Criticality of lower dimensional AdS_d black holes

Aditya Dhumuntarao^{1,2,*} and Robert Mann^{1,3,†}

¹*Perimeter Institute, 31 Caroline St. N. Waterloo, Ontario N2L 2Y5, Canada*

²*School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, USA*

³*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

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In lower dimensions, charged anti-de Sitter (AdS) black holes in an extended phase space, where the cosmological constant is interpreted as the thermodynamic pressure, are typically absent of liquid/gas phase transitions. We investigate the criticality of lower-dimensional, charged, dilatonic, asymptotically AdS (CDAdS_d) black holes generated from consistent truncations of RNAdS_{d+2} black objects. We demonstrate that CDAdS_d black holes in $d < 4$ can exhibit rich van der Waals behavior and confirm that the associated critical exponents match those expected from mean field theory.

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I. INTRODUCTION

In recent years, black hole chemistry has substantially sharpened the connection between the thermodynamics of anti-de Sitter (AdS) black holes and ordinary fluid systems. For rotating, charged black holes with mass M , angular momentum J , and charge Q , it is well known that the first law of black hole mechanics

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \quad (1)$$

is identical to the first law of thermodynamics provided one makes the identifications and temperature $T = \kappa/2\pi$, between surface gravity κ and temperature T , and $S = A/4$ between horizon area A and entropy S [1–4]. However, a notable omission in the first law of black hole thermodynamics is a pressure-volume term, which canonically lacks a gravitational interpretation in spacetimes without a cosmological constant Λ . The fundamental idea of black hole chemistry is to extend the thermodynamic phase space by promoting the cosmological constant of AdS spacetimes to a thermodynamic variable

$$P = -\frac{\Lambda}{8\pi} \quad (2)$$

with a corresponding conjugate volume, V [5–9].

*dhumu002@umn.edu

†rbmann@uwaterloo.ca

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The motivation for this identification is twofold. First, it was observed that the Smarr relation [10] is no longer satisfied in spacetimes with a cosmological constant [6,11]. By extending the thermodynamic phase space, the resulting pressure-volume term was found to saturate the Smarr relation. Secondly, the phase diagram of Einstein-Maxwell- AdS_d theory has long been known to contain classic swallowtail and cusp behavior suggestive of a van der Waals liquid-gas phase transition for $d \geq 4$ [12,13]. The program of black hole chemistry has completed this analogy by demonstrating that the equation of state for charged Reissner-Nördstrom AdS_d (RN- AdS_d) black holes shares identical critical behavior at the second-order transition point and universal compressibility ratio in $d = 4$ to a van der Waals fluid [14,15]. To date, the correspondence between black hole chemistry and real world fluid systems has yielded an interesting array of thermodynamic features in $d \geq 4$, including reentrant phase transitions [15,16], and tricritical points analogous to triple points in the phase diagram of water [17].

In contrast to the numerous explorations in $d \geq 4$, black hole chemistry in lower dimensions is not as thermodynamically rich. Indeed, the phase diagram of Einstein-Maxwell- AdS_d revealed that charged black holes in $d < 4$ have ideal gas like behavior [15,18,19]. In $d = 3$, the charged BTZ₃ black hole solution was observed to be absent of critical behavior [15] and shown to violate the Smarr relation unless additional thermodynamic work terms are introduced [20]. Furthermore, in $d = 2$, the lack of an area also significantly impacts the canonical definitions of entropy, and thermodynamic volume [20]. In general, pure gravity in lower dimensions is topological and lacks the necessary degrees of freedom to support critical behavior. Hence, an outstanding challenge is to find

lower-dimensional black holes that support nontrivial van der Waals behavior and obey canonical Smarr scalings.

In this paper, we resolve this tension by identifying thermodynamically rich, charged, dilatonic, asymptotically AdS black hole solutions to the action

$$\mathcal{I} = - \int \left[\frac{1}{2} (\nabla \psi)^2 + \frac{\psi^2}{4} (R - 2\Lambda - \mathcal{L}) + \frac{\lambda}{4} \right] \sqrt{-g} d^d x$$

in $d < 4$. In Sec. II, we first review criticality of charged black holes in $d \geq 4$ paying close attention to the emergent van der Waals behavior. In Sec. III, we consider consistent truncations of higher-dimensional gravity theories admitting RNAdS $_{d+2}$ black objects and search for black hole solutions to the effective action in $d < 4$. Using lower-dimensional techniques in conjunction with Euclidean methods, we explore the thermodynamics of these solutions in the extended phase space. We confirm that the Smarr relation is satisfied in each case and that the equations of state contain local minima indicative of a van der Waals phase transition.

II. PRESSURE-VOLUME CRITICALITY OF CHARGED AdS $_d$ BLACK HOLES IN $d \geq 4$

To set the stage for studying black hole chemistry in lower dimensions, we review some salient features of the Reissner-Nördstrom solution to Einstein-Maxwell-AdS $_d$ theory in the extended phase space. We describe the thermodynamics in $d \geq 4$ and study the corresponding equation of state. We then conclude by remarking on the critical behavior and exponents.

A. Criticality of RN-AdS $_d$ black holes in $d \geq 4$

To demonstrate criticality of charged black holes, the natural starting point is to study the thermodynamics of Einstein-Maxwell-AdS $_d$ theory in $d \geq 4$. This theory has been thoroughly analyzed in diverse dimensions with the on-shell regularized Euclidean action first calculated in [12,13]. Explorations of these black holes in an extended phase space were considered in [15,21]. The bulk action in d -dimensions reads

$$\mathcal{I} = - \frac{1}{16\pi} \int [R - 2\Lambda - F^2] d^d x \sqrt{-g}, \quad (3)$$

with $\Lambda = -(d-2)(d-1)/2\ell^2$ being the cosmological constant. The RN-AdS $_d$ solution, in static coordinates, and the purely electric $U(1)$ gauge field are then

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \quad (4)$$

$$F = dA, \quad A = \frac{1}{c} \frac{q}{r^{d-3}} dt, \quad (5)$$

where $d\Omega_n^2$ indicates the metric on the n -dimensional sphere, $c = \sqrt{2(d-3)/(d-2)}$, and the metric function has the form

$$f(r) = 1 - \frac{m}{r^{(d-3)}} + \frac{q^2}{r^{2(d-3)}} + \frac{r^2}{\ell^2}. \quad (6)$$

Here, the outer horizon r_+ is located at the largest positive root of $f(r_+) = 0$. The parameters m and q appearing in the metric function are related to the Arnowitt-Deser-Misner (ADM) mass and the total charge [12] of the black hole

$$M = \frac{(d-2)\omega_{d-2}}{16\pi} m, \quad (7)$$

$$Q = \sqrt{2(d-2)(d-3)} \frac{\omega_{d-2}}{8\pi} q, \quad (8)$$

where ω_n is the volume of the unit n -dimensional sphere,

$$\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \quad (9)$$

The standard thermodynamic quantities are then

$$T = \frac{f'(r_+)}{4\pi} = \frac{d-3}{4\pi r_+} + \frac{(d-1)r_+}{4\pi \ell^2} - \frac{(d-3)q^2}{4\pi r_+^{2d-5}}, \quad (10)$$

$$S = \frac{A(\mathcal{H})}{4} = \frac{\omega_{d-2} r_+^{d-2}}{4}, \quad \Phi = \frac{1}{c} \frac{q}{r_+^{d-3}}. \quad (11)$$

In the formalism of the extended phase space [14,15,21], the cosmological constant Λ is identified with the thermodynamic pressure of the system via

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-2)(d-1)}{16\pi \ell^2}. \quad (12)$$

It is then natural to reinterpret the black hole mass as the enthalpy rather than the internal energy and consider a thermodynamic volume conjugate to the pressure

$$V = \left. \frac{\partial M}{\partial P} \right|_{S,Q} = \frac{\omega_{d-2} r_+^{d-1}}{d-1}. \quad (13)$$

In $d > 4$, these quantities satisfy the Smarr formula as well as the (extended phase-space) first law of black hole thermodynamics

$$M = \frac{d-2}{d-3} TS + \Phi Q - \frac{2}{d-3} VP, \quad (14)$$

$$dM = TdS + \Phi dQ + VdP. \quad (15)$$

The identification of a pressure term invites a reinterpretation of the black hole temperature, Eq. (10), as an equation of state [15]

$$P = \frac{(d-2)T}{4r_+} - \frac{(d-3)(d-2)}{16\pi r_+^2} + \frac{(d-3)(d-2)q^2}{16\pi r_+^{2(d-2)}}, \quad (16)$$

where we have used Eq. (12). To make contact with the van der Waals equation of state, one must identify the specific volume of the corresponding fluid [14]. In the context of black hole chemistry, the specific volume v is defined by the total volume of the black hole via [15,21]

$$V = \frac{\omega_{d-2}}{d-1} \left(\frac{d-2}{4} \right)^{d-1} v^{d-1}. \quad (17)$$

Hence, by employing the definitions of P , T , and v , one finds an equation of state for charged black holes

$$P = \frac{T}{v} - \frac{(d-3)}{(d-2)\pi v^2} + \frac{(d-3)q^2}{4\pi v^{2(d-2)}\kappa^{2d-5}}, \quad (18)$$

where $\kappa = (d-2)/4$ [15]. The liquid/gas phase transition experienced by van der Waals fluids is now mapped onto a small-large black hole phase transition in the system for temperatures $T < T_c$ for all $d \geq 4$. The critical points in the equation of state occur at stationary points of inflection in the $P-v$ diagram given by

$$\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0. \quad (19)$$

The critical triple (P_c, T_c, v_c) have the values

$$T_c = \frac{(d-1)^2}{\pi \kappa v_c (2d-1)}, \quad P_c = \frac{(d-1)^2}{16\pi \kappa^2 v_c^2}, \quad v_c = \frac{1}{\kappa} [q^2 d (2d-1)]^{\frac{1}{2(d-1)}}. \quad (20)$$

Analogous to the construction of the universal ratio found for van der Waals fluids, the above triple yields

$$\frac{P_c v_c}{T_c} = \frac{2d-5}{4d-8}. \quad (21)$$

For RN-AdS₄ (i.e., when $d=4$) one recovers the canonical result of the van der Waals compressibility ratio $P_c v_c / T_c = 3/8$.

B. Critical exponents

Given that the equation of state not only reproduces van der Waals behavior but also the universal compressibility ratio, it is natural to consider the critical exponents calculated in the vicinity of the critical point. To compute critical exponents, we define

$$p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}, \quad (22)$$

which translates the equation of state into a ‘law of corresponding states’ analogous to the same function appearing in van der Waals theory, we have [15]

$$p = \frac{4(d-2)\tau}{(2d-5)\nu} - \frac{(d-2)}{(d-3)\nu^2} + \frac{\nu^{-2(d-2)}}{(d-3)(2d-5)}. \quad (23)$$

Note that the only functional dependence on the dimension appears in the last term. As we will demonstrate shortly, the critical exponents are independent of the dimension in $d \geq 4$. This is analogous to the situation in mean field theory where the critical exponents are again independent of the dimension above the upper critical dimension. In this context, this suggests that any equation of state which admits a ‘law of corresponding states’ of the above form will have critical exponents consistent with mean field theory.

To observe this, we may consider an expansion of this equation near the critical point

$$\tau = 1 + t, \quad \nu = (1 + \omega)^{1/z}, \quad (24)$$

where $z = d-1 > 0$. It may be shown that the pressure admits the following expansion

$$p = 1 + A_d t - B_d t \omega - C_d \omega^3 + \mathcal{O}(t \omega^2, \omega^4), \quad (25)$$

where the coefficients satisfy $A_d, B_d, C_d > 0$ and depend on the dimension in the following way

$$A_d = \frac{4d-8}{2d-5}, \quad B_d = \frac{4d-8}{(2d-5)(d-3)}, \quad C_d = \frac{2d-4}{3(d-1)^3}. \quad (26)$$

Differentiating the series at a fixed $t < 0$, we obtain the following differential of the pressure

$$\frac{dp}{d\omega} = -(B_d t + 3C_d \omega^2). \quad (27)$$

Then employing Maxwell’s equal area law (see [14]) the following system of equations emerge for the ‘volume’ of small and large black holes, ω_s and ω_l respectively,

$$p = 1 + A_d t - B_d t \omega_l - C_d \omega_l^3 = 1 + A_d t - B_d t \omega_s - C_d \omega_s^3, \quad 0 = - \int_{\omega_l}^{\omega_s} \omega (B_d t + 3C_d \omega^2) d\omega. \quad (28)$$

The unique solution relates the two volumes via

$$\omega_s = -\omega_l = \sqrt{\frac{B_d |t|}{C_d}}. \quad (29)$$

Given these definitions, we are now in a position to define the critical exponents α , β , γ , and δ . The exponent α describes the behavior of the specific heat at constant volume via

$$C_v = T \left. \frac{\partial S}{\partial T} \right|_v \propto |t|^{-\alpha}. \quad (30)$$

Using the volume, Eq. (13), in conjunction with the entropy $S \equiv S(T, V)$, Eq. (11), we have

$$S(T, V) = \omega_{d-2}^{\frac{1}{d-1}} [(d-1)V]^{\frac{(d-2)}{(d-1)}}, \quad (31)$$

which sets $\alpha = 0$ as the entropy is independent of the temperature in the canonical ensemble. Next, β describes the behavior of the order parameter η , which is the difference of the volume of the gas v_g phase and the volume of the liquid phase v_l , on a given isotherm

$$\eta = V_c(\omega_l - \omega_s) \propto |t|^\beta \Rightarrow \beta = \frac{1}{2}. \quad (32)$$

Using Eq. (29), we have $\eta = 2V_c\omega_l \propto \sqrt{|t|}$ thereby setting $\beta = (1/2)$. Next, the exponent γ determines the behavior of the isothermal compressibility defined by

$$\kappa_T = -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right|_T \propto |t|^{-\gamma}, \quad (33)$$

which yields $\gamma = 1$ after a direct computation using Eq. (27). Lastly, δ governs the behavior of the critical isotherm via

$$|p - 1| \propto |v - v_c|^\delta. \quad (34)$$

The ‘shape of the critical isotherm’ is then straightforward to obtain by evaluating the series, Eq. (25), at $t = 0$ yielding $\delta = 3$. Collecting our results, we see that RN-AdS_d black holes have the critical exponents

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \quad (35)$$

matching precisely with a mean field theory analysis of a van der Waals fluid. This behavior may also be extended to include rotations [15].

III. PRESSURE-VOLUME CRITICALITY IN LOWER DIMENSIONS

Pressure-volume (PV) criticality is sensitive to the horizon topology. To make this observation sharp, consider the equation of state for RN-AdS_d black holes with maximally symmetric horizons in $d \geq 4$

$$P = \frac{T}{v} - \frac{(d-3)\hat{k}}{(d-2)\pi v^2} + \frac{(d-3)q^2}{4\pi v^{2(d-2)}\kappa^{2d-5}}. \quad (36)$$

For nonspherically symmetric horizons ($\hat{k} \neq 1$), the equation of state no longer contains a local minimum [14] in which case these black hole solutions do not satisfy the condition [Eq. (19)] thereby precluding critical behavior.

In lower dimensions, Einstein-Maxwell-AdS_d gravity has insufficient spherical symmetry to support PV criticality. In $d = 3$, for example, the theory admits the charged Bañados-Teitelboim-Zanelli (BTZ) black hole solution [18,19] with a planar horizon topology, $\hat{k} = 0$, which may be compactified onto an S^1 by coordinate transformations. Indeed, studies of the charged BTZ equation of state [15], and its rotating generalization [20], confirms the lack of critical behavior is tied to the horizon topology rather than the logarithmic behavior of electromagnetism. In $d = 2$, Einstein-Maxwell-AdS_d gravity is purely topological as the Einstein-Hilbert term in the action [Eq. (3)] is related to the Euler characteristic of the spacetime and the metric is conformally flat without a black hole present. Therefore the theory is not as thermodynamically rich, and it becomes necessary to study dilaton gravity models to source black holes in $d = 2$. A general class of dilaton-gravity actions coupled to additional scalar fields Φ_M fall within the category [22,23]

$$\mathcal{I} = - \int [H(\psi)(\nabla\psi)^2 + D(\psi)R + V(g, \psi, \Phi_M)] \sqrt{-g} d^2x. \quad (37)$$

Presently, the dilaton-gravity model with $H(\psi) = (1/2)$, $D(\psi) = \psi$, and $V(\psi, \Phi_M) = -2\Lambda$ is the only known example to us to be studied in the context of black hole chemistry [20]. Although this model was observed to admit asymptotically AdS_d black holes with constant charge, the pressure equation of state was nonetheless absent of critical behavior due to the point-like structure of the horizon.

Clearly, gravity in lower dimensions requires, at a minimum, additional degrees of freedom equal to those provided by an S^2 to support nontrivial van der Waals behavior. One natural way that lower-dimensional gravity can inherit such degrees of freedom is through consistent truncations of higher-dimensional gravity theories. Consider the simplest theory of Einstein-AdS_{d+2} gravity coupled to a matter Lagrangian

$$\mathcal{I} = -\frac{1}{16\pi} \int [\tilde{R} - 2\Lambda - \mathcal{L}] \sqrt{-\tilde{g}} d^{d+2}x, \quad (38)$$

where $\Lambda = -d(d+1)/2\ell^2$, $\tilde{g}_{MN}$ is the metric, and $\tilde{R}$ is the Ricci scalar. We assume the above action admits spacetimes that fiber into $\mathcal{M}_d \times S^2$ according to the ansatz

$$ds_{d+2}^2 = \tilde{g}_{MN} dx^M dx^N = g_{ab} dx^a dx^b + \psi^2 (x^a) d\Omega_2^2 \quad (39)$$

with matter fields taking values on $\mathcal{M}_d$. Integrating out the spherical directions, the truncated theory is (see Appendix)

$$\mathcal{I} = - \int \left[\frac{1}{2} (\nabla\psi)^2 + \frac{\psi^2}{4} (R - 2\Lambda - \mathcal{L}) + \frac{\lambda}{2} \right] \sqrt{-g} d^d x. \quad (40)$$

The spherical directions now descend into lower dimensions via dilaton-gravity couplings. Here, $\lambda = R_{S^2}/2$, where the R_{S^2} is the scalar curvature of the round unit two-sphere and henceforth, we set $\lambda = 1$. Clearly, this action is a special case of Eq. (37) extended to d -dimensions. The quadratic mass term $\sim \psi^2 (R - 2\Lambda - \mathcal{L})$ preserves the transverse spherical degrees of freedom allowing for metric and gauge fields to have a leading order fall off as $\mathcal{O}(1/r^{d-1})$ akin to RN-AdS_{d+2} black holes.

In what follows, we will concentrate on the action Eq. (40) in $d = 2, 3$ and search for charged, dilatonic, asymptotically AdS_d (CDAdS_d) solutions to the equations of motion,

$$0 = \square\psi - \frac{\psi}{2} (R - 2\Lambda - \mathcal{L}), \quad (41)$$

$$0 = \frac{\psi^2}{4} G_{ab} - T_{ab}, \quad (42)$$

where $G_{ab} = R_{ab} - (R/2)g_{ab}$ is the Einstein tensor and the stress-energy tensor is

$$T_{ab} = -\frac{1}{2} \left(\nabla_a \psi \nabla_b \psi - \frac{1}{2} g_{ab} [(\nabla\psi)^2 - \Lambda\psi^2] \right) + \frac{1}{4} [\nabla_a \nabla_b \psi^2 - g_{ab} \square\psi^2] + \frac{\psi^2}{2} T_{ab}^{\mathcal{L}} + \frac{1}{4} g_{ab}. \quad (43)$$

We compute the thermodynamics using a combination of lower-dimensional techniques and the on-shell Euclidean action. Using the extended phase space, we demonstrate that CDAdS_d black holes admit PV criticality and have critical exponents which match those expected from mean field theory.

A. CDAdS₂ black holes

Here, we consider an Einstein-Maxwell-Dilaton-AdS action with $\mathcal{L} = F^2$ in $d = 2$. The uplift of this theory

to 4d corresponds to Eq. (3) for which the RN-AdS_d black hole, Eq. (4) is a solution. We search for the consistently truncated black hole solution in $d = 2$ and concentrate on static, purely electric, and radial solutions to the field equations (41)–(43) with the ansatz

$$ds^2 = g_{ab} dx^a dx^b = -f(r) dt^2 + f(r)^{-1} dr^2, \quad (44)$$

$$A \equiv A(r) dt, \quad \psi \equiv \psi(r). \quad (45)$$

For the Maxwell-Lagrangian, the stress-energy contribution and equation of motion are

$$T_{ab}^{\mathcal{L}} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F^2, \quad \nabla_a (\psi^2 F^{ab}) = 0. \quad (46)$$

Given this ansatz, the general solution reads

$$f(r) = 1 - \frac{\Lambda r^2}{3} - \frac{\mu}{r} + \frac{\rho^2}{r^2}, \quad (47)$$

$$F = dA, \quad A = -\frac{\rho}{r} dt, \quad \psi(r) = r. \quad (48)$$

Inserting $\Lambda = -3/\ell^2$, the horizon is given by the largest, real, positive root of

$$f(r_+) = 1 + \frac{r_+^2}{\ell^2} - \frac{\mu}{r_+} + \frac{\rho^2}{r_+^2} = 0, \quad (49)$$

which imposes $\mu > 0$.

In lower dimensions, computing the thermodynamics requires some care. We follow the procedure outlined in [22,23] to determine the mass and charge in terms of μ and ρ in $d = 2$. The chemical potential may be computed using standard holographic arguments [12]. The entropy is more subtle, as the area of the horizon is reduced to a point. We use the prescription outlined by Wald which may be formally applied in any dimension $d \geq 2$ [24,25]. We compare these quantities against an on-shell evaluation of the Euclidean action to confirm their interpretation and then study PV criticality.

For dilaton-gravity theories of the form Eq. (37), it was demonstrated that in $d = 2$ the current $\mathcal{J}_a \equiv T_{ab} \xi^b$ is conserved regardless of whether the stress energy is conserved or if ξ^a is a Killing vector [22]. Such a conserved current exists provided that $\xi^a \equiv \epsilon^{ab} \nabla_b \mathcal{F}(D, H)$ is chosen to conserve T_{ab} along its flow lines. Then, since a divergenceless current is always dual to the gradient of a scalar, the black hole mass may be computed via [22]

$$M = \int \epsilon_{ab} \mathcal{J}^a dx^b = \int \epsilon_{ab} T^a{}_{c \xi^c} dx^b. \quad (50)$$

For Eq. (40) it is possible to show $\xi^a = (1, 0)$. Then evaluating Eq. (43) along Eq. (47), the mass reads

$$M = \frac{\mu}{2}. \quad (51)$$

Next, it has been shown [23] that for $2d$ dilaton-gravity theories that are coupled to a $U(1)$ gauge field, the total charge may be computed via

$$Q = \frac{\psi^2}{\sqrt{-g}} F_{10} = \rho. \quad (52)$$

The corresponding chemical potential is then given by considering the electrostatic potential difference between the horizon and infinity via [12]

$$\Phi = \int_{r_+}^{\infty} dr F_{10} = \frac{\rho}{r_+}. \quad (53)$$

Finally, it is well known that an entropy can be associated with $2d$ black holes by considering an ‘area’ associated with the codimension two surface to the $t-r$ disk [20]. A collapsing fluid in $2d$ contains a single point as its boundary on the half line $r > 0$; analogously the codimension two boundary of a black hole corresponds to a horizon point, r_+ , which has area unity. The Bekenstein-Hawking entropy of the horizon-point is $S_{\text{BH}} = (1/4)$. However, this entropy conflicts with the one obtained from the Euclidean action.

An alternative definition of the entropy may be provided by the Wald formula [24]. In $2d$, the Wald entropy may formally be computed on the codimension two surface to the $t-r$ disk [25], i.e., the horizon. We find

$$S_{\text{Wald}} = -2\pi \frac{\delta \mathcal{I}}{\delta R^{abcd}} \epsilon_{ac} \epsilon_{bd} \Big|_{r_+} = \pi r_+^2. \quad (54)$$

In order to confirm the interpretation of these quantities, we consider the on-shell Euclidean action in the canonical ensemble. Rotating the solution into the Euclidean section, $t \rightarrow i\tau$, we find that the counterterm method yields the on-shell Euclidean action (see Appendix)

$$\bar{\mathcal{I}} = \frac{\beta}{4} \left(r_+ - \frac{r_+^3}{\ell^2} + 3 \frac{\rho^2}{r_+} \right), \quad (55)$$

where β denotes the period the Euclidean section

$$\beta = \frac{4\pi}{\partial_r f} \Big|_{r=r_+} = \frac{4\pi r_+^3 \ell^2}{(r_+^2 - \rho^2) \ell^2 + 3r_+^4}. \quad (56)$$

The corresponding thermodynamic quantities are

$$M = \frac{\partial \bar{\mathcal{I}}}{\partial \beta} \Big|_Q = \frac{\mu}{2}, \quad (57)$$

$$\Phi = \frac{1}{\beta} \frac{\partial \bar{\mathcal{I}}}{\partial Q} \Big|_{\beta} = \frac{\rho}{r_+}, \quad (58)$$

$$S = \frac{\partial \bar{\mathcal{I}}}{\partial \beta} \Big|_Q - \bar{\mathcal{I}} = \pi r_+^2, \quad (59)$$

precisely matching Eqs. (51)–(54).

This completes the identification of the standard thermodynamic parameters in $2d$. We now turn our attention to the extended phase space. Asymptotically, $f \rightarrow \frac{r^2}{\ell^2}$, and hence the spacetime is AdS_2 with the characteristic length scale ℓ . According to the definitions of the extended phase space, we interpret the pressure as

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2}. \quad (60)$$

Correspondingly, the conjugate volume may be computed from

$$V = \frac{\partial M}{\partial P} \Big|_{\beta, Q} = \frac{4\pi}{3} r_+^3. \quad (61)$$

It is then straightforward to check that the first law and the following Smarr relation hold

$$dM = TdS + \Phi dQ + VdP, \quad (62)$$

$$M = 2(TS - VP) + \Phi Q. \quad (63)$$

The Smarr scaling may be understood by considering the length dimension of mass. From Eq. (47), the mass has dimension $[M] = L$ due to the quadratic dilaton-gravity couplings in the action. This mass dimension is identical to the RN- AdS_4 black hole solution and hence the $d = 2$ theory has a similar Smarr scaling.

The PV equation of state is obtained from expressing the Euclidean period, Eq. (56), in terms of temperature and pressure. In this case, the fluid volume is related to the horizon via $v = 2r_+$ and hence the $d = 2$ equation of state reads

$$P(T, v) = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2\rho^2}{\pi v^4}, \quad (64)$$

which has the critical point

$$P_c = \frac{1}{96\pi \rho^2}, \quad T_c = \frac{1}{3\sqrt{6}\pi \rho}, \quad v_c = 2\sqrt{6}\rho. \quad (65)$$

The ratio $(P_c v_c / T_c) = (3/8)$ precisely coincides with the universal van der Waals ratio, analogous to the RN- AdS_4 case. Hence, we have determined a class of $2d$ charged, black hole solutions to the dilaton-gravity

model [Eq. (40)] which supports nontrivial van der Waals behavior.

Using the definitions $p = \frac{P}{P_c}$, $\nu = \frac{v}{v_c}$, and $\tau = \frac{T}{T_c}$, we have the ‘law of corresponding states’

$$p(\tau, \nu) = \frac{8}{3} \frac{\tau}{\nu} - \frac{2}{\nu^2} + \frac{1}{3\nu^4}. \quad (66)$$

Following the analysis given above, it is straightforward to conclude that the critical exponents are those of mean field theory;

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3. \quad (67)$$

B. CDAdS₃ black holes

We now turn our attention to 3d black hole solutions that result from consistent truncations of the bulk action in 5d, Eq. (38). It was shown in [26] that homogeneous uncharged, AdS_d black strings are permissible geometries that solve the field equations of the action (38) with $\mathcal{L} = (\nabla\Sigma)^2$. The massless scalar field modifies the cosmological constant to support the black string along flat directions. Here, we construct charged, dilatonic, AdS₃ planar black holes using $\mathcal{L} = F^2 + (\nabla\Sigma)^2$. The uplift of this solution to $d = 5$ dimensions corresponds to charged AdS₅ black strings with horizon topologies $S^2 \times \mathbb{R}^1$.

We search for planar 3d black hole solutions with purely electric solutions to the field equations (41)–(43) with the ansatz

$$ds_3^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + h(r)^2dx^2, \quad (68)$$

$$A \equiv A(r)dt, \quad \psi \equiv \psi(r), \quad \Sigma \equiv \Sigma(x). \quad (69)$$

In this case, the equations of motion for the gauge and scalar fields are

$$\nabla_a(\psi^2 F^{ab}) = 0, \quad \nabla_a(\psi^2 \nabla^a \Sigma) = 0, \quad (70)$$

and the corresponding stress-energy contribution is

$$T_{ab}^{\mathcal{L}} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2 - \nabla_a\Sigma\nabla_b\Sigma + \frac{1}{2}g_{ab}(\nabla\Sigma)^2. \quad (71)$$

Given this ansatz, it is straightforward to demonstrate that the equations of motion are satisfied for the solution

$$f(r) = \frac{1}{2\varphi^2} - \frac{\Lambda r^2}{6} - \frac{\mu}{r^2} + \frac{\rho^2}{r^4}, \quad (72)$$

$$\psi(r) = \varphi r, \quad h(r) = r, \quad (73)$$

$$A = -\sqrt{\frac{3}{4}} \frac{\rho}{r^2} dt, \quad \Sigma(x) = \frac{x}{\sqrt{2}\varphi}. \quad (74)$$

Here, the horizon is given by the largest positive root of

$$f(r_+) = \frac{1}{2\varphi^2} + \frac{r_+^2}{\ell^2} - \frac{\mu}{r_+^2} + \frac{\rho^2}{r_+^4} = 0, \quad (75)$$

where we have used $\Lambda = -6/\ell^2$. Note that for a real root, the bound $\mu > 0$ must be saturated. The free parameter is set by demanding $2\varphi^2 = 1$.

As we will demonstrate, the parameters μ and ρ are again respectively related to the black hole mass and charge. When computing the thermodynamics, the presence of a planar horizon leads to infinities. To circumvent this, we consider thermodynamic densities with respect to the planar direction.

We first compute the ADM mass using the background subtraction procedure outlined in [27,28]. The total charge is given by the weighted flux across the planar horizon and the chemical potential is determined by considering the electrostatic potential difference from the horizon and infinity [12]. In 3d, the spherically symmetric horizon allows for a direct computation of both the Bekenstein-Hawking and Wald entropies. We check these quantities against the on-shell Euclidean action and identify the correct entropy before studying the PV behavior.

To determine the ADM mass, we use the background subtraction procedure outlined in [27,28]. In 3d, the mass function reads

$$\mathcal{M}(r) = 2\sqrt{f}(\sqrt{f_{\text{ref}}} - \sqrt{f}) \frac{d}{dr} \left(\frac{\psi^2}{4} \mathcal{A} \right), \quad (76)$$

where $\mathcal{A} = xr$ is the area function of planar horizon. In this case, the reference spacetime

$$ds_{\text{ref}}^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{(1 + \frac{r^2}{\ell^2})} + r^2dx^2 \quad (77)$$

is simply the asymptotically¹ AdS₃ solution obtained by $\mu = \rho = 0$. The ADM mass density, computed at the radial boundary, is

$$m = \lim_{r \rightarrow \infty} \frac{1}{x} \mathcal{M}(r) = \frac{3\mu}{8}. \quad (78)$$

For our purposes, it is important to compute the total charge using a weighted flux across the planar horizon. To motivate this calculation, we consider the charge of the uplifted solution. In 5d, the electric flux across the $S^2 \times \mathbb{R}$

¹Provided the toroidal identification $x \sim x + 2\pi$, the space is locally AdS₃ at the expense of a discontinuity in $\Sigma(x)$.

horizon determines the total charge of an RN-AdS₅ black string. Inserting Eq. (68) into Eq. (39), the total charge reads

$$Q = \frac{1}{4\pi} \int \star F = \frac{1}{4\pi} \int F_{10}(\psi^2 h) d\Omega dx. \quad (79)$$

Integrating out the spherical directions, we interpret the charge density in 3d as

$$q = \frac{1}{x} \int \star(\psi^2 F) = \frac{\sqrt{3}}{2} \rho. \quad (80)$$

The corresponding chemical potential follows the standard definition [12],

$$\Phi = \int_{r_+}^{\infty} dr F_{10} = \sqrt{\frac{3}{4}} \frac{\rho}{r_+^2}. \quad (81)$$

In 3d, we have the freedom to either consider the Bekenstein-Hawking entropy or the Wald entropy. Using the standard formulas, one has the entropy densities

$$s_{\text{BH}} = \frac{1}{x} \frac{\mathcal{A}}{4} = \frac{r_+}{4}, \quad (82)$$

$$s_{\text{Wald}} = \frac{2\pi}{x} \int \left(\frac{\delta \mathcal{L}}{\delta R^{abcd}} \epsilon^{ac} \epsilon^{bd} \right) dx = \frac{\pi r_+^3}{2}. \quad (83)$$

To single out the correct thermal entropy, once again we can appeal to the on-shell Euclidean action. Using the counterterm subtraction method [29], the on-shell Euclidean action in the canonical ensemble reads

$$\bar{\mathcal{I}} = \frac{\beta x}{8\ell^2} \left(\frac{r_+^4 \ell^2 - r_+^6 + 5\rho^2 \ell^2}{r_+^2} \right), \quad (84)$$

where β denotes the period of the Euclidean section

$$\beta = \frac{4\pi}{\partial_r f} \Big|_{r=r_+} = \frac{2\pi r_+^5 \ell^2}{(r_+^4 - \rho^2) \ell^2 + 2r_+^6}. \quad (85)$$

In the fixed charge ensemble, the mass, chemical potential, and entropy are straightforward to compute

$$m = \frac{\partial \bar{\mathcal{I}}}{\partial \beta} \Big|_Q = \frac{3\mu}{8}, \quad (86)$$

$$\Phi = \frac{1}{\beta} \frac{\partial \bar{\mathcal{I}}}{\partial Q} \Big|_\beta = \sqrt{\frac{3}{4}} \frac{\rho}{r_+^2}, \quad (87)$$

$$s = \frac{\partial \bar{\mathcal{I}}}{\partial \beta} \Big|_Q - \bar{\mathcal{I}} = \frac{\pi r_+^3}{2\pi}, \quad (88)$$

where we have used the action density $\bar{\mathcal{I}} = (\bar{\mathcal{I}}/x)$. Comparing these quantities against Eq. (78), Eq. (81), and Eq. (83) respectively confirms the interpretation of the charge and selects the Wald entropy as the correct thermodynamic entropy.

We are now in the position to study PV criticality. Clearly, the spacetime is asymptotically AdS₃ with respect to the length scale ℓ , and hence the pressure in the extended phase space formalism may be identified with

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{4\pi\ell^2}. \quad (89)$$

In the fixed charge ensemble, this invites a reinterpretation of the ADM mass as the enthalpy $m \equiv m(P, S)$ and we can compute the thermodynamic volume

$$V = \frac{\partial m}{\partial P} \Big|_{S,Q} = \pi^2 r_+^4. \quad (90)$$

One can check that the first law and Smarr relation

$$dM = TdS + \Phi dQ + VdP, \quad (91)$$

$$M = \frac{3}{2} TS - VP + \Phi Q, \quad (92)$$

hold with these identifications. The corresponding equation of state may be found from inverting the periodicity, Eq. (85), and using the definition of the fluid volume. In this case, the fluid volume is related to the horizon via $v = 4r_+ \ell_p/3$ resulting in the equation of state

$$P(T, v) = \frac{T}{v} - \frac{2}{3\pi v^2} + \frac{512\rho^2}{243\pi v^6}, \quad (93)$$

where we have set $\ell_p = 1$. The critical point of the equation of state yields the triple

$$P_c = \frac{1}{4\sqrt{15}\pi\rho}, \quad T_c = \frac{4}{5\pi(15\rho^2)^{\frac{1}{4}}}, \quad v_c = 4(5\rho^2/3)^{\frac{1}{4}}, \quad (94)$$

and satisfies the ratio $P_c v_c / T_c = 5/12$. Note that this ratio is also shared by RN-AdS₅ black holes [Eq. (21)]. This suggests that the charged AdS₅ black string uplift of CDAdS₃ black holes belongs in the same universality class as RN-AdS₅ black holes.

Constructing the law of corresponding states follows the same procedure as in higher dimensions. Employing the definitions $p = \frac{P}{P_c}$, $\nu = \frac{v}{v_c}$, and $\tau = \frac{T}{T_c}$, the law of corresponding states can be written as

$$p = \frac{12\tau}{5\nu} - \frac{3}{2\nu^2} + \frac{1}{10\nu^6}, \quad (95)$$

which coincides precisely Eq. (23) for $d = 5$. Hence, we conclude that the critical exponents are again consistent with mean-field theory and this completes the identification of lower-dimensional black holes demonstrating critical behavior.

IV. CONCLUSIONS

Given the lack of lower-dimensional charged black hole solutions admitting van der Waals behavior, we constructed gravity theories in $d = 2, 3$ via consistent truncations and explored the extended phase space of black hole solutions using lower-dimensional techniques and Euclidean methods. We have found classic van der Waals structures in the equation of state of charged, dilatonic, asymptotically AdS black hole solutions in $d = 2, 3$ and confirmed that the critical exponents match those expected from mean field theory.

In $d = 2$, the mass and charge of the CDAdS₂ solution, Eq. (47), were computed with lower-dimensional methods outlined in [22,23]. We found, rather than the standard definition of Bekenstein-Hawking entropy, the Wald entropy yields an expression that agrees with the entropy calculated from the on-shell Euclidean action. Furthermore, this solution has several features reminiscent of RN-AdS₄ black hole solutions of Einstein-Maxwell-AdS₄ theory, especially the similarity between the respective equations of state and the coincident van der Waals ratios. The CDAdS₂ solution, however, does not satisfy the canonical Smarr relation in $d = 2$ but instead saturates the formula with $d = 4$. This is due to the quadratic dilaton-gravity coupling in the action, which raises the mass dimension of the black hole to that of $4d$.

In $d = 3$, searching for black hole solutions required some care. We performed consistent truncations of theories admitting charged AdS₅ black strings with horizon topologies $S^2 \times \mathbb{R}$ along the spherical directions. The dimensionally-reduced theory had an additional scalar field that supported planar charged, asymptotically AdS black strings in $d = 3$. To regulate infinities arising from the planar direction, we computed the thermodynamic densities and confirmed that the mass and charge density have a standard interpretation. Again, the Wald entropy density matches the entropy density computed from the on-shell Euclidean action, confirming that the Wald entropy, rather than the Bekenstein-Hawking entropy, is the correct thermal entropy. Similar to the case in $d = 2$, the canonical Smarr relation was not satisfied with $d = 3$. Instead, the scaling properties of these black hole solutions were identical to RN-AdS₅ black strings. The Smarr scaling in our work contrasts an earlier analysis [20] where new work terms were required to satisfy Smarr laws for Einstein-Maxwell-AdS₃ theory without additional scalar fields. We confirmed that the equation of state contains critical phenomena and the van der Waals ratio is identical to that of RN-AdS₅ black holes. The latter observation

suggests that compactification of black objects with various horizon topologies preserves the van der Waals universality class.

There are several interesting directions to pursue. We were here primarily concerned with searching for rich van der Waals behavior in lower dimensions with a dynamical scalar field. However other phase transitions may be available. Indeed by imposing a constant scalar field, the action reduces to

$$\mathcal{I}[\psi_0] = - \int \left[\frac{\psi_0^2}{4} (R - 2\Lambda - \mathcal{L}) + \frac{\lambda}{2} \right] \sqrt{-g} d^d x, \quad (96)$$

with an effective cosmological constant $\Lambda_{\text{eff}} = \frac{\psi_0^2}{4} \Lambda + \frac{\lambda}{4}$. The action in $d = 3$ has known black hole solutions, namely the charged BTZ solution with a planar horizon. Hence the scalar behaves analogous to an order parameter that supports the CDAdS₃ string phase with pressure corresponding to Λ , or to the charged BTZ₃ string phase with pressure given by Λ_{eff} . It would be fruitful to explore an ensemble which allows for such Landau-Ginzburg phase transitions and study if the CDAdS₃ solution lies within the BTZ mass gap.

Similar arguments can be made for the action in $d = 2$. In this case, it is convenient to recall that the near horizon extremal geometry of an RN-AdS₄ black hole fibers into $\text{AdS}_2 \times S_{\psi_0}^2$, i.e., the Robinson-Bertolli spacetime. Upon compactification, this spacetime should appear as a classical solution to the above action if one treats ψ_0 as a Lagrange multiplier. Again the dynamics of the scalar provides access to different black hole phases of the theory. It would be very interesting to precisely chart out the complete phase space of solutions to the general compactified action (40).

Finally, another possible direction is to consider our approach in $d > 3$. Here several new features are available, such as magnetic black hole solutions, p -form electromagnetism, Gauss-Bonnet and Lovelock sectors of gravity, and novel phase transitions in addition to van der Waals.

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APPENDIX: SPHERICAL REDUCTION

We review the dimensional reduction procedure we employ to obtain Eq. (40) in detail. Consider Einstein gravity coupled to a cosmological constant with additional matter degrees of freedom. The bulk action is

$$\mathcal{I}_{\text{bulk}} = -\frac{1}{16\pi} \int [\tilde{R} - 2\Lambda - \mathcal{L}] \sqrt{-\tilde{g}} d^{d+2}x, \quad (\text{A1})$$

where $\mathcal{L}$ is the matter Lagrangian, $\Lambda = -d(d+1)/2\ell^2$ is the cosmological constant, and $\tilde{g}$ is the metric in $(d+2)$ -dimensions with Ricci curvature $\tilde{R}$. Evaluating the action on shell leads to well-known infinite volume divergences [12,14,29] and hence the theory must be regulated with additional surface terms given by

$$\mathcal{I} = \mathcal{I}_{\text{bulk}}[\tilde{g}] + \mathcal{I}_{\text{surf}}[\tilde{h}] - \mathcal{I}_{\text{ct}}[\tilde{h}], \quad (\text{A2})$$

where the induced metric on the boundary is defined by $\tilde{h}_{MN} = \tilde{g}_{MN} - \tilde{n}_M \tilde{n}_N$ with $\tilde{n}_M$ being the outward pointing unit normal. The remaining surface and counterterm actions (up to $d=6$) are defined on a spacelike hypersurface via

$$\mathcal{I}_{\text{surf}} = -\frac{1}{8\pi} \int [\tilde{\mathcal{K}} + 2\tilde{n}_M F^{MN} A_N] \sqrt{-\tilde{h}} d^{d+1}x, \quad (\text{A3})$$

$$\mathcal{I}_{\text{ct}} = -\frac{1}{8\pi} \int \left[\frac{d}{\ell} + \frac{\ell c_1(d)}{2(d-1)} \tilde{\mathcal{R}} + \frac{\ell^3 c_2(d)}{2(d-1)^2} \right. \\ \left. \times \left(\tilde{\mathcal{R}}^{MN} \tilde{\mathcal{R}}_{MN} - \frac{d+1}{4d} \tilde{\mathcal{R}}^2 \right) \right] \sqrt{-\tilde{h}} d^{d+1}x, \quad (\text{A4})$$

where $\tilde{\mathcal{K}}_{MN} = (1/2)\tilde{n}_{(M}\tilde{n}_{N)}$ and $\tilde{\mathcal{R}}_{MN}$ are the extrinsic and Ricci curvatures of the boundary. The coefficients $c_i(d)$ are present to regulate Casimir contributions when computing the on-shell action for black string geometries.

We will consider metrics of the form

$$ds_{d+2}^2 = \tilde{g}_{MN} dx^M dx^N = g_{ab} dx^a dx^b + \psi(x^a)^2 d\Omega_2^2 \quad (\text{A5})$$

and study compactifications of the round S^2 . Note that these metrics are solutions to Eq. (A1). To see this, consider first $d=2$ and $\mathcal{L} = F^2$. Then the static, spherically symmetric, charged black hole solution is the RN-AdS₄ with $\psi = r$. Note that when $d=3$, the horizon has the topology $S^2 \times S^1$ or $S^2 \times \mathbb{R}$ corresponding to a closed or extended black string. Recently, homogeneous asymptotically AdS _{n} $\times \mathbb{R}^p$ black strings have been stabilized using p minimally-coupled scalar fields that depend on the extended p -directions [26]. We consider here $p=1$ in order to support inhomogeneous, asymptotically AdS,

charged black strings in $d=3$ which admit horizon topologies $S^2 \times \mathbb{R}$ (or S^1).

We are now in a position to perform a consistent truncation of the theory along the S^2 of the horizon to d -dimensions. To do so, one further requires the data

$$\sqrt{-\tilde{g}} = \psi^2 \det(\Omega_2) \sqrt{-g}, \quad (\text{A6})$$

$$\tilde{R} = R + \frac{2}{\psi^2} - \frac{4\Box\psi}{\psi} - \frac{2(\nabla\psi)^2}{\psi^2}, \quad (\text{A7})$$

$$\sqrt{-\tilde{h}} = \psi^2 \det(\Omega_2) \sqrt{-h}, \quad (\text{A8})$$

$$\tilde{\mathcal{K}} = \mathcal{K} + \frac{2}{\psi} \sqrt{-g} \tilde{n}^a \nabla_a \psi, \quad (\text{A9})$$

$$\tilde{\mathcal{R}}^{MN} \tilde{\mathcal{R}}_{MN} = \frac{1}{2} \tilde{\mathcal{R}}^2 = \frac{2}{\psi^4}, \quad (\text{A10})$$

where the derivative operators are computed on the base g_{ab} , $\mathcal{K} = n_{,a}^a$, and $\tilde{n}_a = n_a$. Using this data, the spherical directions and the surface term in $\tilde{R}$ can be integrated away. The effective bulk theory is

$$\mathcal{I}_{\text{bulk}} = - \int \left[\frac{1}{2} (\nabla\psi)^2 + \frac{\psi^2}{4} (R - 2\Lambda - \mathcal{L}) + \frac{\lambda}{2} \right] \sqrt{-g} d^d x, \quad (\text{A11})$$

where $\lambda=1$ is related to the curvature of the round unit S^2 . The remaining surface and counterterm actions

$$\mathcal{I}_{\text{surf}} = -\frac{1}{2} \int \psi^2 [\mathcal{K} + 2n_a F^{ab} A_b] \sqrt{-h} d^{d-1}x, \quad (\text{A12})$$

$$\mathcal{I}_{\text{ct}} = -\frac{1}{2} \int \psi^2 \left[\frac{d}{\ell} + \frac{\ell}{(d-1)} \frac{c_1(d)}{\psi^2} \right. \\ \left. + \frac{\ell^3}{2d(d-1)^2} \frac{c_2(d)}{\psi^4} \right] \sqrt{-h} d^{d-1}x, \quad (\text{A13})$$

guarantee that the total action is finite when computing the value on-shell. For $d=2, 3$, the coefficients take the form

$$c_1(d) = \frac{d}{2}, \quad c_2(d) = -\left(\frac{d-2}{2}\right) \left(\frac{d}{d-1}\right)^2. \quad (\text{A14})$$

This choice naturally eliminates any additional Casimir contributions along the noncompact directions for $d \leq 6$.

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