_0 \langle \mathcal{O}_2 \mathcal{O}_4 \mathcal{O}_{\Delta,J} \rangle_0}{\langle \mathcal{O}_{\Delta,J} \mathcal{O}_{\Delta,J} \rangle_0}, \quad (5.18)$$

where $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_{\Delta,J} \rangle_0$ and $\langle \mathcal{O}_{\Delta,J} \mathcal{O}_{\Delta,J} \rangle_0$ are stripped conformal three- and two-point structures. The conformal

block associated with light-ray exchanges then corresponds schematically to the structure

$$\frac{\langle \mathcal{O}_1 L[\mathcal{O}_{\Delta,J}] \mathcal{O}_3 \rangle \langle \mathcal{O}_2 L[\mathcal{O}_{\Delta,J}] \mathcal{O}_4 \rangle}{\langle L[\mathcal{O}_{\Delta,J}] L[\mathcal{O}_{\Delta,J}] \rangle}. \quad (5.19)$$

It was shown in [22] that

$$\begin{aligned} \langle \mathcal{O}_1 L[\mathcal{O}_{\Delta,0}] \mathcal{O}_3 \rangle &= -2\pi i \frac{1}{B(\frac{\Delta+\Delta_{13}}{2}, \frac{\Delta-\Delta_{13}}{2})} \frac{1}{\Delta-1} \\ &\times \langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_{1,1-\Delta} \rangle_0 \equiv C_{13\alpha}^L, \\ \frac{1}{\langle L[\mathcal{O}_{\Delta,0}] L[\mathcal{O}_{\Delta,0}] \rangle} &= -\frac{\Delta-1}{2\pi i} \frac{1}{\langle \mathcal{O}_{1,1-\Delta} \mathcal{O}_{1,1-\Delta} \rangle_0} \equiv D^{L,\alpha\alpha}. \end{aligned} \quad (5.20)$$

It then follows that

$$\frac{\langle \mathcal{O}_1 L[\mathcal{O}_{\Delta,0}] \mathcal{O}_3 \rangle \langle \mathcal{O}_2 L[\mathcal{O}_{\Delta,0}] \mathcal{O}_4 \rangle}{\langle L[\mathcal{O}_{\Delta,0}] L[\mathcal{O}_{\Delta,0}] \rangle} = -\frac{2\pi i}{B(\frac{\Delta+\Delta_{13}}{2}, \frac{\Delta-\Delta_{13}}{2}) B(\frac{\Delta+\Delta_{24}}{2}, \frac{\Delta-\Delta_{24}}{2}) (\Delta-1)} k_{1,1-\Delta}^{\Delta_i}(z, \bar{z}). \quad (5.21)$$

A contribution to the massless scalar four-point function from exchanges of light-transformed massive scalars with $\Delta = 1 + 2\alpha, \alpha \in i\mathbb{R}$ should then take the form¹²

$$\frac{1}{2} \int_{-i\infty}^{i\infty} d\alpha D^{\mathcal{O}\mathcal{O}} C_{13\mathcal{O}} C_{24\mathcal{O}} \frac{-2\pi i}{B(\frac{\Delta+\Delta_{13}}{2}, \frac{\Delta-\Delta_{13}}{2}) B(\frac{\Delta+\Delta_{24}}{2}, \frac{\Delta-\Delta_{24}}{2}) (\Delta-1)} k_{1,-2\alpha}^{\Delta_i}(z, \bar{z}) = \frac{1}{2} \int_{-i\infty}^{i\infty} d\alpha C_{13\alpha}^L C_{24\alpha}^L D^{L,\alpha\alpha} k_{1,-2\alpha}^{\Delta_i}(z, \bar{z}) \quad (5.22)$$

and agrees with the block associated with continuous spin in (5.4). In conclusion, the second term in (5.4) can be identified with the exchange of light-ray primaries including the expected square of OPE coefficients, up to the $\csc \frac{\pi\beta}{2}$ factor.

VI. THE CONTACT AMPLITUDE

Using (3.8) and (5.4), we deduce that the four-point contact amplitude (3.9) has the conformal block representation,

$$\begin{aligned} \mathcal{A}_4^c &= I_{13-24}(z_i, \bar{z}_i) \frac{1}{2 \sin \pi\beta/2} \lim_{m \rightarrow \infty} \left[\frac{1}{2} \int_{-i\infty}^{i\infty} d\alpha c_{13\alpha}^L c_{24\alpha}^L D^{L,\alpha\alpha} k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) \right. \\ &\quad \left. + \sum_{n=0}^{\infty} c_{13n} c_{24n} D^{nn} \cos \pi \left(h_{13} + \frac{n}{2} \right) \cos \pi \left(h_{24} + \frac{n}{2} \right) k_{\frac{1+n}{2}}(z) k_{\frac{1+n}{2}}(\bar{z}) \right], \end{aligned} \quad (6.1)$$

where

$$\begin{aligned} c_{ijn} &= \sqrt{\lambda} \frac{m^{\Delta_i+\Delta_j-3}}{2^{\Delta_i+\Delta_j-1}} B\left(\frac{1+n}{2} + h_{ij}, \frac{1+n}{2} - h_{ij}\right), \\ c_{ij\alpha}^L &= -\pi i \frac{2\sqrt{\lambda}}{m^3} \left(\frac{m}{2}\right)^{2h_i+2h_j} \frac{1}{\alpha}, \\ D^{nn} &= m^2 \frac{n}{2\pi}, \quad D^{L,\alpha\alpha} = -\frac{m^2 \alpha^2}{\pi^2 i}. \end{aligned} \quad (6.2)$$

¹²The overall factor of $\frac{1}{2}$ also appears in [32] in the relation between (5.21) and the analytic continuation of the conformal block to the Regge regime (cf. Eqs. 5.22, 5.23 therein).

These coefficients are consistent with the three-point functions of massless scalars of dimensions Δ_1, Δ_3 (or Δ_2, Δ_4) and a massive scalar of mass m and dimension $\Delta = 1 + n, n \in \mathbb{Z}_+$.

The decomposition (6.1) implies that the theory of a massless scalar ϕ with a ϕ^4 interaction alone has no celestial transform: one must allow for a $\phi^2\Psi$ coupling to a massive scalar Ψ with mass m which is exchanged in conformal blocks. The conformal block decomposition then relates the three-point coupling g to the four-point coupling λ ,

$$g = m\sqrt{\lambda/3}. \quad (6.3)$$

Conversely, the tree-level amplitude in ϕ^4 theory is fully determined by the OPE coefficients of two massless scalars of arbitrary dimensions and a massive scalar of $\Delta = 1 + n$, with n a positive integer. In Appendix D we use the equivalent representation (D2) of (5.4) to compute the conformal block expansion of the tree-level four-point scalar amplitude mediated by a scalar of mass m in (3.2). The formulas there at special values of β also capture the decompositions of some tree-level and one-loop four-point gluon amplitudes [27].

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APPENDIX A: CHECKING THE CONFORMAL BLOCK DECOMPOSITIONS

In this Appendix, we perform two nontrivial checks that the conformal block decomposition (5.4) is correct.

We first show that for $z, \bar{z} \ll 1$ the expansion of the four-point contact amplitude for arbitrary external weights correctly reproduces the $z^2\delta(z - \bar{z})$ behavior of the contact amplitude. The resolution of the four-point scalar contact celestial amplitude takes the form

$$z^2(1-z)^{h_{13}-h_{24}}\delta(z-\bar{z}) = \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) + \sum_{n=1}^{\infty} \frac{n}{2\pi^2} c_{13n} c_{24n} \cos\pi\left(h_{13} + \frac{n}{2}\right) \cos\pi\left(h_{24} + \frac{n}{2}\right) k_{\frac{1+n}{2}}(z) k_{\frac{1+n}{2}}(\bar{z}) \quad (A1)$$

where $c_{13n} c_{24n}$ are the stripped three-point coefficients

$$\begin{aligned} c_{13n} &= B\left(\frac{1}{2} + \frac{n}{2} - h_{13}, \frac{1}{2} + \frac{n}{2} + h_{13}\right), \\ c_{24n} &= B\left(\frac{1}{2} + \frac{n}{2} - h_{24}, \frac{1}{2} + \frac{n}{2} + h_{24}\right). \end{aligned} \quad (A2)$$

When $z, \bar{z} \ll 1$, the leading term is the integral over continuous α , whose contributions have scaling dimension $\Delta = 1$. The remaining blocks have higher scaling dimension and thus contribute higher subleading powers in z . Using the small- z limit of the conformal blocks, we get

$$\begin{aligned} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) &\rightarrow \frac{1}{(2\pi i)} \int_{-i\infty}^{i\infty} d\alpha \sqrt{z\bar{z}} \left(\frac{z}{\bar{z}}\right)^\alpha \\ &= \sqrt{z\bar{z}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi(\log z - \log \bar{z})} \\ &= z^2 \delta(z - \bar{z}). \end{aligned} \quad (A3)$$

This is precisely the four-point contact amplitude in the small z limit.

We now extend the prior argument to show that the conformal block expansion vanishes when $z \neq \bar{z}$ for generic

values of $z, \bar{z}$ between 0 and 1. The continuum term contains an integral over states with $(h, \bar{h}) = (1/2 + \alpha, 1/2 - \alpha)$, $\alpha \in i\mathbb{R}$. Because of the growth conditions on these conformal blocks as a function of α , for $z = \bar{z}$, the integral cannot be closed and is formally divergent. For $z > \bar{z}$, the integral can be closed in the left-hand side of the complex plane, while for $z < \bar{z}$ the integral can be closed to the right. Without loss of generality, we focus on the case when $z < \bar{z}$. Then, the continuum term is

$$\begin{aligned} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \sqrt{z\bar{z}} \left(\frac{z}{\bar{z}}\right)^\alpha &{}_2F_1\left[\begin{matrix} \frac{1}{2} + \alpha - h_{13}, \frac{1}{2} + \alpha + h_{24} \\ 1 + 2\alpha \end{matrix}; z\right] \\ &\times {}_2F_1\left[\begin{matrix} \frac{1}{2} - \alpha - h_{13}, \frac{1}{2} - \alpha + h_{24} \\ 1 - 2\alpha \end{matrix}; \bar{z}\right]. \end{aligned} \quad (A4)$$

For $z < \bar{z}$, we see that integrand is suppressed exponentially at large, positive $\text{Re}(\alpha)$, so the integral can be closed to the right. The conformal block $k_h(z)$ has a pole when h is a negative integer, so we will find poles at $1 - 2\alpha = -m, m \in \mathbb{Z}_{\geq 0}$. Using a residue formula for the hypergeometric function

$$\text{Res}_{c=-m} \left({}_2F_1 \left[\begin{matrix} a & b \\ c \end{matrix}; z \right] \right) = \frac{(-1)^m z^{m+1} (a)_{m+1} (b)_{m+1}}{m! (m+1)!} {}_2F_1 \left[\begin{matrix} 1+a+m, 1+b+m \\ 2+m \end{matrix}; z \right] \quad (\text{A5})$$

where $(a)_{m+1} = a(a+1)\dots(a+m)$, (A4) takes the form

$$\begin{aligned} \frac{1}{(2\pi i)} \int_{-i\infty}^{i\infty} da k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}-\alpha}(\bar{z}) &= - \sum_{m=0}^{\infty} k_{1+\frac{m}{2}}(z) \bar{z}^{-m/2} \text{Res}_{\alpha=\frac{1+m}{2}} \left({}_2F_1 \left[\begin{matrix} \frac{1}{2}-\alpha-h_{13}, \frac{1}{2}-\alpha+h_{24} \\ 1-2\alpha \end{matrix}; \bar{z} \right] \right) \\ &= \frac{1}{2} \sum_{m=0}^{\infty} k_{1+\frac{m}{2}}(z) k_{1+\frac{m}{2}}(\bar{z}) \frac{(-1)^m (-\frac{m}{2}-h_{13})_{m+1} (-\frac{m}{2}+h_{24})_{m+1}}{m! (m+1)!}. \end{aligned} \quad (\text{A6})$$

The factor of $1/2$ arises from

$$\lim_{\alpha \rightarrow \frac{1+m}{2}} \Gamma(1-2\alpha) = \frac{(-1)^m}{2m! (\frac{1+m}{2} - \alpha)} \quad (\text{A7})$$

and the sign from the clockwise contour cancels the sign from the residue at the pole in α . We see that the residue of the conformal block $k_{1/2-\alpha}(\bar{z})$ at $\alpha = \frac{1+m}{2}$ reproduces the conformal block $k_{1+\frac{m}{2}}(z) k_{1+\frac{m}{2}}(\bar{z})$. Writing $m+1 = n$ for $n \in \mathbb{Z}, n \geq 1$ the coefficient of the conformal block $k_{1+\frac{n}{2}}(z) k_{1+\frac{n}{2}}(\bar{z})$ is

$$\begin{aligned} &-\frac{1}{2} (-1)^n n \frac{\Gamma(\frac{1+n}{2}-h_{13}) \Gamma(\frac{1+n}{2}+h_{24})}{\Gamma(\frac{1-n}{2}-h_{13}) \Gamma(\frac{1-n}{2}+h_{24}) (n!)^2} \\ &= -\frac{n}{2\pi^2} \cos \pi \left(\frac{n}{2} + h_{13} \right) \cos \pi \left(\frac{n}{2} + h_{24} \right) B \left(\frac{1+n}{2} + h_{13}, \frac{1+n}{2} - h_{13} \right) B \left(\frac{1+n}{2} + h_{24}, \frac{1+n}{2} - h_{24} \right). \end{aligned} \quad (\text{A8})$$

This is exactly the negative of the coefficient multiplying the corresponding conformal block in the discrete part of the decomposition. Hence, the continuum term generates the δ function in the four-point contact amplitude, together with a convergent sum which is exactly canceled by the discrete sum in the conformal block expansion.

APPENDIX B: CELESTIAL TWO-POINT FUNCTIONS OF MASSIVE SCALARS

In this Appendix we derive the normalization of the celestial two-point function of massive scalars χ of mass m . We start with the momentum space-formula¹³

$$(2\pi)^4 \langle \chi(p_1) \chi(p_2) \rangle = 2p_1^0 (2\pi)^3 \delta^{(3)}(p_1 + p_2), \quad (\text{B1})$$

where

$$p_i = m \hat{p}_i, \quad \hat{p}_i^2 = -1. \quad (\text{B2})$$

The celestial two-point function is found by integrating against AdS_3 bulk-to-boundary propagators [2],

$$\begin{aligned} (2\pi)^4 \langle \chi_{\Delta_1}(z_1, \bar{z}_1) \chi_{\Delta_2}(z_2, \bar{z}_2) \rangle &= m^{-2} \int \frac{d^3 \hat{p}_1}{\hat{p}_1^0} \int \frac{d^3 \hat{p}_2}{\hat{p}_2^0} G_{\Delta_1}(\hat{p}_1; z_1, \bar{z}_1) G_{\Delta_2}(\hat{p}_2; z_2, \bar{z}_2) 2\hat{p}_1^0 (2\pi)^3 \delta^{(3)}(\hat{p}_1 - \hat{p}_2) \\ &= 2(2\pi)^3 m^{-2} \int \frac{d^3 \hat{p}_1}{\hat{p}_1^0} G_{\Delta_1}(\hat{p}_1; z_1, \bar{z}_1) G_{\Delta_2}(\hat{p}_1; z_2, \bar{z}_2). \end{aligned} \quad (\text{B3})$$

The orthogonality of bulk-to-boundary propagators

$$\int \frac{d^3 \hat{p}_1}{\hat{p}_1^0} G_{\Delta_1}(\hat{p}_1; z_1, \bar{z}_1) G_{\Delta_2}(\hat{p}_1; z_2, \bar{z}_2) = -2\pi^3 \frac{1}{(\Delta_1 - 1)^2} \delta(\Delta_1 + \Delta_2 - 2) \delta^{(2)}(z_1 - z_2) + 2\pi^2 \frac{1}{(\Delta_1 - 1)} \delta(\Delta_1 - \Delta_2) \frac{1}{|z_1 - z_2|^{2\Delta_1}} \quad (\text{B4})$$

¹³We follow the convention of [24] where momentum-space amplitudes differ from S -matrix elements by a factor of $i(2\pi)^4$.

then implies that

$$\begin{aligned} \langle \chi_{\Delta_1}(z_1, \bar{z}_1) \chi_{\Delta_2}(z_2, \bar{z}_2) \rangle &= \frac{2\pi}{(\Delta_1 - 1)m^2} \delta(\Delta_1 - \Delta_2) \frac{1}{|z_1 - z_2|^{2\Delta_1}} \\ &\equiv \delta(\Delta_1 - \Delta_2) \frac{D_{12}}{|z_1 - z_2|^{2\Delta_1}}. \end{aligned} \quad (\text{B5})$$

We have used the fact that the conformal primary basis for massive particles consists of wave functions with $\Delta = 1 + i\lambda$ with $\lambda \geq 0$ [2] for which the first term on the right-hand side of (B4) vanishes. We can read off

$$D_{12} = \frac{2\pi}{(\Delta_1 - 1)m^2} \Rightarrow D^{12} = \frac{(\Delta_1 - 1)m^2}{2\pi} \quad (\text{B6})$$

as promised.

APPENDIX C: OFF-SHELL EXCHANGES AND FACTORIZATION

z -dependent functions such as (3.4) and (3.6) are ubiquitous. Indeed, Poincaré symmetry constrains

any celestial four-point function to be multiplied by a β -dependent prefactor. This function arises due to the integral over exchanged off-shell momenta in the bulk Feynman rules. This can be seen from the momentum-space amplitude as follows. Consider the momentum-space four-point contact amplitude

$$\mathcal{A}(p_i) = \lambda \delta^{(4)}(p_1 + p_2 + p_3 + p_4). \quad (\text{C1})$$

We can factor this into a pair of delta functions by integrating over an intermediate off-shell momentum. Due to the kinematic assignment of p_1 and p_2 ingoing and p_3 and p_4 outgoing, the intermediate momenta p must have spacelike momentum, so that we integrate over $-p^2 = M^2 \in (-\infty, 0]$.

We now Mellin transform the external legs to arrive at a correlator on the celestial sphere, where we are left with

$$\begin{aligned} \tilde{\mathcal{A}} &= \frac{\lambda}{2} \int_{-\infty}^0 dM^2 M^2 \int_{-\infty}^{\infty} d\nu \mu(\nu) \int d^2w \tilde{\mathcal{A}}_3(z_1, z_3, w) \tilde{\mathcal{A}}_3^*(z_2, z_4, w), \\ \tilde{\mathcal{A}}_3(z_1, z_3, w) &= \frac{C(\Delta_1, \Delta_3, \Delta)}{|z_{13}|^{\Delta_1 + \Delta_3 - \Delta} |z_3 - w|^{\Delta + \Delta_3 - \Delta_1} |z_1 - w|^{\Delta_1 + \Delta - \Delta_3}}, \\ C(\Delta_1, \Delta_3, \Delta) &= \frac{M^{\Delta_1 + \Delta_3 - 4}}{2^{\Delta_1 + \Delta_3 - 1}} B\left(\frac{\Delta_{13} + \Delta}{2}, \frac{\Delta_{31} + \Delta}{2}\right), \quad \Delta = 1 + i\nu. \end{aligned} \quad (\text{C2})$$

We can now see how the factor of $\delta(\beta)$ arises. The integrals over ν, w generate the appropriate z dependence, while the integral over M^2 , the mass of the exchanged off-shell momentum, becomes

$$\int_{-\infty}^0 dM^2 M^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 6} = 2\pi \delta\left(\sum_i h_i - 2\right) = 4\pi \delta(\beta). \quad (\text{C3})$$

This is precisely the function of β present in the four-point contact amplitude.

We can also show this in the case of a massive scalar exchange. We start with the t -channel scalar amplitude

$$\mathcal{A}(p_i) = -\frac{g^2}{t - m^2} \delta^{(4)}(p_1 + p_2 + p_3 + p_4). \quad (\text{C4})$$

As before, this can be written as

$$\begin{aligned} \mathcal{A}(p_i) &= g^2 \int d^4p \frac{1}{p^2 + m^2} \delta^{(4)}(p_1 + p_3 - p) \delta^{(4)}(p_2 + p_4 + p) \\ &= g^2 \int dM^2 \int d^4p \frac{\delta(p^2 + M^2)}{m^2 - M^2} \delta^{(4)}(p_1 + p_3 - p) \delta^{(4)}(p_2 + p_4 + p) \\ &= \frac{g^2}{2} \int dM^2 M^2 \int \widetilde{d^3\hat{p}} \int \widetilde{d^3\hat{p}'} \frac{1}{m^2 - M^2} \delta^{(4)}(p_1 + p_3 - p) \delta^{(4)}(p_2 + p_4 + p') \\ &\quad \times \int_{-\infty}^{\infty} d\nu \mu(\nu) \int d^2w G_{1+i\nu}(\hat{p}; w) G_{1-i\nu}(\hat{p}'; w), \end{aligned} \quad (\text{C5})$$

where the integral over M^2 runs over the negative real values,

$$\widetilde{d^3 p} \equiv \frac{d^3 \vec{p}}{p^0} \quad (\text{C6})$$

and in the last line we used completeness of the bulk-to-boundary propagators,

$$\int_{-\infty}^{\infty} d\nu \mu(\nu) \int d^2 w G_{1+i\nu}(\hat{p}; w) G_{1-i\nu}(\hat{p}'; w) = \hat{p}^0 \delta^{(3)}(\hat{p} - \hat{p}'). \quad (\text{C7})$$

Just as before, upon taking Mellin transforms with respect to the external lines we find

$$\begin{aligned} \tilde{A} &= \frac{g^2}{2} \int_{-\infty}^0 dM^2 M^2 \frac{1}{m^2 - M^2} \int d\nu \mu(\nu) \\ &\times \int d^2 w \tilde{A}_3(z_1, z_3, w) \tilde{A}_3^*(z_2, z_4, w). \end{aligned} \quad (\text{C8})$$

The celestial three-point functions are given by (C2). We see now that the integral over M^2 generates the $1/\sin(\pi\beta/2)$ factor as upon a change of variables and contour integration

$$\int_0^\infty dM^2 \frac{M^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 6}}{m^2 + M^2} \propto \frac{\pi m^{\beta-2}}{\sin(\pi\beta/2)}, \quad \beta = \sum_{i=1}^4 \Delta_i - 4. \quad (\text{C9})$$

APPENDIX D: AMPLITUDES MEDIATED BY MASSIVE EXCHANGE

In this Appendix we extend our analysis to four-point scalar interactions mediated by a massive exchange. For simplicity, we also set $h_{12} = h_{34} = 0$. Taking particles 1,3 incoming and 2,4 outgoing with dimensions on the principal series, (3.2) reduces to

$$\begin{aligned} \tilde{A}_4^{13 \leftrightarrow 24} &= I_{13-24}(z_i, \bar{z}_i) \frac{g^2 \pi}{8m^2} \frac{(m/2)^\beta}{\sin \pi\beta/2} \delta(z - \bar{z}) |z|^2 \\ &\times \left[1 + e^{i\pi\beta/2} |z|^{\beta/2} + \left| \frac{z}{z-1} \right|^{\beta/2} \right]. \end{aligned} \quad (\text{D1})$$

The first term in (D1) is proportional to the four-point contact amplitude decomposed before so we focus on the remaining two terms. The trick is to replace the prefactor in (A1) by its conformal block representation (5.4) with $h_{13} = h_{24} = 0$ when it multiplies $|z|^{\beta/2}$ and by its equivalent representation

$$\begin{aligned} z^2 \delta(z - \bar{z}) &= \left[\sum_{k=0}^{\infty} \frac{2k}{2\pi^2} c_{13(2k)} c_{24(2k)} k_{\frac{1+2k}{2}} \left(\frac{z}{z-1} \right) k_{\frac{1+2k}{2}} \left(\frac{\bar{z}}{\bar{z}-1} \right) \right. \\ &\left. + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\alpha k_{\frac{1}{2}+\alpha} \left(\frac{z}{z-1} \right) k_{\frac{1}{2}-\alpha} \left(\frac{\bar{z}}{\bar{z}-1} \right) \right] \end{aligned} \quad (\text{D2})$$

when it multiplies the $|z|^{\beta/2}$ term. Equation (D2) is obtained by letting $z \rightarrow \frac{z}{z-1}$, $\bar{z} \rightarrow \frac{\bar{z}}{\bar{z}-1}$ in (5.4) and noticing that the left-hand side is invariant under this change of variables. k_h are the functions in (4.9) with $h_{13} = h_{24} = 0$ and $c_{13(2k)}, c_{24(2k)}$ are (A2) evaluated at $h_{13} = h_{24} = 0$, $n = 2k$. The problem then reduces to finding the partial wave representations of

$$|z|^{\beta/4} k_h(z), \left| \frac{z}{z-1} \right|^{\beta/4} k_h \left(\frac{z}{z-1} \right). \quad (\text{D3})$$

Combining these with their antiholomorphic counterparts and deforming the contour then yields the conformal block decomposition of (D1). Using the Mellin-Barnes representation of the hypergeometric function (4.9)

$$k_h(z) = \frac{\Gamma(2h)}{\Gamma(h)^2} \int_{-\infty}^{\infty} \frac{ds}{2\pi i} \frac{\Gamma(h+s)\Gamma(h+s)\Gamma(-s)}{\Gamma(2h+s)} z^{h+s} (-1)^s, \quad (\text{D4})$$

the inner products [35]

$$\begin{aligned} \langle z^q, \Psi_\alpha \rangle &= \frac{\Gamma(q - \frac{1}{2} \pm \alpha)}{\Gamma(q)^2}, \\ \left\langle \left(\frac{z}{1-z} \right)^q, \Psi_\alpha \right\rangle &= \frac{\Gamma(1-q)\Gamma(-\frac{1}{2} \pm \alpha + q)}{\Gamma(\frac{1}{2} \pm \alpha)\Gamma(q)} \end{aligned} \quad (\text{D5})$$

and performing the contour integrals, we find

$$\langle z^{\beta/4} k_h(z), \Psi_\alpha \rangle = \frac{\Gamma(h + \beta/4 - \frac{1}{2} \pm \alpha)}{\Gamma(h + \beta/4)^2} \times {}_4F_3 \left[\begin{matrix} h, h, h + \beta/4 - \frac{1}{2} + \alpha, h + \beta/4 - \frac{1}{2} - \alpha \\ 2h, h + \beta/4, h + \beta/4 \end{matrix}; 1 \right]. \quad (\text{D6})$$

Here

$$\Gamma(\pm x) \equiv \Gamma(x)\Gamma(-x). \quad (\text{D7})$$

The lengthier formula for $\langle (\frac{z}{1-z})^{\beta/4} k_h(\frac{z}{1-z}), \Psi_\alpha \rangle$ is given by

$$\begin{aligned} |z|^{\beta/2+2} \delta(z-\bar{z}) &= \frac{1}{2\pi^2} \sum_{n,k,k'=0}^{\infty} \mathcal{C}_{13(2n)}^k \mathcal{C}_{24(2n)}^{k'} k_{\frac{2n+1}{2}+\beta/4+k}(z) k_{\frac{2n+1}{2}+\beta/4+k'}(\bar{z}) \\ &+ \frac{1}{2\pi i} \sum_{k,k'=0}^{\infty} \int_{-i\infty}^{i\infty} d\alpha \mathcal{C}_{13\alpha}^k \mathcal{C}_{24-\alpha}^{k'} k_{\frac{1}{2}+\beta/4+\alpha+k}(z) k_{\frac{1}{2}+\beta/4-\alpha+k'}(\bar{z}), \end{aligned} \quad (\text{D8})$$

and

$$\begin{aligned} \left| \frac{z}{1-z} \right|^{\beta/2} |z|^2 \delta(z-\bar{z}) &= \frac{1}{2\pi^2} \sum_{n,k,k'=0}^{\infty} \mathcal{D}_{13(2n)}^k \mathcal{D}_{24(2n)}^{k'} k_{\frac{2n+1}{2}+\beta/4+k}(z) k_{\frac{2n+1}{2}+\beta/4+k'}(\bar{z}) \\ &+ \frac{1}{2\pi i} \sum_{k,k'=0}^{\infty} \int_{-i\infty}^{i\infty} d\alpha \mathcal{D}_{13\alpha}^k \mathcal{D}_{24-\alpha}^{k'} k_{\frac{1}{2}+\alpha+\beta/4+k}(z) k_{\frac{1}{2}-\alpha+\beta/4+k'}(\bar{z}). \end{aligned} \quad (\text{D9})$$

The OPE coefficients $\mathcal{C}$ and $\mathcal{D}$ are given in (D18) and (D25). We describe the details on the computation of these two contributions below.

1. $z^{\beta/2}$ term

We start off by decomposing

$$z^{\beta/4} k_h(z) \quad (\text{D10})$$

in conformal partial waves. We use the Mellin-Barnes representation of the conformal block

$$k_h(z) = \underbrace{\frac{\Gamma(2h)}{\Gamma(h)^2}}_r \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(h+s)^2 \Gamma(-s)}{\Gamma(2h+s)} z^{h+s} (-1)^s \quad (\text{D11})$$

and the inversion formula of z^q

$$\langle z^q, \Psi_\alpha \rangle = \frac{\Gamma(q - \frac{1}{2} \pm \alpha)}{\Gamma(q)^2}. \quad (\text{D12})$$

Putting everything together, we find

$$\langle z^{\beta/4} k_h(z), \Psi_\alpha \rangle = r \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(h+s)^2 \Gamma(-s)}{\Gamma(2h+s)} \frac{\Gamma(h+s+\beta/4-\frac{1}{2} \pm \alpha)}{\Gamma(h+s+\beta/4)^2} (-1)^s. \quad (\text{D13})$$

We now close the contour in the right-hand plane to find

$$\langle z^{\beta/4} k_h(z), \Psi_\alpha \rangle = \frac{\Gamma(h+\beta/4-\frac{1}{2} \pm \alpha)}{\Gamma(h+\beta/4)^2} {}_4F_3 \left[\begin{matrix} h, h, h+\beta/4-\frac{1}{2}+\alpha, h+\beta/4-\frac{1}{2}-\alpha \\ 2h, h+\beta/4, h+\beta/4 \end{matrix}; 1 \right]. \quad (\text{D14})$$

Notice that the only poles in α come from $\Gamma(h+\frac{\beta}{4}-\frac{1}{2} \pm \alpha)$. Moreover (5.4) is symmetric in $z \leftrightarrow \bar{z}$ so the alpha space transform will be symmetric under $\alpha \leftrightarrow \bar{\alpha}$. We find

$$z^{\beta/2+2} \delta(z-\bar{z}) = \int_{-i\infty}^{i\infty} \frac{d\alpha d\bar{\alpha}}{Q(-\alpha)Q(-\bar{\alpha})} f(\alpha, \bar{\alpha}) k_{\frac{1}{2}+\alpha}(z) k_{\frac{1}{2}+\bar{\alpha}}(\bar{z}), \quad (\text{D15})$$

with

$$f(\alpha, \bar{\alpha}) = \frac{1}{2\pi^2} \sum_{n=0}^{\infty} 2nc_{13(2n)} c_{24(2n)} \langle z^{\beta/4} k_{\frac{2n+1}{2}}(z), \Psi_{\frac{1}{2}+\alpha} \rangle \langle z^{\beta/4} k_{\frac{2n+1}{2}}(\bar{z}), \Psi_{\frac{1}{2}+\bar{\alpha}} \rangle \\ + \frac{1}{(2\pi i)} \int_{-i\infty}^{i\infty} d\alpha' \langle z^{\beta/4} k_{\frac{1}{2}+\alpha'}(z), \Psi_{\frac{1}{2}+\alpha} \rangle \langle z^{\beta/4} k_{\frac{1}{2}-\alpha'}(\bar{z}), \Psi_{\frac{1}{2}+\bar{\alpha}} \rangle. \quad (\text{D16})$$

In both terms we can deform the contour in the right-hand $\alpha, \bar{\alpha}$ planes to only pick up poles at $\alpha = h + \frac{\beta}{4} - \frac{1}{2} + k$ with $h = \frac{1+2n}{2}, \frac{1}{2} + \alpha'$ respectively (and similar equations for $\bar{\alpha}$). We conclude that

$$z^{\beta/2+2} \delta(z - \bar{z}) = \frac{1}{2\pi^2} \sum_{n,k,k'=0}^{\infty} 2nC_{13(2n)}^k C_{24(2n)}^{k'} k_{\frac{2n+1}{2}+\frac{\beta}{4}+k}(z) k_{\frac{2n+1}{2}+\frac{\beta}{4}+k'}(\bar{z}) \\ + \frac{1}{2\pi i} \sum_{k,k'=0}^{\infty} \int_{-i\infty}^{i\infty} d\alpha C_{13\alpha}^k C_{24-\alpha}^{k'} k_{\frac{1}{2}+\alpha+\frac{\beta}{4}+k}(z) k_{\frac{1}{2}-\alpha+\frac{\beta}{4}+k'}(\bar{z}), \quad (\text{D17})$$

where

$$C_{13n}^k = c_{13n} \frac{(-1)^k \Gamma(n + \beta/2 + k)}{k! \Gamma(\frac{n+1}{2} + \frac{\beta}{4})^2} {}_4F_3 \left[\begin{matrix} \frac{n+1}{2}, \frac{n+1}{2}, n + \beta/2 + k, -k \\ n + 1, \frac{n+1}{2} + \frac{\beta}{4}, \frac{n+1}{2} + \frac{\beta}{4} \end{matrix}; 1 \right] \frac{1}{Q(-\frac{n}{2} - \frac{\beta}{4} - k)}, \\ C_{13\alpha}^k = \frac{(-1)^k}{k!} \frac{\Gamma(2\alpha + \beta/2 + k)}{\Gamma(\frac{1}{2} + \alpha + \frac{\beta}{4}) \Gamma(\frac{1}{2} + \alpha + \frac{\beta}{4})} {}_4F_3 \left[\begin{matrix} \frac{1}{2} + \alpha, \frac{1}{2} + \alpha, 2\alpha + \beta/2 + k, -k \\ 2\alpha + 1, \frac{1}{2} + \alpha + \frac{\beta}{4}, \frac{1}{2} + \alpha + \frac{\beta}{4} \end{matrix}; 1 \right] \frac{1}{Q(-\alpha - \frac{\beta}{4} - k)}. \quad (\text{D18})$$

It is not difficult to see that for $\beta = 0$, the generalized hypergeometric function collapses to a ratio of gamma functions which vanishes for $k > 0$ as expected. Note that the discrete tower of exchanges now appear at positive integer dimensions shifted by $\beta/2$, consistent with the behavior of the celestial four-point amplitude in the small z limit.

2. $|\frac{z}{z-1}|^{\beta/2}$ term

We begin by computing

$$\left\langle \left(\frac{z}{1-z} \right)^q, \Psi_{\alpha} \right\rangle = \int_0^1 \frac{dz}{z^2} \left(\frac{z}{1-z} \right)^q \frac{1}{\Gamma(\frac{1}{2} \pm \alpha)} \int \frac{ds}{2\pi i} \frac{\Gamma(-s) \Gamma(\frac{1}{2} + s \pm \alpha)}{\Gamma(1+s)} \left(\frac{1-z}{z} \right)^s \\ = \int \frac{ds}{2\pi i} \frac{\Gamma(-s) \Gamma(\frac{1}{2} + s \pm \alpha) \Gamma(q-s-1) \Gamma(-q+s+1)}{\Gamma(1+s) \Gamma(\epsilon)} \frac{1}{\Gamma(\frac{1}{2} \pm \alpha)} \\ = \frac{\Gamma(\frac{1}{2} \pm \alpha) \Gamma(-q+1) \Gamma(-\frac{1}{2} \pm \alpha + q)}{\Gamma(\frac{1}{2} \pm \alpha) \Gamma(q)} \frac{1}{\Gamma(\frac{1}{2} \pm \alpha)} \\ = \frac{\Gamma(-q+1) \Gamma(-\frac{1}{2} \pm \alpha + q)}{\Gamma(\frac{1}{2} \pm \alpha) \Gamma(q)}. \quad (\text{D19})$$

We can now repeat the analysis of Sec. D 1 using the alternate form of the delta function resolution (D2). We first notice that

$$\left\langle \left| \frac{z}{z-1} \right|^{\beta/4} k_h \left(\frac{z}{z-1} \right), \Psi_{\alpha} \right\rangle = (-1)^h r \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(h+s)^2 \Gamma(-s)}{\Gamma(2h+s)} \left\langle \left(\frac{z}{1-z} \right)^{\beta/4+h+s}, \Psi_{\alpha} \right\rangle \\ = r(-1)^h \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-(\beta/4+h+s)+1) \Gamma(-\frac{1}{2} \pm \alpha + \beta/4+h+s) \Gamma(h+s)^2 \Gamma(-s)}{\Gamma(\frac{1}{2} \pm \alpha) \Gamma(\beta/4+h+s) \Gamma(2h+s)}. \quad (\text{D20})$$

This formula has two pole sequences at

$$s = k, \quad s = -h - \beta/4 + 1 + k. \quad (\text{D21})$$

The first gives

$$(-1)^h \frac{\Gamma(-\frac{1}{2} \pm \alpha + \frac{\beta}{4} + h) \Gamma(1 - h - \beta/4)}{\Gamma(\frac{\beta}{4} + h) \Gamma(\frac{1}{2} \pm \alpha)} {}_4F_3 \left[\begin{matrix} h, h, -\frac{1}{2} \pm \alpha + \frac{\beta}{4} + h \\ \frac{\beta}{4} + h, 2h, h + \frac{\beta}{4} \end{matrix}; 1 \right], \quad (\text{D22})$$

while the second yields a term with no poles in α , hence does not contribute to the conformal block decomposition. As before, we can close the contour in the right-hand side $\alpha, \bar{\alpha}$ planes picking up poles at

$$\alpha = \frac{\beta - 2}{4} + h + k, \quad (\text{D23})$$

with $h = \frac{1}{2} + n, \frac{1}{2} + \alpha'$ respectively and similarly for $\bar{\alpha}$. The conformal block decomposition looks similar to that in the previous section, namely

$$\begin{aligned} \left| \frac{z}{1-z} \right|^{\beta/2} z^2 \delta(z - \bar{z}) &= \frac{1}{2\pi^2} \sum_{n,k,k'=0}^{\infty} 2n \mathcal{D}_{13(2n)}^k \mathcal{D}_{24(2n)}^{k'} k_{\frac{2n+1}{2} + \frac{\beta}{4} + k}(z) k_{\frac{2n+1}{2} + \frac{\beta}{4} + k'}(\bar{z}) \\ &+ \frac{1}{2\pi i} \sum_{k,k'=0}^{\infty} \int_{-i\infty}^{i\infty} d\alpha \mathcal{D}_{13\alpha}^k \mathcal{D}_{24-\alpha}^{k'} k_{\frac{1}{2} + \alpha + \frac{\beta}{4} + k}(z) k_{\frac{1}{2} - \alpha + \frac{\beta}{4} + k'}(\bar{z}), \end{aligned} \quad (\text{D24})$$

with

$$\mathcal{D}_{13n}^k = c_{13n} \frac{(-1)^{\frac{n+1}{2}+k}}{k!} \frac{\Gamma(n + \beta/2 + k) \Gamma(-\beta/4 - \frac{n-1}{2})}{\Gamma(\frac{\beta}{4} + \frac{n+1}{2}) \Gamma(\frac{1}{2} \pm (\frac{2n+\beta}{4} + k))} {}_4F_3 \left[\begin{matrix} \frac{n+1}{2}, \frac{n+1}{2}, k + \beta/2 + n, -k \\ \frac{\beta}{4} + \frac{n+1}{2}, n + 1, \frac{n+1}{2} + \frac{\beta}{4} \end{matrix}; 1 \right] \frac{1}{Q(-\frac{\beta}{4} - \frac{n}{2} - k)} \quad (\text{D25})$$

and an analogous expression for $\mathcal{D}_{13\alpha}^k$. The exchanges are again at $\Delta \sim \beta/2 + k + k', J \sim k - k'$ [which is what we would get if the amplitude was not multiplied by $\delta(z - \bar{z})$] together with a tower of exchanges with Δ shifted by positive even integers as a consequence of translation invariance.

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