D-brane actions in nonrelativistic string theory and T duality

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(Received 14 December 2020; accepted 13 September 2021; published 5 October 2021)

We study D-brane actions in nonrelativistic string theory. We consider single D-brane and analyze its properties under T duality. We also derive an action for N D8-branes and also for lower dimensional D-branes through T duality transformations.

DOI: [10.1103/PhysRevD.104.086009](https://doi.org/10.1103/PhysRevD.104.086009)

I. INTRODUCTION AND SUMMARY

Nonrelativistic string theory is specific form of two dimensional quantum field theory defined on the string world sheet [1,2]. The target space geometry of this two-dimensional field theory is stringy Newton-Cartan geometry, see also [3–10].

Stringy Newton-Cartan geometry can be physically defined as a theory where probe of the geometry is two dimensional object (string). As a result target space-time naturally splits into two dimensional longitudinal sector and 8-dimensional transverse sector where we implicitly work with the parent string theory with critical dimension equal to ten. As it is well known from the study of nonrelativistic string theory [1,2] in the flat space-time longitudinal sector is Lorentzian while transverse sector is Euclidean.

It is very interesting to analyze open string theory in this background where now boundary conditions imposed at the end of string world sheet play an important role. Very careful analysis of open string theory in nonrelativistic background was performed recently in [3,4]. It was shown there that imposing Dirichlet boundary conditions in longitudinal sector leads to nonrelativistic open string theory on D8-brane. It was shown in [4] that in the flat space-time low energy effective action of open strings on N -nonrelativistic D8-branes is described by $U(N)$ -Yang-Mills theory which is invariant under Galilean transformation and hence corresponds to nonrelativistic effective field theory. T duality of nonrelativistic open string theory was further analyzed in [3], first in the flat space-time and then in general space-time with the help of sigma model or using

Dirac-Born-Infeld (DBI) description. The goal of this paper is to extend the analysis of T duality of nonrelativistic open string using DBI description and its non-Abelian generalization.

We begin this paper with the construction of nonrelativistic Dp-brane which is similar to the definition of nonrelativistic theories by dimensional reduction of the background with light-like isometry. In more details, we consider DBI action for Dp-brane in the background with light-like isometry and we presume that Dp-brane is extended along this direction. Then we perform dimensional reduction along this direction.¹ As a result we obtain nonrelativistic D(p-1)-brane which is generalization of the construction presented in [3] when some of the directions in transverse sector obey Dirichlet boundary conditions. Note that using this construction we get D(p-1)-brane that is localized at light-like direction. We further study T duality along k -spatial directions that are longitudinal to the world-volume of D(p-1)-brane and we obtain an action for nonrelativistic D(p-1-k)-brane in the background that is related to the original one by Buscher's rules [12,13] which are generalization of standard Buscher's prescription [14,15] to the case of T duality along k -directions.² However it is important to stress that the resulting D(p-1-k)-brane action has the same form as nonrelativistic D(p-1)-brane on condition that the components of time form τ along spatial directions that we dualize, are zero. Note that this is the same condition that was imposed in [10] when T duality of closed string along transverse directions was analyzed.

As the next step we proceed to the construction of an action for N nonrelativistic Dp-branes. To do this we closely follow [17]. Explicitly, we start with the space-time filling DBI

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¹We could presume that lightlike direction is compact. This can be defined as the limit of a compactification on a space-like circle which is almost lightlike. Finally such almost lightlike circle is related by large boost to spatial compactification [11].

²For earlier review of T duality, see [16].

action for N D9-branes in the background with light-like isometry. Performing dimensional reduction along light-like direction we obtain an action for N nonrelativistic D8-branes. An action for N D(8-k)-branes is derived by performing T duality along k -spatial directions. This non-Abelian action for N D(8-k)-branes is crucial for the analysis of T duality along direction transverse to the D(8-k)-brane world-volume as is known from the study of transverse T duality in relativistic case. In more details, in order to study T duality of Dp-brane along directions transverse to its world volume we should consider an infinite array of Dp-branes on the covering space when all world volume fields obey quotient conditions [18–20], see also [21] for generalization of this analysis to the case of general background. We firstly apply this approach to the case of nonrelativistic D8-brane transverse to lightlike direction. Following [18–20] we replace nonrelativistic D8-brane action with the non-Abelian action for N D8-branes on the covering space where N goes to infinity and where world volume fields obey appropriate quotient conditions. Then we find that this configuration is equivalent to D9-brane in the background with lightlike direction. This can be interpreted as inverse operation to the definition of nonrelativistic D8-brane and also as a nice consistency check. Then we generalize this approach to the case of T duality along lightlike direction transverse to D(8-k)-brane and we again show that it is equivalent to D(9-k)-brane where this D(9-k)-brane is extended along lightlike direction.

Let us outline our results and suggest possible extension of this work. We studied effective actions for nonrelativistic Dp-branes in torsional Newton-Cartan background. We found their form and we analyzed their transformation properties under T duality transformations. Then we generalized this analysis to the case of N nonrelativistic D8 and D(8-k)-branes. We argued that these non-Abelian generalizations are crucial for analysis how nonrelativistic D(8-k)-brane transforms under T duality along transverse lightlike direction when we show that nonrelativistic D(8-k)-brane maps to relativistic D(9-k)-brane in the background with lightlike isometry. We mean that this is nice consistency check of the proposal of the action for N nonrelativistic D(8-k)-branes.

The natural extension of this work would be to analyze properties of the Wess-Zumino term which describes how Dp-brane couples to Ramond-Ramond forms. This is a nontrivial task since in order to study this problem we should know the nonrelativistic limit of Ramond-Ramond fields which has not been found yet. On the other hand, it would be natural to start with the fact that nonrelativistic D(p-1)-brane is defined using T duality of relativistic Dp-brane along lightlike directions. Then we should perform the same T duality transformations in case of Wess-Zumino term and we could find term that expresses

coupling of nonrelativistic D(p-1)-brane to Ramond-Ramond form. We hope to return to this problem in future.

This paper is organized as follows. In the next section II we introduce nonrelativistic D(p-1)-brane. Then in Sec. III we study its properties under T duality. In Sec. IV we perform its non-Abelian generalization. Finally in Sec. V we study how nonrelativistic D-branes transform under T duality performed along lightlike directions which is transverse to their world volumes.

II. NONRELATIVISTIC D-BRANE AND T DUALITY

In this section we review and extend construction of Dp-branes in nonrelativistic background, following recent work [3].

We start with Dp-brane action in the relativistic background with lightlike isometry along y -direction which means that the metric component $g_{yy} = 0$ and all space-time fields do not depend on y . Let us now consider DBI action for Dp-brane in general background

$$S = -T_p \int d^{p+1} \xi e^{-\phi} \sqrt{-\det \mathbf{A}_{\alpha\beta}},$$

$$\mathbf{A}_{\alpha\beta} = g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + b_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + \lambda F_{\alpha\beta}, \quad (1)$$

where $\xi^\alpha, \alpha, \beta = 0, 1, \dots, p$ label world volume of p-brane and where $x^\mu(\xi), \mu = 0, 1, \dots, 9$ are world-volume fields that describe embedding of Dp-brane in the target space-time with the background fields which are dilaton ϕ , metric $g_{\mu\nu}$ and NSNS two form field $b_{\mu\nu}$. Further, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, where A_α is gauge field propagating on the world volume of Dp-brane. Finally $\lambda = 2\pi\alpha'$ and $T_p = \frac{1}{\lambda^{(p+1)/2}}$ is Dp-brane tension.

Let us consider a Dp-brane that is extended along $y \equiv x^9$ direction so that we impose the following static gauge

$$y = \xi^p. \quad (2)$$

The next step is to perform dimensional reduction when we presume that all world volume fields do not depend on y . As a result we obtain following form of the matrix $\mathbf{A}_{\alpha\beta}$

$$\begin{aligned} \mathbf{A}_{yy} &= 0, & \mathbf{A}_{y\hat{\beta}} &= g_{yj} \partial_{\hat{\beta}} x^j + b_{yj} \partial_{\hat{\beta}} x^j - \lambda \partial_{\hat{\beta}} A_y, \\ \mathbf{A}_{\hat{\alpha}y} &= \partial_{\hat{\alpha}} x^i g_{iy} + \partial_{\hat{\alpha}} x^i b_{iy} + \lambda \partial_{\hat{\alpha}} A_y, \\ \mathbf{A}_{\hat{\alpha}\hat{\beta}} &= g_{ij} \partial_{\hat{\alpha}} x^i \partial_{\hat{\beta}} x^j + b_{ij} \partial_{\hat{\alpha}} x^i \partial_{\hat{\beta}} x^j + \lambda F_{\hat{\alpha}\hat{\beta}}, \end{aligned} \quad (3)$$

where $\xi^{\hat{\alpha}}, \hat{\alpha}, \hat{\beta} = 0, 1, \dots, p-1$ label world-volume coordinates on D(p-1)-brane and where $i, j = 0, 1, \dots, 8$.

Using (3) we find that the action (1) has the form

$$S_{NR} = -T_p \int dy \int d^p \xi e^{-\phi} \sqrt{-\det \begin{pmatrix} 0 & (g_{yj} + b_{yj})\partial_{\hat{\beta}} x^j - \lambda \partial_{\hat{\beta}} A_y \\ \partial_{\hat{\alpha}} x^i (g_{iy} + b_{iy}) + \lambda \partial_{\hat{\alpha}} A_y & \mathbf{A}_{\hat{\alpha}\hat{\beta}} \end{pmatrix}}. \quad (4)$$

Note that the determinant in the action (4) has the form

$$\det \begin{pmatrix} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha}y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} \end{pmatrix} \quad (5)$$

that, using properties of the determinant and also the fact that $\mathcal{M}_{yy} = 0$ is equal to

$$\det \begin{pmatrix} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha}y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} \end{pmatrix} = \det \begin{pmatrix} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha}y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} + V_{\hat{\alpha}} \mathcal{M}_{y\hat{\beta}} + \mathcal{M}_{\hat{\alpha}y} W_{\hat{\beta}} \end{pmatrix}, \quad (6)$$

where $V_{\hat{\alpha}}$ and $W_{\hat{\beta}}$ are arbitrary functions. To proceed further we define T -dual coordinate $\tilde{y}$ as

$$\tilde{y} = \lambda A_y \quad (7)$$

and, following [3] we define τ_{μ}^A , $A = 0, 1$ as

$$\tau_i^0 = C g_{yi}, \quad \tau_{\tilde{y}}^0 = 0, \quad \tau_i^1 = C b_{yi}, \quad \tau_{\tilde{y}}^1 = C \quad (8)$$

so that we can write

$$\mathcal{M}_{\hat{\alpha}y} = \tau_{\hat{\alpha}}, \quad \mathcal{M}_{y\hat{\beta}} = \bar{\tau}_{\hat{\beta}}, \quad \tau_{\hat{\alpha}} = \tau_{\hat{\alpha}}^0 + \tau_{\hat{\alpha}}^1, \quad \bar{\tau}_{\hat{\alpha}} = \tau_{\hat{\alpha}}^0 - \tau_{\hat{\alpha}}^1, \quad (9)$$

where $\tau_{\hat{\alpha}} = \tau_{\mu} \partial_{\hat{\alpha}} \tilde{x}^{\mu} = \tau_i \partial_{\hat{\alpha}} x^i + \tau_{\tilde{y}} \partial_{\hat{\alpha}} \tilde{y}$. Let us now define $V_{\hat{\alpha}}$ and $W_{\hat{\alpha}}$ as

$$\begin{aligned} V_{\hat{\alpha}} &= \partial_{\hat{\alpha}} \tilde{x}^{\mu} V_{\mu} = \partial_{\hat{\alpha}} x^i (C_i^0 + C_i^1) + \partial_{\hat{\alpha}} \tilde{y} (C_{\tilde{y}}^0 + C_{\tilde{y}}^1), \\ W_{\hat{\alpha}} &= \partial_{\hat{\alpha}} \tilde{x}^{\mu} W_{\mu} = \partial_{\hat{\alpha}} x^i (C_i^0 - C_i^1) + \partial_{\hat{\alpha}} \tilde{y} (C_{\tilde{y}}^0 - C_{\tilde{y}}^1). \end{aligned} \quad (10)$$

Now using (10) we obtain

$$\begin{aligned} \mathcal{M}_{\hat{\alpha}\hat{\beta}} + V_{\hat{\alpha}} \mathcal{M}_{y\hat{\beta}} + \mathcal{M}_{\hat{\alpha}y} W_{\hat{\beta}} \\ = H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}}, \end{aligned} \quad (11)$$

where $H_{\hat{\alpha}\hat{\beta}} = H_{\mu\nu} \partial_{\hat{\alpha}} \tilde{x}^{\mu} \partial_{\hat{\beta}} \tilde{x}^{\nu}$ where $H_{\mu\nu}$ is equal to

$$\begin{aligned} H_{ij} &= g_{ij} + C_i^0 \tau_j^0 + \tau_i^0 C_j^0 - C_i^1 \tau_j^1 - \tau_i^1 C_j^1, \\ H_{\tilde{y}\tilde{y}} &= 2(C_{\tilde{y}}^0 \tau_{\tilde{y}}^0 - C_{\tilde{y}}^1 \tau_{\tilde{y}}^1), \\ H_{\tilde{y}j} &= C_{\tilde{y}}^0 \tau_j^0 + \tau_{\tilde{y}}^0 C_j^0 - C_{\tilde{y}}^1 \tau_j^1 - \tau_{\tilde{y}}^1 C_j^1, \\ H_{i\tilde{y}} &= C_i^0 \tau_{\tilde{y}}^0 + \tau_i^0 C_{\tilde{y}}^0 - C_i^1 \tau_{\tilde{y}}^1 - \tau_i^1 C_{\tilde{y}}^1. \end{aligned} \quad (12)$$

In the same way we find that $\mathcal{B}_{\mu\nu}$ is equal to

$$\begin{aligned} \mathcal{B}_{ij} &= b_{ij} + C_i^1 \tau_j^0 - C_i^0 \tau_j^1 + \tau_i^1 C_j^0 - C_i^0 \tau_j^1, \\ \mathcal{B}_{i\tilde{y}} &= C_i^1 \tau_{\tilde{y}}^0 - C_i^0 \tau_{\tilde{y}}^1 + \tau_i^1 C_{\tilde{y}}^0 - \tau_i^0 C_{\tilde{y}}^1, \\ \mathcal{B}_{\tilde{y}j} &= C_{\tilde{y}}^1 \tau_j^0 + \tau_{\tilde{y}}^1 C_j^0 - C_{\tilde{y}}^0 \tau_j^1 - \tau_{\tilde{y}}^0 C_j^1 \end{aligned} \quad (13)$$

so that the action for nonrelativistic D(p-1)-brane (4) has the form

$$S_{NR} = -T_{p-1} \int d^p \xi e^{-\hat{\phi}} \sqrt{-\det \begin{pmatrix} 0 & \tau_\mu \partial_{\hat{\beta}} \tilde{x}^\mu \\ \partial_{\hat{\alpha}} \tilde{x}^\nu \bar{\tau}_\nu & H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}} \end{pmatrix}}, \quad (14)$$

where we identified T_{p-1} as

$$T_{p-1} = T_p \int dy \quad (15)$$

and also $\hat{\phi}$ as

$$\hat{\phi} = \phi + \ln |C| \quad (16)$$

which is the same result as was found in [3]. Finally using properties of determinant (6) we see that there is natural redefinition of the background fields given by prescription

$$\tilde{H}_{\hat{\alpha}\hat{\beta}} + \tilde{\mathcal{B}}_{\hat{\alpha}\hat{\beta}} = H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + X_{\hat{\alpha}} \bar{\tau}_{\hat{\beta}} + \tau_{\hat{\alpha}} Y_{\hat{\beta}} \quad (17)$$

that can be written in equivalent form

$$\begin{aligned} \tilde{H}_{\mu\nu} &= H_{\mu\nu} - Z_\mu^A \tau_\nu^B \eta_{AB} - T_\mu^A Z_\nu^B \eta_{AB}, \\ \tilde{\mathcal{B}}_{\mu\nu} &= \mathcal{B}_{\mu\nu} - Z_\mu^A \tau_\nu^B \epsilon_{AB} - T_\mu^A Z_\nu^B \epsilon_{AB}, \end{aligned} \quad (18)$$

where we defined $Z_\mu^A, A = 0, 1$ as

$$X_\mu = Z_\mu^0 + Z_\mu^1, \quad Y_\mu = Z_\mu^0 - Z_\mu^1, \quad (19)$$

and where $\eta_{AB} = \text{diag}(-1, 1)$ and $\epsilon_{01} = -\epsilon_{10} = 1$. These transformations are known as Stuckelberg transformations of the nonrelativistic fields [3–5, 10]. Finally it is important to stress that the nonrelativistic D(p-1)-brane action (14) has similar form as the nonrelativistic Dp-brane actions that were studied in [22, 23] since they were derived by the same procedure. On the other hand these actions differ by their final form since (14) was derived by manipulation with the determinant and redefinition of the background fields in

order to have contact with [3, 4] while the final form of the actions found in [22, 23] have been elaborated by different way. In fact, the action (14) can be rewritten to the form of the actions studied in [3, 4] when we introduce matrix $\mathbf{A}_{\hat{\alpha}\hat{\beta}} = H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}}$ and its inverse $\mathbf{A}^{\hat{\alpha}\hat{\beta}}$. Then (14) can be rewritten as

$$S_{NR} = -T_{p-1} \int d^p \xi e^{-\hat{\phi}} \sqrt{\bar{\tau}_\nu \partial_{\hat{\alpha}} \tilde{x}^\nu \mathbf{A}^{\hat{\alpha}\hat{\beta}} \partial_{\hat{\beta}} \tilde{x}^\mu \bar{\tau}_\mu \sqrt{\det \mathbf{A}_{\hat{\alpha}\hat{\beta}}}} \quad (20)$$

that has similar form as nonrelativistic Dp-brane actions studied in [22, 23]. On the other hand nonrelativistic D(p-1)-brane action (14) is more tractable for the analysis of T duality and hence we will use this form in what follows.

III. T DUALITY OF NONRELATIVISTIC D(p-1)-BRANE

In this section we study how nonrelativistic D(p-1)-brane action (14) transforms under T duality. We start with situation when we perform T duality along k –longitudinal spatial dimensions where nonrelativistic D(p-1)-brane wraps them. Then it is natural to perform gauge fixing

$$\xi^m = x^m, \quad m = 9 - k, \dots, 8, \quad (21)$$

where now all world-volume fields do not depend on ξ^m . Instead they are functions of remaining world-volume coordinates $\xi^{\bar{\alpha}}$ where $\bar{\alpha} = 0, \dots, p-1-k$. Further, let us denote remaining coordinates as $x^{\bar{\mu}}, \bar{\mu}, \bar{\nu} = 0, 1, \dots, 9-k$. Finally we introduce $E_{\mu\nu} = H_{\mu\nu} + \mathcal{B}_{\mu\nu}$. In this case the action (14) has the form

$$\begin{aligned} S_{NR} &= -T_{(p-1)} \int d^k \xi \int d^{p-k} \xi e^{-\hat{\phi}} \sqrt{-\det \begin{pmatrix} 0 & \tau_{\bar{\mu}} \partial_{\bar{\beta}} x^{\bar{\mu}} & \tau_n \\ \partial_{\bar{\alpha}} \tilde{x}^{\bar{\mu}} \bar{\tau}_{\bar{\mu}} & E_{\bar{\alpha}\bar{\beta}} + \lambda F_{\bar{\alpha}\bar{\beta}} & E_{\bar{\alpha}n} + \lambda \partial_{\bar{\alpha}} A_n \\ \bar{\tau}_m & E_{m\bar{\beta}} - \lambda \partial_{\bar{\beta}} A_m & E_{mn} \end{pmatrix}} \\ &= -T_{(p-1-k)} \int d^{p-k} \xi e^{-\hat{\phi}} \sqrt{\det E_{mn}} \\ &\quad \times \sqrt{-\det \begin{pmatrix} -\tau_n \tilde{E}^{nm} \bar{\tau}_m & \tau_{\bar{\mu}} \partial_{\bar{\beta}} x^{\bar{\mu}} - \tau_m \tilde{E}^{mn} (E_{n\bar{\beta}} - \lambda \partial_{\bar{\beta}} A_n) \\ \partial_{\bar{\alpha}} \tilde{x}^{\bar{\mu}} \bar{\tau}_{\bar{\mu}} - (E_{\bar{\alpha}n} + \lambda \partial_{\bar{\alpha}} A_n) \tilde{E}^{nk} \bar{\tau}_k & E_{\bar{\alpha}\bar{\beta}} + \lambda F_{\bar{\alpha}\bar{\beta}} - (E_{\bar{\alpha}n} + \lambda \partial_{\bar{\alpha}} A_n) \tilde{E}^{nm} (E_{m\bar{\beta}} - \lambda \partial_{\bar{\beta}} A_m) \end{pmatrix}}, \end{aligned} \quad (22)$$

where $\tilde{E}^{mn}$ is matrix inverse to $H_{mn} + \mathcal{B}_{mn}$. We see that the form of the action for nonrelativistic D(p-1-k)-brane (22) is not functionally the same as the action (14) due to the presence of the expression $-\tau_n \tilde{E}^{nm} \tilde{\tau}_m$. However when we derived the action (22) we presumed that nonrelativistic D(p-1)-brane wraps x^k directions which are pure spatial. Then it is natural to presume that

$$\tau_m = \tilde{\tau}_m = 0. \quad (23)$$

Note also that these conditions were imposed when T duality along pure spatial dimension was analyzed in [10].

With the help of conditions (23) we obtain that the action for nonrelativistic D(p-1-k)-brane has the form

$$S_{NR} = -T_{p-1-k} \int d^{p-k} \xi e^{-\tilde{\phi}} \sqrt{\det \begin{pmatrix} 0 & \tau_{\tilde{\mu}} \partial_{\tilde{\beta}} x^{\tilde{\mu}} \\ \partial_{\tilde{\alpha}} \tilde{x}^{\tilde{\mu}} \tilde{\tau}_{\tilde{\mu}} & P[\tilde{E}_{\tilde{\alpha}\tilde{\beta}}] + \lambda F_{\tilde{\alpha}\tilde{\beta}} \end{pmatrix}}, \quad (24)$$

where $\tilde{\phi}$ is transformed dilaton field defined by equation

$$e^{-\tilde{\phi}} = e^{-\hat{\phi}} \sqrt{\det E_{mn}}, \quad (25)$$

and where $P[\tilde{E}_{\tilde{\alpha}\tilde{\beta}}]$ is pull-back of the T dual metric to the world volume of D(p-1-k)-brane defined as

$$P[\tilde{E}_{\tilde{\alpha}\tilde{\beta}}] = \partial_{\tilde{\alpha}} x^{\tilde{\mu}} \tilde{E}_{\tilde{\mu}\tilde{\nu}} \partial_{\tilde{\beta}} x^{\tilde{\nu}} + \partial_{\tilde{\alpha}} x^{\tilde{\mu}} \tilde{E}_{\tilde{\mu}}^m \partial_{\tilde{\beta}} \tilde{x}_m + \partial_{\tilde{\alpha}} \tilde{x}_m \tilde{E}_m^{\tilde{\nu}} \partial_{\tilde{\beta}} x^{\tilde{\nu}} + \partial_{\tilde{\alpha}} \tilde{x}_m \tilde{E}^{mn} \partial_{\tilde{\beta}} \tilde{x}_n, \quad (26)$$

where T dual coordinates $\tilde{x}_m$ are defined as

$$\tilde{x}_m = \lambda A_m. \quad (27)$$

Finally components of T dual metric have the form

$$\tilde{E}_{\tilde{\mu}\tilde{\nu}} = E_{\tilde{\mu}\tilde{\nu}} - E_{\tilde{\mu}m} \tilde{E}^{mn} E_{n\tilde{\nu}}, \quad \tilde{E}_{\tilde{\mu}}^m = E_{\tilde{\mu}m} \tilde{E}^{mn}, \quad \tilde{E}_{\tilde{\nu}}^n = -\tilde{E}^{nm} E_{m\tilde{\nu}}. \quad (28)$$

Note that (28) are standard T duality rules as are known in relativistic string theories [12,13].

As the natural step we should analyze T duality properties of nonrelativistic D(p-1)-brane when we dualize along directions transverse to the world volume of D(p-1)-brane. However as was shown in [18,19] this can be done when we consider infinite number of D(p-1)-branes on the covering space whose world-volume fields obey appropriate quotient conditions. In order to do this we should first find an action for N - nonrelativistic Dp-branes in non-relativistic background.

IV. NON-ABELIAN GENERALIZATION

In order to find non-Abelian action for N nonrelativistic Dp-branes in nonrelativistic background we will follow [17] when we start with the non-Abelian action for N -space-time filling D9-branes. This action has the form

$$S = -s \text{Tr} T_9 \int d^{10} \xi e^{-\phi} \sqrt{-\det(E_{\mu\nu} + \lambda F_{\mu\nu})}, \quad (29)$$

where

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i[A_{\mu}, A_{\nu}], \quad (30)$$

where A_{μ} are $N \times N$ Hermitian matrices and where $s\text{Tr}$ means symmetrized trace. Now as in Sec. II we presume that the background has lightlike isometry and we label this direction by $\xi^9 \equiv y$. We further presume that all world-volume fields do not depend on them. Then we get

$$\lambda F_{\hat{\mu}y} = D_{\hat{\mu}} \Phi_{\tilde{y}} = -\lambda F_{y\hat{\mu}}, \quad (31)$$

where we introduced $\Phi_{\tilde{y}}$ as $N \times N$ matrix through the formula

$$\Phi_{\tilde{y}} = \lambda A_y, \quad (32)$$

and where $\hat{\mu} = 0, 1, \dots, 8$. Note that $\Phi_{\tilde{y}}$ describes embedding of D8-branes in transverse $\tilde{y}$ - directions. Using this notation we get

$$\det(E_{\mu\nu} + \lambda F_{\mu\nu}) = \det \begin{pmatrix} 0 & E_{y\hat{\nu}} - D_{\hat{\nu}} \Phi_{\tilde{y}} \\ E_{\hat{\mu}y} + D_{\hat{\mu}} \Phi_{\tilde{y}} & E_{\hat{\mu}\hat{\nu}} + \lambda F_{\hat{\mu}\hat{\nu}} \end{pmatrix}. \quad (33)$$

Then performing the same manipulation with the determinant as in the second section (II) we obtain an action for N nonrelativistic D8-branes in nonrelativistic background in the form

$$S_{NR}^N = -T_8 \int d^9 \xi \text{Stre}^{-\hat{\phi}} \sqrt{-\det \begin{pmatrix} 0 & \tau_{\bar{\nu}} + \tau_{\bar{y}} D_{\bar{\nu}} \Phi_{\bar{y}} \\ \bar{\tau}_{\bar{\mu}} + D_{\bar{\mu}} \Phi_{\bar{y}} \bar{\tau}_{\bar{y}} & H_{\bar{\mu}\bar{\nu}} + \mathcal{B}_{\bar{\mu}\bar{\nu}} + \lambda F_{\bar{\mu}\bar{\nu}} \end{pmatrix}}. \quad (34)$$

Let us now perform T duality along k -directions which means that all world-volume fields do not depend on ξ^m , $m = (9 - k), \dots, 8$. Note that all world-volume matrix valued fields depend on $\xi^{\bar{\mu}}, \bar{\mu}, \bar{\nu} = 0, 1, \dots, 8 - k$. As a result we obtain

$$F_{\bar{\mu}n} = D_{\bar{\mu}} \Phi_n, \quad F_{m\bar{\nu}} = -D_{\bar{\nu}} \Phi_m, \quad F_{mn} = i\lambda^{-1} [\Phi_m, \Phi_n] \quad (35)$$

and consequently

$$\begin{aligned} & \det \begin{pmatrix} 0 & \tau_{\bar{\nu}} + \tau_{\bar{y}} D_{\bar{\nu}} \Phi_{\bar{y}} \\ \bar{\tau}_{\bar{\mu}} + D_{\bar{\mu}} \Phi_{\bar{y}} \bar{\tau}_{\bar{y}} & H_{\bar{\mu}\bar{\nu}} + \mathcal{B}_{\bar{\mu}\bar{\nu}} + \lambda F_{\bar{\mu}\bar{\nu}} \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & \tau_{\bar{\nu}} + \tau_{\bar{y}} D_{\bar{\nu}} \Phi_{\bar{y}} & \tau_n + \tau_{\bar{y}} i\lambda^{-1} [\Phi_n, \Phi_{\bar{y}}] \\ \bar{\tau}_{\bar{\mu}} + D_{\bar{\mu}} \Phi_{\bar{y}} \bar{\tau}_{\bar{y}} & H_{\bar{\mu}\bar{\nu}} + \mathcal{B}_{\bar{\mu}\bar{\nu}} + \lambda F_{\bar{\mu}\bar{\nu}} & H_{\bar{\mu}n} + \mathcal{B}_{\bar{\mu}n} + D_{\bar{\mu}} \Phi_n \\ \bar{\tau}_m + i\lambda^{-1} [\Phi_m, \Phi_{\bar{y}}] \bar{\tau}_{\bar{y}} & H_{m\bar{\nu}} + \mathcal{B}_{m\bar{\nu}} - D_{\bar{\nu}} \Phi_m & H_{mn} + \mathcal{B}_{mn} + i\lambda^{-1} [\Phi_m, \Phi_n] \end{pmatrix}. \end{aligned} \quad (36)$$

For simplicity we introduce notation $E_{\mu\nu} = H_{\mu\nu} + \mathcal{B}_{\mu\nu}$. Further, let us now presume that the matrix E_{mn} has an inverse matrix $\tilde{E}^{mn}$. This is certainly always possible to define, for example in the simplest case when $k = 1$ so that H_{zz} is one-dimensional. Then we can write the determinant (36) as

$$\det \begin{pmatrix} \mathbf{A}_{\bar{y}\bar{y}} & \mathbf{A}_{\bar{y}\bar{\nu}} \\ \mathbf{A}_{\bar{\mu}\bar{y}} & \mathbf{A}_{\bar{\mu}\bar{\nu}} \end{pmatrix}, \quad (37)$$

where

$$\begin{aligned} \mathbf{A}_{\bar{y}\bar{y}} &= -(\tau_m + i\lambda^{-1} [\Phi_m, \Phi_{\bar{y}}] \tau_{\bar{y}}) (Q^{-1})^{mn} (\bar{\tau}_n + i\lambda^{-1} [\Phi_n, \Phi_{\bar{y}}] \bar{\tau}_{\bar{y}}), \\ \mathbf{A}_{\bar{y}\bar{\nu}} &= \tau_{\bar{\nu}} + \tau_{\bar{y}} D_{\bar{\nu}} \Phi_{\bar{y}} - (\tau_n + i\lambda^{-1} \tau_{\bar{y}} [\Phi_n, \Phi_{\bar{y}}]) (Q^{-1})^{nm} (E_{m\bar{\nu}} - D_{\bar{\nu}} \Phi_m), \\ \mathbf{A}_{\bar{\mu}\bar{y}} &= \bar{\tau}_{\bar{\mu}} + D_{\bar{\mu}} \Phi_{\bar{y}} \bar{\tau}_{\bar{y}} - (E_{\bar{\mu}m} + D_{\bar{\mu}} \Phi_m) (Q^{-1})^{mn} (\bar{\tau}_n + i\lambda^{-1} [\Phi_n, \Phi_{\bar{y}}] \bar{\tau}_{\bar{y}}), \\ \mathbf{A}_{\bar{\mu}\bar{\nu}} &= E_{\bar{\mu}\bar{\nu}} - (E_{\bar{\mu}m} + D_{\bar{\mu}} \Phi_m) (Q^{-1})^{mn} (E_{n\bar{\nu}} - D_{\bar{\nu}} \Phi_n), \end{aligned} \quad (38)$$

where

$$Q_{mn} = E_{mn} + i\lambda^{-1} [\Phi_m, \Phi_n], \quad (39)$$

and where $(Q^{-1})^{mn}$ is its inverse $Q_{mk} (Q^{-1})^{kn} = \delta_m^n$.

We showed in previous section that T duality of nonrelativistic D(p-1)-brane along k -spatial directions is again nonrelativistic D(p-1-k)-brane on condition when components of τ along directions we dualize are equal to zero as it is explicitly stressed in (23). Let us then impose the same condition in case of non-Abelian action for N nonrelativistic D(8-k)-branes. Further, let us express final form of (38) using T-dual form of the metric given in (28) and we obtain

$$\begin{aligned} \mathbf{A}_{\bar{y}\bar{y}} &= -\lambda^{-2} \tau_{\bar{y}} [\Phi_{\bar{y}}, \Phi_m] (Q^{-1})^{mn} [\Phi_n, \Phi_{\bar{y}}] \bar{\tau}_{\bar{y}}, \\ \mathbf{A}_{\bar{y}\bar{\nu}} &= \tau_{\bar{\nu}} + \tau_{\bar{y}} D_{\bar{\nu}} \Phi_{\bar{y}} - i\lambda^{-1} \tau_{\bar{y}} [\Phi_{\bar{y}}, \Phi_n] (Q^{-1})^n_m (\tilde{E}^m_{\bar{\nu}} + \tilde{E}^{mk} D_{\bar{\nu}} \Phi_k), \\ \mathbf{A}_{\bar{\mu}\bar{y}} &= \bar{\tau}_{\bar{\mu}} + D_{\bar{\mu}} \Phi_{\bar{y}} \bar{\tau}_{\bar{y}} - i\lambda^{-1} (\tilde{E}^m_{\bar{\mu}} + D_{\bar{\mu}} \Phi_k \tilde{E}^{km}) (Q^{-1})^n_m [\Phi_n, \Phi_{\bar{y}}] \bar{\tau}_{\bar{y}}, \\ \mathbf{A}_{\bar{\mu}\bar{\nu}} &= P[\tilde{E}_{\bar{\mu}\bar{\nu}}] + P[\tilde{E}_{\bar{\mu}r} \tilde{E}^{rs} ((Q^{-1})^t_s - \delta^t_s) \tilde{E}_{t\bar{\nu}}], \end{aligned} \quad (40)$$

where $P[\tilde{E}_{\bar{\mu}\bar{\nu}}]$ is pull-back of the T-dual metric defined as

$$P[\tilde{E}_{\bar{\mu}\bar{\nu}}] = \tilde{E}_{\bar{\mu}\bar{\nu}} + D_{\bar{\mu}}\Phi_m\tilde{E}^m_{\bar{\nu}} + \tilde{E}_{\bar{\mu}}^n D_{\bar{\nu}}\Phi_n + D_{\bar{\mu}}\Phi_m\tilde{E}^{mn}D_{\bar{\nu}}\Phi_m, \quad (41)$$

and where

$$\begin{aligned} P[\tilde{E}_{\bar{\mu}r}\tilde{E}^{rs}((Q^{-1})_s^t - \delta_s^t)\tilde{E}_{t\bar{\nu}}] &= \tilde{E}_{\bar{\mu}}^m\tilde{E}_{nk}((Q^{-1})_r^k - \delta_r^k)\tilde{E}_v^r + \tilde{E}_{\bar{\mu}}^n\tilde{E}_{nl}((Q^{-1})_k^l - \delta_k^l)\tilde{E}^{kn}D_{\bar{\nu}}\Phi_n \\ &+ D_{\bar{\mu}}\Phi_n\tilde{E}^{nr}\tilde{E}_{rk}((Q^{-1})_s^k - \delta_s^k)\tilde{E}_v^s + D_{\bar{\mu}}\Phi_k\tilde{E}^{kr}\tilde{E}_{rs}((Q^{-1})_i^s - \delta_i^s)\tilde{E}^{in}D_{\bar{\nu}}\Phi_n. \end{aligned} \quad (42)$$

Finally, $\tilde{\phi}$ is given in (25). Collecting all these terms together we obtain an action for N nonrelativistic D(8-k)-branes in the form

$$S_{NR}^N = -T_{8-k}\text{Str} \int d^{9-k}\xi e^{-\tilde{\phi}} \sqrt{-\det\left(\begin{smallmatrix} \mathbf{A}_{\tilde{y}\tilde{y}} & \mathbf{A}_{\tilde{y}\bar{\nu}} \\ \mathbf{A}_{\bar{\mu}\tilde{y}} & \mathbf{A}_{\bar{\mu}\bar{\nu}} \end{smallmatrix}\right)} \det Q_k^l. \quad (43)$$

This is the final for of the action for N D(8-k)-branes. Observe that there is general nonzero $\mathbf{A}_{\tilde{y}\tilde{y}}$ as opposite to the case of single D(8-k)-branes. However note that for collection N D(8-k)-branes that are localized at single point $\tilde{y}_0$ we have that $\Phi_{\tilde{y}} = \tilde{y}_0 \mathbf{I}_{N \times N}$ where $\mathbf{I}_{N \times N}$ is unit matrix. As a result $\Phi_{\tilde{y}}$ commutes with all matrices and hence $\mathbf{A}_{\tilde{y}\tilde{y}} = 0$. Then the action (43) has similar form as the action for single D(8-k)-brane in nonrelativistic background which is nice consistency check.

V. T DUALITY ALONG $\tilde{y}$ -DIRECTION

Now we are ready to study how nonrelativistic D(8-k)-brane transforms under T duality along directions transverse to its world volume. Since T duality along spatial directions that are transverse to its world volume is the same as in case of relativistic Dp-brane we skip this analysis and recommend [21] for more details. Instead we focus on T duality along $\tilde{y}$ -direction.

We begin with the simpler case which is D8-brane transverse to $\tilde{y}$ -direction and perform T duality along it. It is well known that in order to perform T duality along transverse direction we should consider configuration of infinite number of D8-branes on covering space. To do this we should presume that $\tilde{y}$ coordinate is compact so that the

covering space is real line. Then the world-volume matrix valued fields obey following quotient condition [20]

$$\begin{aligned} \mathcal{U}\Phi_{\tilde{y}}\mathcal{U}^{-1} &= \lambda^{1/2} + \Phi_{\tilde{y}}, \\ \mathcal{U}A_{\bar{\mu}}\mathcal{U}^{-1} &= A_{\bar{\mu}}. \end{aligned} \quad (44)$$

In order to solve the quotient equation it is natural to introduce an auxiliary Hilbert space of functions $f(y)$ on which $\Phi_{\tilde{y}}$ and $\mathcal{U}$ act. Then $\mathcal{U}$ is generator of the functions on this covering space with coordinate y in the form

$$\mathcal{U} = e^{i\frac{y}{\sqrt{\lambda}}}, \quad (45)$$

where y is coordinate of the space on which functions $f(y)$ are defined. Then $\Phi_{\tilde{y}}$ has to be equal to

$$\Phi_{\tilde{y}} = i\lambda\partial_y - A_y(y), \quad (46)$$

where now $A_y(y)$ is ordinary function that acts on Hilbert space by ordinary multiplication. Using (46) we obtain

$$D_{\bar{\mu}}\Phi_{\tilde{y}} = -\lambda\mathcal{F}_{\bar{\mu}y}, \quad \mathcal{F}_{\bar{\mu}y} = \partial_{\bar{\mu}}A_y - \partial_yA_{\bar{\mu}} \quad (47)$$

so that the action for N D8-branes can be written as

$$\begin{aligned} S_{NR}^N &= -\frac{T_8}{\sqrt{\lambda}} \int d^9\xi dy e^{-\tilde{\phi}} \sqrt{-\det\left(\begin{smallmatrix} 0 & \tau_{\bar{\nu}} + \lambda\tau_{\tilde{y}}\mathcal{F}_{y\bar{\nu}} \\ \bar{\tau}_{\bar{\mu}} - \lambda\mathcal{F}_{\bar{\mu}y}\bar{\tau}_{\tilde{y}} & H_{\bar{\mu}\bar{\nu}} + \mathcal{B}_{\bar{\mu}\bar{\nu}} + \lambda\mathcal{F}_{\bar{\mu}\bar{\nu}} \end{smallmatrix}\right)} \\ &= -T_9 \int d^9\xi dy e^{-\tilde{\phi}} \sqrt{-\bar{\tau}_{\tilde{y}}\tau_{\tilde{y}}} \sqrt{-\det\left(\begin{smallmatrix} 0 & \tau_{\bar{\nu}}\tau_{\tilde{y}}^{-1} + \lambda\mathcal{F}_{y\bar{\nu}} \\ -\bar{\tau}_{\bar{\mu}}\bar{\tau}_{\tilde{y}}^{-1} + \lambda\mathcal{F}_{\bar{\mu}y} & E_{\bar{\mu}\bar{\nu}} + \lambda\mathcal{F}_{\bar{\mu}\bar{\nu}} \end{smallmatrix}\right)} \end{aligned} \quad (48)$$

using also the fact that

$$\text{Tr} = \frac{1}{\sqrt{\lambda}} \int dy. \quad (49)$$

If we use definition of $\tau_{\tilde{y}}, \bar{\tau}_{\tilde{y}}$ as was given in Sec. II we obtain that the action (48) can be written as

$$S = -T_9 \int d^{10}\xi e^{-\phi} \sqrt{-\det \begin{pmatrix} 0 & E_{y\hat{v}} + \lambda \mathcal{F}_{y\hat{v}} \\ E_{\hat{\mu}y} + \lambda \mathcal{F}_{\hat{\mu}y} & E_{\hat{\mu}\hat{v}} + \lambda \mathcal{F}_{\hat{\mu}\hat{v}} \end{pmatrix}} \quad (50)$$

which is the original form of DBI action for D9-brane in the relativistic background with lightlike isometry.

A. T duality in case of D(8-k)-brane

Now we proceed to the most interesting problem which is T duality along $\hat{y}$ direction that is transverse to the world-volume of nonrelativistic D(8-k)-brane. As in previous section we consider an array of infinite number of D(8-k)-branes whose world-volume fields obey quotient conditions (44) and (47). Note that there is additional condition on the matrix Φ_m in the form

$$\mathcal{U}\Phi_m\mathcal{U}^{-1} = \Phi_m. \quad (51)$$

Now $\Phi_m(\xi)$ and $A_{\hat{\mu}}(\xi)$ should be considered as ordinary functions $\phi_m(\xi, y)$ and $A_{\hat{\mu}}(\xi, y)$ defined on the space labeled by y that act on test function $f(y)$ by ordinary multiplications.

Then we again find that

$$D_{\hat{\mu}}\Phi_{\hat{y}} = -\lambda \mathcal{F}_{\hat{\mu}y} \quad (52)$$

together with

$$i\lambda^{-1}[\Phi_{\hat{y}}, \Phi_m] = -\partial_y \phi_m \quad (53)$$

and also $[\Phi_m, \Phi_n] = [\phi_m(y), \phi_n(y)] = 0$ since this is commutator of ordinary functions. As a result we get that $Q_{mn} = E_{mn}$ and $(Q^{-1})^{mn} = \tilde{E}^{mn}$. Using these results we obtain that components of the matrix $\mathbf{A}$ have the form

$$\begin{aligned} \mathbf{A}_{\hat{y}\hat{y}} &= -\tau_{\hat{y}}\bar{\tau}_{\hat{y}}\partial_y\phi_m\tilde{E}^{mn}\partial_y\phi_n, \\ \mathbf{A}_{\hat{y}\hat{v}} &= \bar{\tau}_{\hat{v}} + \lambda\tau_{\hat{y}}\mathcal{F}_{y\hat{v}} + \partial_y\phi_n(\tilde{E}_{\hat{v}}^n + \tilde{E}^{nk}\partial_{\hat{v}}\phi_k)\tau_{\hat{y}}, \\ \mathbf{A}_{\hat{\mu}\hat{y}} &= \bar{\tau}_{\hat{\mu}} - \lambda\mathcal{F}_{\hat{\mu}y}\bar{\tau}_{\hat{y}} - (\tilde{E}_{\hat{\mu}}^m + \partial_{\hat{\mu}}\phi_k\tilde{E}^{km})\partial_y\phi_m\bar{\tau}_{\hat{y}}, \\ \mathbf{A}_{\hat{\mu}\hat{v}} &= P[\tilde{E}_{\hat{\mu}\hat{v}}]. \end{aligned} \quad (54)$$

Inserting these results into the action (43) we obtain that it has the form

$$\begin{aligned} S_{NR}^N &= -\frac{T_{8-k}}{\lambda^{1/2}} \int dy d^{9-k}\xi e^{-\tilde{\phi}} \sqrt{-\tau_{\hat{y}}\tau_{\hat{y}} \det Q_j^i} \\ &\times \sqrt{-\det \begin{pmatrix} \partial_y\phi_m\tilde{E}^{mn}\partial_y\phi_n & \tau_{\hat{y}}^{-1}\bar{\tau}_{\hat{v}} + \partial_y\phi_n(\tilde{E}_{\hat{v}}^n + \tilde{E}^{nk}\partial_{\hat{v}}\phi_k) + \lambda\mathcal{F}_{y\hat{v}} \\ -\bar{\tau}_{\hat{y}}^{-1}\bar{\tau}_{\hat{\mu}} + (\tilde{E}_{\hat{\mu}}^m + \partial_{\hat{\mu}}\phi_k\tilde{E}^{km})\partial_y\phi_m + \lambda\mathcal{F}_{\hat{\mu}y} & P[\tilde{E}_{\hat{\mu}\hat{v}}] + \lambda\mathcal{F}_{\hat{\mu}\hat{v}} \end{pmatrix}}. \end{aligned} \quad (55)$$

Finally using $\tau_{\hat{y}} = C$, $\bar{\tau}_{\hat{y}} = -C$ we get that the action (55) corresponds to the relativistic DBI D(9-k)-brane action that wraps y -direction that is lightlike as can be seen from the absence of the metric component $\tilde{E}_{yy}$ in (55). This result is nice consistency check of T duality relations between nonrelativistic D-branes and relativistic D-branes with lightlike isometry. Note also that the background fields are given by T duality rules (28).

ACKNOWLEDGMENTS

This work is supported by the grant Integrable Deformations (GA20-04800S) from the Czech Science Foundation (GACR).

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