Prescriptive unitarity with elliptic leading singularities

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We investigate the consequences of elliptic leading singularities for the unitarity-based representations of two-loop amplitudes in planar, maximally supersymmetric Yang-Mills theory. We show that diagonalizing with respect to these leading singularities ensures that the integrand basis is termwise pure (suitably generalized, to the elliptic multiple polylogarithms, as necessary). We also investigate an alternative strategy based on diagonalizing a basis of integrands on differential forms; this strategy, while neither termwise Yangian-invariant nor pure, offers several advantages in terms of complexity.

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I. INTRODUCTION AND OVERVIEW

Generalized unitarity has proven an extremely powerful framework for the representation of scattering amplitudes at large multiplicity and/or loop order. The basic idea is that any loop *integrand*—a rational differential form on the space of internal loop momenta—can be viewed as an element of a basis of standardized Feynman loop integrands. Provided the basis of integrands (chosen as a subspace of the infinite-dimensional space of rational functions of loop momenta) is large enough, it may be used to represent all the scattering amplitudes of any theory and spacetime dimension. This idea has a long history (see e.g. [1,2]); it was formalized and used to famous effect in e.g. [3–9], and has been recently refined, generalized, and put to use for many impressive applications (see e.g. [10–15]).

Among the many advantages of this approach is that the basis of integrands, so long as it is large enough for the theory in question, will be sufficient to represent literally all amplitudes (arbitrary multiplicity and states) in a wide class of theories at any loop order. See e.g. [16] for a thorough discussion of how these spaces of loop integrands may be defined and constructed. Thus, the integrands in the basis need only be integrated once and for all—reusable for any

process of interest. As loop integration has been (and remains) among the hardest problems in perturbative quantum field theory, this is a very important feature. This makes clear the importance of choosing “good” integrands for a basis—the precise measures of which have evolved greatly with time. (Roughly speaking, a good basis would consist of integrands which can be integrated “most easily” or which result in the “simplest” expressions.)

Another advantage to unitarity is that loop-independent coefficients of particular amplitudes with respect to a basis can be computed in terms of mostly (and often wholly) on-shell data—specifically, on-shell functions [8,9,17–22]. When these on-shell functions are leading singularities, they have no internal degrees of freedom. Historically, leading singularities have been defined as maximal codimension residues (of polylogarithmic differential forms); more recently, this definition has been modified and generalized to include any full-dimensional compact contour integral of a scattering amplitude integrand [23]. Leading singularities have played a key role in the development of our modern understanding of quantum field theory (see e.g. [5,15,24–31]), and many of the remarkable aspects of scattering amplitudes (their simplicity, and wide range of symmetries) were discovered in this context. For example, the Britto-Cachazo-Feng-Witten (BCFW) recursion relations for tree amplitudes were first discovered in this setting [3,32,33], as was the infinite-dimensional Yangian symmetry of planar maximally supersymmetric Yang-Mills theory (sYM) [34–36], and the correspondence between on-shell functions in sYM and residues of the positroid volume-form in Grassmannian manifolds [37,38].

When leading singularities are used to determine the coefficients of loop amplitudes with respect to some integrand basis, generalized unitarity becomes a relatively

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simple problem of linear algebra—matching the “cuts” of field theory against the corresponding cuts of the integrand basis. Until recently, however, it was unclear if leading singularities represented *complete* information about perturbative scattering amplitudes even in the simplest theories. The reason for this uncertainty lies in the fact that, for sufficiently large multiplicity and/or loop order, scattering amplitude integrands in most theories are not “ $d \log$ ” differential forms [39–47] and cannot be characterized by (maximal codimension) residues alone. For such cases, the traditional definition of leading singularity becomes incomplete; and the most typical strategy to deal with nonpolylogarithmic contributions has been to use the highest codimension residues that exist (sub^{*n*}-leading singularities), and then use sufficient numbers of off-shell evaluations to match a loop integrand functionally on the remaining degrees of freedom. (Examples of such strategies being used can be found in [15,17].) The result of this approach, however, has many obvious disadvantages; in particular, it results in representations of amplitudes that (at least term-by-term) involve references to arbitrary choices (for the off-shell evaluation) which can break many of the niceties that scattering amplitudes are known to possess.

We should remind the reader that a prescriptive basis of integrands is one which has been diagonalized on a particular, spanning set of cuts so that the coefficients of amplitudes in this basis are specific on-shell functions (as

opposed to the basis being chosen semiarbitrarily, and the coefficients being linear combinations thereof). When amplitudes are polylogarithmic, the most natural choice of cuts to define a prescriptive basis would be a (spanning) set of maximal codimension contour integrals. When elliptic (or worse) nonpolylogarithmic obstructions exist (as in the works of [15,17]), these maximal codimension cuts have historically included also a number of of particular “evaluations”; in this work, we describe two alternative, novel ways to arrange integrand bases to match amplitudes.

Before we discuss any concrete examples, it is worth highlighting a conventional difference between this work relative to virtually all existing literature: we have chosen to write all loop integration measures in terms of $\vec{d}^{4L} \vec{\ell}$ where $\vec{d} := d/(2\pi)$ (by analogy to “ $\hbar$ ”). As such, many of our results differ by powers of $(2\pi i)$ relative to those found elsewhere. This choice is motivated by the fact that an integrand normalized to have unit *residues* with respect to the measure $\vec{d}^{4L} \vec{\ell}$ will have unit *contour* integrals with respect to $\vec{d}^{4L} \vec{\ell}$. As such, most formulas appear identical to other literature; a notable exception, however, is the case in which subleading singularities are considered, for which our convention requires relative factors of i .

To illustrate how prescriptive unitarity can work when there are elliptic contributions, consider the elliptic double-box integrand for massless, scalar φ^4 -theory in four dimensions:

$$\begin{array}{c}
 \begin{array}{ccccc}
 & 10 & 1 & 2 & \\
 9 & \bullet & \bullet & \bullet & 3 \\
 | & | & | & | & \\
 8 & \bullet & \bullet & \bullet & 4 \\
 | & | & | & | & \\
 & 7 & 6 & 5 &
 \end{array}
 \Leftrightarrow \frac{\vec{d}^4 \ell_1 \vec{d}^4 \ell_2}{(\ell_1|2)(\ell_1|4)(\ell_1|6)(\ell_1|\ell_2)(\ell_2|7)(\ell_2|9)(\ell_2|1)} =: \mathcal{I}_{\text{db}}^{\varphi^4}.
 \end{array} \quad (1.1)$$

Above, $(\ell_i|a)$ represents an ordinary, scalar inverse propagator expressed in dual-momentum coordinates (the details of which we review below) and $\vec{d} = \frac{d}{2\pi}$ as defined above. At any rate, (1.1) is an eight-dimensional (rational) differential form on the space of loop momenta. Taking a contour integral in loop-momentum space which “encircles” all seven propagators (putting them on-shell), however, results in an elliptic differential form on the remaining variable¹:

$$\oint_{\substack{\{(\ell_i|a)=\epsilon\} \\ |(\ell_1|\ell_2)|=\epsilon}} \mathcal{I}_{\text{db}}^{\varphi^4} = \frac{-ic_y}{(2|6)(7|1)(4|9)y(\alpha)} \vec{d}\alpha \quad (1.2)$$

¹There are two solutions to these seven (quadratic) equations; here we write the contour on one of them.

where we have used α to represent the final loop momentum variable, $y^2(\alpha)$ is an irreducible quartic (with coefficients that depend on the momenta of the particles involved), and c_y is a factor introduced to render $y^2(\alpha)$ *monic*. [The factor of c_y is required by the fact that $y^2(\alpha)$ is typically not monic. We will give a precise form of c_y in this parametrization later.] The precise details are not important to us now; but it is easy to see that (1.2) represents an elliptic differential form, without any further “residues” on which we may define a traditional leading singularity.

Recently [23], a broader definition of leading singularity has been introduced to include any full-dimensional compact contour-integral of a scattering amplitude integrand. With this new definition, we can in fact define a leading singularity for the double-box integrand by integrating (1.2) over, for example, the a -cycle, Ω_a , of the elliptic curve:

$$\oint_{\Omega_a} \oint_{\{(\ell_i|a)=\epsilon\} \atop {(\ell_1|\ell_2)=\epsilon}} \mathcal{I}_{\text{db}}^{\varphi^4} = \frac{2}{\pi} \frac{c_y}{(2|6)(7|1)(4|9)} \frac{1}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} K[\varphi] =: \mathbf{e}_a^{\varphi^4}; \quad (1.3)$$

here, φ is a cross-ratio in the roots r_i of the quartic (the details of which are not important to us now). If we wanted to choose a basis of integrands which would be normalized to “match” this leading singularity prescriptively, we would need to normalize the double-box integrand accordingly:

$$\mathcal{I}_{\text{db}}^{\varphi^4} \mapsto \mathcal{I}_{\text{db}} := \mathcal{I}_{\text{db}}^{\varphi^4} / \mathbf{e}_a^{\varphi^4} = \mathfrak{d}^4 \ell_1 \mathfrak{d}^4 \ell_2 \frac{\pi(2|6)(7|1)(4|9) \sqrt{(r_3 - r_2)(r_4 - r_1)} / K[\varphi]}{2c_y(\ell_1|2)(\ell_1|4)(\ell_1|6)(\ell_1|\ell_2)(\ell_2|7)(\ell_2|9)(\ell_2|1)}. \quad (1.4)$$

(As far as the elliptic integral is concerned, this is the standard normalization of algebraic geometry—used for much the same reasons.) Thus, a prescriptive representation of the ten-particle scattering amplitude in this theory at two loops would involve a term

$$\mathcal{I}_{\text{db}} \times \mathbf{e}_a^{\varphi^4} = (\mathcal{I}_{\text{db}}^{\varphi^4} / \mathbf{e}_a^{\varphi^4}) \times \mathbf{e}_a^{\varphi^4} = \mathcal{I}_{\text{db}}^{\varphi^4}. \quad (1.5)$$

While this example may seem overly trivial [especially considering that the original scalar integrand in (1.1) is literally a term in the Feynman expansion], the rewriting of it according to prescriptive unitarity according to (1.5) has a remarkable feature: the now-normalized basis integrand $\mathcal{I}_{\text{db}}$ is in fact dual-conformally invariant and pure in the sense defined by the authors of [48,49]—as such, it is arguably the simplest possible form of the integral (and, presumably, the easiest to integrate).

To be clear, any function (or Feynman integral) is called “pure” if differentiation with respect to external kinematics always results in a lowering of some notion of (transcendental) “weight.” Thus, for example, $\log(g(z))$ would be pure, while $f(z)\log(g(z))$ would generally not be. Most two-loop Feynman integrals with loop-independent numerators are impure—typically they result in expressions of the form $f(z)(\text{Li}_4(q_1(z)) + \dots) + g(z)(\text{Li}_4(q_2(z)) + \dots)$; as such, they cannot be rendered pure by any rescaling. However, in most cases, these integrals can be seen simple as linear combinations of pure integrals (those in parentheses) where $f(z)$ and $g(z)$ are merely the coefficients appearing at the integral level. [For a broader discussion of integrand “purity,” we suggest the reader consult Refs. [48–50]; for the present, we merely mention this to emphasize that such integrands, and the differential equations that they satisfy (see e.g. [51]) have been defined so as to manifest many remarkable properties.]

A. Organization and outline

In this work, we generalize and expand upon the discussion above to the case of two-loop amplitudes in planar, maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory (sYM). A closed formula for all such amplitude integrands was first derived in Ref. [15] (representing an

early application of what became known as “prescriptive” unitarity [17]), which succeeded despite the presence of elliptic integrals due to carefully made (but arbitrary) choices for off-shell evaluations in combinations of leading and subleading singularities. The resulting representations given in [15,52,53] involved terms that were neither Yangian-invariant nor integrands that were pure.

In Sec. II we review the salient elements of two-loop prescriptive unitarity, as well as the novel generalization of elliptic leading singularities introduced in [23]. In Sec. III, the main result of this paper, we revisit the prescriptive unitarity story in the light of our recent work. In particular, we derive two novel representations of amplitudes in planar sYM at two loops, both defined completely prescriptively and unambiguously. The first, described in Sec. III A, involves a prescriptive integrand basis chosen by diagonalization on leading singularities (in the new, broader sense); it is intrinsically homological—one which has been diagonalized with respect to a particular set of contour integrals—and results in a representation of amplitudes that, term-by-term, involves Yangian-invariant coefficients and pure integrals. In Sec. III B, we describe an alternative representation based instead on a cohomological diagonalization of the integrand basis—one in which integrand basis has been diagonalized with respect to independent differential forms. The resulting form is simpler in many ways (especially algebraically), but involves coefficients that are not Yangian-invariant and a basis of loop integrands that are not generally pure.

II. REVIEW: PRESCRIPTIVE INTEGRAND BASES FOR TWO-LOOP sYM

In this section, we outline how unitarity can be (and has been) used to construct dual-conformal representations of two-loop amplitudes in planar sYM. For the sake of clarity and completeness, we describe the ingredients involved in such a construction somewhat differently than what was presented in Ref. [15], in which a basis of integrands was constructed in a way to manifest certain infrared features of scattering amplitudes; here, we take a more pedagogical approach: one whose construction is more transparent, and

more directly highlights the challenges faced before elliptic leading singularities were understood.

A. Bases of dual-conformal integrands: General structure

In order to represent scattering amplitudes in planar sYM at the integrand level, we would like to first construct a complete basis of dual-conformal Feynman integrands. Once such a basis is constructed, the coefficients of scattering amplitudes in this basis follows from generalized unitarity. The starting point, therefore, is a definition of the space of Feynman integrands required, and a rule by which a *particular* set of basis integrands may be chosen.

A very useful (and arguably accidental) feature of planar integrands at two loops is that a complete and *not over-complete* basis of dual-conformal integrands exists. This is

in contrast with the space of dual-conformal integrands at one loop, for which, beginning at six external points, the space of all conformal pentagon integrands begins to satisfy rational relations (and could therefore be described as overcomplete) (see e.g. [54–56]); such overcompleteness is a source of technical problems and inelegance, because choices of independent integrands must be made in order to form a basis; and these choices are reflected in the elegance of (or lack thereof in) the resulting representations of amplitudes. This is in contrast to one loop or three or more loops, for which dual-conformality apparently requires overcompleteness.

At two loops, dual-conformality requires that each loop momentum has four *net* powers of propagators; in the language of Ref. [16], this means that we are interested in the space $\mathfrak{B}_4^{(4)}$. A spanning set of such integrands can be constructed from the following three topologies:

(2.1)

called double-pentagons ($\Gamma_{4,1,4}$), pentaboxes ($\Gamma_{4,1,3}$), and double-boxes ($\Gamma_{3,1,3}$), respectively. To render these integrands dual-conformal, each must be endowed with a particular vector space of numerators:

$$\mathfrak{N}_4(\Gamma_{4,1,4}) = [\ell_1][\ell_2], \quad \mathfrak{N}_4(\Gamma_{4,1,3}) = [\ell_1], \quad \mathfrak{N}_4(\Gamma_{3,1,3}) = 1, \quad (2.2)$$

where

$$[\ell_i]^q := \text{span}_{\oplus Q_j} \left\{ \prod_{j=1}^q (\ell_i | Q_j) \right\}, \quad Q_j \in \mathbb{R}^4. \quad (2.3)$$

We have used dual-momentum coordinates for which the massless external momenta are given by $p_a = x_{a+1} - x_a$ (with cyclic labeling understood), and with the association of $\ell_i \Leftrightarrow x_{\ell_i}$. In terms of dual-momentum coordinates, the Lorentz invariants are defined by $(a|b) := (x_a - x_b)^2$.

It is straightforward to verify the three vector spaces in (2.2) have dimension 36, 6, and 1, respectively. It is worth

noting that another class of dual-conformal integrands—the “kissing boxes”

(2.4)

—are in fact spanned by the double pentagons. Moreover, it is a standard result that all other dual-conformal integrands at two loops can be expanded (similarly to Passarino-Veltman reduction [57] at one loop) into the three integrand topologies (with their associated numerator spaces) shown in (2.2).

For each topology, it is natural to divide the vector space of numerators into so-called “top-level” and “contact-term” degrees of freedom. Thus, we can give particular representatives for these numerators reflecting this division in the following way. For the double-pentagon, we have

$$\begin{aligned} \mathfrak{N}_4(\Gamma_{4,1,4}) = \text{span}\{ & (\ell_1 | Q_{a,b,c,d}^1)(\ell_2 | Q_{e,f,g,h}^1), \dots, (\ell_1 | Q_{a,b,c,d}^2)(\ell_2 | Q_{e,f,g,h}^2), \\ & (\ell_1 | Q_{a,b,c,d}^1)e^2, \dots, (\ell_1 | Q_{a,b,c,d}^2)h^2, \dots, d^2(\ell_2 | Q_{e,f,g,h}^2), a^2e^2, \dots, d^2h^2\}. \end{aligned} \quad (2.5)$$

Here, $Q_{a,b,c,d}^i$ represents the i th solution to the *quad-cut* involving propagators a, b, c, d —that is, the i th solution to $(\ell|a) = (\ell|b) = (\ell|c) = (\ell|d) = 0$. The breakdown (2.5) makes manifest the fact that of the 36 degrees of freedom required to specify a *particular* dual-conformal double-pentagon integrand, only four of these “top-level” degrees of freedom are independent of the pentabox and double-box integrands. Similarly, of the six-dimensional space of dual-conformal pentabox integrands, all but two

of these dimensions are spanned by double-box “contact-terms”:

$$\mathfrak{N}_4(\Gamma_{4,1,3}) = \text{span}\{(\ell_1|Q_{a,b,c,d}^1), (\ell_1|Q_{a,b,c,d}^2), a^2, \dots, d^2\}. \quad (2.6)$$

Graphically, this decomposition reflects the fact that the double-box topology can be obtained by contracting one of the edges $\{a, b, c, d\}$,

Thus, each pentabox integrand,

$$\text{Pentabox} \Leftrightarrow \bar{d}^4 \ell_1 \bar{d}^4 \ell_2 \frac{\mathbf{n}_0^i(\ell_1|Q_{a,b,c,d}^i) + \mathbf{n}_a^i(\ell_1|a) + \mathbf{n}_b^i(\ell_1|b) + \mathbf{n}_c^i(\ell_1|c) + \mathbf{n}_d^i(\ell_1|d)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)} \quad (2.7)$$

is not fully specified until the four (loop-independent, but kinematic-dependent) “constants” $\mathbf{n}^i$ have been specified. The top-level normalization, $\mathbf{n}_0^i$, is determined by requiring that (some combination of) the *polylogarithmic* leading singularity which encircles the eight Feynman propagators of the pentabox is unity.

To be clear, *any* (independent) choice of four top-level numerators for the double-pentagons, and two top-level numerators for the pentaboxes, together with a set of dual-conformally normalized double-boxes, will define a *particular* basis of integrands into which any amplitude can be expanded (using traditional methods of generalized unitarity).

Suppose that one were to choose a spanning set of maximal-dimensional, compact cycles of integration $\{\Omega_i\}$; then provided the basis of integrands $\{\mathcal{I}_i\}$ satisfied the condition of *prescriptivity*,

$$\int_{\Omega_i} \mathcal{I}_j = \delta_{i,j}, \quad (2.8)$$

then in the representation of an amplitude, the coefficient of any particular integrand $\mathcal{I}_i$ would be trivial:

$$\mathcal{A} = \sum_{\mathcal{I}_i \in \mathfrak{B}} a_i \mathcal{I}_i \quad \text{where } a_i := \int_{\Omega_i} \mathcal{A}. \quad (2.9)$$

These coefficients—amplitudes integrated over maximal-dimensional, compact cycles of integration—were called “leading singularities” in Ref. [23]; but this definition of “leading singularity” is substantially broader than previous definitions, which included only integrals over maximal-dimensional compact contours *corresponding to residues around polylogarithmic poles*. In particular, the basis described in Ref. [15] did not meet this criterion of prescriptivity because it was unclear how the then-existing notion of leading singularity should be used in the case of elliptic integrals (for which no maximal codimension residues exist).

Although we do not wish to review the particular contour choices made by the authors of Ref. [15] in their construction of the elliptic-free part of their basis, it is worth reviewing how the issue of elliptics were normalized (without identifying a contour Ω_j for which they would be dual). To best understand this concern—and how this present work resolves this problem much more satisfactorily—it is worthwhile to review how integrands satisfying (2.8) (or some version thereof) can be constructed.

One point that is worth emphasizing here is that for any prescriptive basis of integrands, the matching of all leading singularities a_i in (2.9) is trivial; *however*, it is not obvious that the representation (2.9) matches the integral of the amplitude over any cycle $\tilde{\Omega}$ not explicitly used in the basis. These nonmanifestly matched leading singularities can

always be understood to work as a consequence of completeness of the basis; or, they can be viewed as an independent check that (2.9) is correct.

B. Triangular structure of contours and contact-terms

A *prescriptive* basis of integrands is one satisfying (2.8); that is, the integrands $\{\mathcal{I}_i\}$ are constructed as the cohomological duals of a particular set of cycles. To see how this

can work (and has been done previously), it is worthwhile to first consider the case where all integrals are elliptic-free—as would be the case, for example, for a basis of n -particle integrands where $n < 10$ is the number of external legs.

A spanning set of cycles can be chosen according to the following pictures—referred to as “kissing boxes” $\Omega_{\text{kb},ij}$, “pentaboxes” $\Omega_{\text{pb},i}$, and “composite double-boxes” Ω_{db} , respectively:

$$\left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} \right\}, \quad (2.10)$$

composite

which indicate which eight poles are to be “encircled” by the contour, and where $i, j \in \{1, 2\}$ denote which solutions to the cut equations are taken; for the double-boxes, when $n < 10$, every diagram has at least one three-point vertex; the “composite” cut corresponds to cutting that vertex *and* taking the momenta to be collinear (which corresponds to an eighth pole that can be encircled). Thus, for example, the following are particular elements of the schematic class of double-box contours for $n < 10$,

$$\text{composite} \supset \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \vdots \end{array} \right\}, \quad (2.11)$$

where the notation $\odot$ indicates the composite cut is to be taken at the corresponding three-particle vertex.

Consider an arbitrary dual-conformal double-pentagon integral $\mathcal{I}_{\text{dp}}^0$ chosen from the 36-dimensional vector space thereof. It will have support on all four contours of the kissing-box topology (corresponding to cutting all of the double-pentagon’s external propagators), $8 \times 2 = 16$ pentabox contours, and on $4 \times 4 = 16$ of the double-box contours. Thus, integrating $\mathcal{I}_0$ on these 36 cycles will produce a vector with 36 nonvanishing entries, schematically,

$$\int_{\Omega_i} \mathcal{I}_{\text{dp}}^0 = \left(\int_{\Omega_{\text{kb},ij}} \mathcal{I}_{\text{dp}}^0 \quad \int_{\Omega_{\text{pb},i}} \mathcal{I}_{\text{dp}}^0 \quad \int_{\Omega_{\text{db}}} \mathcal{I}_{\text{dp}}^0 \right) \quad (2.12)$$

Now consider an arbitrary conformal pentabox integrand $\mathcal{I}_{\text{pb}}^0$; it vanishes on any kissing-box contour, because no pentabox has this subset of Feynman propagators; but it will generically have a nonvanishing integral on both of the pentabox cycles, and on the four double-box contours which correspond to its contact terms.

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 1: Two adjacent hexagons sharing a vertical edge. Left hexagon labeled } i, \text{ right labeled } j. \end{array} & \begin{array}{c} \text{Diagram 2: A single hexagon labeled } i. \end{array} & \begin{array}{c} \text{Diagram 3: A composite of two hexagons sharing a vertical edge.} \\ \text{composite} \end{array} \\
 \int_{\Omega_i} \mathcal{I}_{\text{pb}}^0 = \begin{pmatrix} & & \\ & 0 & \\ & \int_{\Omega_{\text{pb},i}} \mathcal{I}_{\text{pb}}^0 & \\ & \int_{\Omega_{\text{db}}} \mathcal{I}_{\text{pb}}^0 & \end{pmatrix} & &
 \end{array} \quad (2.13)$$

Finally, the double-box integrals only have support on the double-box contours with the same set of external propagators; thus, regardless of this integrand's normalization, we have

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 1: Two adjacent hexagons sharing a vertical edge. Left hexagon labeled } i, \text{ right labeled } j. \end{array} & \begin{array}{c} \text{Diagram 2: A single hexagon labeled } i. \end{array} & \begin{array}{c} \text{Diagram 3: A composite of two hexagons sharing a vertical edge.} \\ \text{composite} \end{array} \\
 \int_{\Omega_i} \mathcal{I}_{\text{db}}^0 = \begin{pmatrix} & & \\ & 0 & \\ & 0 & \\ & \int_{\Omega_{\text{db}}} \mathcal{I}_{\text{db}}^0 & \end{pmatrix} & &
 \end{array} \quad (2.14)$$

This system is obviously *triangular*. Considering the space of all integrands (and all combinations of external propagators allowed for each topology), one would find a “period matrix” of the form

$$\int_{\Omega_i} \mathcal{I}_j^0 = \mathbf{M}_{i,j}, \quad (2.15)$$

where the matrix $\mathbf{M}_{i,j}$ is independent of the loop momenta, and may be formed by horizontal concatenation of the row vectors in (2.12)–(2.14); it is generally upper triangular in structure (as illustrated above)—with each integral only having support on contours corresponding to subsets of its propagators.

A *prescriptive* basis corresponding to this set of cycles could then be defined as

$$\{\mathcal{I}_i\} \quad \text{with} \quad \mathcal{I}_i := (\mathbf{M}^{-1})_{ij} \mathcal{I}_j^0, \quad (2.16)$$

which would satisfy (2.8) by construction. Inverting a triangular matrix is fairly trivial. Importantly, the resulting

integrand is *not* one for which the coefficients $\mathbf{n}_i$ of each integrand's contact term degrees of freedom are zero, but instead one where they are fixed as the unique solution to the linear system of equations

$$\int_{\Omega_i} \mathcal{I}_j = \delta_{i,j}. \quad (2.17)$$

The problem, of course, is what to do when there are elliptic integrals—which do not support any residues of maximal codimension. In the work of Ref. [15] a peculiar choice of how to deal with these integrals was made, which we review presently.

C. Elliptic double-box integrands

When no *residue* contours exist, the authors of Ref. [15] chose instead to use *evaluations*. To be concrete, each double-box integral was evaluated at some arbitrary point x^* along the codimension-7 residue, parametrized in terms of a single degree of freedom x ,

$$\text{Res}_{7\text{-cut}}(\mathcal{I}_{\text{db}}^0) = \text{Res}_{7\text{-cut}} \left(\begin{array}{c} \text{Diagram: A double-box contour with four vertices and four internal edges.} \end{array} \right) = \frac{\mathbf{n}_{\text{db}}^0 dx}{y(x)}, \quad \text{where } y^2(x) = Q(x), \quad (2.18)$$

where $Q(x)$ is an irreducible quartic in x , and $\mathbf{n}_{\text{db}}^0$ denotes the normalization. In lieu of any pole on which a residue of maximal codimension could be defined, the normalization of these integrals was simply fixed by the condition that the corresponding integrands *evaluated to 1* at some arbitrary point x^* . That is, $\mathbf{n}_{\text{db}}^0 = \sqrt{Q(x^*)}$ in the example above. This uniquely (if arbitrarily) fixed the double-box integrals in the basis to be

$$\mathcal{I}_{\text{db}} \Leftrightarrow \begin{array}{c} \begin{array}{c} g \quad a \\ \bullet \quad \bullet \\ \ell_2 \quad \ell_1 \\ \bullet \quad \bullet \\ e \quad c \end{array} \end{array} \Leftrightarrow \vec{d}^4 \ell_1 \vec{d}^4 \ell_2 \frac{\mathbf{n}_{\text{db}}^0(a|c)(d|f)(b|e)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|\ell_2)(\ell_2|d)(\ell_2|e)(\ell_2|f)}, \quad (2.19)$$

where we have included a conventional factor in the numerator which ensures dual-conformal invariance.

What about the other integrands in the basis? Naturally enough, the other integrals were uniquely specified by the criteria that they *vanish* at these particular points x^* (corresponding to each double-box subtopology of the integrand). In this basis, the coefficient of each double box would be field theory evaluated on the corresponding seven-cut, and evaluated at the particular point x^* . Obviously, the particular points x^* were arbitrary, and the final results for any amplitude were x^* -independent.

This resulted in a basis which was neither prescriptive nor pure—and for which many integrands depended explicitly on the arbitrarily chosen points x^* used to normalize the double-box integrands. Importantly, the coefficients of the double boxes were not Yangian-invariant (although the entire amplitude integrand would be).

This is a fairly unsatisfactory representation. In this work, we show how incorporating elliptic contours (and elliptic leading singularities) can naturally lead to a directly prescriptive basis. And we also provide one alternative strategy—based not on matching field theory on cycles, but on particular differential forms—for representing amplitudes; this second form, while neither termwise prescriptive nor pure in the sense above, does have some algebraic advantages over the first.

D. Elliptic leading singularities (and other on-shell functions)

Let us start with the subleading singularity associated with a contour in field theory with the topology of a double-box integral as shown in (2.19). That is, we would like to define the subleading singularity associated with²

$$\mathfrak{db}_{\pm}(\alpha) := \begin{array}{c} \begin{array}{c} F \quad f \quad A \quad a \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ E \quad D \quad C \quad b \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ e \quad d \quad c \end{array} \end{array}, \quad (2.20)$$

where α represents the single on-shell degree of freedom of the subleading singularity, and $\pm$ denotes which of the two

²As described in [15], it suffices for us to consider only maximally helicity violating (MHV) amplitudes tree-amplitudes at the vertices—as general amplitudes can then be generated by multiplication by the corresponding on-shell amplitudes.

branches of the seven-cut equations is chosen. This could easily be represented in the Grassmannian according to [58,59], but we prefer to disambiguate its structure by writing it explicitly. To do this, we express $\mathfrak{db}_{\pm}(\alpha)$ as a simple-pole-enclosing contour integral over the codimension-6 “kissing-triangle” function, for which the nonvanishing term involves three R -invariants,

$$\frac{\vec{d}\alpha \vec{d}\beta}{\alpha\beta} R[1, \hat{a}, \hat{c}, \hat{d}, \hat{f}] R[\hat{a}, B, b, C, c] R[\hat{d}, E, e, F, f], \quad (2.21)$$

which is a special case of the general expression given in Appendixes A and B of [15], written here in terms of R -invariants with (shifted) momentum supertwistor [60] arguments defined as

$$\begin{aligned} \hat{a}(\alpha) &:= \mathcal{Z}_a + \alpha \mathcal{Z}_A, & \hat{d}(\beta) &:= \mathcal{Z}_d + \beta \mathcal{Z}_D, \\ \hat{c}(\alpha) &:= (Cc) \cap (\hat{a}Bb), & \hat{f}(\beta) &:= (Ff) \cap (\hat{d}Ee). \end{aligned} \quad (2.22)$$

To compute the double-box subleading singularity, we consider a “residue” contour encircling the pole at $\langle \hat{a} \hat{c} \hat{d} \hat{f} \rangle = 0$, which is a quadratic condition involving both α and β . Without any loss of generality, we are free to solve this condition for β ; the (two) branches of solutions to $\langle \hat{a} \hat{c} \hat{d} \hat{f} \rangle = 0$ are then given by

$$\beta_{\pm}^* := \frac{\langle \hat{a} \hat{c} d(DEe) \cap (Ff) \rangle + \langle \hat{a} \hat{c} D(dEe) \cap (Ff) \rangle \pm y(\alpha)/c_y}{2\langle \hat{a} \hat{c} (DEe) \cap (Ff) D \rangle}, \quad (2.23)$$

where $y^2(\alpha)$ is a quartic polynomial defined as

$$\begin{aligned} \frac{1}{c_y^2} y^2(\alpha) &:= (\langle \hat{a} \hat{c} d(DEe) \cap (Ff) \rangle + \langle \hat{a} \hat{c} D(dEe) \cap (Ff) \rangle)^2 \\ &\quad - 4\langle \hat{a} \hat{c} D(dEe) \cap (Ff) \rangle \langle \hat{a} \hat{c} d(DEe) \cap (Ff) \rangle, \end{aligned} \quad (2.24)$$

and the factor of $1/c_y^2$ is included to render the quartic $y^2(\alpha)$ monic by construction. The residue of the kissing-triangle hexacut function at $\beta \mapsto \beta_{\pm}^*$ follows by noting that

$$\oint_{\langle \hat{a} \hat{c} \hat{d} \hat{f} \rangle=0} \left(\frac{\vec{d}\beta}{\langle \hat{a} \hat{c} \hat{d} \hat{f} \rangle} \right) = \pm \frac{ic_y}{y(\alpha)}. \quad (2.25)$$

Notice that the $\pm$ sign on the right-hand side reflects the choice of the branch of the cut equations, and $y(\alpha)$ is a square root of the quartic (2.24).

Since there are two different solutions to $\langle \hat{a} \hat{c} \hat{d} \hat{f} \rangle = 0$, it makes sense to define “the” double-box subleading singularity $\mathfrak{db}(\alpha)$ as the difference of the kissing-triangle on the two contours. We therefore define

$$[\mathfrak{db}_+(\alpha) - \mathfrak{db}_-(\alpha)] =: \mathfrak{db}(\alpha) =: \frac{d\alpha}{y(\alpha)} \widehat{\mathfrak{db}}(\alpha), \quad (2.26)$$

where in the second equality we have defined a useful “hatted” double-box function where the inverse of the square root of the quartic has been factored out.

Let us now return to (2.20). The double-box on-shell function has various factorization channels corresponding to the six amplitudes appearing in its definition. Each of these channels corresponds to a simple pole of $\mathfrak{db}(\alpha)$ at, say, $\alpha = a_i \in \mathbb{C}$ on which we may define an additional contour about a $d \log$ pole. According to prescriptive unitarity, each such residue is topologically equivalent to a “pentabox” leading singularity, which we denote by $\mathfrak{pb}_i$. We can, therefore, expand our next-to-leading singularity in a basis of those factorizations into pentaboxes according to

$$\mathfrak{db}(\alpha) =: \frac{d\alpha}{y(\alpha)} \mathfrak{db}_0 + \frac{d\alpha}{y(\alpha)} \sum_i \frac{y(a_i)}{(\alpha - a_i)} \mathfrak{pb}_i, \quad (2.27)$$

where the α -independent coefficient of the basis element without any extra simple poles, denoted by $\mathfrak{db}_0$, is defined as

$$\mathfrak{db}_0 := \widehat{\mathfrak{db}}(\alpha) - \sum_i \frac{y(a_i)}{(\alpha - a_i)} \mathfrak{pb}_i. \quad (2.28)$$

While it is not manifest in (2.28), the coefficient $\mathfrak{db}_0$ is, by construction, independent of α . This trivial—yet important—point follows directly from Liouville’s theorem of elementary complex analysis: as we have removed every pole, including the pole at $\alpha = \infty$, the “function” $\mathfrak{db}_0$ is entire on the Riemann sphere, that is, it must be a constant. At the risk of being overly illustrative, let us emphasize that this implies that we may equivalently write

$$\mathfrak{db}_0 := \widehat{\mathfrak{db}}(\alpha^*) - \sum_i \frac{y(a_i)}{(\alpha^* - a_i)} \mathfrak{pb}_i, \quad (2.29)$$

where α^* is an arbitrarily chosen point.

The recent work [23] defines an “elliptic” leading singularity by integrating the double-box seven-cut differential form $\mathfrak{db}(\alpha)$ over either the a or b “cycle” of the elliptic curve defined by $y^2(\alpha)$,

$$e_{a,b} := \oint_{\Omega_{a,b}} \mathfrak{db}(\alpha) = \oint_{\Omega_{a,b}} \frac{d\alpha}{y(\alpha)} \widehat{\mathfrak{db}}(\alpha) = \pm 2 \oint_{\Omega_{a,b}} \mathfrak{db}_{\pm}(\alpha). \quad (2.30)$$

To perform the integral over the a -cycle (or any cycle for that matter), it is quite useful to factorize the monic quartic polynomial defined in (2.24) in terms of its roots,

$$y^2(\alpha) =: (\alpha - r_1)(\alpha - r_2)(\alpha - r_3)(\alpha - r_4), \quad (2.31)$$

where, for positive kinematics [61], the roots form two complex conjugate pairs $\{r_1, r_2\}$ and $\{r_3, r_4\}$, and are ordered so that $\Re(r_1) > \Re(r_3)$ and $\Im(r_{1,3}) > 0$. With this particular ordering in mind, the a , b -cycle integrals (2.30) can be written in terms of elliptic integrals involving the dual-conformal invariant cross-ratio

$$\varphi := \frac{(r_2 - r_1)(r_3 - r_4)}{(r_2 - r_3)(r_1 - r_4)} =: \frac{r_{21}r_{34}}{r_{23}r_{14}}, \quad (2.32)$$

which, in our conventions, is always within the interval $[0, 1]$ for positive kinematics. Choosing the branch cuts to connect each complex conjugate pair of roots, we now define the a -cycle contour, Ω_a , to enclose the cut between $r_{1,2}$. To compute the full a -cycle integral we use the two formulas

$$\oint_{\Omega_a} \frac{d\alpha}{y(\alpha)} = \frac{2i}{\pi \sqrt{r_{32}r_{41}}} K[\varphi] \quad (2.33)$$

and

$$\begin{aligned} \oint_{\Omega_a} \frac{d\alpha y(a_i)}{(\alpha - a_i)y(\alpha)} &= \frac{2i}{\pi \sqrt{r_{32}r_{41}}} \frac{y(a_i)}{(r_4 - a_i)} \\ &\times \left(K[\varphi] + \frac{r_{42}}{(r_2 - a_i)} \Pi \left[\frac{(r_4 - a_i)r_{21}}{(r_2 - a_i)r_{41}}, \varphi \right] \right), \end{aligned} \quad (2.34)$$

where our definitions of the complete elliptic integrals of the first and third kinds, $K[\varphi]$ and $\Pi[q, \varphi]$, respectively, are in agreement with *Mathematica*’s. We have provided a different—and perhaps more easily generalizable—representation of these elliptic period integrals in terms of Lauricella functions in the Appendix. The attentive reader will notice that these two integral formulas differ from the results quoted in [23]; this discrepancy is a consequence of our use of $d = d/(2\pi)$ throughout this work. Following the discussion in [23], observe that in the definition (2.27) of the double-box one-form, the coefficient of the pentabox $\mathfrak{pb}_i$ is the same as in (2.29), but for the appearance of α^* in place of α . Upon performing the integration, if we choose $\alpha \rightarrow r_4$ the terms involving $K[\varphi] \mathfrak{pb}_i$ thus cancel, and we are left with

$$\mathbf{e}_a = \frac{2i}{\pi\sqrt{r_{32}r_{41}}} \left(K[\varphi] \widehat{\mathbf{db}}(\alpha^* \rightarrow r_4) + \sum_i \frac{y(a_i)r_{42}}{(r_2 - a_i)(r_4 - a_i)} \right. \\ \left. \times \Pi \left[\frac{(r_4 - a_i)r_{21}}{(r_2 - a_i)r_{41}}, \varphi \right] \mathbf{pb}_i \right). \quad (2.35)$$

If instead we consider the b -cycle, say, the one encircling a branch cut which connect two roots with different real part (e.g. r_1 and r_3), we have

$$\mathbf{e}_b = \frac{2i}{\pi\sqrt{r_{23}r_{41}}} \left(K[1 - \varphi] \widehat{\mathbf{db}}(\alpha^* \rightarrow r_4) \right. \\ \left. + \sum_i \frac{y(a_i)r_{43}}{(r_3 - a_i)(r_4 - a_i)} \Pi \left[\frac{(r_4 - a_i)r_{31}}{(r_3 - a_i)r_{41}}, 1 - \varphi \right] \mathbf{pb}_i \right), \quad (2.36)$$

which is the same as the first expression after the exchange $r_2 \leftrightarrow r_3$. As discussed at greater length in [23], both expressions (2.35) and (2.36) are Yangian-invariant (although nontrivially, as it is represented as a sum of many terms which are not individually Yangian-invariant), as may be verified by direct computation using, for example, the level 1 generators written in momentum supertwistor space.

III. NEW PRESCRIPTIVE REPRESENTATIONS: TWO APPROACHES

We shall now argue that the results of the previous section suggest two natural prescriptions for the normalization of the double-box integrand which avoid entirely the

arbitrary choices of the original prescriptive unitarity program.

A. Homological diagonalization—With respect to contours

One natural choice for the normalization of the double-box integrand $\mathbf{n}_{\text{db}}$ would be analogous to the choice made in the introduction for scalar φ^4 theory. That is, we may choose to normalize the integrand so that it integrates to 1 on the contour associated with, say, the a -cycle of the elliptic curve; since the loop-momentum-dependent part of the integrand is proportional to $1/y(\alpha)$, this fixes its normalization, using (2.33), to be

$$\mathbf{n}_{\text{db}} := \frac{\pi}{2} \frac{\sqrt{(r_3 - r_2)(r_4 - r_1)}}{c_y K[\varphi]}. \quad (3.1)$$

This choice of normalization ensures that we match the elliptic leading singularity in field theory, $\mathbf{e}_a$, manifestly with the double-box integrand directly

$$\oint_{\Omega_a} \oint_{\substack{\{(\mathcal{E}_i|a)\}=\epsilon \\ |(\mathcal{E}_1|\mathcal{E}_2)|=\epsilon}} \mathcal{A} = \mathbf{e}_a \oint_{\Omega_a} \oint_{\substack{\{(\mathcal{E}_i|a)\}=\epsilon \\ |(\mathcal{E}_1|\mathcal{E}_2)|=\epsilon}} \mathcal{I}_{\text{db}} = \mathbf{e}_a. \quad (3.2)$$

To be clear, here $\mathcal{A}$ represents an amplitude integrand (for some multiplicity and helicity degree in planar sYM).

Notice that this representation, however, does not match the b -cycle leading singularity of the amplitude manifestly at all. Although we have normalized the double-boxes so that the a -cycle contours of field theory are manifest, the result on the b -cycle is, rather,

$$\mathbf{e}_a \oint_{\Omega_b} \oint_{\substack{\{(\mathcal{E}_i|a)\}=\epsilon \\ |(\mathcal{E}_1|\mathcal{E}_2)|=\epsilon}} \mathcal{I}_{\text{db}} = -i\mathbf{e}_a \frac{K[1 - \varphi]}{K[\varphi]} \\ = \frac{2}{\pi\sqrt{r_{23}r_{41}}} \left(K[1 - \varphi] \widehat{\mathbf{db}}(\alpha^* \rightarrow r_4) + \sum_i \frac{y(a_i)r_{42}}{(r_2 - a_i)(r_4 - a_i)} \frac{K[1 - \varphi]}{K[\varphi]} \Pi \left[\frac{(r_4 - a_i)r_{21}}{(r_2 - a_i)r_{41}}, \varphi \right] \mathbf{pb}_i \right) \quad (3.3)$$

which is not at all equal to

$$\oint_{\Omega_b} \oint_{\substack{\{(\mathcal{E}_i|a)\}=\epsilon \\ |(\mathcal{E}_1|\mathcal{E}_2)|=\epsilon}} \mathcal{A} = \mathbf{e}_b \quad (3.4)$$

which was given in (2.36). This is not in fact a problem: as we will see, the pentaboxes (and double-pentagon integrands) *do* have support on the b -cycle integrals (even after their contact terms have been fixed), and exactly give the necessary contributions to reproduce the correct b -cycle leading singularity in (2.36). (With hindsight, this “magic”

can be seen to follow directly from the completeness of the integrand basis.)

Let us now discuss the implications of prescriptive unitarity for the contact-term rules of the pentabox and double-pentagon integrands. As always, prescriptivity requires that our integrands be *diagonal* in a choice of contours; therefore, in the homological scheme, the contact terms are determined by the requirement that all other integrands in our basis *vanish* identically on all elliptic Ω_a -cycle contours associated with double-box subtopologies.

To see how this works in practice, consider a pentabox integrand which contains a double-box contact-term:

$$\text{pentabox} \subset \text{pentabox} \Leftrightarrow \frac{\tilde{d}^4 \ell_1 \tilde{d}^4 \ell_2 [\mathbf{n}_0^i(\ell_1|\mathcal{Y}^i) + \mathbf{n}_\rho^i(\ell_1|\rho) + \dots]}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|\rho)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)} \quad (3.5)$$

where $\mathbf{n}_\rho^i$ is the coefficient of the term in the numerator proportional to $(\ell_1|\rho)$. Schematically, the full contribution of the pentabox on the a -cycle contour can be written as

$$\oint_{\Omega_a} \oint_{\substack{|\ell_i|a|=\epsilon \\ |(\ell_1|\ell_2)|=\epsilon}} \text{pentabox} = \oint_{\Omega_a} \tilde{d}\alpha \left[\frac{y(a_\rho)}{(\alpha - a_\rho)y(\alpha)} + \frac{\mathbf{n}_\rho^i}{y(\alpha)} \right], \quad (3.6)$$

where a_ρ corresponds to the pole where $(\ell_1|\rho) = 0$. Notice that the leading term follows directly from the fact that the integral is normalized to have unit leading singularity on some pentabox contour. Diagonalization of the basis according to homology—leading singularities—fixes this contact-term coefficient, $\mathbf{n}_\rho^i$, of the pentabox by the criterion that (3.6) vanishes. Namely, we must choose

$$\mathbf{n}_\rho^i := -\frac{y(a_\rho)}{(r_4 - a_\rho)} \left[1 + \frac{r_{42}}{(r_2 - a_\rho)} \Pi \left[\frac{(r_4 - a_i)r_{21}}{(r_2 - a_i)r_{41}}, \varphi \right] / K[\varphi] \right]. \quad (3.7)$$

One important thing to note is that the pentabox integrands with these contact-terms chosen *do not vanish* on the b -cycle contours. This is a good thing—as the normalization we have chosen for the double-boxes makes the correctness of the amplitude integrand on the b -cycle extremely nonmanifest in this representation. It is not hard to see, however, that when these contact terms are used, they generate on the b -cycle exactly the terms needed to cancel the “wrong” elliptic- Π terms involving the pentabox leading singularities that arise from the double-box integrands in (3.3). Furthermore, the contact terms of the pentabox integrands also contribute the correct pieces involving the pentabox leading singularities appearing in e_b (2.36) once the corresponding contributions from the double-pentagons are included (so as to match all pentabox leading singularities).

This diagonalization strategy has several obvious advantages. For one thing, it is morally the direct realization of “prescriptive unitarity” according to a choice of leading

singularities. Moreover, as emphasized in the Introduction, it should have the property that all integrands defined in this way are *pure*, and all coefficients are Yangian-invariant.

Nevertheless, there are several reasons to be dissatisfied with this basis of integrands. For example, it deeply obscures the fact that the integrand is a rational differential form in loop momenta. Choosing basis integrands whose normalization depends on the roots of quartics makes this representation fairly unwieldy in practice (at least for most computer algebra packages). Therefore, we are motivated to consider a slightly different strategy, with huge advantages in terms of (algebraic) complexity, but which abandons the desire for a pure integrand basis—and requires the use of non-Yangian-invariant coefficients.

B. Cohomological diagonalization— With respect to forms

The attentive reader may already have guessed an alternative strategy for matching amplitudes in sYM—namely, according to the various differential forms that appear in the double-box subleading singularity $\mathfrak{db}(\alpha)$ in (2.27):

$$\mathfrak{db}(\alpha) =: \frac{d\alpha}{y(\alpha)} \mathfrak{db}_0 + \frac{d\alpha}{y(\alpha)} \sum_i \frac{y(a_i)}{(\alpha - a_i)} \mathfrak{pb}_i. \quad (3.8)$$

Considering the fact that the codimension-7 contour of the scalar double-box integrand (2.19) which encircles its seven propagators results in

$$\oint_{\substack{|\ell_i|a|=\epsilon \\ |(\ell_1|\ell_2)|=\epsilon}} \tilde{d}^4 \ell_1 \tilde{d}^4 \ell_2 \frac{\mathbf{n}_{\text{db}}(a|c)(d|f)(b|e)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|\ell_2)(\ell_2|d)(\ell_2|e)(\ell_2|f)} = \mathbf{n}_{\text{db}}(-ic_y) \frac{d\alpha}{y(\alpha)}, \quad (3.9)$$

it would be natural to choose $\mathbf{n}_{\text{db}}$ to be $+i$, and to match its coefficient in the representation of field theory amplitudes to be $c_y \mathbf{db}_0$. [With this factor of c_y —implicit in the definition (3.8)—this on-shell function is little-group neutral (if not Yangian-invariant.) Moreover, it is easy to see that this combination of integrand and coefficient automatically matches the *leading* term in $\mathbf{e}_b$ in (2.36). The only

terms missing from both leading singularities are those involving the pentabox leading singularities $\mathbf{pb}_i$.

These not-yet-matched pieces of $\mathbf{e}_{a,b}$ can easily be seen to arise from the pentabox and double-pentagon contributions. Consider again how the contact terms appear in the pentabox integrands' contributions to the a -cycle, say,

$$\oint_{\Omega_a} \oint_{\{|\ell_1|a|=\epsilon\} \atop {|\ell_1|\ell_2|=\epsilon}} \oint_{\text{diagram}} = \oint_{\Omega_a} d\alpha \left[\frac{y(a_\rho)}{(\alpha - a_\rho)y(\alpha)} + \frac{\mathbf{n}_\rho^i}{y(\alpha)} \right]. \quad (3.10)$$

Taking the contact-term numerator $\mathbf{n}_\rho^i \mapsto 0$ will leave a nonvanishing contribution from the first term in (3.10) on both cycles. It is easy to see that the missing differential form (which results in the combination of the elliptic functions K and Π) in (2.34) is exactly that which appears in the pentabox integrands already—and the coefficients of these integrands are precisely the $\mathbf{pb}_i$ needed to reproduce the “missing” pieces in the elliptic leading singularities $\mathbf{e}_{a,b}$.

This definition of the double-box and the corresponding rule for the contact terms of the pentabox and double-pentagon integrands results in an extremely simple prescription for the integrand. Moreover, it is morally equivalent to a choice of diagonalization with respect to the various (local) differential-forms in loop-momentum space. As such, we call such a prescription a *cohomological* choice for our basis.

Despite the obvious advantages, we have checked that the coefficient of the double-box normalized in this way (namely, $c_y \mathbf{db}_0$) is *not* Yangian-invariant. Moreover, the double-box integrands are not pure. We strongly suspect that the pentabox and double-pentagon integrands (that contain elliptic contact-term components) are similarly nonpure. Thus, despite the algebraic and conceptual simplicity of the cohomological approach just described, we suspect that the *homological* prescriptive representation will ultimately prove the superior one for integration.

C. Consistency checks for amplitude integrands

As already mentioned in Sec. II, the pentabox integrands lack the requisite number of degrees of freedom to match all pentabox cuts of field theory term-by-term. However, the expressions for the elliptic leading singularities $\mathbf{e}_{a,b}$ involve them separately—and arise at poles in different locations in the α -plane. Thus, our description above regarding the pentaboxes' roles in these leading singularities does not alone ensure that we have matched everything in field theory. The missing ingredients, of course, are the kissing-box leading singularities times double-pentagons. These terms when combined with those of the pentaboxes ensure

that all the pentabox leading singularities *do* get matched correctly, and individually. Thus, we have been somewhat schematic in our analysis above, relying on the fact that these parts of the amplitude integrands are guaranteed to work correctly once taken in combination.

While ensured to work, we have checked this completely in the case of the ten-particle $N^3\text{MHV}$ amplitude. Specifically, we have checked that both diagonalization procedures described above—the homological and the cohomological—result in integrand representations that exactly match the results of BCFW recursion, say. Thus, we are confident this procedure is free of any overlooked subtleties.

D. Smooth degenerations

The reader may have considered our preference for normalizing integrands with respect to the a -cycle Ω_a rather arbitrary. It was not entirely so. As discussed in [23], the a -cycle integral of the elliptic curve smoothly degenerates to 1 in all relevant cases (or kinematic limits). Thus, our analysis above, if applied to the case of double-box integrands that *do* support polylogarithmic contour integrals, reduces naturally to ordinary polylogarithmic ones. (Recall that a unit-residue integrand $d^{4L}\vec{\ell}$ is equivalent to a *unit-contour* integrand for $d^{4L}\vec{\ell}$.) Thus, we can apply the results of this work to truly general double-boxes, and thereby construct pure-integrand bases that happen to be “ $d \log$ ” whenever the double-boxes' elliptic curves turn out to be degenerate.

IV. GENERALIZATIONS: MORE LOOPS AND CALABI-YAU MANIFOLDS

Beyond two loops (and for nonplanar theories at two loops), nonpolylogarithmic structures beyond elliptic integrals abound [40,45–47,62]. Examples of scalar integrals with such structures include the three-loop train-track and wheel integrals,

$$\left\{ \text{[Left Diagram]}, \text{[Right Diagram]} \right\}, \quad (4.1)$$

the maximal cuts (subleading singularities encircling all propagators) which are known to involve Calabi-Yau two- and threefolds, respectively.

To see how the corresponding nonpolylogarithmic leading singularities can be incorporated analogously to what we have described for the elliptic case, let us briefly outline the structure expected for the train-track contribution.

On the maximal cut surface encircling all ten propagators of the train track, the sub²-leading singularity should take the form

$$\text{tr}(\alpha, \beta) =: \frac{\hat{d}\alpha \hat{d}\beta}{y(\alpha, \beta)} \hat{\text{tr}}(\alpha, \beta), \quad (4.2)$$

where $y^2(\alpha, \beta)$ is an irreducible quartic in both variables $\{\alpha, \beta\}$ simultaneously [45]. As with the double-box subleading singularity (2.20), the train track in sYM will have multiple codimension-1, simple poles around which there are elliptic subleading singularities. We may express this by decomposing (4.2) according to

$$\begin{aligned} \hat{\text{tr}}(\alpha, \beta) =: & \hat{\text{tr}}_0 + \sum_{a_i} \frac{y^2(a_i, \beta)}{(\alpha - a_i)y(a_i, \beta)} \text{e}\mathcal{I}_{a_i}(\beta) \\ & + \sum_{b_i} \frac{y^2(\alpha, b_i)}{(\beta - b_i)y(\alpha, b_i)} \text{e}\mathcal{I}_{b_i}(\alpha), \end{aligned} \quad (4.3)$$

where the terms in the sum $\text{e}\mathcal{I}_{a_i}$ and $\text{e}\mathcal{I}_{b_i}$ are elliptic subleading singularities arising from single-pole factorizations of the original train track. These in turn can be further decomposed analogously to (2.27), resulting in an expression of the form

$$\begin{aligned} \hat{\text{tr}}(\alpha, \beta) =: & \hat{\text{tr}}_0 + \sum_{a_i} \frac{y^2(a_i, \beta)}{(\alpha - a_i)y(a_i, \beta)} \hat{\text{e}}\mathcal{I}_{a_i} \\ & + \sum_{b_i} \frac{y^2(\alpha, b_i)}{(\beta - b_i)y(\alpha, b_i)} \hat{\text{e}}\mathcal{I}_{b_i} \\ & + \sum_{a_i, b_j} \frac{y(a_i, b_j)}{(\alpha - a_i)(\beta - b_j)} \text{p}\mathcal{I}_{a_i, b_j}, \end{aligned} \quad (4.4)$$

where $\text{p}\mathcal{I}_{a_i, b_j}$ are the “pentalladder” polylogarithmic leading singularities associated with simultaneous factorizations in two different loops.

The number and detailed form of terms appearing in this decomposition will depend on the number of factorization

channels of the train track (which depends on multiplicity), but the basic structure is clear: (4.4) is nothing but a decomposition of the sub²-leading singularity (4.2) into a basis of differential forms involving one or two simple poles, respectively—with superfunction coefficients. [For the scalar train-track contribution to the three-loop 12-particle amplitude in scalar φ^4 theory, only the leading term $\hat{\text{tr}}_0$ is required—which happens to be (-1) —as there are no factorization channels to the participating φ^4 tree-amplitudes.]

The generalization of this analysis to the three-loop wheel integral—which involves a Calabi-Yau threefold surface—is relatively straightforward, resulting in a decomposition of the sub³-leading singularity into a top-level, irreducible volume form times some “CY₃” leading singularity, three separate sums of $K3$ -volume forms with single simple poles times “ $K3$ ” leading singularities; three double-nested sums of elliptic integrals with two simple poles times “elliptic” leading singularities; and, finally, a triple-sum of terms with simple poles times polylogarithmic leading singularities.

It is worth mentioning that, unlike the three-loop train track which is known to have support in sYM (as argued in [45]), the three-loop wheel is not any single component of an amplitude in planar sYM; as such, it is possible that the CY₃ leading singularity vanishes. Thus, this makes its evaluation an important open challenge—left for future work. [While we are unaware of closed analytic formulas for the period integrals that would be required for such a check—analogueous to those in (2.33) and (2.34) for the elliptic case—we are relatively optimistic that numerical integration will work for these low-dimensional cases.]

The implications of these higher-dimensional Calabi-Yau leading singularities for prescriptive unitarity should be clear. In particular, we suspect that if the three-loop basis of planar integrands outlined in [17] were diagonalized *homologically*, the result would be a complete representation of amplitudes involving termwise “pure” integrals times Yangian-invariants; and the *cohomological* diagonalization of this basis into separate forms should be extremely straightforward to implement from the decompositions as in (4.4). To be clear, the analysis described above should allow for integrand bases to be constructed at three or higher loops according to *either* a homological or cohomological strategy. However, for the homological approach, it would be necessary to have better knowledge of the precise mathematical forms of the period integrals over the differential forms appearing in (4.4), for example. We must leave this to future work.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we have made use of the new, broadened definition of leading singularities (beyond the polylogarithmic case) to derive two new prescriptive representations

of two-loop scattering amplitude integrands in planar sYM. This analysis was illustrative of a more general strategy, with applications well beyond the planar limit and to theories with less or no supersymmetry. In many ways, our results directly reflect the primary goals of prescriptive unitarity: constructing loop-integrand bases that are diagonal in a spanning set of *contours*. For scattering amplitudes free of nonpolylogarithmic structures, this strategy directly reproduces the $d\log$ differential forms of the traditional approach; but as we have seen, the generalization beyond polylogarithms is extremely natural. We have also motivated a different strategy: how to construct a prescriptive integrand basis diagonal with respect to *cohomology*; the resulting basis may not be pure and the coefficients required may not be Yangian-invariant, but the ultimate representation of loop integrands is dramatically simpler from an algebraic point of view.

We have described how this story illustrates a broader one—with applications well beyond the case of elliptic leading singularities in planar theories at two loops. It would be extremely interesting to apply these lessons more widely to generate (purportedly) “pure” master integrals for applications beyond the planar limit, and to theories without supersymmetry. Although we have not proven the “purity” of this broader class of integrals, and although the strategies and techniques required to efficiently exploit the differential structure of pure integrals are still being developed (even in the elliptic case—but see e.g. [63–65]), we strongly suspect that the prescriptive bases we have constructed will prove computationally valuable as master integrals for diverse applications. For example, constructing such bases for massive theories now appears straightforward, and undoubtedly has more immediate applications for real physical applications (see e.g. [64–68]). But we leave such analyses to future work.

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APPENDIX: HYPERGEOMETRIC REPRESENTATIONS OF THE ELLIPTIC INTEGRALS

Although the representations for the complete elliptic integrals given in (2.33) and (2.34) above are fairly standard ones (with relatively efficient implementations in *Mathematica*, for example), it is worthwhile to outline an alternative form for these integrals—which we hope has some promise to generalize beyond the elliptic case. In this appendix, we outline how these elliptic periods can be expressed in terms of Lauricella hypergeometric functions. (We refer the reader to e.g. [69–71] for some discussions on these functions in the context of Feynman integrals.)

Consider first the elliptic period integral given in (2.33):

$$I_a^{(1)} := \oint_{\Omega_a} \frac{d\alpha}{y(\alpha)} = \frac{2i}{\pi\sqrt{r_{32}r_{41}}} K[\varphi], \quad (\text{A1})$$

where $y^2(\alpha) := (\alpha - r_1)(\alpha - r_2)(\alpha - r_3)(\alpha - r_4)$, $r_{ij} := (r_i - r_j)$, and

$$\varphi := \frac{(r_2 - r_1)(r_3 - r_4)}{(r_2 - r_3)(r_1 - r_4)} =: \frac{r_{21}r_{34}}{r_{23}r_{14}}. \quad (\text{A2})$$

(We refer the reader to Sec. IID for our conventions regarding the ordering of the roots r_i .) This integral can be recast slightly by the change of variables

$$x := \frac{\alpha - r_1}{r_2 - r_1}, \quad (\text{A3})$$

upon which (A1) becomes

$$I_a^{(1)} := \oint_{\Omega_a} \frac{d\alpha}{y(\alpha)} = \frac{i}{\pi\sqrt{r_{32}r_{41}}} \int_0^1 dx \frac{1}{\sqrt{x(1-x)(1-(21;31)x)(1-(21;41)x)}}, \quad (\text{A4})$$

where we have introduced the shorthand

$$(ab;cd) := \frac{r_a - r_b}{r_c - r_d}. \quad (\text{A5})$$

From the definition of the Lauricella hypergeometric function

$$F^{(n)}(a, b_1, \dots, b_n, c | x_1, \dots, x_n) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dx x^{a-1} (1-x)^{c-a-1} (1-x_1 x)^{-b_1} \dots (1-x_n x)^{-b_n} \quad (\text{A6})$$

we have the identification

$$I_a^1 = \frac{i\Gamma(\frac{1}{2})^2}{\pi\sqrt{r_{31}r_{41}}} F^{(2)}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \middle| (21; 31), (21; 41)\right). \quad (\text{A7})$$

Applying the same transformation to the second fundamental period integral (2.34),

$$I_a^{(2)} := \oint_{\Omega_a} \frac{d\alpha y(p)}{(\alpha - p)y(\alpha)}, \quad (\text{A8})$$

results in a representation

$$I_a^{(2)} = \frac{-iy(p)\Gamma(\frac{1}{2})^2}{\pi(p-r_1)\sqrt{r_{31}r_{41}}} F^{(3)}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1 \middle| (21; 31), (21; 41), (21; p1)\right). \quad (\text{A9})$$

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