Study of the non-Hermitian $\mathcal{PT}$ -symmetric $g\phi^2(i\phi)^\epsilon$ theory: Analysis of all orders in ϵ and resummations

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In a recent work Bender *et al.* [Phys. Rev. D **98**, 125003 (2018)] proposed to study non-Hermitian $\mathcal{PT}$ -symmetric quantum field theories within the framework of the logarithmic expansion. For a single component scalar field theory, the $\mathcal{PT}$ -symmetric interaction term is written as $g\phi^2(i\phi)^\epsilon$ (generic values of ϵ). The logarithmic expansion is obtained by considering ϵ as a small parameter, and expanding $(i\phi)^\epsilon$ in powers of ϵ . Approximations to Green's functions $\mathcal{G}_n$ are obtained by truncating the interaction at finite orders in ϵ , and the first order contribution was calculated by Bender *et al.* The value of ϵ that corresponds to the original theory is then replaced in the final results. It was also suggested that this expansion could be used to implement a systematic renormalization of the theory order by order in ϵ . Using techniques that we developed recently for the analysis of a Hermitian scalar theory, in the present work we calculate Green's functions at $\mathcal{O}(\epsilon^2)$, pushing also the analysis to higher orders. We find that, at each finite order in ϵ , the theory is noninteracting for any dimension $d \geq 2$. Therefore, differently from the weak-coupling and the $\hbar$ expansions, a systematic renormalization of the theory order by order in ϵ cannot be realized. Actually, as long as we limit ourselves to finite orders in ϵ , it is not even possible to define the theory. We are then lead to consider resummations, and we start by resumming the ultraviolet leading diagrams. Unfortunately, the results are quite poor, as these resummations simply give the lowest order results of the weak-coupling expansion. We also consider the resummation of subleading terms, but the outcome does not improve much: we get nothing more than the $\mathcal{O}(g^2)$ weak-coupling results.

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I. INTRODUCTION

The non-Hermitian $\mathcal{PT}$ -symmetric scalar quantum field theory $g\phi^2(i\phi)^\epsilon$ was recently studied within the framework of the logarithmic expansion [1], that is an expansion in powers of ϵ , and it was suggested to use this expansion to implement a sensible definition of the theory, together with its systematic renormalization order by order in ϵ . In [1], Green's functions $\mathcal{G}_n$ were calculated up to first order in ϵ .

In recent years, there has been a lot of activity on the subject of non-Hermitian $\mathcal{PT}$ -symmetric quantum field theories [2–9], also in connection with possible phenomenological applications. It is then worth further investigating the structural aspects on which these theories are founded.

In non-Hermitian $\mathcal{PT}$ -symmetric theories, the definition of the inner product generally requires the knowledge of the $\mathcal{C}$ operator [10], but for quantum field theories it is a very hard task to calculate $\mathcal{C}$, and only perturbative results in $d = 2$ dimensions are known [11,12]. For the calculation of Green's functions, however, there is no need to know $\mathcal{C}$, as the $\mathcal{G}_n$ involve vacuum expectation values, and the vacuum state is an eigenvector of the $\mathcal{C}$ operator, with eigenvalue 1 [1,13].

The theory is defined by the (Euclidean) Lagrangian in d dimensions ($\epsilon \geq 0$):

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g\mu^2\phi^2(i\mu^{1-d/2}\phi)^\epsilon, \quad (1)$$

and Green's functions are given by

$$\mathcal{G}_n(x_1, \dots, x_n) = \frac{1}{Z} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{-\int d^d u \mathcal{L}}. \quad (2)$$

The logarithmic expansion is realized once the interaction is written as

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$$\frac{1}{2}g\mu^2\phi^2(i\mu^{1-d/2}\phi)^\epsilon = \frac{1}{2}g\mu^2\phi^2\sum_p \frac{\epsilon^p}{p!}\log^p(i\mu^{1-d/2}\phi), \quad (3)$$

where, if we can restrict ourselves to real values of ϕ , the logarithm is

$$\log(i\mu^{1-d/2}\phi) = \frac{1}{2}\left[\log(\mu^{2-d}\phi^2) + i\pi\frac{|\phi|}{\phi}\right]. \quad (4)$$

Equation (4) is one of the crucial ingredients needed to define the method introduced in [1]. In this respect we note that, for a generic non-Hermitian, $\mathcal{PT}$ -symmetric, one-dimensional quantum mechanical model, the boundary conditions on the Schrodinger equation are imposed in complex Stokes sectors, that might not include the real axis [10]. In the case of a quantum field theory, the search for Stokes wedges for the integration over *infinitely many* field variables is an insurmountable task [1], and the restriction to real values of ϕ is of great help for the calculation of Green's functions.

Such a restriction is implemented by considering small values of ϵ . Inserting then the expansion (3) in (2), the contributions to Green's functions at each power of ϵ ultimately reduce to functional integrals whose calculation is traced back to an application of the Wick theorem.

Clearly, for a thorough analysis of this expansion, we need to go beyond the first order in ϵ . The goal of the present work is to study Green's functions of the non-Hermitian $\mathcal{PT}$ -symmetric $g\phi^2(i\phi)^\epsilon$ theory at higher orders in ϵ . To perform this analysis, we will use techniques that we developed in a previous paper of ours [14], where we studied ordinary, i.e. Hermitian, scalar theories.

We begin by calculating Green's functions at $\mathcal{O}(\epsilon^2)$, and find a closed form for all the $\mathcal{G}_n$ in terms of hypergeometric functions. Analyzing the ultraviolet (UV) behavior of these Green's functions, however, we find that up to this order in ϵ the theory is noninteracting, as all Green's functions with $n \geq 3$ vanish.

We then move to consider higher orders. The result does not change: the analysis of the UV behavior of the $\mathcal{G}_n$ shows that at each finite order in ϵ , the theory is noninteracting for any dimension $d \geq 2$. This poses serious problems for the definition of $\mathcal{PT}$ -symmetric theories in terms of the expansion in ϵ . Actually, the original idea [1] of implementing the renormalization of the theory order by order in ϵ cannot be realized.

In view of these “negative findings”, we are lead to consider resummations. To this end, we first identify and then resum the UV leading contributions at each order in ϵ . However, the results are quite deceptive: these resummations give rise to the lowest order, i.e. $\mathcal{O}(g)$, weak-coupling diagrams for the $\mathcal{G}_n$. Successively we move to consider the resummation of subleading terms, but the results do not

really improve: we get nothing more than the $\mathcal{O}(g^2)$ weak-coupling diagrams. In order to better elucidate these findings, we specify to the physically relevant $ig\phi^3$ model, introduced several years ago [15–17] to study the density of Lee-Yang zeros of the partition function $Z[h]$ on the imaginary axis of h (h is the magnetic field).

The rest of the paper is organized as follows. In Sec. II we begin by setting up the tools for our analysis, review some previous results, and analyze the UV behavior of the $\mathcal{O}(\epsilon)$ contributions to the $\mathcal{G}_n$. In Sec. III we move to the $\mathcal{O}(\epsilon^2)$, obtain general expressions for all Green's functions $\mathcal{G}_n$, and study their UV behavior. In Sec. IV we extend the analysis to higher orders. Section V is devoted to the resummation of the UV-leading contributions at each order in ϵ , while in Sec. VI we consider the resummation of subleading terms. Section VII is for the conclusions.

II. GREEN'S FUNCTIONS AT $\mathcal{O}(\epsilon)$ AND THEIR UV BEHAVIOR

In the present section, after sketching the steps for the calculation of the $\mathcal{G}_n$ at each order in ϵ , we review the first order calculations, thus obtaining the results of [1], with some generalizations detailed below. We then analyze the UV behavior of Green's functions, showing that at this order the theory is noninteracting. We also introduce “effective vertices” that will play a very important role for the analysis of the theory at higher orders.

Referring to the theory defined by the Lagrangian (1), and using the expansion (3), we can write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \equiv \mathcal{L}_0 + \sum_{p=1}^{\infty} \mathcal{L}_p, \quad (5)$$

where $\mathcal{L}_0$ and $\mathcal{L}_p$ are defined by

$$\mathcal{L}_0 \equiv \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(m^2 + g\mu^2)\phi^2, \quad (6)$$

$$\mathcal{L}_p \equiv \frac{1}{2}g\mu^2 \frac{\epsilon^p}{p!} \phi^2 \log^p(i\mu^{1-d/2}\phi), \quad (7)$$

and we included the ϵ^0 term in the “free Lagrangian”. Using Eq. (4), $\mathcal{L}_p$ is split as

$$\begin{aligned} \mathcal{L}_p &= \frac{g\mu^2}{2p!} \left(\frac{\epsilon}{2}\right)^p \sum_{k=0}^p \binom{p}{k} \phi^2 \log^{p-k}(\mu^{2-d}\phi^2) \left(i\pi\frac{|\phi|}{\phi}\right)^k \\ &\equiv \sum_{k=0}^p \mathcal{L}_{pk}. \end{aligned} \quad (8)$$

The expansion of Green's functions in the parameter ϵ is obtained by treating $\mathcal{L}_{\text{int}}$ as a perturbation, and expanding as usual the exponential $e^{-\mathcal{S}_{\text{int}}}$ in the path integral that

defines the $\mathcal{G}_n$. We will consider only connected Green's functions.

From (8) we see that all the “interaction terms” $\mathcal{L}_{pk}$ contain factors of the kinds $\phi^2 \log^k(\mu^{2-d}\phi^2)$ and $i\pi|\phi|/\phi$. Therefore, in addition to the terms that appear in the corresponding ordinary theory $g\phi^2(\phi^2)^\epsilon$ considered in [14], in the present case also powers of $|\phi|/\phi$ appear. The calculation of the different orders in ϵ of the connected Green's functions can be done as sketched below [1]. The powers of $\log \phi^2$ are replaced by the formal identity (that is an adaptation of the replica trick [18])

$$\mu^{2-d}\phi^2 \log^k(\mu^{2-d}\phi^2) = \lim_{N \rightarrow 1} \frac{d^k}{dN^k} \mu^{(2-d)N} \phi^{2N}, \quad (9)$$

and the factors $\phi|\phi|$ are written with the help of the integral formula

$$\begin{aligned} \frac{|\phi|}{\phi} &= \frac{2}{\pi} \int_0^\infty dt \frac{\sin(t\phi)}{t} \\ &= \frac{2}{\pi} \int_0^\infty dt \sum_{r=0}^\infty \frac{(-1)^r t^{2r}}{(2r+1)!} \phi^{2r+1}. \end{aligned} \quad (10)$$

The variables N_i are initially treated as positive integer numbers, so that from both (9) and (10) the functional integrals for the calculation of the $\mathcal{G}_n$ are traced back to an application of Wick's theorem. The results are then analytically extended to real values of the N_i 's.

A. Green's functions at $\mathcal{O}(\epsilon)$: Effective vertices

To put to work the above machinery, we begin by calculating the $\mathcal{O}(\epsilon)$ contribution to the connected Green's functions, thus recovering results similar to those of [1], but relative to the Lagrangian (1). From (5) we have (we need only the free partition function Z_0 as we consider connected $\mathcal{G}_n$)

$$\mathcal{G}_n^{(\epsilon)}(x_1, \dots, x_n) = -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \mathcal{L}_1[\phi(u)]. \quad (11)$$

From (8) we see that $\mathcal{L}_1 = \mathcal{L}_{10} + \mathcal{L}_{11}$. Since $\mathcal{L}_{10}$ is even in ϕ while $\mathcal{L}_{11}$ is odd, they contribute to even and odd $\mathcal{G}_n$ respectively. We treat them separately.

1. Odd Green's functions

The $\mathcal{O}(\epsilon)$ contributions to the odd Green's functions come from $\mathcal{L}_{11}$. From (8) and (10) we then have

$$\begin{aligned} \mathcal{G}_n^{(\epsilon)}(x_1, \dots, x_n) &= -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \frac{\epsilon}{4} g\mu^2 i\pi |\phi(u)| \phi(u) \\ &= -i \frac{\epsilon}{2} g\mu^2 \int d^d u \int_0^\infty dt \sum_{r=0}^\infty \frac{(-1)^r t^{2r}}{(2r+1)!} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi^{2r+3}(u). \end{aligned} \quad (12)$$

For $r < \frac{n-3}{2}$ the functional integral vanishes, as we cannot draw connected diagrams, while for $r \geq \frac{n-3}{2}$ we get

$$\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi^{2r+3}(u) = B_n(r) \Delta(0)^{r-\frac{n-3}{2}} \prod_{i=1}^n \Delta(x_i - u) \quad (13)$$

where $B_n(r)$ is the combinatorial factor coming from the different contractions

$$B_n(r) = (2r+3) \dots (2r+4-n)(2r+2-n)!! = 2^{\frac{n-1}{2}} (r+1)_{\frac{n-1}{2}} (2r+3)!!, \quad (14)$$

and $\Delta(x-u)$ is the free propagator ($M^2 \equiv m^2 + g\mu^2$)

$$\Delta(x-u) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2} e^{-ip \cdot (x-u)} \rightarrow \Delta(0) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2}. \quad (15)$$

The loop integral $\Delta(0)$ diverges for $d \geq 2$, and for the sake of definiteness we can regularize this expression by introducing a physical momentum cutoff:

$$\Delta(0) \sim \begin{cases} \Lambda^{d-2} & \text{for } d > 2, \\ \log \Lambda & \text{for } d = 2. \end{cases} \quad (16)$$

We observe here that any other regularization would work equally well. The results of the present work do not depend in any way on the choice of the regularization.

Due to the presence of the falling factorial $(r+1)_{\frac{n-1}{2}}$, the coefficient $B_n(r)$ vanishes for $r < \frac{n-3}{2}$, so that (13) holds true for all positive integers r . Therefore, the $\mathcal{O}(\epsilon)$ contribution (12) to the odd Green's functions can be written as

$$\mathcal{G}_n^{(\epsilon)}(x_1, \dots, x_n) = \Pi_n^{(\epsilon)} \int d^d u \prod_{i=1}^n \Delta(x_i - u), \quad (17)$$

where, following our previous work on the corresponding ordinary theory[14], we have defined the effective vertices $\Pi_n^{(\epsilon)}$ (with an odd number n of external legs) as

$$\Pi_n^{(\epsilon)} \equiv -i \frac{\epsilon}{2} g \mu^2 \int_0^\infty dt \sum_{r=0}^\infty \frac{(-1)^r t^{2r}}{(2r+1)!} B_n(r) \Delta(0)^{r-\frac{n-3}{2}}. \quad (18)$$

The series over r and the integral over t can be performed, and we finally get

$$\begin{aligned} \mathcal{G}_n^{(\epsilon)}(x_1, \dots, x_n) &= -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \frac{\epsilon}{4} g \mu^2 \phi^2(u) \log(\mu^{2-d} \phi^2(u)) \\ &= -\frac{\epsilon}{4} g \mu^2 \int d^d u \lim_{N \rightarrow 1} \frac{d}{dN} \mu^{(2-d)(N-1)} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi^{2N}(u). \end{aligned} \quad (20)$$

As explained before, the functional integral is calculated by considering first integer values of N . For $N < \frac{n}{2}$ the path integral vanishes, as we cannot draw connected diagrams, while for $N \geq \frac{n}{2}$ we have

$$\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi^{2N}(u) = C_n(N) \Delta(0)^{N-\frac{n}{2}} \prod_{i=1}^n \Delta(x_i - u), \quad (21)$$

where $C_n(N)$ is the combinatorial factor coming from the contractions:

$$C_n(N) = 2N(2N-1) \dots (2N-n+1)(2N-n-1)!! = 2^{\frac{n}{2}}(N)_{\frac{n}{2}}(2N-1)!!. \quad (22)$$

As for $N < \frac{n}{2}$ the falling factorial $(N)_{\frac{n}{2}}$ vanishes, Eq. (21) holds true even for $N < \frac{n}{2}$, and then for all positive integers N . Starting from this observation, the required analytic extension of (21) to real values of N is obtained with the help of the identity $(2N-1)!! = 2^{N-1} \frac{\Gamma(N+\frac{1}{2})}{\Gamma(\frac{3}{2})}$, and this finally allows to take the derivative with respect to N and the limit $N \rightarrow 1$.

The $\mathcal{O}(\epsilon)$ contribution to the even Green's functions $\mathcal{G}_n$ is then

$$\mathcal{G}_n^{(\epsilon)}(x_1, \dots, x_n) = \Pi_n^{(\epsilon)} \int d^d u \prod_{i=1}^n \Delta(x_i - u), \quad (23)$$

$$\Pi_n^{(\epsilon)} = \frac{\epsilon}{2} g \mu^2 (-i)^{n+2} \Gamma\left(\frac{n}{2} - 1\right) \left[\frac{\Delta(0)}{2}\right]^{1-\frac{n}{2}}. \quad (19)$$

Even though (19) is obtained for odd values of n , we will see below that the calculation of even Green's functions will also result in effective vertices $\Pi_n^{(\epsilon)}$ that are again given by (19) (with the exception of the special case $n=2$).

From (17) we see that Green's functions at $\mathcal{O}(\epsilon)$ are written in terms of coordinate-independent (and then momentum-independent) effective vertices $\Pi_n^{(\epsilon)}$ of order ϵ , to which n external legs are attached. The effective vertex $\Pi_n^{(\epsilon)}$ is obtained once (at least part of) the interaction is integrated out, and as such it is a dressed vertex function. Later we will also introduce higher order effective vertices $\Pi_n^{(\epsilon^p)}$.

2. Even Green's functions

The $\mathcal{O}(\epsilon)$ contributions to the even Green's functions come from $\mathcal{L}_{10}$, and coincide with the terms of the same order of the corresponding $g\phi^2(\phi^2)^\epsilon$ Hermitian theory [14]. Using the replica trick for the logarithm in $\mathcal{L}_{10}$ we have

where we defined the even n -legs $\mathcal{O}(\epsilon)$ effective vertex $\Pi_n^{(\epsilon)}$ as

$$\Pi_n^{(\epsilon)} \equiv -\frac{\epsilon}{4} g \mu^2 \lim_{N \rightarrow 1} \frac{d}{dN} [\mu^{(2-d)(N-1)} C_n(N) \Delta(0)^{N-\frac{n}{2}}]. \quad (24)$$

Performing the derivative and the limit we finally obtain:

$$\Pi_2^{(\epsilon)} = -\frac{\epsilon}{2} g \mu^2 K, \quad (25)$$

$$\Pi_n^{(\epsilon)} = \frac{\epsilon}{2} g \mu^2 (-i)^{n+2} \Gamma\left(\frac{n}{2} - 1\right) \left[\frac{2}{\Delta(0)}\right]^{\frac{n}{2}-1} \text{ for } n \neq 2 \quad (26)$$

where

$$K \equiv 1 + \log [2\mu^{2-d}\Delta(0)] + \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}. \quad (27)$$

Comparing (26) with (19) we see that, as anticipated above, the two expressions for $\Pi_n^{(\epsilon)}$ with n even and odd coincide.

The introduction of these effective vertices provides a great advancement for several reasons. As for the corresponding ordinary theory [14], we will see that, at any order in ϵ , Green's function can be written in terms of effective vertices and loop integrals. Moreover, we will show that the effective vertices are extremely useful for analyzing the UV behavior of Green's functions at each order in ϵ . In the next subsection we start with the $\mathcal{O}(\epsilon)$.

B. UV behavior of Green's functions at $\mathcal{O}(\epsilon)$

When studying a quantum field theory, it is of the greatest importance to analyze the UV behavior of Green's functions. We now perform such an analysis, and this is another novelty of the present section. To this end, we observe that the loop integral $\Delta(0)$, that appears in the $\Pi_n^{(\epsilon)}$, diverges for $d \geq 2$.

Starting with the one-point Green's function $\mathcal{G}_1$, that is nothing but the vacuum expectation value $\mathcal{G}_1 = \langle 0|\phi|0 \rangle = v$, from (17) for $n = 1$ we see that

$$\mathcal{G}_1^{(\epsilon)} \sim -i\epsilon \Delta(0)^{\frac{1}{2}}. \quad (28)$$

Needless to say, this divergence is trivially renormalized inserting in the Lagrangian a linear counterterm $\delta v\phi$. In this respect, we observe that, differently to what is claimed in [1], having received a nonvanishing negative imaginary result (actually negative imaginary *infinite*) for the unrenormalized $\mathcal{G}_1^{(\epsilon)}$ is totally irrelevant in connection with the expectation that $\mathcal{G}_1$ (as a result of the $\mathcal{PT}$ -symmetric nature of the theory) should be a negative imaginary number. The renormalization can produce any possible imaginary value, including $\mathcal{G}_1^{\text{ren}} = 0$.

Let us move now to the two-point Green's function $\mathcal{G}_2$. Going to momentum space, and adding to (23) the order ϵ^0 free propagator, up to $\mathcal{O}(\epsilon)$ we have (remember that $M^2 = m^2 + g\mu^2$)

$$\tilde{\mathcal{G}}_2(p) = \frac{1}{p^2 + M^2} - \frac{1}{p^2 + M^2} \left(\frac{\epsilon g \mu^2 K}{2} \right) \frac{1}{p^2 + M^2}. \quad (29)$$

From (27) and (29) we see that the correction to the tree-level result for $\mathcal{G}_2$ goes as

$$\mathcal{G}_2 \sim \log \Delta(0). \quad (30)$$

The term in the round brackets of (29) is easily recognized as the lowest order one-particle irreducible (1PI) self-energy diagram within the expansion in ϵ . The resummation of the geometric series (in increasing powers of ϵ) can be then performed, thus obtaining for $\tilde{\mathcal{G}}_2$:

$$\tilde{\mathcal{G}}_2(p) = \frac{1}{p^2 + M^2 + \frac{1}{2}\epsilon g \mu^2 K}. \quad (31)$$

Inserting (27) in (31), we see that the radiatively corrected mass m_R^2 at this order is

$$m_R^2 = M^2 + \frac{\epsilon g \mu^2}{2} \left\{ \log [2\mu^{2-d}\Delta(0)] + 1 + \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} \right\}, \quad (32)$$

that diverges as $\log \Lambda$ for any $d > 2$, and as $\log(\log \Lambda)$ for $d = 2$. This divergence is trivially canceled once the mass counterterm contained in M^2 is tuned accordingly.

Moreover it is worth to note that as compared to the algebraic divergence encountered at first order of the weak-coupling expansion, we here observe a much milder logarithmic divergence. Actually we have shown [14] that, when $\mathcal{G}_2$ is calculated at the same order of approximation, this result for δm^2 is also present in the ordinary theory. This shows that such a mild UV behavior is peculiar of the expansion in ϵ .

Going now to the $\mathcal{G}_n$ with $n > 2$ [actually for $n \neq 2$ the equation below incorporates also the case in (28) for $n = 1$], from (26) we see that

$$\mathcal{G}_n \sim \Delta(0)^{1-\frac{n}{2}} \quad (33)$$

so that, for $d \geq 2$, at $\mathcal{O}(\epsilon)$ all the $\mathcal{G}_n$ for $n \geq 3$ vanish.

At this order then the theory turns out to be non-interacting, and this sounds as a surprising and disturbing result. This point clearly needs to be further investigated, and to this end we have to consider higher orders in the expansion. In the next section we systematically analyze the $\mathcal{O}(\epsilon^2)$.

III. GREEN'S FUNCTIONS AT $\mathcal{O}(\epsilon^2)$ AND THEIR UV BEHAVIOR

The $\mathcal{O}(\epsilon^2)$ contributions to Green's functions are given by

$$\mathcal{G}_n^{(\epsilon^2)}(x_1, \dots, x_n) = \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \times \left\{ - \int d^d u \mathcal{L}_2[\phi(u)] + \frac{1}{2} \iint d^d u_1 d^d u_2 \mathcal{L}_1[\phi(u_1)] \mathcal{L}_1[\phi(u_2)] \right\}. \quad (34)$$

The $\mathcal{L}_2$ term gives diagrams with one vertex, the other one diagrams with two vertices. We treat these two contributions separately, denoting them as $\mathcal{G}_n^{(\epsilon^2,1)}$ and $\mathcal{G}_n^{(\epsilon^2,2)}$ respectively.

A. Diagrams with one effective vertex

From (8) we know that $\mathcal{L}_2 = \mathcal{L}_{20} + \mathcal{L}_{21} + \mathcal{L}_{22}$. Since $\mathcal{L}_{20}$ and $\mathcal{L}_{22}$ are even in ϕ , while $\mathcal{L}_{21}$ is odd, they contribute

only to the even and to the odd Green's functions respectively, so that it is useful to treat the two cases separately.

1. Odd Green's functions

Taking $\mathcal{L}_{21}$ from (8), and using (9) and (10), we have

$$\begin{aligned} \mathcal{G}_n^{(\epsilon^2,1)}(x_1, \dots, x_n) &= -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \frac{\epsilon^2}{8} g \mu^2 \phi^2(u) \log(\mu^{2-d} \phi^2(u)) i\pi \frac{|\phi(u)|}{\phi(u)} \\ &= -i \frac{\epsilon^2}{4} g \mu^2 \int d^d u \lim_{N \rightarrow 1} \frac{d}{dN} \mu^{(2-d)(N-1)} \int_0^\infty dt \sum_{r=0}^\infty \frac{(-t^2)^r}{(2r+1)!} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi(u)^{2N+2r+1} \\ &= -i \frac{\epsilon^2}{4} g \mu^2 \left[\frac{2}{\Delta(0)} \right]^{\frac{n-1}{2}} \lim_{N \rightarrow 1} \frac{d}{dN} \mu^{(2-d)(N-1)} \int_0^\infty dt \sum_{r=0}^\infty \frac{(-t^2)^r}{(2r+1)!} \Delta(0)^{r+N} (r+N)_{\frac{n-1}{2}} (2N+2r+1)!! \\ &\quad \times \int d^d u \prod_{i=1}^n \Delta(x_i - u). \end{aligned} \quad (35)$$

2. Even Green's functions

The term $\mathcal{L}_{22}$ is quadratic in ϕ , and then contributes only to $\mathcal{G}_2$, the calculation being trivial. Concerning the contribution of $\mathcal{L}_{20}$, from (8) and (9) we get

$$\begin{aligned} \mathcal{G}_n^{(\epsilon^2,1)}(x_1, \dots, x_n) &= -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \frac{\epsilon^2}{16} g \mu^2 \phi^2(u) \log^2(\mu^{2-d} \phi^2(u)) \\ &= -\frac{\epsilon^2}{16} g \mu^2 \int d^d u \lim_{N \rightarrow 1} \frac{d^2}{dN^2} \left\{ \mu^{(2-d)(N-1)} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi(u)^{2N} \right\} \\ &= -\frac{\epsilon^2}{16} g \mu^2 \lim_{N \rightarrow 1} \frac{d^2}{dN^2} \left\{ \mu^{(2-d)(N-1)} [\Delta(0)]^{N-\frac{n}{2}} C_n(N) \right\} \int d^d u \prod_{i=1}^n \Delta(x_i - u). \end{aligned} \quad (36)$$

Performing in (35) and (36) the derivative with respect to N , the limit $N \rightarrow 1$, the sum over r and the integral over t , for generic n (either even or odd) we get

$$\mathcal{G}_n^{(\epsilon^2,1)}(x_1, \dots, x_n) = \Pi_n^{(\epsilon^2)} \int d^d u \prod_{i=1}^n \Delta(x_i - u), \quad (37)$$

where we have defined the $\mathcal{O}(\epsilon^2)$ effective vertices

$$\Pi_2^{(\epsilon^2)} = -\frac{\epsilon^2}{8} g \mu^2 \left[K^2 - 1 + \psi' \left(\frac{3}{2} \right) + \pi^2 \right], \quad (38)$$

$$\begin{aligned} \Pi_n^{(\epsilon^2)} &= \frac{\epsilon^2}{4} g \mu^2 (-i)^{n+2} \Gamma \left(\frac{n}{2} - 1 \right) \\ &\quad \times \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-1} (K - H_{\frac{n}{2}-2}) \quad \text{for } n \neq 2, \end{aligned} \quad (39)$$

and K is given in (27), while H_n stands for the n th Harmonic number. In the expression for $n = 2$, we have also included the contribution from $\mathcal{L}_{22}$.

3. Analysis of the UV behavior

Equation (37) shows that the UV behavior of the generic $\mathcal{G}_n^{(\epsilon^2,1)}$ is entirely given by the UV behavior of the $\Pi_n^{(\epsilon^2)}$ in (38) and (39), so that

$$\mathcal{G}_2^{(\epsilon^2,1)} \sim \log^2 \Delta(0), \quad (40)$$

$$\mathcal{G}_n^{(\epsilon^2,1)} \sim \Delta(0)^{1-\frac{n}{2}} \log \Delta(0) \quad \text{for } n \neq 2. \quad (41)$$

Comparing (40) and (41) with the behavior of the $\mathcal{G}_n$ at $\mathcal{O}(\epsilon)$ [Eqs. (30) and (33)], we see that the leading behavior of this one-vertex contribution is enhanced only by a power

of $\log \Delta(0)$. We note that $\mathcal{G}_1$ and $\mathcal{G}_2$ get a divergent contribution, while $\mathcal{G}_n^{(\epsilon^2, 1)}$ vanishes when $n \geq 3$.

However, before drawing any conclusion, we have to consider the other $\mathcal{O}(\epsilon^2)$ diagrams, namely the diagrams with two vertices. We do that in the next subsection.

B. Diagrams with two effective vertices

At $\mathcal{O}(\epsilon^2)$, the diagrams with two vertices come from $\mathcal{L}_1 \cdot \mathcal{L}_1$ terms. From (8) we see that $\mathcal{L}_1 = \mathcal{L}_{10} + \mathcal{L}_{12}$, and then the product $\mathcal{L}_1 \cdot \mathcal{L}_1$ splits in the sum of three contributions, $\mathcal{L}_{10} \cdot \mathcal{L}_{10} + 2\mathcal{L}_{10} \cdot \mathcal{L}_{11} + \mathcal{L}_{11} \cdot \mathcal{L}_{11}$ (the mixed

term gets a factor 2 as there are two contributions that are equivalent under the exchange $u_1 \Leftrightarrow u_2$). The $\mathcal{L}_{10} \cdot \mathcal{L}_{11}$ term is odd in ϕ , and gives nonvanishing contributions only to the odd Green's functions, while the first and the third terms are even, and contribute only to the even Green's functions. Let us treat the two cases separately.

1. Odd Green's functions

Taking from (8) the expression for $\mathcal{L}_{10} \cdot \mathcal{L}_{11}$, and applying (9) and (10), we obtain

$$\begin{aligned} \mathcal{G}_n^{(\epsilon^2, 2)}(x_1, \dots, x_n) &= \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \times \left(\frac{\epsilon}{4} g \mu^2 \right)^2 i\pi \int d^d u_1 d^d u_2 \phi^2(u_1) \log(\mu^{2-d} \phi^2(u_1)) \phi(u_2) |\phi(u_2)| \\ &= i \frac{\epsilon^2}{8} g^2 \mu^4 \int d^d u_1 d^d u_2 \lim_{N \rightarrow 1} \frac{d}{dN} \mu^{(2-d)(N-1)} \int_0^\infty dt \sum_{r=0}^\infty \frac{(-t^2)^r}{(2r+1)!} \\ &\quad \times \frac{1}{Z_0} \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{-S_0} \phi(u_1)^{2N} \phi(u_2)^{2r+3}. \end{aligned} \quad (42)$$

Let us evaluate the path integral in (42), starting by contracting the first $s(\leq n)$ fields $\phi(x_i)$ with $\phi(u_1)$ and the remaining $n-s$ fields with $\phi(u_2)$. The other diagrams are obtained by permutations of the external legs. We have to distinguish the case with an even number of internal lines, corresponding to an even number of external legs attached to the vertex u_1 ($s = 2j$), from the case with an odd number of external legs attached to u_1 ($s = 2j+1$). These contributions will be denoted with $\mathcal{G}_{n,A}^{(\epsilon^2, 2)}$ and $\mathcal{G}_{n,B}^{(\epsilon^2, 2)}$ respectively.

Let us start with the A-type contribution. The connected diagrams that can be drawn from the path integral are of the kind ($2l$ is the even number of internal lines)

Note that $0 \leq j \leq \frac{n-1}{2}$. Moreover for generic integers $r \geq 0$ and $N > 0$, connected diagrams are obtained only for $l \geq 1$, and for

$$\begin{cases} N \geq j + l \\ r \geq \frac{n-3}{2} - j + l \end{cases} \quad (44)$$

from which we also have that the number of internal lines is limited by

$$l \leq \text{Min} \left[N - j, r - \frac{n-3}{2} + j \right] \equiv \text{min}. \quad (45)$$

Under these conditions, the path integral in (42), i.e. the sum of diagrams of the kind (43), gives

$$\sum_{j=0}^{\frac{n-1}{2}} \sum_{l=1}^{\text{min}} C_{njl}(N, r) \Delta(0)^{N-j-l} \Delta(0)^{r-\frac{n-3}{2}+j-l} \Delta(u_1 - u_2)^{2l} \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \text{perm}. \quad (46)$$

where $C_{njl}(N, r)$ is the combinatorial factor coming from the contractions:

$$\begin{aligned} C_{njl}(N, r) &= (2N) \dots (2N - 2j - 2l + 1) (2N - 2j - 2l - 1)!! \\ &\quad \times (2r + 3) \dots (2r + 4 - n + 2j - 2l) (2r + 2 - n + 2j - 2l)!! \frac{1}{(2l)!} \\ &= [(2N - 1)!! (N)_{j+l} 2^{j+l}] [(2r + 3)!! (r + 1)_{\frac{n-1}{2}-j+l} 2^{\frac{n-1}{2}-j+l}] \frac{1}{(2l)!} \\ &= C_{2j+2l}(N) B_{n-2j+2l}(r) \frac{1}{(2l)!}. \end{aligned} \quad (47)$$

In (47) the terms $C_{2j+2l}(N)$ and $B_{n-2j+2l}(r)$ are the combinatorial coefficients found at $\mathcal{O}(\epsilon)$, when we were dealing with one-vertex diagrams with even and odd number of legs respectively. Concerning the conditions (44), it is important to make the following observations. Due to the presence of the falling factorial, the factor $C_{2j+2l}(N)$, defined in (22), vanishes when the condition (44)₁ is not satisfied. We can then consider also values of

$N < l + j$, and this allows us to reach the value $N = 1$, that is necessary for the subsequent analytic extension. Moreover the factor $B_{n-2j+2l}(r)$, defined in (14), vanishes when $0 \leq r \leq \frac{n-1}{2} - j + l$, so that, despite the condition (44)₂, we can take the sum over r in (42) starting from $r = 0$. Finally, thanks to these two latter observations, the sum over l in (46) can be extended up to infinity, as (47) vanishes when $l > \min$. Then we can write

$$\begin{aligned} \mathcal{G}_{n,A}^{(\epsilon^2,2)} &= \sum_{j=0}^{\frac{n-1}{2}} \int d^d u_1 d^d u_2 \sum_{l=1}^{\infty} \left[-\frac{\epsilon}{4} g \mu^2 \lim_{N \rightarrow 1} \frac{d}{dN} \mu^{(2-d)(N-1)} C_{2j+2l}(N) \Delta(0)^{N-j-l} \right] \\ &\times \left[-i \frac{\epsilon}{2} g \mu^2 \int_0^{\infty} dt \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{(2r+1)!} B_{n-2j+2l}(r) \Delta(0)^{r-\frac{n-3}{2}+j-l} \right] \frac{\Delta(u_1 - u_2)^{2l}}{(2l)!} \\ &\times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \binom{n}{2j} - 1 \text{ perm.} \end{aligned} \quad (48)$$

We easily recognize in the two square brackets the $\mathcal{O}(\epsilon)$ effective vertices $\Pi_k^{(\epsilon)}$ with even and odd number of legs, Eqs. (24) and (18) respectively, so that

$$\mathcal{G}_{n,A}^{(\epsilon^2,2)} = \sum_{j=0}^{\frac{n-1}{2}} \int d^d u_1 d^d u_2 \sum_{l=1}^{\infty} \Pi_{2j+2l}^{(\epsilon)} \Pi_{n-2j+2l}^{(\epsilon)} \frac{\Delta(u_1 - u_2)^{2l}}{(2l)!} \times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \binom{n}{2j} - 1 \text{ perm.} \quad (49)$$

Equation (49) is a compact and elegant form for the A -type contribution, written in terms of diagrams built with couples of effective vertices. Actually we will see that, at each order in ϵ , the $\mathcal{G}_n$ are all written in terms of effective vertices (those of higher order in ϵ will be defined later) and loop integrals connecting them. This shows the great advantage induced by the effective vertices.

The series over l can be easily resummed; however, we have to distinguish the case $n = 1$ from the others. Starting with the former we get

$$\mathcal{G}_{1,A}^{(\epsilon^2,2)} = \frac{i\sqrt{\pi}\epsilon^2 g^2 \mu^4}{12} \left[\frac{2}{\Delta(0)} \right]^{-\frac{3}{2}} \int d^d u_1 d^d u_2 \left[6 K z - z^2 {}_3F_2 \left(1, 1, \frac{3}{2}; \frac{5}{2}, 3; z \right) \right] \Delta(x_1 - u_1) \quad (50)$$

while for the others odd values of n :

$$\begin{aligned} \mathcal{G}_{n,A}^{(\epsilon^2,2)} &= \frac{\epsilon^2}{2} g^2 \mu^4 (-i)^{n+2} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}\right) \times \int d^d u_1 d^d u_2 \left[K z - \frac{n}{6} z^2 {}_3F_2 \left(1, 1, 1 + \frac{n}{2}; \frac{5}{2}, 3; z \right) \right] \prod_{i=1}^n \Delta(x_i - u_1) \\ &+ (-i)^n \frac{\epsilon^2}{2} g^2 \mu^4 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}-1\right) \times \int d^d u_1 d^d u_2 z {}_3F_2 \left(1, 1, \frac{n}{2}-1; \frac{3}{2}, 2; z \right) \prod_{i=1}^2 \Delta(x_i - u_1) \prod_{h=3}^n \Delta(x_h - u_2) + \text{perm.} \\ &+ (-i)^n \frac{\epsilon^2}{4} g^2 \mu^4 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=2}^{\frac{n-1}{2}} \Gamma(j-1) \Gamma\left(\frac{n}{2}-j-1\right) \\ &\times \int d^d u_1 d^d u_2 \left[{}_2F_1 \left(j-1, \frac{n}{2}-j-1; \frac{1}{2}; z \right) - 1 \right] \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{h=2j+1}^n \Delta(x_h - u_2) + \text{perm.} \end{aligned} \quad (51)$$

where the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ are defined by

$${}_2F_1(a_1, a_2; b; z) = \sum_{l=0}^{\infty} \frac{a_1^{(l)} a_2^{(l)}}{b^{(l)}} \frac{z^l}{l!}, \quad (52)$$

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{l=0}^{\infty} \frac{a_1^{(l)} a_2^{(l)} a_3^{(l)}}{b_1^{(l)} b_2^{(l)}} \frac{z^l}{l!} \quad (53)$$

and we denoted with z :

$$z \equiv \left[\frac{\Delta(u_1 - u_2)}{\Delta(0)} \right]^2. \quad (54)$$

Equations (50) and (51) show that the series over l give rise to hypergeometric functions that depend on the spacetime integration variables u_1 and u_2 . In this respect, we note that, as long as $u_1 \neq u_2$, we have $|z| < 1$, and the convergence of the hypergeometric functions is guaranteed. However, when $u_1 = u_2$, i.e. $z = 1$, convergence issues could arise. More precisely, for ${}_2F_1$ the convergence is guaranteed if

$$\text{Re}(b - a_1 - a_2) > 0 \quad (55)$$

while for ${}_3F_2$ the condition is

$$\text{Re}(b_1 + b_2 - a_1 - a_2 - a_3) > 0. \quad (56)$$

Considering the arguments of the hypergeometric functions in (50) and (51) we see that both conditions (55) and (56) reduce to

$$n < 5. \quad (57)$$

This means that, while $\mathcal{G}_{1,A}^{(\epsilon^2,2)}$ and $\mathcal{G}_{3,A}^{(\epsilon^2,2)}$ are certainly well behaved, for $n \geq 5$ we need to carefully analyze the

singularities in $\mathcal{G}_{n,A}^{(\epsilon^2,2)}$ induced by the limit $(u_1 - u_2) \rightarrow 0$, and to investigate whether they are integrable or not.

Performing the explicit calculation for the cases $d = 3$ and $d = 4$, we have verified that starting from $\mathcal{G}_{5,A}^{(\epsilon^2,2)}$ more and more severe nonintegrable singularities in the limit $(u_1 - u_2) \rightarrow 0$ do actually appear. However, when the B -type contribution, i.e. diagrams with an odd number of internal lines, is also considered (see below), it turns out that the divergences introduced by these “ A -diagrams” are exactly canceled by opposite divergences due to the “ B -diagrams.” Similar cancellations also occur for the even Green’s functions that we will consider later. For this reason, we do not need to consider any longer the singularities arising from the hypergeometric functions.

Let us move now to the B -type contribution. The connected diagrams that can be drawn are of the kind

Following the same steps made for the A -type contribution, we obtain as before a final expression in terms of diagrams built with two $\mathcal{O}(\epsilon)$ effective vertices $\Pi_k^{(\epsilon)}$, one with even and one with odd number of legs:

$$\mathcal{G}_{n,B}^{(\epsilon^2,2)} = \sum_{j=0}^{\frac{n-1}{2}} \int d^d u_1 d^d u_2 \sum_{l=0}^{\infty} \Pi_{2j+2l+2}^{(\epsilon)} \Pi_{n-2j+2l}^{(\epsilon)} \frac{\Delta(u_1 - u_2)^{2l+1}}{(2l+1)!} \times \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{k=2j+2}^n \Delta(x_k - u_2) + \left(\binom{n}{2j+1} - 1 \right) \text{perm.} \quad (59)$$

The series over l can be easily resummed. However, as before, we have to distinguish the case $n = 1$ from the others. Starting with the former we get

$$\mathcal{G}_{1,B}^{(\epsilon^2,2)} = \frac{i \sqrt{\pi} \epsilon^2 g^2 \mu^4}{6\sqrt{2}} [\Delta(0)]^{\frac{3}{2}} \int d^d u_1 d^d u_2 \left[3 K \sqrt{z} + z^{\frac{3}{2}} {}_3F_2 \left(1, 1, \frac{1}{2}; \frac{5}{2}, 2; z \right) \right] \Delta(x_1 - u), \quad (60)$$

while for the other odd values of n

$$\begin{aligned} \mathcal{G}_{n,B}^{(\epsilon^2,2)} &= \frac{(-i)^n \epsilon^2 g^2 \mu^4}{2} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2} - 1\right) \\ &\times \int d^d u_1 d^d u_2 \left[K \sqrt{z} - \left(\frac{n}{3} - \frac{2}{3}\right) z^{\frac{3}{2}} {}_3F_2 \left(1, 1, \frac{n}{2}; \frac{5}{2}, 2; z \right) \right] \Delta(x_1 - u_1) \prod_{h=2}^n \Delta(x_h - u_2) + \text{perm.} \\ &- \frac{(-i)^n \epsilon^2 g^2 \mu^4}{2} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=1}^{\frac{n-1}{2}} \Gamma(j) \Gamma\left(\frac{n}{2} - j - 1\right) \\ &\times \int d^d u_1 d^d u_2 \sqrt{z} {}_2F_1 \left(j, \frac{n}{2} - j - 1; \frac{3}{2}; z \right) \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{h=2j+2}^n \Delta(x_h - u_2) + \text{perm}, \end{aligned} \quad (61)$$

and the final expressions for the odd Green's functions is obtained summing up $\mathcal{G}_{n,A}^{(\epsilon^2,2)}$ and $\mathcal{G}_{n,B}^{(\epsilon^2,2)}$. Let us move now to the even Green's functions.

2. Even Green's functions

Reading from (8) the expression for the terms $\mathcal{L}_{10} \cdot \mathcal{L}_{10}$ and $\mathcal{L}_{11} \cdot \mathcal{L}_{11}$ we obtain

$$\mathcal{G}_n^{(\epsilon^2,2)}(x_1, \dots, x_n) = \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \times \left\{ \frac{\epsilon^2}{32} g^2 \mu^4 \int d^d u_1 d^d u_2 \phi^2(u_1) \log[\mu^{2-d} \phi^2(u_1)] \phi^2(u_2) \log[\mu^{2-d} \phi^2(u_2)] \right. \\ \left. - \frac{\epsilon^2 \pi^2}{32} g^2 \mu^4 \int d^d u_1 d^d u_2 \phi(u_1) |\phi(u_1)| \phi(u_2) |\phi(u_2)| \right\}. \quad (62)$$

We already encountered the first of the two terms in (62) when we calculated the $\mathcal{G}_n$ at $\mathcal{O}(\epsilon^2)$ for the ordinary scalar theory $g\phi^2(\phi^2)^\epsilon$ in [14]. The contribution from this term is given by the sum of all the connected diagrams built with a couple of $\mathcal{O}(\epsilon)$ effective vertices $\Pi_k^{(\epsilon)}$ with an even number of legs. Labeling with A_1 and B_1 the diagrams in which the number of external legs attached to each vertex is even and odd respectively, we have

$$\mathcal{G}_{n,A_1}^{(\epsilon^2,2)} = \frac{1}{2} \sum_{j=0}^{n/2} \int d^d u_1 d^d u_2 \sum_{l=1}^{\infty} \Pi_{2j+2l}^{(\epsilon)} \Pi_{n-2j+2l}^{(\epsilon)} \frac{\Delta(u_1 - u_2)^{2l}}{(2l)!} \times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{h=2j+1}^n \Delta(x_h - u_2) + \binom{n}{2j} - 1 \text{ perm}, \quad (63)$$

$$\mathcal{G}_{n,B_1}^{(\epsilon^2,2)} = \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} \int d^d u_1 d^d u_2 \sum_{l=0}^{\infty} \Pi_{2j+2l+2}^{(\epsilon)} \Pi_{n-2j+2l}^{(\epsilon)} \frac{\Delta(u_1 - u_2)^{2l+1}}{(2l+1)!} \times \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{h=2j+2}^n \Delta(x_h - u_2) + \binom{n}{2j+1} - 1 \text{ perm}. \quad (64)$$

We now replace (25) and (26) for the effective vertices in (63) and (64), and sum the series over l . Starting with the two-point Green's function we get

$$\mathcal{G}_{2,A_1}^{(\epsilon^2,2)} = \frac{\epsilon^2}{12} g^2 \mu^4 \Delta(0) \int d^d u_1 d^d u_2 \left[3 K z - z^2 {}_3F_2\left(1, 1, 2; \frac{5}{2}, 3; z\right) \right] \Delta(x_1 - u_1) \Delta(x_2 - u_1), \quad (65)$$

$$\mathcal{G}_{2,B_1}^{(\epsilon^2,2)} = \frac{\epsilon^2}{12} g^2 \mu^4 \Delta(0) \int d^d u_1 d^d u_2 \left[3 K^2 \sqrt{z} + 2 z^{\frac{3}{2}} {}_3F_2\left(1, 1, 1; \frac{5}{2}, 2; z\right) \right] \Delta(x_1 - u_1) \Delta(x_2 - u_2), \quad (66)$$

while for the even Green's functions with $n \geq 4$ we have

$$\mathcal{G}_{n,A_1}^{(\epsilon^2,2)} = \frac{\epsilon^2}{6} g^2 \mu^4 (-1)^{\frac{n}{2}+1} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}\right) \int d^d u_1 d^d u_2 \left[3 K z - \frac{n}{2} z^2 {}_3F_2\left(1, 1, \frac{n}{2}+1; \frac{5}{2}, 3; z\right) \right] \prod_{i=1}^n \Delta(x_i - u_2) \\ + \frac{\epsilon^2}{2} g^2 \mu^4 (-1)^{\frac{n}{2}} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}-1\right) \int d^d u_1 d^d u_2 z {}_3F_2\left[\left(1, 1, \frac{n}{2}-1\right); \left(\frac{3}{2}, 2\right); z\right] \prod_{i=1}^2 \Delta(x_i - u_1) \prod_{k=3}^n \Delta(x_k - u_2) + \text{perm}. \\ + \frac{\epsilon^2}{8} g^2 \mu^4 (-1)^{\frac{n}{2}} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=2}^{\frac{n}{2}-2} \Gamma(j-1) \Gamma\left(\frac{n}{2}-j-1\right) \int d^d u_1 d^d u_2 \left[{}_2F_1\left(j-1, \frac{n}{2}-j-1; \frac{1}{2}; z\right) - 1 \right] \\ \times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \text{perm}. \quad (67)$$

$$\begin{aligned}
\mathcal{G}_{n,B_1}^{(\epsilon^2,2)} &= \frac{\epsilon^2}{6} g^2 \mu^4 (-1)^{\frac{n}{2}} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}-1\right) \int d^d u_1 d^d u_2 \left[3 K \sqrt{z} - (n-2) z^{\frac{3}{2}} {}_3F_2\left(1, 1, \frac{n}{2}; \frac{5}{2}, 2; z\right) \right] \\
&\times \Delta(x_1 - u_1) \prod_{i=2}^n \Delta(x_i - u_2) + \text{perm.} \\
&+ \frac{\epsilon^2}{4} g^2 \mu^4 (-1)^{\frac{n}{2}+1} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=1}^{\frac{n}{2}-2} \Gamma(j) \Gamma\left(\frac{n}{2}-j-1\right) \int d^d u_1 d^d u_2 \sqrt{z} {}_2F_1\left(j, \frac{n}{2}-j-1; \frac{3}{2}; z\right) \\
&\times \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{k=2j+2}^n \Delta(x_k - u_2) + \text{perm.}
\end{aligned} \tag{68}$$

We have to consider now the second of the two terms in (62). Using (10), we write this contribution to $\mathcal{G}_n^{(\epsilon^2,2)}$ as

$$-\frac{\epsilon^2}{8} g^2 \mu^4 \int d^d u_1 d^d u_2 \int_0^\infty dt_1 \sum_{r_1=0}^\infty \frac{(-1)^{r_1} t_1^{2r_1}}{(2r_1+1)!} \int_0^\infty dt_2 \sum_{r_2=0}^\infty \frac{(-1)^{r_2} t_2^{2r_2}}{(2r_2+1)!} \times \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi^{2r_1+3}(u_1) \phi^{2r_2+3}(u_2). \tag{69}$$

As before, we need first to calculate the connected component of the functional integral in (69) for each $r_1, r_2 \geq 0$. There are two classes of diagrams, depending on whether the number of external legs attached to each vertex is even (let us call them A_2 -type diagrams) or odd (B_2 -type diagrams), and we will treat these two cases separately.

Starting with the A_2 -type contribution, we have diagrams of the kind

$$\tag{70}$$

where for any couple $r_1, r_2 \geq 0$, the only possible diagrams are those with values of $0 \leq j \leq \frac{n}{2}$ and $l \geq 0$, satisfying the conditions

$$\begin{aligned}
r_1 &\geq j-1+l \\
r_2 &\geq \frac{n}{2}-j-1+l
\end{aligned} \tag{71}$$

which imply that the number of internal lines is limited by

$$l \leq \text{Min} \left[r_1 + 1 - j, r_2 + 1 - \frac{n}{2} + j \right] \equiv \text{Min}. \tag{72}$$

Under these conditions, the sum of the diagrams of the type (70) gives

$$\sum_{j=0}^{n/2} \sum_{l=0}^{\text{Min}} C_{njl}(r_1, r_2) \Delta(0)^{r_1+1-j-l} \Delta(0)^{r_2+1-\frac{n}{2}+j-l} \Delta(u_1 - u_2)^{2l+1} \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \text{perm.} \tag{73}$$

where $C_{njl}(r_1, r_2)$ is the combinatorial factor coming from the contractions:

$$\begin{aligned}
C_{njl}(r_1, r_2) &= [(2r_1+3)!!(r_1+1)_{j+l} 2^{j+l}] [(2r_2+3)!!(r_2+1)_{\frac{n}{2}-j+l} 2^{\frac{n}{2}-j+l}] \frac{1}{(2l+1)!} \\
&= B_{2j+2l+1}(r_1) B_{n-2j+2l+1}(r_2) \frac{1}{(2l+1)!}.
\end{aligned} \tag{74}$$

In (74) the terms $B_{2j+2l+1}(r_1)$ and $B_{n-2j+2l+1}(r_2)$ are the combinatorial coefficients (14) found when we were dealing with $\mathcal{O}(\epsilon)$ one-vertex diagrams with an odd number of legs. As for the odd Green's functions, this makes it possible to express the final result for the even $\mathcal{G}_n$ in terms of diagrams with two effective-vertices $\Pi_k^{(\epsilon)}$.

Moreover, for the same reasons seen for the odd Green's functions, we can extend both the series over r_1 and r_2 starting from $r_1 = r_2 = 0$, and the sum over l up to infinity, regardless the conditions (71) and (72). We can then write (in this case there is no need to consider $\mathcal{G}_2$ separately)

$$\begin{aligned}
\mathcal{G}_{n,A_2}^{(\epsilon^2,2)} &= \frac{1}{2} \sum_{j=0}^{n/2} \int d^d u_1 d^d u_2 \sum_{l=0}^{\infty} \left[-i \frac{\epsilon}{2} g \mu^2 \int_0^{\infty} dt_1 \sum_{r_1=0}^{\infty} \frac{(-1)^{r_1} t^{2r_1}}{(2r_1+1)!} B_{2j+2l+1}(r_1) \Delta(0)^{r_1+1-j-l} \right] \\
&\times \left[-i \frac{\epsilon}{2} g \mu^2 \int_0^{\infty} dt_2 \sum_{r_2=0}^{\infty} \frac{(-1)^{r_2} t^{2r_2}}{(2r_2+1)!} B_{n-2j+2l+1}(r_2) \Delta(0)^{r_2+1-\frac{n}{2}+j-l} \right] \frac{\Delta(u_1-u_2)^{2l+1}}{(2l+1)!} \\
&\times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \binom{n}{2j} - 1 \text{ perm.}
\end{aligned} \tag{75}$$

where we easily recognize in the square brackets the $\mathcal{O}(\epsilon)$ effective vertices (18) with an odd number of legs. Then

$$\mathcal{G}_{n,A_2}^{(\epsilon^2,2)} = \frac{1}{2} \sum_{j=0}^{n/2} \int d^d u_1 d^d u_2 \sum_{l=0}^{\infty} \Pi_{2j+2l+1}^{(\epsilon)} \Pi_{n-2j+2l+1}^{(\epsilon)} \frac{\Delta(u_1-u_2)^{2l+1}}{(2l+1)!} \times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \binom{n}{2j} - 1 \text{ perm.} \tag{76}$$

Inserting in (76) the $\Pi_k^{(\epsilon)}$ as given in (19), and performing the sum over l , we get

$$\begin{aligned}
\mathcal{G}_{n,A_2}^{(\epsilon^2,2)} &= \frac{\epsilon^2}{4} g^2 \mu^4 (-i)^{n+2} \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=0}^{n/2} \Gamma\left(j-\frac{1}{2}\right) \Gamma\left(\frac{n}{2}-j-\frac{1}{2}\right) \times \int d^d u_1 d^d u_2 \sqrt{z} {}_2F_1\left(j-\frac{1}{2}, \frac{n}{2}-j-\frac{1}{2}; \frac{3}{2}; z\right) \\
&\times \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{k=2j+1}^n \Delta(x_k - u_2) + \binom{n}{2j} - 1 \text{ perm.}
\end{aligned} \tag{77}$$

Moving to the B_2 contribution, we now see that the diagrams are of the kind

$$\tag{78}$$

Following steps similar to those made for the A_2 contributions, we obtain as before an expression in terms of diagrams built with two $\mathcal{O}(\epsilon)$ effective vertices $\Pi_k^{(\epsilon)}$ with an odd number of legs:

$$\mathcal{G}_{n,B_2}^{(\epsilon^2,2)} = \frac{1}{2} \sum_{j=0}^{n/2-1} \int d^d u_1 d^d u_2 \sum_{l=1}^{\infty} \Pi_{2j+2l+1}^{(\epsilon)} \Pi_{n-2j+2l-1}^{(\epsilon)} \frac{\Delta(u_1-u_2)^{2l}}{(2l)!} \times \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{k=2j+2}^n \Delta(x_k - u_2) + \binom{n}{2j+1} - 1 \text{ perm.} \tag{79}$$

Inserting as before the $\Pi_k^{(\epsilon)}$ given in (19), and performing the sum over l :

$$\begin{aligned}
\mathcal{G}_{n,B_2}^{(\epsilon^2,2)} &= \frac{\epsilon^2}{8} g^2 \mu^4 (-i)^n \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-2} \sum_{j=0}^{\frac{n}{2}-1} \Gamma\left(j-\frac{1}{2}\right) \Gamma\left(\frac{n}{2}-j-\frac{3}{2}\right) \times \int d^d u_1 d^d u_2 \left[{}_2F_1\left(j-\frac{1}{2}, \frac{n}{2}-j-\frac{3}{2}; \frac{1}{2}; \left(\frac{\Delta(u_1-u_2)}{\Delta(0)}\right)^2\right) - 1 \right] \\
&\times \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{k=2j+2}^n \Delta(x_k - u_2) + \binom{n}{2j+1} - 1 \text{ perm.}
\end{aligned} \tag{80}$$

The $\mathcal{O}(\epsilon^2)$ contribution to the even Green's functions from the two-vertex diagrams is then obtained summing up the four terms A_1 , A_2 , B_1 , and B_2 .

3. Analysis of the UV behavior

The analysis of the UV behavior of the two-vertex contributions to Green's functions is easily performed if we refer to (49), (59), (63), (64), (76), and (79), where the $\mathcal{G}_n^{(\epsilon^2, 2)}$ are expressed in terms of effective vertices $\Pi_k^{(\epsilon)}$ and loop integrals. From the UV behavior of the $\Pi_k^{(\epsilon)}$ [that is

$$\mathcal{G}_{n,A}^{(\epsilon^2, 2)} \sim D_s \Delta(0)^{-\frac{n}{2}} \log \Delta(0), \quad \mathcal{G}_{n,B}^{(\epsilon^2, 2)} \sim \Delta(0)^{1-\frac{n}{2}} \log \Delta(0), \quad (81)$$

$$\mathcal{G}_{n,A_1}^{(\epsilon^2, 2)} \sim D_s \Delta(0)^{-\frac{n}{2}} \log \Delta(0), \quad \mathcal{G}_{n,B_1}^{(\epsilon^2, 2)} \sim \begin{cases} \Delta(0)^{1-\frac{n}{2}} \log \Delta(0) & \text{for } n \neq 2, \\ \log^2 \Delta(0) & \text{for } n = 2, \end{cases} \quad (82)$$

$$\mathcal{G}_{n,A_2}^{(\epsilon^2, 2)} \sim \begin{cases} \Delta(0)^{1-\frac{n}{2}} & \text{for } n \neq 2, \\ \log \Delta(0) & \text{for } n = 2, \end{cases} \quad \mathcal{G}_{n,B_2}^{(\epsilon^2, 2)} \sim D_s \Delta(0)^{-\frac{n}{2}} \quad (83)$$

where D_s is the superficial degree of divergence of the loop integral with two propagators:

$$D_s \sim \begin{cases} \text{const} & \text{for } d \leq 3, \\ \log \Lambda & \text{for } d = 4, \\ \Lambda^{d-4} & \text{for } d > 4. \end{cases} \quad (84)$$

The superficial degree of divergence D_s comes from the loop integrals generated by the lines that connect the two effective vertices. In (84) we have given the explicit expression of D_s in terms of an UV momentum cutoff, but, as we have already observed, even in this case we could use any kind of regularization. The results do not depend on this choice.

Let us consider now the results (40), (41), (81), (82), and (83) for the UV behavior of all the one-vertex and two-vertex $\mathcal{O}(\epsilon^2)$ contributions to Green's functions. We see that $\mathcal{G}_1$ and $\mathcal{G}_2$ diverge. As it was the case at $\mathcal{O}(\epsilon)$, these divergences are trivially removed with the introduction of $\mathcal{O}(\epsilon^2)$ linear and quadratic counterterms, together with an additional $\mathcal{O}(\epsilon^2)$ wave function renormalization. This would be the starting point of a systematic renormalization of the theory at each order in ϵ .

On the other hand, we also see that the $\mathcal{G}_n$ with $n \geq 3$ vanish. Therefore, putting these results together with those obtained at $\mathcal{O}(\epsilon)$, we conclude that up to $\mathcal{O}(\epsilon^2)$ the theory is noninteracting. However, before drawing any conclusion on the original program of realizing a systematic renormalization of the theory within the framework of the logarithmic expansion, we have to consider the higher orders.

immediately read from (19) and (26)], and from the superficial degree of divergence of the loop integrals, we observe that in each of the series over l the first term is the dominant one, so that the leading behavior of the different contributions is

IV. HIGHER ORDER CONTRIBUTIONS TO THE $\mathcal{G}_n$

In this section we extend the analysis on the UV behavior of Green's functions to higher orders in ϵ . When moving to a generic order ϵ^p , we encounter different contributions coming from diagrams having a number of vertices $V \leq p$.

First of all we note that, referring to Eq. (8), the terms that contribute to $\mathcal{G}_n^{(\epsilon^p)}$ are given by products of $\mathcal{L}_{p_i k_i}[\phi(u_i)]$, with $i = 1, \dots, V$. Each $\mathcal{L}_{p_i k_i}$ brings a factor either of the type $\phi^2 \log^k \phi^2$ or of the type $\phi |\phi| \log^k \phi^2$, that have to be treated with (9) and (10). When performing the contractions, the combinatorial factors always split in coefficients of the kind B_n and C_n in (14) and (22), encountered when considered one-vertex diagrams [see (47) and (74) for the $\mathcal{O}(\epsilon^2)$ two-vertex contributions]. This gives rise to a factorization that finally leads to express the $\mathcal{G}_n^{(\epsilon^p)}$ as a sum of all the possible diagrams built in terms of effective vertices. This general result is the great advancement brought by the introduction of the effective vertices.

As a consequence, the analysis of the UV behavior of Green's functions becomes an easy task even at higher orders in ϵ . We only need to derive the UV behavior of the generic $\mathcal{O}(\epsilon^p)$ effective vertex $\Pi_k^{(\epsilon^p)}$, and the superficial degree of divergence of the loop integrals due to the propagators that connect these vertices.

To this end, let us first calculate the generic $\mathcal{O}(\epsilon^p)$ effective vertices $\Pi_n^{(\epsilon^p)}$ by considering the one-vertex contribution to the $\mathcal{G}_n$ at order ϵ^p . From (8) we have

$$\begin{aligned} \mathcal{G}_n^{(\epsilon^p, 1)}(x_1, \dots, x_n) &= -\frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u \mathcal{L}_p \\ &= -\frac{g\mu^2}{2p!} \left(\frac{\epsilon}{2}\right)^p \int d^d u \sum_{k=0}^p \binom{p}{k} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi(u)^2 \log^{p-k} (\mu^{2-d} \phi(u)^2) \left(i\pi \frac{|\phi(u)|}{\phi(u)}\right)^k. \end{aligned} \quad (85)$$

Let us begin by considering even Green's functions, for which we only have even values of k , i.e. even powers of $\frac{|\phi|}{\phi}$. In this case,

$$\mathcal{G}_n^{(\epsilon^p, 1)}(x_1, \dots, x_n) = -\frac{1}{2}g\mu^2 \sum_{\substack{k=0 \\ \text{even}}}^p \left(\frac{\epsilon}{2}\right)^p \frac{(i\pi)^k}{(p-k)!k!} \times \int d^d u \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi(u)^2 \log^{p-k}(\mu^{2-d} \phi(u)^2). \quad (86)$$

By following similar steps to those employed in the previous sections, the path integral in (86) is immediately calculated. We get

$$\mathcal{G}_n^{(\epsilon^p, 1)}(x_1, \dots, x_n) = \Pi_n^{(\epsilon^p)} \int d^d u \prod_{i=1}^n \Delta(x_i - u), \quad (87)$$

that diagrammatically is written as

(88)

with the effective vertex $\Pi_n^{(\epsilon^p)}$ at $\mathcal{O}(\epsilon^p)$ given by

$$\Pi_n^{(\epsilon^p)} \equiv -\frac{g\mu^2}{p!} \left(\frac{\epsilon}{2}\right)^p \left[\frac{2}{\Delta(0)}\right]^{\frac{n}{2}-1} \sum_{\substack{k=0 \\ \text{even}}}^p \binom{p}{k} (i\pi)^k \lim_{N \rightarrow 1} \frac{d^{(p-k)}}{dN^{(p-k)}} \left[(2\mu^{2-d} \Delta(0))^{N-1} \frac{\Gamma(N + \frac{1}{2}) \Gamma(N + 1)}{\Gamma(\frac{3}{2}) \Gamma(N + 1 - \frac{n}{2})} \right]. \quad (89)$$

Concerning the odd Green's functions, in (85) we only have odd values of k , i.e. odd powers of $\frac{|\phi|}{\phi}$, so that in this case

$$\mathcal{G}_n^{(\epsilon^p, 1)}(x_1, \dots, x_n) = -\frac{1}{2}g\mu^2 \sum_{\substack{k=0 \\ \text{odd}}}^p \left(\frac{\epsilon}{2}\right)^p \frac{(i\pi)^k}{(p-k)!k!} \times \int d^d u \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \phi(u) |\phi(u)| \log^{p-k}(\mu^{2-d} \phi(u)^2). \quad (90)$$

Performing the path integral in (90) by applying the same techniques developed in the previous sections, we find that the odd $\mathcal{G}_n^{(\epsilon^p, 1)}$ are again given by (87), with an effective vertex that has the same expression as in (89), with the difference that k is odd rather than even.

From (89), and from the equivalent equation with k odd, we find that the UV behavior of the $\Pi_n^{(\epsilon^p)}$ is given by

$$\Pi_n^{(\epsilon^p)} \sim \begin{cases} \log^p \Delta(0) & \text{for } n = 2, \\ \Delta(0)^{1-\frac{n}{2}} \log^{p-1} \Delta(0) & \text{for } n \neq 2, \end{cases} \quad (91)$$

where (as we already know) $\Delta(0)$ goes as $\log \Lambda$ for $d = 2$, and as Λ^{d-2} for $d > 2$. We note that, when $p = 1, 2$, Eq. (91) reduces to the previously studied cases.

Let us consider now the UV behavior of $\mathcal{O}(\epsilon^p)$ diagrams containing more than one effective vertex. As one-particle reducible diagrams contribute only to dress legs, in the following we limit ourselves to consider only 1PI diagrams.

Indicating with N_I the number of internal lines, the superficial degree of divergence D of a generic loop integral is

$$D = d(N_I - V + 1) - 2N_I, \quad (92)$$

and the latter is (superficially) convergent when $D < 0$.

Concerning the contribution of the effective vertices, we begin by noting that the total number of legs is $n + 2N_I$. Therefore the UV behavior of the effective vertices is

$$\prod_{i=1}^V \Pi_{m_i}^{(k_i)} \sim \prod_{i=1}^V \Delta(0)^{1-\frac{m_i}{2}} \log^{k_i-1} \Delta(0) \sim \Delta(0)^{V-\frac{n}{2}-N_I} \quad (93)$$

where in the last member of (93) we retain only the algebraic dependence on $\Delta(0)$. Actually we neglected a factor containing a power of $\log \Delta(0)$, as it is harmless for our analysis.

Let us distinguish the two cases $D < 0$ and $D \geq 0$.

- (i) *Case $D < 0$.* In this case the UV behavior dependence comes uniquely from the effective vertices, i.e. from (93). For any fixed value of V , the condition $D < 0$ is verified when $V \leq N_I < \frac{d}{d-2}(V-1)$, where the lower limit comes from the fact that we are considering only 1PI diagrams. The leading diagram (i.e. the one that is less suppressed in terms of Λ) is obtained when N_i takes its minimal value, that is $N_I = V$. It goes as

$$\Delta(0)^{-\frac{n}{2}}. \quad (94)$$

- (ii) *Case $D \geq 0$.* This case does not exist in $d = 2$ dimensions. For $d > 2$, the UV behavior of the diagram with n external lines, V vertices, and N_I internal lines comes from both the effective vertices (93) and the loop integrals¹ [see (92)]. It is

$$\Lambda^{\frac{2-d}{2}n+d-2V} \quad \text{for } d > 2. \quad (95)$$

From Eqs. (87), (91), (94), and (95) we see that at any order ϵ^p the dominant contribution to each of Green's functions $\mathcal{G}_n$ comes from the diagram with $V = 1$, that is from the one-vertex diagram given by (87) and (91). In Secs. II and III we have already seen that this is the case at order ϵ and ϵ^2 .

Moreover, from (87) and (91) we see that Green's functions $\mathcal{G}_1$ and $\mathcal{G}_2$ diverge at each order ϵ^p . This is what we already found for the cases $p = 1, 2$. These divergences are trivially removed with the introduction of linear and quadratic counterterms at each order ϵ^p .

Finally, (87) and (91) also tell us that Green's functions $\mathcal{G}_n$ with $n \geq 3$ vanish for any dimension $d \geq 2$. Therefore, at any finite order in ϵ , and for any $d \geq 2$, the theory is noninteracting.

This is a disturbing, and at first sight surprising, result. However it is not difficult to understand its origin. Actually, any finite order in ϵ gives nothing but an approximation to the interaction Lagrangian that is truncated at a finite power of $\log(\phi)$. Due to the mild behavior of the logarithm (and of its powers), we can easily understand that this is a “too poor” truncation of the physical interaction, not sufficient to grasp enough of the quantum fluctuations, i.e. to guarantee the existence of nontrivial S -matrix elements. We believe that this shows an intrinsic weakness of the logarithmic expansion.

Moreover, the fact that at any order in ϵ the theory is noninteracting shows that the original program of a systematic renormalization of the theory order by order in [1] cannot be realized, actually loses its meaning. In particular, the renormalization of the vacuum expectation value of the field and of its mass, obtained through the introduction in the Lagrangian of the linear and quadratic counterterms $\delta v\phi$ and $\frac{1}{2}\delta m^2\phi^2$ (considered above) has no physical significance.

Physically speaking, there is no way of giving a meaning to the theory if we truncate it to a finite order in ϵ .

We might still hope that a sensible definition could be given after resumming diagrams from each order in ϵ . In this respect, it is clear that the first thing to do is to resum the leading contributions to Green's functions, that as we have previously shown are the one-vertex diagrams (87). This amounts to resumming the ladder of leading logarithms in (91).

Before ending this section we observe that, in order to perform the resummations, it is useful, although not necessary, to go back to Eq. (89) (and the analogous one for odd values of k) and close the sum over k . To this end, defining

$$f_n(N) = (2\mu^{2-d}\Delta(0))^{N-1} \frac{\Gamma(N+\frac{1}{2})\Gamma(N+1)}{\Gamma(\frac{3}{2})\Gamma(N+1-\frac{n}{2})} \times \begin{cases} \cos(N\pi) & \text{for } n \text{ even,} \\ i \sin(N\pi) & \text{for } n \text{ odd,} \end{cases} \quad (96)$$

and noting that

$$\lim_{N \rightarrow 1} \frac{d^k}{dN^k} \sin(N\pi) = \begin{cases} 0 & \\ i(i\pi)^k & \end{cases} \quad \lim_{N \rightarrow 1} \frac{d^k}{dN^k} \cos(N\pi) = \begin{cases} -(i\pi)^k & \text{for } k \text{ even,} \\ 0 & \text{for } k \text{ odd,} \end{cases} \quad (97)$$

we can write the generic n -legs effective vertex at $\mathcal{O}(\epsilon^p)$ as

$$\Pi_n^{(\epsilon^p)} = g\mu^2 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-1} \left(\frac{\epsilon}{2} \right)^p \frac{1}{p!} \lim_{N \rightarrow 1} \frac{d^p}{dN^p} f_n(N). \quad (98)$$

Let us proceed now to the resummations.

¹Once again, for the sake of definiteness, we give the results in term of an ultraviolet momentum cutoff. However the following results do not depend on this choice, and any regularization would lead to the same conclusions.

V. RESUMMATION OF THE ONE-VERTEX DIAGRAMS

In the previous sections we have seen that, at any given order ϵ^p , the leading contributions to Green's functions $\mathcal{G}_n$ come from the one-vertex diagrams (88).

Indicating their resummation with $\mathcal{G}_n^{(1)}$:

$$\begin{aligned}\mathcal{G}_n^{(1)}(x_1, \dots, x_n) &= \sum_{p=1}^{\infty} \mathcal{G}_n^{(\epsilon^p, 1)}(x_1, \dots, x_n) \\ &= \left(\sum_{p=1}^{\infty} \Pi_n^{(\epsilon^p)} \right) \int d^d u \prod_{i=1}^n \Delta(x_i - u). \quad (99)\end{aligned}$$

Equation (99) shows that the resummation of the $\mathcal{G}_n^{(\epsilon^p, 1)}$ actually amounts to resumming the $\Pi_n^{(\epsilon^p)}$, thus providing an x -independent (and then p -independent) approximation $\Gamma_n^{(1)}$ to the n -points vertex function Γ_n . From Eq. (98) we see that

$$\begin{aligned}\Gamma_n^{(1)} &\equiv \sum_{p=1}^{\infty} \Pi_n^{(\epsilon^p)} \\ &= g\mu^2 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-1} \sum_{p=1}^{\infty} \left(\frac{\epsilon}{2} \right)^p \frac{1}{p!} \left(\frac{d^p f_n(N)}{dN^p} \right)_{N=1} \\ &= g\mu^2 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-1} \left[f_n \left(1 + \frac{\epsilon}{2} \right) - f_n(1) \right] \quad (100)\end{aligned}$$

where the last step is possible as the function $f_n(N)$ is analytic on the positive real axis. From (96), we finally get

$$\begin{aligned}\Gamma_n^{(1)} &= g\mu^2 \delta_{n,2} - g\mu^2 \left[\frac{2}{\Delta(0)} \right]^{\frac{n}{2}-1} (2\mu^{2-d} \Delta(0))^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon+3}{2}) \Gamma(\frac{\epsilon+4}{2})}{\Gamma(\frac{3}{2}) \Gamma(\frac{\epsilon+4-n}{2})} \\ &\times \begin{cases} \cos(\frac{\epsilon\pi}{2}) & \text{for } n \text{ even,} \\ i \sin(\frac{\epsilon\pi}{2}) & \text{for } n \text{ odd.} \end{cases} \quad (101)\end{aligned}$$

A few comments are in order. First of all we note that the resummation of the leading logarithms has drastically changed the UV behavior of Green's functions. In particular, the dependence on powers of logarithms $[\log^p(\Delta(0))]$ or $\log^{p-1}(\Delta(0))$, see Eq. (91)] has been traded by the algebraic dependence $\Delta(0)^{\frac{\epsilon}{2}}$. Interestingly, while at any finite order in ϵ the UV behavior of $\mathcal{G}_n$ does not depend on ϵ , after resummation it does.

Restricting ourselves to integer values of ϵ , from Eq. (101) we see that, due to the presence of the functions $\cos(\frac{\epsilon\pi}{2})$ and $\sin(\frac{\epsilon\pi}{2})$, the $\mathcal{G}_n$ for which n has not the same parity of ϵ vanish. In addition, as the denominator contains the function $\Gamma(\frac{\epsilon+4-n}{2})$, only those $\mathcal{G}_n$ with $n < \epsilon + 4$ are nonvanishing.

However, the condition $\epsilon < 2$ is necessary to restrict the field variable ϕ in the path integral that defines Green's

functions to real values, as required by the method followed in the present work. Therefore, among the possible integer values of ϵ , we can consider only $\epsilon = 1$. As it corresponds to the well-known $ig\phi^3$ model, that was introduced several years ago [15–17] for studying the density of Lee-Yang zeros of the partition function $Z[h]$ on the imaginary axis of h (where h is the magnetic field), this case is particularly relevant to consider.

Specifying then Eq. (101) to the $\epsilon = 1$ case, and considering the amputated Green's functions, i.e. the vertex functions $\Gamma_n^{(1)}$, we see that the only nonvanishing $\mathcal{G}_n$ are $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$. In this case

$$\mathcal{G}_1^{(1)} = \Gamma_1^{(1)} = -\frac{3}{2} ig\mu^{3-\frac{d}{2}} \Delta(0) = \text{---} \bullet \text{---} \quad (102)$$

$$\mathcal{G}_2^{(1)} = \Gamma_2^{(1)} = g\mu^2 = \text{---} \otimes \text{---} \quad (103)$$

$$\mathcal{G}_3^{(1)} = \Gamma_3^{(1)} = -3ig\mu^{3-\frac{d}{2}} = \text{---} \bullet \text{---} \quad (104)$$

$$\mathcal{G}_n^{(1)} = \Gamma_n^{(1)} = 0 \quad \text{for } n \geq 4. \quad (105)$$

From Eqs. (102)–(105) we see that for the theory $\frac{1}{2}ig\phi^3$ the resummation that we have just performed gives nothing but the trivial $\mathcal{O}(g)$ results for the vertex functions Γ_n . This is why in the right-hand side of these equations we have introduced ordinary weak-coupling diagrams (not to be confused with the diagrams of the logarithmic expansion considered up to now).

More specifically: (i) $\Gamma_3^{(1)}$ is just the three-point vertex; (ii) $\Gamma_1^{(1)}$ is the usual shift in the vev generated by the $\mathcal{O}(g)$ tadpole diagram; (iii) $\Gamma_2^{(1)}$ is the two-point vertex that simply comes from having inserted the $\mathcal{O}(\epsilon^0)$ of the interaction term in the free Lagrangian [see Eq. (6)], so generating the well-known trivial reshuffling in the expansion.

The resummation of the leading contributions to Green's functions has then produced just trivial results, and this is quite deceptive. At the same time, these results show that, once we resort to resummations, physically sensible (albeit up to now trivial) results can be obtained. Here we resummed diagrams with one effective vertex $\Pi_k^{(\epsilon^p)}$. In the next section we proceed to the resummation of diagrams with two effective vertices.

VI. RESUMMATION OF TWO-VERTEX DIAGRAMS

The contribution to Green's function $\mathcal{G}_n$ at $\mathcal{O}(\epsilon^p)$ ($p \geq 2$) from diagrams with two vertices comes from terms of the kind $\int d^d u_1 \mathcal{L}_k \int d^d u_2 \mathcal{L}_{p-k}$. Indicating this term with $\mathcal{G}_n^{(\epsilon^p, 2)}$ we have

$$\mathcal{G}_n^{(\epsilon^p, 2)}(x_1, \dots, x_n) = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0} \phi(x_1) \dots \phi(x_n) \int d^d u_1 d^d u_2 \mathcal{L}_k[\phi(u_1)] \mathcal{L}_{p-k}[\phi(u_2)]. \quad (106)$$

In order to perform the path integral in (106), we need to distinguish the odd from the even Green's functions. Let us begin by considering the odd $\mathcal{G}_n$.

A. Odd Green's functions

When the number of external legs is odd, the only possible two-vertex diagrams are those containing one effective vertex with an even number of legs and one with an odd number. Obviously each of them can come either

from $\mathcal{L}_k$ or from $\mathcal{L}_{p-k}$. However, under the sum over k these two cases bring the same contribution, thus giving a factor 2 that cancels the factor $\frac{1}{2}$ in (106). Therefore (as done in Sec. III B) we distinguish two classes of diagrams, namely those where the two effective vertices are connected with an even number of internal lines, and those connected by an odd number of lines, indicating them as *A*-type and *B*-type diagrams respectively. We have

$$\begin{aligned} \mathcal{G}_{n,A}^{(\epsilon^p, 2)} &= \sum_{k=1}^{p-1} \sum_{j=0}^{\frac{n-1}{2}} \sum_{l=1}^{\infty} \Pi_{2j+2l}^{(\epsilon^k)} \Pi_{n-2j+2l}^{(\epsilon^{p-k})} \frac{1}{(2l)!} \\ &\times \left[\int d^d u_1 d^d u_2 \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{h=2j+1}^n \Delta(x_h - u_2) \Delta(u_1 - u_2)^{2l} + \binom{n}{2j} - 1 \text{ perm} \right] \end{aligned} \quad (107)$$

and

$$\begin{aligned} \mathcal{G}_{n,B}^{(\epsilon^p, 2)} &= \sum_{k=1}^{p-1} \sum_{j=0}^{\frac{n-1}{2}} \sum_{l=0}^{\infty} \Pi_{2j+2l+2}^{(\epsilon^k)} \Pi_{n-2j+2l+2}^{(\epsilon^{p-k})} \frac{1}{(2l+1)!} \\ &\times \left[\int d^d u_1 d^d u_2 \prod_{i=1}^{2j+1} \Delta(x_i - u_1) \prod_{h=2j+2}^n \Delta(x_h - u_2) \Delta(u_1 - u_2)^{2l+1} + \binom{n}{2j+1} - 1 \text{ perm} \right] \end{aligned} \quad (108)$$

that correspond to diagrams of the kind:

Let us call $\mathcal{G}_n^{(2)} \equiv \sum_{p=2}^{\infty} \mathcal{G}_n^{(\epsilon^p, 2)}$ the sum of all these two-vertex diagrams, and perform separately the resummation of *A*- and *B*-type diagrams. Let us begin with the former:

$$\mathcal{G}_{n,A}^{(2)} = \sum_{j=0}^{\frac{n-1}{2}} \sum_{l=1}^{\infty} \left\{ \sum_{p=2}^{\infty} \sum_{k=1}^{p-1} \Pi_{2j+2l}^{(\epsilon^k)} \Pi_{n-2j+2l}^{(\epsilon^{p-k})} \right\} \frac{1}{(2l)!} \left[I_{2l}(x_1, \dots, x_{2j}; x_{2j+1}, \dots, x_n) + \binom{n}{2j} - 1 \text{ perm} \right], \quad (110)$$

where we introduced the notation

$$I_r(x_1, \dots, x_{2j}; x_{2j+1}, \dots, x_n) = \int d^d u_1 d^d u_2 \prod_{i=1}^{2j} \Delta(x_i - u_1) \prod_{h=2j+1}^n \Delta(x_h - u_2) \Delta(u_1 - u_2)^r. \quad (111)$$

The double series in the curly brackets is nothing but the Cauchy product:

$$\sum_{p=2}^{\infty} \sum_{k=1}^{p-1} \Pi_{2j+2l}^{(\epsilon^k)} \Pi_{n-2j+2l}^{(\epsilon^{p-k})} = \left(\sum_{k=1}^{\infty} \Pi_{2j+2l}^{(\epsilon^k)} \right) \left(\sum_{q=1}^{\infty} \Pi_{n-2j+2l}^{(\epsilon^q)} \right), \quad (112)$$

where $q = p - k$, and we need that the two series in the right-hand side are convergent, and at least one of them absolutely convergent.

We already summed in (101) the series in the round brackets. They are Taylor expansions of functions analytic in the complex half-plane $\text{Re}(\epsilon) > -3$, due to the presence of $\Gamma(\frac{\epsilon+3}{2})$ in (101), and then their radius of convergence in ϵ

is 3. Therefore the validity of (112) is guaranteed for $\epsilon < 3$: in this case both series are absolutely convergent.

For our scopes Eq. (112) is crucial, as it converts the resummation of diagrams with two effective vertices in the product of series whose terms are single effective vertices. Then, thanks to (100), Eq. (112) results in the product of vertex functions $\Gamma_n^{(1)}$ [given in (101)]. From (110) and (112) we then have

$$\mathcal{G}_{n,A}^{(2)} = \sum_{j=0}^{\frac{n-1}{2}} \sum_{l=1}^{\infty} \frac{\Gamma_{2j+2l}^{(1)} \Gamma_{n-2j+2l}^{(1)}}{(2l)!} \left[I_{2l}(x_1, \dots, x_{2j}; x_{2j+1}, \dots, x_n) + \binom{n}{2j} - 1 \text{ perm} \right]. \quad (113)$$

Following similar steps, the contribution from B -type diagrams is

$$\mathcal{G}_{n,B}^{(2)} = \sum_{j=0}^{\frac{n-1}{2}} \sum_{l=0}^{\infty} \frac{\Gamma_{2j+2l+2}^{(1)} \Gamma_{n-2j+2l}^{(1)}}{(2l+1)!} \left[I_{2l+1}(x_1, \dots, x_{2j+1}; x_{2j+2}, \dots, x_n) + \binom{n}{2j+1} - 1 \text{ perm} \right] \quad (114)$$

and summing (113) and (114), we finally get the odd $\mathcal{G}_n^{(2)}$.

B. Even Green's functions

When the number of external legs is even, there are two classes of diagrams: diagrams where the two effective vertices have an even number of legs (type 1), and diagrams where both the effective vertices have an odd number of legs (type 2). Moreover, in both cases there are two possibilities: each of the effective vertices can be connected

either with an even (A -type diagrams) or with an odd (B -type) number of external legs. Then we have 4 different classes of diagrams: A_1 , B_1 , A_2 , and B_2 (the same classes that we encountered in Sec. III B).

Summing these contributions from all orders in ϵ , and following similar steps to those made for the odd Green's functions, we find that all these contributions are easily expressed in terms of the resummed vertex functions $\Gamma_p^{(1)}$

$$\mathcal{G}_{n,A_1}^{(2)} = \frac{1}{2} \sum_{j=0}^{\frac{n}{2}} \sum_{l=1}^{\infty} \frac{\Gamma_{2j+2l}^{(1)} \Gamma_{n-2j+2l}^{(1)}}{(2l)!} \left[I_{2l}(x_1, \dots, x_{2j}; x_{2j+1}, \dots, x_n) + \binom{n}{2j} - 1 \text{ perm} \right], \quad (115)$$

$$\mathcal{G}_{n,B_1}^{(2)} = \frac{1}{2} \sum_{j=0}^{\frac{n}{2}-1} \sum_{l=0}^{\infty} \frac{\Gamma_{2j+2l+2}^{(1)} \Gamma_{n-2j+2l}^{(1)}}{(2l+1)!} \left[I_{2l+1}(x_1, \dots, x_{2j+1}; x_{2j+2}, \dots, x_n) + \binom{n}{2j+1} - 1 \text{ perm} \right], \quad (116)$$

$$\mathcal{G}_{n,A_2}^{(2)} = \frac{1}{2} \sum_{j=0}^{\frac{n}{2}} \sum_{l=0}^{\infty} \frac{\Gamma_{2j+2l+1}^{(1)} \Gamma_{n-2j+2l+1}^{(1)}}{(2l+1)!} \left[I_{2l+1}(x_1, \dots, x_{2j}; x_{2j+1}, \dots, x_n) + \binom{n}{2j} - 1 \text{ perm} \right], \quad (117)$$

$$\mathcal{G}_{n,B_2}^{(2)} = \frac{1}{2} \sum_{j=0}^{\frac{n}{2}-1} \sum_{l=1}^{\infty} \frac{\Gamma_{2j+2l+1}^{(1)} \Gamma_{n-2j+2l-1}^{(1)}}{(2l)!} \left[I_{2l}(x_1, \dots, x_{2j+1}; x_{2j+2}, \dots, x_n) + \binom{n}{2j+1} - 1 \text{ perm} \right]. \quad (118)$$

The sum of (115)–(118) finally gives the even $\mathcal{G}_n^{(2)}$.

As in the previous section, we now specify to the case of the $\frac{1}{2}ig\phi^3$ theory. We will see below that, as it was the case for the resummation of diagrams with one effective vertex, again we get simple and trivial results of the weak-coupling expansion.

Going to momentum space, and considering as usual amputated Green's functions, from each of the contributions in (113)–(118) we get (the vertices with three and two legs of the weak-coupling expansion were already identified in the previous section)

$$\mathcal{G}_{1,A}^{(2)} = \frac{1}{2} (-3ig\mu^{3-\frac{d}{2}}) (g\mu^2) \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + M^2)^2} = \text{diagram} \quad (119)$$

$$\mathcal{G}_{1,B}^{(2)} = \frac{1}{2} (g\mu^2) \frac{1}{M^2} (-3ig\mu^{3-\frac{d}{2}}) \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2} = \text{diagram} \quad (120)$$

$$\begin{aligned} \mathcal{G}_{3,B}^{(2)}(p_1, p_2, p_3) &= (g\mu^2) \frac{1}{p_1^2 + M^2} (-3ig\mu^{3-\frac{d}{2}}) + (p_1 \rightarrow p_2) + (p_1 \rightarrow p_3) \\ &= \text{diagram} + (p_1 \rightarrow p_2) + (p_1 \rightarrow p_3) \end{aligned} \quad (121)$$

$$\mathcal{G}_{2,B_1}^{(2)}(p, -p) = (g\mu^2) \frac{1}{p^2 + M^2} (g\mu^2) = \text{diagram} \quad (122)$$

$$\mathcal{G}_{2,A_2}^{(2)} = \frac{1}{2} (-3ig\mu^{3-\frac{d}{2}}) \frac{1}{M^2} (-3ig\mu^{3-\frac{d}{2}}) \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2} = \text{diagram} \quad (123)$$

$$\mathcal{G}_{2,B_2}^{(2)}(p, -p) = \frac{1}{2} (-3ig\mu^{3-\frac{d}{2}})^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2} \frac{1}{(k+p)^2 + M^2} = \text{diagram} \quad (124)$$

$$\begin{aligned} \mathcal{G}_{4,A_2}^{(2)}(s, t, u) &= \frac{(-3ig\mu^{3-\frac{d}{2}})^2}{s + M^2} + (s \rightarrow t) + (s \rightarrow u) \\ &= \text{diagram} + (s \rightarrow t) + (s \rightarrow u) \end{aligned} \quad (125)$$

where $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$, and $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$ with the conservation of the external momenta implied.

The above results are very deceptive. Despite the additional effort needed to perform this second resummation, again we recover quite trivial weak-coupling results. The first four diagrams give corrections to the diagrams (102)–(104) of the previous section, that are due to the insertion of $\Gamma_2^{(1)}$ in the propagators. The last three diagrams are typical contributions of the weak-coupling expansion at $\mathcal{O}(g^2)$. More specifically, (123) and (124) are loop corrections to the propagator, while (125) is the tree level $2 \rightarrow 2$ particle scattering diagram.

VII. SUMMARY AND CONCLUSIONS

In the present work we study the non-Hermitian $\mathcal{PT}$ -symmetric $g\phi^2(i\phi)^\epsilon$ quantum field theory in d dimensions, and apply the logarithmic expansion (that is an expansion in powers of ϵ) to the calculation of Green's functions $\mathcal{G}_n$. The first order of this expansion was considered in [1], where it was also suggested that such an expansion could be used to implement a systematic renormalization of the theory, order by order in ϵ .

We start our analysis by reviewing the general framework and tools of this expansion, referring in particular to techniques that we developed in a previous work of ours [14], where we studied an ordinary (i.e. Hermitian) scalar theory.

Handling the divergences that appear in the loop integrals (when $d \geq 2$) by introducing a momentum cutoff, we show that at $\mathcal{O}(\epsilon)$ the theory is noninteracting. We stress that this result does not depend on the specific choice made for regularizing the loop divergences, and can be equally well obtained within the framework of dimensional regularization, or of any other regularization.

We then move to the calculation of Green's functions at $\mathcal{O}(\epsilon^2)$. The mathematical tools needed for this analysis are more sophisticated than those required for calculating the $\mathcal{O}(\epsilon)$ contribution. In this respect, techniques that we introduced and tested in our previous work [14] turn out to be of great help, and allow us to write the results in a very compact and elegant form: all the $\mathcal{G}_n$ are expressed in terms of the hypergeometric functions ${}_2F_1$ and ${}_3F_2$.

However, these Green's functions contain divergent loop integrals, and as usual the calculation is performed by considering regularized expressions. Using again a momentum cutoff, we show that even at $\mathcal{O}(\epsilon^2)$ the theory is noninteracting, as all Green's functions with $n \geq 3$ vanish.

As for the order ϵ , these results do not depend on the choice made for regularizing the loop integrals. We also considered higher orders in ϵ , and finally found that at each finite order in ϵ the theory is noninteracting.

This result looks very surprising. However, we believe that it is such only in appearance. In fact, if we truncate the interaction to a finite power of $\log \phi$ (this is what we do when we consider a finite order in ϵ), we implement a “too poor” truncation of the physical interaction, not sufficient to guarantee the existence of nontrivial S -matrix elements: powers of $\log \phi$ are “too mild” to support interaction.

The conclusion is that the original program (hope) of implementing the renormalization of the theory order by order in ϵ loses its meaning. Actually, the vanishing of Green's functions at each finite order in ϵ shows that within such an expansion the theory cannot even be defined.

Nevertheless, the analysis of the UV behavior of Green's functions shows that, as ϵ increases, the contributions to the $\mathcal{G}_n$ with $n \geq 3$ are less and less suppressed. We then moved to consider the resummation of different contributions to the $\mathcal{G}_n$ at each order in ϵ , starting with the UV leading ones. However, the results are quite deceptive. The resummation of the leading diagrams simply lead to the $\mathcal{O}(g)$ weak-coupling results for the $\mathcal{G}_n$. We then considered the resummation of subleading terms, but the results did not improve much: we got the $\mathcal{O}(g^2)$ weak-coupling results. We elucidated these findings by resorting to the $ig\phi^3$ theory.

Therefore, even after putting at work the heavy artillery of resummations, we got results that are much quicker obtained within the framework of the weak-coupling expansion.

There might still be the possibility that other resummations could provide less trivial results, although it is not clear to us how this could be achieved. In any case, what we have definitely established with the present work is that the original idea of implementing a systematic renormalization of the theory order by order in ϵ cannot be realized; actually it has no sense as the theory is noninteracting at any finite order in ϵ .

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