Hypersymplectic geometry and supersymmetric solutions in (t, s) 5D supergravity

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Relying on the method of spinorial geometry, purely bosonic supersymmetric solutions in $N = 2$, five-dimensional supergravity theories coupled to vector multiplets in all space-time signatures are found. Explicit examples of some new solutions are presented.

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I. INTRODUCTION

The five-dimensional $N = 2$ supergravity theories with Lorentzian signature coupled to vector multiplets was constructed many years ago in [1]. In recent years, there has been some interest in supergravity theories in various space-time signatures. The Euclidean versions of the supergravity theories of [1] were considered in [2], where it was demonstrated that the Lagrangian of the Euclidean theory has the kinetic terms of the gauge fields with the nonconventional sign. Euclidean $N = 2$ theories in four dimensions were first considered in [3–6]. The Euclidean four-dimensional $N = 2$ supergravity theories were obtained as dimensional reductions of $N = 2$, $D = 5$ supergravity theories on a timelike circle. The reduction of the five-dimensional Euclidean theory on a circle produces Euclidean $N = 2$ four-dimensional supergravity with the nonconventional signs of the gauge fields kinetic terms. The four-dimensional Euclidean supergravity theory with vector and hypermultiplets was also obtained via the dimensional reduction of Euclidean ten-dimensional type IIA supergravity over a Calabi-Yau threefold, CY_3 [7]. A class of Lorentzian five-dimensional $N = 2$ supergravity theories constructed in [1] is obtainable via the dimensional reduction of the standard eleven-dimensional supergravity [8] on a CY_3 [9]. Recently, in [10], $N = 2$ four and five-dimensional supergravity theories in space-time signatures (t, s) , where t and s are respectively the number of time and spatial dimensions, were constructed by reducing Hull's eleven-dimensional supergravity [11] on CY_3 . For a detailed analysis on supersymmetry algebras in arbitrary space-time dimension and signature we refer the reader to [12].

The eleven dimensional supergravity theories of Hull with space-time signatures $(1, 10)$, $(5, 6)$ and $(9, 2)$ have actions with the standard conventional sign for the 3-form gauge kinetic term. The mirror theories with signatures $(10, 1)$, $(6, 5)$, and $(2, 9)$ all have the non-conventional sign for the 3-form gauge fields kinetic terms. In the reduction of the theories with signatures $(1, 10)$, $(5, 6)$, and $(2, 9)$, the CY_3 is taken to be of signature $(0, 6)$. For the reduction of theories with signatures $(10, 1)$, $(6, 5)$, and $(9, 2)$, the CY_3 is of signature $(6, 0)$.

By employing the methods of [13], a systematic classification of supersymmetric solutions of the $(1, 4)$ five-dimensional minimal supergravity was given in [14]. In this approach, the existence of at least one Killing spinor is assumed and differential forms as bilinears in terms of this spinor are constructed. The algebraic and differential constraints satisfied by the bilinears can be used to fix the solution of the space-time metric in addition to the bosonic fields of the supersymmetric solution. It was found in [14] that half-supersymmetric solutions with time-like Killing vectors have a four-dimensional base space given by a hyper-Kähler manifold. These findings for the time-like solutions were generalized to supergravity theories coupled to arbitrary many Abelian vector multiplets in [15] where also a uniqueness theorem for asymptotically flat supersymmetric black holes with regular horizons was given.

The goal of our present work is the generalization of the results unique to all $N = 2$, five-dimensional supergravity theories coupled to vector multiplets in all space-time signatures. The Killing spinor equations shall be analysed using the spinorial geometry methods which were first employed in the analysis of supersymmetric solutions in ten and eleven dimensions in [16]. Spinorial geometry [17] has been very useful and efficient in the classifications of solutions with various fractions of supersymmetry in all space-time dimensions [18].

We organize our work as follows. In Sec. II, a review of some of the basic properties of the ungauged

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five-dimensional supergravity coupled to arbitrary many vector multiplets is given. Section III contains the analysis of supersymmetric solutions where the set of rules for the construction of these solutions is given. Some examples and a summary are given in Sec. IV.

II. (t, s) FIVE-DIMENSIONAL SUPERGRAVITY

Ignoring hypermultiplets, the bosonic action of the theory for all $N = 2$, $D = 5$ supergravity contains the gravity multiplet and vector multiplets and is given by [1,10]

$$S_5 = \int_{M_5} \frac{1}{2} R \star 1 - \frac{1}{2} Q_{IJ}(X) dX^I \wedge \star dX^J + \frac{\kappa^2}{2} Q_{IJ}(X) F^I \wedge \star F^J - \frac{1}{12} C_{IJK} A^I \wedge F^J \wedge F^K \quad (2.1)$$

where C_{IJK} are real constants symmetric in I, J, K . We have $\kappa^2 = -1$, for signature (1,4), (5,0) and (3,2) theories and $\kappa^2 = 1$ for signature (4,1), (0,5), and (2,3). Here F^I are two-forms representing the gauge fields. The information about the theory is encoded in the cubic prepotential which describes very special geometry

$$\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1, \quad (2.2)$$

X^I being the very special coordinates, functions of the n real scalar fields belonging to the vector multiplets.

The gauge coupling metric can be derived from the prepotential and is given

$$Q_{IJ} = -\frac{1}{2} (\partial_{X^I} \partial_{X^J} (\ln \mathcal{V}))_{\mathcal{V}=1} = \frac{1}{2} (9X_I X_J - C_{IJK} X^K), \quad (2.3)$$

where the dual fields X_I are defined by

$$X_I = \frac{1}{6} C_{IJK} X^J X^K. \quad (2.4)$$

We also have the useful relations

$$Q_{IJ} X^J = \frac{3}{2} X_I, \quad Q_{IJ} dX^J = -\frac{3}{2} dX_I. \quad (2.5)$$

The Killing spinor equations associated with the above theories are given

$$\left[\nabla_\mu + \frac{\kappa}{8} H_{\rho\sigma} (\Gamma_\mu \Gamma^{\rho\sigma} - 6\delta_\mu^\rho \Gamma^\sigma) \right] \epsilon = 0 \quad (2.6)$$

and

$$(\kappa G_{\mu\nu}^I \Gamma^{\mu\nu} - 2\partial_\mu X^I \Gamma^\mu) \epsilon = 0, \quad (2.7)$$

where

$$\begin{aligned} G_{\mu\nu}^I &= F_{\mu\nu}^I - X^I X_J F_{\mu\nu}^J, \\ \nabla_\mu &= \partial_\mu + \frac{1}{4} \omega_{\mu,\rho\sigma} \Gamma^{\rho\sigma}, \\ H_{\mu\nu} &= X_I F_{\mu\nu}^I. \end{aligned} \quad (2.8)$$

Here Γ_μ are Dirac matrices and $\omega_{\mu,\rho\sigma}$ are the spin connections. For the supergravity theories with space-time signature (1,4), (3,2), and (5,0), we have $\kappa = -i$. For the supergravity theories with space-time signatures (4,1), (2,3), and (0,5), we have $\kappa = 1$.

III. SUPERSYMMETRIC SOLUTIONS

In what follows, we find solutions admitting Killing spinors through the analysis of the Killing spinor equations (2.6) and (2.7) using spinorial geometry methods. We take the Dirac spinors to be the space of complex forms on $\mathbb{R}^2$ spanned over $\mathbb{C}$ by 1, e_1 , e_2 and $e_{12} = e_1 \wedge e_2$. To proceed in the analysis of solutions admitting Killing spinors, we start by writing our metric solutions in the form

$$\begin{aligned} ds_5^2 &= \kappa^2 (\mathbf{e}^5)^2 + \eta_{\alpha\bar{\beta}} \mathbf{e}^\alpha \mathbf{e}^{\bar{\beta}} \\ &= \kappa^2 (\mathbf{e}^5)^2 + 2(\kappa_1^2 \mathbf{e}^1 \mathbf{e}^{\bar{1}} + \kappa_2^2 \mathbf{e}^2 \mathbf{e}^{\bar{2}}) \end{aligned} \quad (3.1)$$

where κ^2 , κ_1^2 and κ_2^2 are chosen to be ± 1 , depending on the space-time signature of the considered theory. For example, if we are considering the supergravity theories with (2,3) signature, we take $\kappa^2 = \kappa_1^2 = 1$, $\kappa_2^2 = -1$ or alternatively $\kappa^2 = \kappa_2^2 = 1$, $\kappa_1^2 = -1$.

The action of the Γ -matrices on spinors is given by

$$\begin{aligned} \Gamma_1 &= \kappa_1 \sqrt{2} e^1 \wedge, & \Gamma_{\bar{1}} &= \kappa_1 \sqrt{2} i_{e^1}, \\ \Gamma_2 &= \sqrt{2} \kappa_2 e^2 \wedge, & \Gamma_{\bar{2}} &= \sqrt{2} \kappa_2 i_{e^2}, \\ \Gamma_5 1 &= \kappa 1, & \Gamma_5 e_1 &= -\kappa e_1, \\ \Gamma_5 e_2 &= -\kappa e_2, & \Gamma_5 e_{12} &= \kappa e_{12}. \end{aligned}$$

We shall find solutions for the Killing spinor $\epsilon = f1$. This Killing spinor orbit corresponds to timelike solutions in the standard supergravity models with signature (1,4). Plugging $\epsilon = f1$ in the Killing spinor equation (2.6), we obtain the following conditions

$$\begin{aligned}
 \partial_\alpha \log f + \frac{1}{2} \omega_{\alpha, \mu}^\mu - \frac{3}{4} H_{\alpha 5} &= 0, \\
 \partial_{\bar{\alpha}} \log f + \frac{1}{2} \omega_{\bar{\alpha}, \mu}^\mu - \frac{1}{4} H_{\bar{\alpha} 5} &= 0, \\
 \partial_5 \log f + \frac{1}{2} \omega_{5, \mu}^\mu + \frac{1}{4} \kappa^2 H_\mu^\mu &= 0, \\
 \kappa^2 \omega_{1, \bar{1} 5} + \frac{1}{2} \kappa_1^2 H_\mu^\mu - \frac{3}{2} H_{1 \bar{1}} &= 0, \\
 \kappa^2 \omega_{2, \bar{2} 5} + \frac{1}{2} \kappa_2^2 H_\mu^\mu - \frac{3}{2} H_{2 \bar{2}} &= 0, \\
 \kappa^2 \omega_{2, \bar{1} 5} - \frac{3}{2} H_{2 \bar{1}} &= 0, \\
 \kappa^2 \omega_{1, \bar{2} 5} - \frac{3}{2} H_{1 \bar{2}} &= 0, \\
 \omega_{2, \bar{1} 5} - \frac{1}{2} \kappa^2 H_{2 \bar{1}} &= 0, \\
 \omega_{1, \bar{2} 5} - \frac{\kappa^2}{2} H_{1 \bar{2}} &= 0, \\
 \omega_{1, \bar{1} \bar{2}} + \frac{1}{2} \kappa_1^2 H_{2 5} &= 0, \\
 \omega_{2, \bar{1} \bar{2}} - \frac{1}{2} \kappa_2^2 H_{1 5} &= 0, \\
 \omega_{\alpha, \beta \gamma} &= 0, \\
 \omega_{\alpha, \alpha 5} &= 0, \\
 \omega_{5, \bar{1} 5} - \kappa^2 H_{5 \bar{1}} &= 0, \\
 \omega_{5, \bar{2} 5} - \kappa^2 H_{5 \bar{2}} &= 0, \\
 \omega_{5, \bar{1} \bar{2}} + \frac{1}{2} \kappa^2 H_{1 \bar{2}} &= 0.
 \end{aligned} \tag{3.2}$$

The analysis of this linear system of equations implies the following conditions

$$\begin{aligned}
 \partial_5 f &= 0, \\
 \omega_{5, S \alpha} - 2\kappa^2 \partial_\alpha \log f &= 0, \\
 \omega_{5, \alpha \beta} + \omega_{\alpha, \beta 5} &= 0, \\
 \omega_{\bar{\beta}, \alpha 5} + \omega_{\alpha, \bar{\beta} 5} &= 0, \\
 \omega_{5, \mu}^\mu - \omega_{\mu, 5}^\mu &= 0, \\
 \omega_{\alpha, \mu}^\mu - \partial_\alpha \log f &= 0, \\
 \omega_{\alpha, \beta \gamma} &= 0, \\
 \omega_{\alpha, \bar{\mu} \bar{\nu}} + \eta_{\alpha \bar{\mu}} \partial_{\bar{\nu}} \log f - \eta_{\alpha \bar{\nu}} \partial_{\bar{\mu}} \log f &= 0,
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 H_{\alpha \beta} &= -2\kappa^2 \omega_{5, \alpha \beta}, \\
 H_{\alpha 5} &= 2\partial_\alpha \log f, \\
 H_{\alpha \bar{\beta}} &= \frac{2}{3} \kappa^2 (\omega_{\alpha, \bar{\beta} 5} - \eta_{\alpha \bar{\beta}} \omega_{5, \mu}^\mu), \\
 H_\mu^\mu &= -2\kappa^2 \omega_{5, \mu}^\mu.
 \end{aligned} \tag{3.4}$$

The analysis of (2.7) gives the conditions

$$\begin{aligned}
 F_\mu^I &= X^I H_\mu^\mu, \\
 F_{5\alpha}^I &= X^I H_{5\alpha} - \partial_\alpha X^I, \\
 F_{\alpha \beta}^I &= X^I H_{\alpha \beta}, \\
 \partial_5 X^I &= 0.
 \end{aligned} \tag{3.5}$$

To proceed, we define the 1-form

$$V = f^2 \mathbf{e}^5, \tag{3.6}$$

and introduce the coordinate τ such that the dual vector field is given by $\kappa^2 f^2 \frac{\partial}{\partial \tau}$. The first four conditions in (3.3) provide the necessary and sufficient conditions for V to define a Killing vector. Also those conditions imply that

$$\mathcal{L}_V \mathbf{e}^5 = 0. \tag{3.7}$$

Furthermore one finds

$$\begin{aligned}
 \mathcal{L}_V \mathbf{e}^1 &= -\kappa_1^2 f^2 [(\omega_{5, \bar{1} 1} - \omega_{1, \bar{1} 5}) \mathbf{e}^1 + (\omega_{5, \bar{1} 2} - \omega_{2, \bar{1} 5}) \mathbf{e}^2] \\
 \mathcal{L}_V \mathbf{e}^2 &= -\kappa_2^2 f^2 [(\omega_{5, \bar{2} 1} - \omega_{1, \bar{2} 5}) \mathbf{e}^1 + (\omega_{5, \bar{2} 2} - \omega_{2, \bar{2} 5}) \mathbf{e}^2]
 \end{aligned} \tag{3.8}$$

By making an appropriate gauge transformation as discussed in [19], we can set

$$\mathcal{L}_V \mathbf{e}^\alpha = 0. \tag{3.9}$$

We can choose coordinates such that

$$\mathbf{e}^5 = f^2 (d\tau + w), \quad \mathbf{e}^\alpha = f^{-1} \mathbf{E}^\alpha. \tag{3.10}$$

where the function f , the one-form w and $\mathbf{E}^\alpha$ are all independent of the coordinate τ . At this stage, we define the following three two-forms:

$$\begin{aligned}
 \mathbf{J}_1 &= \mathbf{E}^1 \wedge \mathbf{E}^2 + \mathbf{E}^{\bar{1}} \wedge \mathbf{E}^{\bar{2}}, \\
 \mathbf{J}_2 &= -i(\mathbf{E}^1 \wedge \mathbf{E}^2 - \mathbf{E}^{\bar{1}} \wedge \mathbf{E}^{\bar{2}}), \\
 \mathbf{J}_3 &= i(\kappa_1^2 \mathbf{E}^1 \wedge \mathbf{E}^{\bar{1}} + \kappa_2^2 \mathbf{E}^2 \wedge \mathbf{E}^{\bar{2}}).
 \end{aligned} \tag{3.11}$$

It can be shown that

$$d\mathbf{J}_1 = d\mathbf{J}_2 = d\mathbf{J}_3 = 0$$

provided

$$\Omega_{\alpha,\bar{\mu}\bar{\nu}} = 0, \quad \Omega_{\alpha,\beta\gamma} = 0, \quad \Omega_{\alpha,\mu}{}^\mu = 0, \quad (3.12)$$

where Ω represent the spin connections of the base manifold with vielbeins $\mathbf{E}^\alpha$. In fact, the conditions in (3.12) are implied by the last three conditions of (3.3). Moreover, $\mathbf{J}_i$, $i = 1, 2, 3$, are covariantly constant two-forms on the base manifold. They also satisfy the following algebra

$$\mathbf{J}_1^2 = \mathbf{J}_2^2 = -\kappa_1^2 \kappa_2^2, \quad \mathbf{J}_3^2 = -1, \quad \mathbf{J}_1 \mathbf{J}_2 = -\mathbf{J}_2 \mathbf{J}_1 = -\kappa_1^2 \kappa_2^2 \mathbf{J}_3. \quad (3.13)$$

For $\kappa_1^2 \kappa_2^2 = -1$, relevant for theories with space-time signatures (2,3) and (3,2), the algebra (3.13) is that of para-quaternions or the so-called split quaternions [20]. We shall refer to the base manifold with such a structure as hyper-symplectic [21]. For the cases with $\kappa_1^2 = \kappa_2^2 = \pm 1$, relevant for space-time signatures (1,4), (4,1), (5,0) and (0,5), the algebra (3.13) defines the algebra of quaternions.

We now turn back to the analysis of the gauge fields. Using (3.4), we have

$$H = -2\kappa^2 \omega_{5,\alpha\bar{\beta}} \mathbf{e}^\alpha \wedge \mathbf{e}^{\bar{\beta}} + 2\partial_\alpha \log f \mathbf{e}^\alpha \wedge \mathbf{e}^5 + \frac{2}{3} \kappa^2 (\omega_{\alpha,\bar{\beta}5} - \eta_{\alpha\bar{\beta}} \omega_{5,\mu}^\mu) \mathbf{e}^\alpha \wedge \mathbf{e}^{\bar{\beta}}. \quad (3.14)$$

Noting that

$$\begin{aligned} \kappa^2 d\mathbf{e}^5 &= -2\kappa^2 \mathbf{e}^5 \wedge d \log f \\ &+ 2(\omega_{1,5\bar{1}} \mathbf{e}^{\bar{1}} \wedge \mathbf{e}^1 + \omega_{2,5\bar{2}} \mathbf{e}^{\bar{2}} \wedge \mathbf{e}^2) \\ &+ 2(\omega_{\bar{1},52} \mathbf{e}^2 + \omega_{\bar{1},5\bar{2}} \mathbf{e}^{\bar{2}}) \wedge \mathbf{e}^{\bar{1}} \\ &+ 2(\omega_{1,52} \mathbf{e}^2 + \omega_{1,5\bar{2}} \mathbf{e}^{\bar{2}}) \wedge \mathbf{e}^1. \end{aligned} \quad (3.15)$$

We obtain

$$\begin{aligned} H - d\mathbf{e}^5 &= -\frac{2}{3} \kappa^2 (\omega_{1,5\bar{1}} \mathbf{e}^{\bar{1}} \wedge \mathbf{e}^1 + \omega_{2,5\bar{2}} \mathbf{e}^{\bar{2}} \wedge \mathbf{e}^2 \\ &+ 2\omega_{1,5\bar{2}} \mathbf{e}^{\bar{2}} \wedge \mathbf{e}^1 + 2\omega_{\bar{1},52} \mathbf{e}^2 \wedge \mathbf{e}^{\bar{1}}) \\ &- \frac{2}{3} \kappa^2 \kappa_1^2 \kappa_2^2 (\omega_{5,2\bar{2}} \mathbf{e}^1 \wedge \mathbf{e}^{\bar{1}} + \omega_{5,1\bar{1}} \mathbf{e}^2 \wedge \mathbf{e}^{\bar{2}}). \end{aligned} \quad (3.16)$$

The right hand of the above equation can be expressed in terms of the self-dual part of dw , and we have

$$H = d\mathbf{e}^5 - \frac{f^2}{3} (dw + *dw), \quad (3.17)$$

where our orientation is such that $\epsilon_{1\bar{1}2\bar{2}} = \kappa_1^2 \kappa_2^2$. If we write

$$f^2 dw = G_+ + G_-, \quad (3.18)$$

then we have

$$H = d\mathbf{e}^5 + \Psi$$

with

$$\Psi = -\frac{2}{3} G_+ \quad (3.19)$$

and thus Ψ is a self-dual 2-form on the base manifold. Using (3.5), we find

$$F^I = d(X^I \mathbf{e}^5) + \Psi^I \quad (3.20)$$

where $\Psi = X_I \Psi^I$. The Bianchi identity then implies

$$d\Psi^I = 0 \quad (3.21)$$

and thus Ψ^I are harmonic self-dual 2-forms on the base manifold with metric $ds_4^2 = \eta_{\alpha\bar{\beta}} \mathbf{e}^\alpha \mathbf{e}^{\bar{\beta}}$. Turning to Maxwell equations

$$d(Q_{IJ} * F^J) = \frac{\kappa^2}{4} C_{IJK} F^J \wedge F^K \quad (3.22)$$

we obtain after some calculation

$$\nabla^2 (f^{-2} X_I) = -\frac{\kappa^2}{6} C_{IJK} \Psi^J \cdot \Psi^K \quad (3.23)$$

where the Laplacian is for the metric $ds_4^2 = \eta_{\alpha\bar{\beta}} \mathbf{E}^\alpha \mathbf{E}^{\bar{\beta}}$ and we have the convention that for two p -forms α and β , we have

$$\alpha \cdot \beta = \frac{1}{p!} \alpha_{n_1 \dots n_p} \beta^{n_1 \dots n_p}. \quad (3.24)$$

Finally we note that the integrability conditions for the Killing spinor equations together with imposing the Bianchi identity and the equations of motion for the gauge fields guarantee that all the equations of motion are satisfied.

IV. EXAMPLES AND DISCUSSION

In [14] solutions for the minimal case (no vector multiplets) with base space $\mathbb{R}^4$ were constructed. The four dimensional base metric can be expressed in terms of the left or right invariant forms of $SU(2)$ given in terms of Euler angles. The right invariant one forms are given by

$$\begin{aligned} \sigma_1 &= \sin \phi d\theta - \cos \phi \sin \theta d\psi, \\ \sigma_2 &= \cos \phi d\theta + \sin \theta \sin \phi d\psi, \quad \sigma_3 = d\phi + \cos \theta d\psi \end{aligned} \quad (3.25)$$

and the left invariant ones are given by

$$\begin{aligned}\chi_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\ \chi_2 &= \cos\psi d\theta + \sin\theta \sin\psi d\phi, \\ \chi_3 &= d\psi + \cos\theta d\phi\end{aligned}\quad (3.26)$$

satisfying

$$d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k, \quad d\chi_i = \frac{1}{2}\epsilon_{ijk}\chi_j \wedge \chi_k. \quad (3.27)$$

In terms of these forms, the flat four-dimensional metric can be written in the form

$$\begin{aligned}ds_4^2 &= dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &= dr^2 + \frac{r^2}{4}(\chi_1^2 + \chi_2^2 + \chi_3^2) \\ &= dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2).\end{aligned}\quad (3.28)$$

Defining

$$\mathbf{E}^1 = \frac{1}{\sqrt{2}}(e^0 + ie^3) \quad \mathbf{E}^2 = \frac{1}{\sqrt{2}}(e^2 + ie^1) \quad (3.29)$$

with

$$e^0 = dr, \quad e^1 = \frac{1}{2}r\sigma_1, \quad e^2 = \frac{1}{2}r\sigma_2, \quad e^3 = \frac{1}{2}r\sigma_3, \quad (3.30)$$

the three complex structures are then can be given by

$$\begin{aligned}\mathbf{J}_1 &= e^0 \wedge e^1 - e^2 \wedge e^3 = \frac{1}{4}d(r^2\sigma_1), \\ \mathbf{J}_2 &= e^0 \wedge e^2 + e^1 \wedge e^3 = \frac{1}{4}d(r^2\sigma_2), \\ \mathbf{J}_3 &= e^0 \wedge e^3 - e^1 \wedge e^2 = \frac{1}{4}d(r^2\sigma_3).\end{aligned}\quad (3.31)$$

For solutions with neutral flat base space, we can express the four-dimensional base metric in terms of the forms

$$\begin{aligned}\sigma'_1 &= \sin\phi d\theta - \cos\phi \sinh\theta d\psi, \\ \sigma'_2 &= \cos\phi d\theta + \sinh\theta \sin\phi d\psi, \\ \sigma'_3 &= d\phi + \cosh\theta d\psi\end{aligned}\quad (3.32)$$

or

$$\begin{aligned}\chi'_1 &= -\sin\psi d\theta + \cos\psi \sinh\theta d\phi, \\ \chi'_2 &= \cos\psi d\theta + \sinh\theta \sin\psi d\phi, \\ \chi'_3 &= d\psi + \cosh\theta d\phi\end{aligned}\quad (3.33)$$

satisfying

$$d\sigma'_i = -\frac{1}{2}f_{ijk}\sigma'_j \wedge \sigma'_k, \quad d\chi'_i = \frac{1}{2}f_{ijk}\chi'_j \wedge \chi'_k \quad (3.34)$$

where f_{ijk} are the structure constants of $SO(2,1)$. The metric takes the form

$$\begin{aligned}ds_4^2 &= dr^2 + \frac{r^2}{4}(-\sigma_1'^2 - \sigma_2'^2 + \sigma_3'^2) \\ &= dr^2 + \frac{r^2}{4}(-\chi_1'^2 - \chi_2'^2 + \chi_3'^2) \\ &= dr^2 + \frac{r^2}{4}(-d\theta^2 - \sinh^2\theta d\phi^2 + (d\psi + \cosh\theta d\phi)^2).\end{aligned}\quad (3.35)$$

In this case the three two-forms satisfying the hypersymplectic algebra can be given by

$$\mathbf{J}_i = \frac{1}{4}d(r^2\sigma'_i), \quad i = 1, 2, 3. \quad (3.36)$$

As an example, we consider the STU model with space-time signatures (2,3) and (3,2) described by the prepotential $\mathcal{V} = X^1 X^2 X^3$. The solutions with signatures (1,4) were considered in [15,22]. The metric is given by the general form

$$ds_5^2 = \kappa^2 f^4 (d\tau + w)^2 + f^{-2} \left[dr^2 + \frac{r^2}{4} (-d\theta^2 - \sinh^2\theta d\phi^2 + (d\psi + \cosh\theta d\phi)^2) \right] \quad (3.37)$$

where $\kappa^2 = 1$ for solutions with (2,3) signature and $\kappa^2 = -1$ for solutions with (3,2) signature. We consider the simple case with vanishing Ψ^I in (3.23) which implies that $f^{-2}X_I$ are given in terms of harmonic functions H_I on the base space and we obtain

$$\begin{aligned}f^{-6} &= H_1 H_2 H_3 \quad X^1 = \left(\frac{H_3 H_2}{H_1^2} \right)^{1/3}, \\ X^2 &= \left(\frac{H_3 H_1}{H_2^2} \right)^{1/3}, \quad X^3 = \left(\frac{H_2 H_1}{H_3^2} \right)^{1/3}.\end{aligned}\quad (3.38)$$

The gauge fields are given by

$$F^I = d(X^I f^2 (d\tau + w)). \quad (3.39)$$

As $G_+ = 0$, we obtain from (3.18) that dw is anti-self-dual and we can set

$$w = \frac{J}{r^2}(d\phi + \cosh \theta d\psi) \quad (3.40)$$

with a constant J .

One can also consider solutions with neutral base given by an analytic continuation of the Eguchi-Hanson metric given by [23]

$$ds_4^2 = W^{-1}dr^2 + \frac{r^2}{4}(-\sigma_1^2 - \sigma_2^2 + W\sigma_3^2) \quad (3.41)$$

with

$$W = 1 - \frac{a^4}{r^4}.$$

In this case, the hypersymplectic structure is defined by

$$\mathbf{J}_i = d\left(\frac{1}{4}r^2 W^{1/2} \sigma'_i\right). \quad (3.42)$$

One can also have analytic continuations of the general hyper-Kähler N -multi-centered Gibbons-Hawking metrics which admit triholomorphic Killing vector field [24,25] and obtain hypersymplectic metrics. Recall that these metrics are described by

$$ds^2 = V^{-1}(dx^4 + \theta) + V((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \\ \nabla \times \theta = \nabla V, \quad V = \eta + \sum_{i=1}^N \frac{\varrho}{|x - x_i|} \quad (3.43)$$

where η and ϱ are constants. The triholomorphic Killing vector is ∂_{x^4} and $\theta = \theta_i dx^i$. For $\eta = 1$ and $N = 0$, we obtain flat space and for $\eta = 0$ and $N = 1$, we obtain Eguchi-Hanson metric. One can analytically continue the metrics (3.43) and obtain hypersymplectic metrics. For example we can consider the metrics

$$ds^2 = V^{-1}(dx^4 + \theta) + V(-(dx^1)^2 - (dx^2)^2 + (dx^3)^2)$$

with the hypersymplectic structure given by

$$\begin{aligned} \mathbf{J}_1 &= (dx^4 + \theta) \wedge dx^1 - V dx^2 \wedge dx^3, \\ \mathbf{J}_2 &= (dx^4 + \theta) \wedge dx^2 - V dx^3 \wedge dx^1, \\ \mathbf{J}_3 &= (dx^4 + \theta) \wedge dx^3 + V dx^1 \wedge dx^2. \end{aligned} \quad (3.44)$$

All the solutions considered in [14] which included generalizations of BMPV black hole solutions [26], rotating Eguchi-Hanson and Taub-NUT solutions and solutions with Gibbons-Hawking base space can be analytically continued to obtain solutions with neutral bases. The

continued hypersymplectic manifold will inherit the Killing fields of the Euclidean metric [23]. However it must be emphasized that neutral manifolds are less rigid than Riemannian manifolds. For example, Killing vectors with zero norms can not exist in the Riemannian case. Not all neutral hypersymplectic metrics can be obtained by analytic continuations.

In general, hypersymplectic metrics can be written in terms of one function in the form

$$ds^2 = \frac{\partial^2 Y}{\partial x \partial w} dx dw + \frac{\partial^2 Y}{\partial y \partial z} dy dz \\ + \frac{\partial^2 Y}{\partial y \partial w} dy dw + \frac{\partial^2 Y}{\partial x \partial z} dx dz \quad (3.45)$$

where the function Y satisfies the so-called the first Heavenly equation [27,28]

$$\frac{\partial^2 Y}{\partial x \partial w} \frac{\partial^2 Y}{\partial y \partial z} - \frac{\partial^2 Y}{\partial y \partial w} \frac{\partial^2 Y}{\partial x \partial z} = 1. \quad (3.46)$$

An alternative representation of hypersymplectic metrics is given by

$$ds^2 = dy \left(dw - \frac{\partial^2 S}{\partial x^2} dy - \frac{\partial^2 S}{\partial w \partial x} dz \right) \\ - dz \left(dx + \frac{\partial^2 S}{\partial w^2} dz + \frac{\partial^2 S}{\partial w \partial x} dy \right) \quad (3.47)$$

with S satisfying the so-called second Heavenly equation [28]

$$\frac{\partial^2 S}{\partial w \partial y} - \frac{\partial^2 S}{\partial z \partial x} + \frac{\partial^2 S}{\partial w^2} \frac{\partial^2 S}{\partial x^2} - \left(\frac{\partial^2 S}{\partial x \partial w} \right)^2 = 0. \quad (3.48)$$

Many interesting four-dimensional hypersymplectic metrics with various types of Killing vectors such as null Killing vectors and conformal Killing vectors have been constructed (see for example [29,30]). Using the Heavenly equation formalism, a notable example of a class of noncompact metrics on the cotangent bundles of Riemann surfaces with genus ≥ 1 was constructed in [31].

In what follows we shall consider the (2,2) analogs of pp-waves [28] which in the notation of [29] take the form

$$ds^2 = dy(dw - Q(x, y)dy) - dzdx \quad (3.49)$$

where Q is an arbitrary function. These metrics have a null Killing vector ∂_w which can be thought of as a neutral signature version of a triholomorphic Killing vector [30]. The metrics (3.49) were also considered in the context of twistors [32] and have also appeared in the analysis of [23] and in the classification of neutral solutions admitting Killing spinors [33]. Using our formalism we rewrite (3.49) in the form

$$ds^2 = 2(\mathbf{E}^1 \mathbf{E}^{\bar{1}} - \mathbf{E}^2 \mathbf{E}^{\bar{2}}) \quad (3.50)$$

with

$$\begin{aligned} \mathbf{E}^1 &= \frac{1}{2\sqrt{2}}[dw + (1 - Q)dy + i(dz - dx)], \\ \mathbf{E}^2 &= \frac{1}{2\sqrt{2}}[dw - (1 + Q)dy + i(dz + dx)]. \end{aligned} \quad (3.51)$$

Then the hypersymplectic structures is expressed in terms of

$$\begin{aligned} \mathbf{J}_1 &= \frac{1}{2}(dy \wedge dw - dz \wedge dx), \\ \mathbf{J}_2 &= \frac{1}{2}(dy \wedge dz + (dw - Qdy) \wedge dx), \\ \mathbf{J}_3 &= \frac{1}{2}(dy \wedge dz - (dw - Qdy) \wedge dx). \end{aligned} \quad (3.52)$$

Again as an example we again consider solutions of the STU model with $G_+ = 0$. In this case we obtain

$$\begin{aligned} f^{-6} &= \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \quad X^1 = \left(\frac{\mathcal{H}_2 \mathcal{H}_3}{\mathcal{H}_1^2} \right)^{1/3}, \\ X^2 &= \left(\frac{\mathcal{H}_3 \mathcal{H}_1}{\mathcal{H}_2^2} \right)^{1/3}, \quad X^3 = \left(\frac{\mathcal{H}_2 \mathcal{H}_1}{\mathcal{H}_3^2} \right)^{1/3} \end{aligned} \quad (3.53)$$

where $\mathcal{H}_i$ are harmonic functions on the base space described by (3.49) which can be arbitrary functions of

the coordinates x and y . As dw is anti-self-dual we can for example set

$$dw = (dy \wedge dw - dz \wedge dx). \quad (3.54)$$

In this paper we have considered a class of solutions admitting Killing spinors of five dimensional ungauged supergravity with Abelian vector multiplets. The base space of solutions with space-time signatures (1,4), (4,1), (5,0) and (0,5) are given in terms of hyper-Kähler manifolds. The solutions of the five dimensional theories with space-time signatures (2,3) and (3,2), the base manifold admits a hypersymplectic structure [21].

Hypersymplectic geometry has a very rich structure and not all hypersymplectic manifolds can be obtained from hyper-Kähler manifolds via analytic continuation. All the examples considered in [14,15] can be analytically continued to obtain solutions with hypersymplectic base manifold. It would be of interest to construct many explicit solutions and generalize our results to gauged five-dimensional supergravity theories. We hope to report on this in a future publication.

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