R parity from string compactification

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The strategy for assigning Z_{4R} parity in the string compactification is presented. For the visible sector, an anti-SU(5) [flipped-SU(5)] grand unification (GUT) model with three families is used to reduce the number of representations compared to the number in the minimal supersymmetric standard models. The SO(32) heterotic string is used to allow a large non-Abelian gauge group SU(N), $N \geq 9$, for the hidden sector such that the number of extra U(1) factors is small. A discrete subgroup of the gauge U(1)'s is defined as the Z_{4R} parity. Spontaneous symmetry breaking of anti-SU(5) GUT is achieved by the vacuum expectation values of two index antisymmetric tensor Higgs fields $\mathbf{10}_{+1}$ and $\overline{\mathbf{10}}_{-1}$ that led to our word “anti-SU(5).” In the illustrated example, the multiplicity 3 in one twisted sector allows the permutation symmetry S_3 that leads us to select the third family members and one MSSM pair of the Higgs quintets.

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I. INTRODUCTION

Grand unified theories (GUTs) attracted a great deal of attention ethetically because they provided unification of gauge couplings and charge quantization [1–3]. But there seems to be a fundamental reason leading to GUTs even at the standard model (SM) level. With the electromagnetic and charged currents (CCs), the leptons need representations which are a doublet or bigger. A left-handed (L-handed) lepton doublet (ν_e, e) alone is not free of gauge anomalies because the observed electromagnetic charges are not $\pm \frac{1}{2}$. The anomalies from the fractional electromagnetic charges of the u and d quarks are needed to make the total anomaly to vanish [4,5]. In view of this necessity for jointly using both leptons and quarks to cancel gauge anomalies even in the SM, we can view that unification of leptons and quarks is fundamentally needed in addition to the esthetic view.

In the SM, the largest number of parameters arises from the Yukawa couplings which form the bases of the family structure. Repetition of fermion families in 4-dimensional (4D) field theory or family-unified GUT (family-GUT) was formulated by Georgi [6], requiring un-repeated chiral representations of SU(N) while not allowing gauge anomalies. Some interesting family-GUT models are the spinor representation of SO(14) [7,8] and $\mathbf{84} \oplus 9 \cdot \overline{\mathbf{9}}$ of SU(9) [9].¹

¹For more attempts of family-GUT s, see references in [10].

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While Refs. [7–9] do not provide interesting nonvanishing flavor quantum number, the SU(11) model [6] allows a possibility for non-vanishing flavor quantum number such as U(1) _{$\mu-\tau$} or U(1) _{$B-L$} [10].

On the other hand, the standard-like models from string have been the main focus of string compactification for the ultraviolet completion of the SM toward the minimal supersymmetric standard model (MSSM) or simply supersymmetric standard models (SSMs) in the last several decades [11–34]. These models use the chiral spectrum from the level-1 construction which leads to unification of gauge couplings [35]. So, the standard-like models from string compactification achieved the goal of gauge coupling unification and hence GUT theories from string have not attracted much attention. Nevertheless, GUTs from strings [36,37] have been discussed sporadically for anti-SU(5) [38] (or flipped SU(5) [39]), dynamical symmetry breaking [40,41], and family unification [42]. In fact, family-GUTs are much easier than standard-like models in discussing the family problem, in particular on the origin of the mixing between quarks/leptons because the number of representations in family-GUTs is generally much smaller than in their (standard-like model) subgroups. However, the demand of “standard-like model” scenario adopted here is quite far from reproducing experimental facts such as the fermion masses, except providing the guiding principles.

In this paper we study the R parity assignment for a family-GUT from string compactification. The motivation for R -parity in the MSSM or SSMs is to allow “proton longevity” allowing the lifetime beyond 10^{36} years. Even though $\Delta B \neq 0$ dimension-3 terms in the superpotential are forbidden, $\Delta B \neq 0$ dimension-4 terms are also required to be forbidden. This invites Z_{4R} parity in the MSSM or

SSMs. So far, most string compactification models used the $E_8 \times E'_8$ heterotic string in which a GUT with rank greater than 8 is impossible. The group $SU(11)$ has rank 10 which cannot arise from compactification of $E_8 \times E'_8$. Therefore, first we formulate the orbifold compactification [43,44] of $SO(32)$ heterotic string [45] whose rank is 16. From the compactification of the $SO(32)$ heterotic string, however, we cannot derive the $SU(11)$ model and the largest possible non-Abelian gauge group we obtain is $SU(9)$ [46].

To discuss the R parity from string compactification, we should consider a specific model. Among compactification schemes, we adopt the orbifold method. Among 13 possibilities listed in Ref. [43,44], we employ Z_{12-I} orbifold because it has the simplest twisted sectors. Twisted sectors are distinguished by Wilson lines [47]. The Wilson line in Z_{12-I} distinguishes three fixed points at a twisted sector. Therefore, it suffices to consider only three cases at a twisted sector. In all the other orbifolds of Ref. [43], consideration of various possibilities of Wilson lines and the accompanying consistency conditions are much more involved.

In Sec. II, we recapitulate the orbifold methods used in this paper for an easy reference to Sec. III. In Sec. III, we construct a specific supersymmetric model possessing non-Abelian groups $SU(9)$ and $SU(5)$. In Sec. IV, we assign Z_{4R} quantum numbers to the massless fields obtained in Sec. III. Section V is a conclusion.

II. ORBIFOLD COMPACTIFICATION

A. $SO(4n+2)$ spinors and tensor representations of $SU(2n+1)$

Orthogonal groups have vector/tensor and spinor representations. Chiral fermions are possible in the spinor representations of group $SO(4n+2)$ which are complex. On the other hand, only vector/tensor representations are possible in the $SU(N)$ groups. The anomaly-free set of chiral representations of $SU(2n+1)$ groups for $n \geq 2$ was first illustrated by the spinor representation of $SO(10)$ [48,49]. One of the maximal subgroups of $SO(10)$ is $SU(5) \times U(1)_\chi$. The spinor representation **16** of $SO(10)$ branches to two, one, and zero index tensors of $SU(5)$, $\mathbf{10}_0 + \bar{\mathbf{5}}_0 + \mathbf{1}_0$ of $SU(5) \times U(1)_\chi$, where the subscript 0 is the χ charge. The simplest GUT of Georgi and Glashow was obtained in this way [3,48]. Spinors of $SO(4n+2)$ for $n \geq 3$ include tensor representations of $SU(2n+1)$ with 3 indices and more. In our discussion where at most two index tensors (and their complex conjugates) are allowed, tensor representations (with $n \geq 3$) are not derived from $SO(4n+2)$.

Some tensor representations of $SU(N)$ are counted by the branching of a spinor of $SO(2N)$. For example, $SO(10)$ spinor is $\mathbf{16} = (\underline{++--}) \oplus (\underline{++++}) \oplus (\underline{----})$ whose numbers are $\mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}$. Therefore, we call $(\underline{++--})$ **10** under $SU(5)$, etc. In other words, **10**

represented by $(\underline{++--})$ is named as s-type-tensor (even though $SU(5)$ does not have spinor representations) while **10** represented by $(\underline{11000})$ is named as v-type-tensor.

All antisymmetric representations $\Phi^A, \Phi^{AB} \equiv \Phi^{[AB]}, \dots, \Phi^{[AB\dots]}$, etc. of $SU(N)$ contain only **3** and $\bar{\mathbf{3}}$ under its subgroup $SU(3)$. Therefore, using only antisymmetric representations for matter in an $SU(N)$ GUT guarantees no high dimensional colored matter particles such as **6**, $\bar{\mathbf{6}}$, etc. The chiral $SU(N)$ representations are possible with one and two index tensors of $SU(N)$, e.g.,

$$SU(9): \quad \mathbf{9} = (\underline{-10^8}); \quad \mathbf{36} = (\underline{-1-10^7}) \quad (1)$$

$$SU(5): \quad \mathbf{5} = (\underline{+10^4}); \quad \mathbf{10} = (\underline{+1+10^3}) \quad (2)$$

where we defined differently for $SU(9)$ and $SU(5)$. It is a matter of convention to call what are **9** and **5**. Here, the first (second) ones are called one-(two)-index tensors. The second ones carry two nonzero numbers. For spinor types, we also use this convention, but spinors fill all the 9 (for $SU(9)$) and 5 (for $SU(5)$) slots such as $(\underline{+ + - - \dots})$. One index spinor is defined to have one sign and the rest the opposite sign. In string compactification, there appear only up to two index tensors. Define $\bar{\mathbf{36}}$ of the s-type in $SU(9)$ such that $\bar{\mathbf{36}} \cdot \mathbf{9} \cdot \mathbf{9}$ is allowed if two times the sum of entries is 0 modulo 9.² Therefore, we have $\bar{\mathbf{36}} = (\underline{+ + - - - - - - -})$. Similarly, we have the s-type $\bar{\mathbf{10}}$ of $SU(5)$ as $\bar{\mathbf{10}} = (\underline{+ + + - -})$, satisfying $+1 + 2 + 2 = 5$ from $\bar{\mathbf{10}} \cdot \mathbf{5} \cdot \mathbf{5}$. Thus, one and two index s-type representations are

$$SU(9): \quad \bar{\mathbf{9}} = (\underline{+ - - - - - - -}); \quad \mathbf{36} = (\underline{- - + + + + + +}). \quad (3)$$

$$SU(5): \quad \mathbf{5} = (\underline{+ - - - -}); \quad \mathbf{10} = (\underline{+ + - - -}) \quad (4)$$

One **36** of (1) and two $\bar{\mathbf{9}}$'s of (3) give $-4 - 7 - 7 = -18$ from $\mathbf{36} \cdot \bar{\mathbf{9}} \cdot \bar{\mathbf{9}}$.

Counting the dimension of $SO(32)$ spinor, we note that both even and odd numbers of $+$ signs should be included. This is in contrast to the E_8 spinor where only even numbers of $+$ signs are considered.

B. Two dimensional orbifolds

In compactifying 10D string theory to 4D effective field theory, the small internal 6D is a compactified three two-tori. So, the basic is a two dimensional torus. A two dimensional orbifold is this two dimensional torus modded by discrete groups for which we adopt the discrete group Z_{12-I} . The shift vector we use here is similar to that of [46]

²The spinor $(\underline{+ + + + + + + +})$ for $+$ $\equiv \frac{1}{2}$ gives two times the sum as 9.

but not the same, and the spectrum we obtain is a bit simpler. Since the strategy of compactifying SO(32) heterotic string is already presented there, we list key formulas used in the next chapter in the Appendix A.

Most interesting spectra in the present paper are arising in the twisted sectors. Geometrically, twisted sectors correspond to fixed points. The moding vector of $\mathbf{Z}_{12-I}$ is ϕ_s [43,44],

$$\phi_s = \frac{1}{12}(5, 4, 1). \quad (5)$$

The multiplicities in the fixed points are three because ϕ_s contains $\frac{1}{3}$ in the second torus. This multiplicity 3 can be distinguished by Wilson lines $a_3 = a_4$ (subscripts 3 and 4 denoting two directions in the second torus) [47]. Entries of a_3 are some integer multiples of $\frac{1}{3}$. But $3a_3$ contains only integer entries, that leads to some conditions at twisted sectors T_3 , T_6 and T_9 .

There is one point to be noted for $\mathbf{Z}_{12}$. If N = even, the opposite chiralities of the $k = 1, \dots, \frac{N}{2} - 1$ sectors are provided in the $k = N - 1, \dots, \frac{N}{2} + 1$ sectors,

$$T_k \leftrightarrow T_{N-k}.$$

Then, the corresponding phases of Eq. (A3) compare as

$$e^{2\pi i \Theta_k} \leftrightarrow e^{2\pi i \Theta_{N-k}} \quad (6)$$

whose difference is $e^{2\pi i(N-2k)/12}$. Thus, if $2k = N$ then T_{N-k} do not provide the charge conjugated fields of T_k , but they are identical. For T_3 or T_9 , therefore, we must provide the additional charge conjugated fields with an extra phase $e^{2\pi i(10/12)} = e^{2\pi i(-2/12)}$, the difference of $\hat{s} \cdot \phi_s$ for $\hat{s} = (+ + +)$ of R and $(- - -)$ of L. We choose T_9 for this phase. In T_6 , however, we do not need this since the charged conjugated fields also appear there.

C. Multiplicities in the twisted sectors

In the compactification of the SO(32) heterotic string, spinors in U are not appearing because it is not possible

to have $P^2 = 2$ from sixteen $\pm \frac{1}{2}$'s. Only vector types are possible in U . The T_k twisted sector has three possibilities

$$T_k^{0,+,-} : kV_a = \begin{cases} kV \equiv kV_0 \\ k(V + a_3) \equiv kV_+ \\ k(V - a_3) \equiv kV_- \end{cases} \quad (7)$$

We select only the even lattices shifted from the untwisted lattices, therefore, we consider even numbers for the sum of absolute value of each element of P . They should be even numbers if the absolute values are added. In Appendix A, Θ_k in Eq. (A3) is defined for the twisted sector T_k . Since different P 's in the same twisted sector may lead to different gauge group representations, there is a need to distinguish them. So, we may use

$$\begin{aligned} \Theta_{\text{Group}} = & -\tilde{s} \cdot \phi_s - k p_{\text{vec}}^{\text{th}} \cdot \phi_s + kP \cdot V_0 \\ & + \frac{k}{2}(\phi_s^2 - V_0^2) + \Delta_k^N - \delta_k^N, \end{aligned} \quad (8)$$

where

$$\delta_k^N = 2\delta_k. \quad (9)$$

III. THE MODEL

The left-hand side (LHS) sector of the heterotic string is the gauge sector. The shift vector V_0 and Wilson line a_3 are restricted to satisfy the $\mathbf{Z}_{12-I}$ orbifold conditions,

$$\begin{aligned} 12(V_0^2 - \phi_s^2) &= 0 \text{ mod even integer,} \\ 12(V_0 \cdot a_3) &= 0 \text{ mod even integer,} \\ 12|a_3|^2 &= 0 \text{ mod even integer.} \end{aligned} \quad (10)$$

Here, $a_3 (= a_4)$ is chosen to allow and/or forbid some spectra, and is composed of fractional numbers with the integer multiples of $\frac{1}{3}$ because the second torus has the $\mathbf{Z}_3$ symmetry. The model is

$$\begin{aligned} V_0 &= \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{12}, \frac{6}{12}, \frac{6}{12}, \frac{6}{12}, \frac{6}{12}, \frac{6}{12}, \frac{6}{12} \right), & V_0^2 &= \frac{234}{144} \rightarrow \frac{-54}{144}, \\ V_+ &= \left(\frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{-5}{12}, \frac{+6}{12}, \frac{+2}{12}, \frac{+2}{12}, \frac{+2}{12}, \frac{+2}{12} \right), & V_+^2 &= \frac{90}{144}, \\ V_- &= \left(\frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+1}{12}, \frac{+11}{12}, \frac{+6}{12}, \frac{-2}{12}, \frac{-2}{12}, \frac{-2}{12}, \frac{-2}{12} \right), & V_-^2 &= \frac{186}{144} \end{aligned} \quad (11)$$

where

$$a_3 = a_4 = \left(0^9; \frac{-2}{3}, 0; \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right). \quad (12)$$

The right-hand side (rhs) sector of the heterotic string is given by the spin lattice $s = (\ominus \text{ or } \oplus; \hat{s})$ with every entry being integer multiples of $\frac{1}{2}$, satisfying $s^2 = 2$. The $\hat{\phi}_s$ for $\mathbf{Z}_{12-I}$ with three entries is Eq. (5),

$$\hat{\phi}_s = \left(\frac{5}{12}, \frac{4}{12}, \frac{1}{12}\right), \quad \text{with} \quad \phi_s^2 = \left(\frac{5}{12}, \frac{4}{12}, \frac{1}{12}\right)^2 = \frac{42}{144}. \quad (13)$$

A. Untwisted sector U

In U , we find the following nonvanishing roots of $\text{SU}(5) \times \text{SU}(9)' \times \text{U}(1)^4$,

$\text{SU}(5)$ gauge multiplet:

$$P \cdot V = 0 \text{ mod. integer and } P \cdot a_3 = 0 \text{ mod. integer} \\ \text{SU}(5): \quad P = (0^9; 0^2; \underline{1-1000}). \quad (14)$$

$\text{SU}(9)'$ gauge multiplet: $P \cdot V = 0 \text{ mod. integer}$

$$\text{SU}(9)': \quad P = (\underline{+1-10000000}; 00; 00000). \quad (15)$$

In addition, there exists $\text{U}(1)^4$ symmetry. The nonsinglet matter fields satisfy

$\text{SU}(5)$ and/or $\text{SU}(9)'$ matter multiplet:

$$P \cdot V = \frac{1, 4, 5}{12}, P \cdot a_3 = 0 \text{ mod. integer}. \quad (16)$$

The conditions (16) allows the $P^2 = 2$ lattice shown in Table I.

B. Twisted sectors

In T_9 , for example, we have

$$9V_0 = \left(\left(\frac{+9}{12}\right)^9; \frac{+27+54}{12}; \left(\frac{+54}{12}\right)^5\right), \quad (17)$$

TABLE I. It is a $\mathbf{9}_L$ in view of Eq. (1).

U_i	P	Chirality	$[p_{\text{spin}}](-p_{\text{spin}} \cdot \phi_s)$
$U_1(p \cdot V = \frac{5}{12})$	$(-10^8; 01; 0^5)$	L	$[\ominus; - - -](\frac{+5}{12})$
$U_2(p \cdot V = \frac{4}{12})$	...		...
$U_3(p \cdot V = \frac{1}{12})$	...		...

$$9V_+ = \left(\left(\frac{+9}{12}\right)^9; \frac{-45+54}{12}; \left(\frac{+18}{12}\right)^5\right), \quad (18)$$

$$9V_- = \left(\left(\frac{+9}{12}\right)^9; \frac{+99+54}{12}; \left(\frac{-18}{12}\right)^5\right). \quad (19)$$

We shift the lattice by adding $\text{SU}(9)' \times \text{SU}(5)$ singlet vector; $v: (v_1, \dots, v_{16})$ satisfying $\sum_i |v_i| = 0 \text{ modulo } 2k$. In the following equations, we list all the shift vectors, $kV_{0,+,-}$, in all twisted sectors. Equations (17), (18), (19) are in Eq. (24).

$$T_1^0 \left(\left(\frac{+1}{12}\right)^9; \frac{+3}{12}, \frac{+6}{12}; \left(\frac{+6}{12}\right)^5\right); \\ T_1^+ \left(\left(\frac{+1}{12}\right)^9; \frac{-5}{12}, \frac{+6}{12}; \left(\frac{+2}{12}\right)^5\right); \\ T_1^- \left(\left(\frac{+1}{12}\right)^9; \frac{+11}{12}, \frac{+6}{12}; \left(\frac{-2}{12}\right)^5\right), \quad (20)$$

$$T_2^0 \left(\left(\frac{+2}{12}\right)^9; \frac{-6}{12}, -1; 0^5\right); \\ T_2^+ \left(\left(\frac{+2}{12}\right)^9; \frac{+2}{12}, -1; \left(\frac{-8}{12}\right)^5\right); \\ T_2^- \left(\left(\frac{+2}{12}\right)^9; \frac{-2}{12}, -1; \left(\frac{-4}{12}\right)^5\right), \quad (21)$$

$$T_4^0 \left(\left(\frac{+4}{12}\right)^9; 0, 0; (+1)^5\right); \\ T_4^+ \left(\left(\frac{+4}{12}\right)^9; \frac{-8}{12}, 0; \left(\frac{-4}{12}\right)^5\right); \\ T_4^- \left(\left(\frac{+4}{12}\right)^9; \frac{+8}{12}, 2; \left(\frac{+4}{12}\right)^5\right), \quad (22)$$

$$T_7^0 \left(\left(\frac{-5}{12}\right)^9; \frac{+9}{12}, \frac{+6}{12}; \left(\frac{+6}{12}\right)^5\right); \\ T_7^+ \left(\left(\frac{+1}{12}\right)^9; \frac{-35}{12}, \frac{+42}{12}; \left(\frac{+2}{12}\right)^5\right); \\ T_7^- \left(\left(\frac{+7}{12}\right)^9; \frac{+5}{12}, \frac{+6}{12}; \left(\frac{-2}{12}\right)^5\right), \quad (23)$$

$$T_9^0 \left(\left(\frac{-3}{12}\right)^9; \frac{-9}{12}, \frac{+6}{12}; \left(\frac{+6}{12}\right)^5\right); \\ T_9^+ \left(\left(\frac{+9}{12}\right)^9; \frac{+3}{12}, \frac{+6}{12}; \left(\frac{-6}{12}\right)^5\right); \\ T_9^- \left(\left(\frac{+9}{12}\right)^9; \frac{+3}{12}, \frac{-6}{12}; \left(\frac{-6}{12}\right)^5\right), \quad (24)$$

$$T_6^0 = \left(\left(\frac{-6}{12}\right)^9; \frac{+18}{12}, +3; 0^5\right). \quad (25)$$

TABLE II. Two indices s-type from $T_9^0: (\mathbf{1}, \mathbf{36}')_L$. This is the abbreviated one of Table XXIII presented in Appendix C. Note that we provided an extra phase $e^{2\pi i(-2/N)}$ as commented below Eq. (6).

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{\text{th}} \cdot \phi_s$	$k P_9 \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_9^N	$-\delta_9^N$	Θ_9	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{+2}{12} + \frac{-2}{12}$	2
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-8}{12} + \frac{-2}{12}$	1

1. Twisted sector $T_9(\delta_3 = \frac{3}{12})$

• Two indices s-type from T_9^0 : with Eq. (17) the spinor forms satisfying the mass-shell condition $(P + 9V_0)^2 = \frac{234}{144} = \frac{13}{8}$ and the Wilson-line condition $12(P + 9V_0) \cdot a_3 = 0$ are possible for SU(9):

$$P_9 = \left(\frac{-3}{2}, \frac{-3}{2}, -7; \frac{-5}{2}, \frac{-9}{2}; \left(\frac{-9}{2} \right)^5 \right). \quad (26)$$

Instead of the above P_9 , with Eq. (24) we have

$$P_9 = (\underline{- - +^7}; + -; -^5) \quad (27)$$

which is a shift from Eq. (26) by $(1^9; 3, 4; 4^5)$. With Eqs. (24) and (27), we present the multiplicity in Table II.

2. Twisted sector $T_4(\delta_4 = 0)$

- One index v-type from T_4^0 :

$$P_9 = (\underline{-10^8}; 00; (-1)^5), \quad (28)$$

satisfies $(P_9 + 4V_0)^2 = \frac{192}{144}$. Thus, we obtain Table III.

- One index s-type from T_4^- : the spinor-form

$$P_9 = \left(\underline{+ -^8}; -, \frac{-3}{2}; -^5 \right), \quad (29)$$

satisfies $(P_9 + 4V_-)^2 = \frac{192}{144}$, which is shown in Table IV.

3. Twisted sector $T_1(\delta_1 = \frac{1}{12})$

In T_1 , we use Eq. (11).

- One index v-type from T_1^+ : the vector

$$P_5 = (0^9; +1, 0; \underline{-1, 0, 0, 0, 0}), \quad (30)$$

satisfies $(P + V_+)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{210}{144}$, and all states are shown in Table V.

- One index v-type from T_1^+ : the vector

$$P_5 = (0^9; 0, -1; \underline{-1, 0, 0, 0, 0}), \quad (31)$$

satisfies $(P + V_+)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{210}{144}$, and all states are shown in Table VI.

TABLE III. One index v-type from T_4^0 : Thus, we obtain $7 \cdot (\mathbf{1}, \mathbf{9}')_R$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{\text{th}} \cdot \phi_s$	$k P_9 \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_4^N	$-\delta_4^N$	Θ_9	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+8}{12}$	$\frac{-4}{12}$	$\frac{84}{144}$	$\frac{+108}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+1}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+8}{12}$	$\frac{-4}{12}$	$\frac{84}{144}$	$\frac{+108}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{-9}{12}$	7

TABLE IV. One index s-type from T_4^- : Thus, we obtain $7 \cdot (\mathbf{1}, \mathbf{9}')_L$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{\text{th}} \cdot \phi_s$	$k P_9 \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_4^N	$-\delta_4^N$	Θ_9	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+8}{12}$	$\frac{-4}{12}$	$\frac{84}{144}$	$\frac{-372}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+9}{12}$	7
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+8}{12}$	$\frac{-4}{12}$	$\frac{84}{144}$	$\frac{-372}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{-1}{12}$	0

TABLE V. One index v-type from T_1^+ : Thus, we obtain $3 \cdot (\mathbf{5}, \mathbf{1})_R$ that is ρ in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{\text{th}} \cdot \phi_s$	$k P_5 \cdot V_+$	$(k/2)\phi_s^2$	$-(k/2)V_+^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+5}{12}$	$\frac{-7}{12}$	$\frac{+21}{144}$	$\frac{-45}{144}, \frac{-2}{12}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{+1}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+5}{12}$	$\frac{-8}{12}$	$\frac{+21}{144}$	$\frac{-45}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{-9}{12}$	3

TABLE VI. One index v-type from T_1^+ : Thus, we obtain $3 \cdot (\mathbf{5}, \mathbf{1})_L$ that is σ in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_5 \cdot V_+$	$(k/2)\phi_s^2$	$-(k/2)V_+^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+5}{12}$	$\frac{-8}{12}$	$\frac{+21}{144}$	$\frac{-45}{144}, \frac{-2}{12}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{0}{12}$	3
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+5}{12}$	$\frac{-8}{12}$	$\frac{+21}{144}$	$\frac{-45}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{-10}{12}$	0

TABLE VII. One index v-type from T_1^- : Thus, we obtain $3 \cdot (\mathbf{1}, \mathbf{9}')_R$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_5 \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+5}{12}$	$\frac{0}{12}$	$\frac{+21}{144}$	$\frac{-93}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{+4}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+5}{12}$	$\frac{0}{12}$	$\frac{+21}{144}$	$\frac{-93}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{-6}{12}$	3

- One index v-type from T_1^- : the vector

$$P_9 = (\underline{-10^8}; -1, 0; 0^5), \quad (32)$$

satisfies $(P + V_-)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{210}{144}$, and the spectrum is shown in Table VII.

- One index v-type from T_1^- : the vector

$$P_5 = (0^9; -1, 0; \underline{+10^4}), \quad (33)$$

satisfies $(P + V_-)^2 = \frac{162}{144}$ which is short by $\frac{4}{12}$ from the target value of $\frac{210}{144}$, and we obtain Table VIII.

4. Twisted sector $T_2(\delta_2 = \frac{0}{12})$

- One index v-type from T_2^0 : the vector

$$P_9 = (\underline{-10^8}; 0, +1; 0^5), \quad (34)$$

which gives $(P + 2V_+)^2 = \frac{168}{144}$ which is short by $\frac{48}{144} = \frac{4}{12}$ from the target value of $\frac{216}{144}$, and massless fields are shown in Table IX.

- One index s-type from T_2^- : the vector

$$P_5 = (-^9; +, +; \underline{-+ + + +}), \quad (35)$$

which gives $(P + 2V_-)^2 = \frac{192}{144}$ which is short by $\frac{2}{12}$ from the target value of $\frac{216}{144}$, and massless fields are shown in Table X.

TABLE VIII. One index v-type from T_1^- : Thus, we obtain $3 \cdot (\mathbf{5}, \mathbf{1})_L + 3 \cdot (\mathbf{5}, \mathbf{1})_R$ that are β and α in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_5 \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+5}{12}$	$\frac{-1}{12}$	$\frac{+21}{144}$	$\frac{-93}{144}$	$\frac{+4}{12}$	$\frac{-2}{12}$	$\frac{+5}{12}$	3
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+5}{12}$	$\frac{-1}{12}$	$\frac{+21}{144}$	$\frac{-93}{144}$	$\frac{+4}{12}$	$\frac{-2}{12}$	$\frac{-5}{12}$	3

TABLE IX. One index v-type from T_2^0 : Thus, we obtain $3 \cdot (\mathbf{1}, \mathbf{9}')_R$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_2^N	$-\delta_2^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+10}{12}$	$\frac{-2}{12}$	$\frac{+42}{144}$	$\frac{+54}{144}, \frac{8}{12}$	$\frac{+4}{12}$	$\frac{0}{12}$	$\frac{+1}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+10}{12}$	$\frac{-2}{12}$	$\frac{+42}{144}$	$\frac{+54}{144}$	$\frac{+4}{12}$	$\frac{0}{12}$	$\frac{-9}{12}$	3

TABLE X. One index s-type from T_2^- : Thus, we obtain $3 \cdot (\bar{\mathbf{5}}, \mathbf{1})_L + 3 \cdot (\bar{\mathbf{5}}, \mathbf{1})_R$ that are η and ξ in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_2^N	$-\delta_2^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+10}{12}$	$\frac{+4}{12}$	$\frac{+42}{144}$	$\frac{-90}{144}, \frac{-4}{12}$	$\frac{+2}{12}$	$\frac{0}{12}$	$\frac{+5}{12}$	3
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+10}{12}$	$\frac{+4}{12}$	$\frac{+42}{144}$	$\frac{-90}{144}$	$\frac{+2}{12}$	$\frac{0}{12}$	$\frac{-5}{12}$	3

TABLE XI. One index s-type from $T_7^0: 3 \cdot (\mathbf{1}, \mathbf{9}')_R$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_9 \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_k^N	$-\delta_k^N$	Θ_9	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{-4}{12}$	$\frac{+147}{144}, \frac{4}{12}$	$\frac{+189}{144}$	$\frac{+6}{12}$	$\frac{-2}{12}$	$\frac{+10}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+1}{12}$	$\frac{-4}{12}$	$\frac{+147}{144}$	$\frac{+189}{144}$	$\frac{+6}{12}$	$\frac{-2}{12}$	$\frac{0}{12}$	3

TABLE XII. One index v-type from $T_7^+: 3 \cdot (\mathbf{1}, \mathbf{9}')_L$.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_9 \cdot V_+$	$(k/2)\phi_s^2$	$-(k/2)V_+^2$	Δ_k^N	$-\delta_k^N$	Θ_9	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{+2}{12}$	$\frac{+147}{144}$	$\frac{-315}{144}, \frac{-2}{12}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{+6}{12}$	3
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+1}{12}$	$\frac{+2}{12}$	$\frac{+147}{144}$	$\frac{-315}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{-4}{12}$	0

5. Twisted sector $T_7(\delta_7 = \frac{1}{12})$

- One index s-type from T_7^0 :

$$P_9 = (\underline{+8}, -, -, -5), \quad (36)$$

gives $(P_9 + 7V_0)^2 = \frac{138}{144}$, which is short of $\frac{6}{12}$ from $\frac{210}{12}$, and we obtain Table XI.

- One index v-type from T_7^+ :

$$P_9 = (\underline{-10^8}, +3, -4, 0^5), \quad (37)$$

gives $(P_9 + 7V_+)^2 = \frac{186}{144}$ which is short of $\frac{2}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XII.

- Two indices s-type from T_7^- : The vector

$$P_5 = (-^9; --; \underline{++++-}), \quad (38)$$

gives $(P_5 + 7V_-)^2 = \frac{186}{144}$ which is short of $\frac{2}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XIII.

- One index s-type from T_7^- : The vector

$$P_5 = (-^9; --; \underline{+++++}), \quad (39)$$

gives $(P_5 + 7V_-)^2 = \frac{138}{144}$ which is short of $\frac{6}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XIV.

6. Twisted sector $T_6(\delta_6 = \frac{0}{12})$

Note that $\frac{k}{2}(V_a^2 - \phi_s^2)$ in Eq. (A3) is not distinguished by Wilson lines, and we just calculate the spectra from $6V_0$ with multiplicity 3 (for $T_6^{0,+,-}$). With 3 and 0⁵ in (25), we obtain only vectorlike representations. To clarify, we list all the possibilities for **5**, **$\bar{5}$** , **10**, and **$\bar{10}$** . All the allowed ones are spinor forms.

- S-type for **10**,

$$P_5 = \left(+^9; \frac{-3}{2}, \frac{-5}{2} \quad \text{or} \quad \frac{-7}{2}; \underline{++---} \right) \quad (40)$$

which saturates the masslessness condition $(P_5 + 6V_0)^2 = \frac{216}{144}$. The spectra are shown in Table XV. Considering Eq. (40) for $\frac{-5}{2}$ and $\frac{-7}{2}$, we obtain 12 each for L and R. Thus, we obtain $36 \cdot (\mathbf{10}_L \oplus \mathbf{10}_R)$.

- S-type for **$\bar{10}$** ,

$$P_5 = \left(+^9; \frac{-3}{2}, \frac{-5}{2} \quad \text{or} \quad \frac{-7}{2}; \underline{+++-} \right) \quad (41)$$

TABLE XIII. Two indices s-type from $T_7^-: 3(\mathbf{\bar{10}}, \mathbf{1})_R$ that is T in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_5 \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{-2}{12}$	$\frac{+147}{144}$	$\frac{-651}{144}, \frac{6}{12}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{+10}{12}$	0
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+1}{12}$	$\frac{-2}{12}$	$\frac{+147}{144}$	$\frac{-651}{144}$	$\frac{+2}{12}$	$\frac{-2}{12}$	$\frac{0}{12}$	3

TABLE XIV. One index s-type from $T_7^-: 3(\mathbf{\bar{5}}, \mathbf{1})_L$ that is $\bar{F}$ in Table XIX.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_5 \cdot V_-$	$(k/2)\phi_s^2$	$-(k/2)V_-^2$	Δ_k^N	$-\delta_k^N$	Θ_5	Mult.
$\ominus = L$	$(---)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{-4}{12}$	$\frac{+147}{144}$	$\frac{-651}{144}$	$\frac{+6}{12}$	$\frac{-2}{12}$	$\frac{0}{12}$	3
$\oplus = R$	$(+++)$	$\frac{-5}{12}$	$\frac{+1}{12}$	$\frac{-4}{12}$	$\frac{+147}{144}$	$\frac{-651}{144}$	$\frac{+6}{12}$	$\frac{-2}{12}$	$\frac{-10}{12}$	0

TABLE XV. T_6 : **10** appears 6 times for L and 6 times for R. The same multiplicities occur also in T_6^{+-} . So, in total, there appear 36 multiplicities of **10** for L and also for R.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-kP_{\text{vec}}^{\text{th}} \cdot \phi_s$	$kP_1 \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_6^N	$-\delta_6^N$	Θ_5	Mult. of 10 and $\overline{\mathbf{10}}$ s.
$\ominus = L$	$(- - -)$	$\frac{+5}{12}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+126}{144}$	$\frac{+162}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+5}{12}$	6
$\oplus = R$	$(+ + +)$	$\frac{-5}{12}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+126}{144}$	$\frac{+162}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{-5}{12}$	6

which saturates the masslessness condition $(P_5 + 6V_0)^2 = \frac{216}{144}$. So, we obtain $36 \cdot (\mathbf{10}_L \oplus \overline{\mathbf{10}}_R)$.

- S-type for **5**,

$$P_5 = \left(+^9; \frac{-3}{2}, \frac{-5}{2} \quad \text{or} \quad \frac{-7}{2}; + - - - - \right) \quad (42)$$

which saturates the masslessness condition $(P_5 + 6V_0)^2 = \frac{216}{144}$, and we obtain $36 \cdot (\mathbf{5}_L \oplus \mathbf{5}_R)$.

- S-type for $\bar{\mathbf{5}}$,

$$P_5 = \left(+^9; \frac{-3}{2}, \frac{-5}{2} \quad \text{or} \quad \frac{-7}{2}; + + + + - \right) \quad (43)$$

which saturates the masslessness condition $(P_5 + 6V_0)^2 = \frac{216}{144}$, and we obtain $36 \cdot (\bar{\mathbf{5}}_L \oplus \bar{\mathbf{5}}_R)$.

Summarizing the non-singlet chiral fields we obtained, $\mathbf{9}'_L(U) + \mathbf{36}'_L(T_9^0) + (\mathbf{36}'_L(T_9^0) + \mathbf{36}'_R(T_9^0)) + 7 \cdot \mathbf{9}'_R(T_4^0) + 7 \cdot \mathbf{9}'_L(T_4^-) + 3 \cdot \mathbf{9}'_R(T_1^-) + 3 \cdot \mathbf{9}'_R(T_2^0) + 3 \cdot \mathbf{9}'_R(T_7^0) + 3 \cdot \mathbf{9}'_L(T_7^+)$ for $\text{SU}(9)'$. After removing vectorlike representations, note that the $\text{SU}(9)'$ anomaly is absent with

$$\overline{\mathbf{36}}'_R + 5 \cdot \mathbf{9}'_R. \quad (44)$$

Similarly after removing vectorlike representations, for $\text{SU}(5)$ we obtain

$$3 \cdot \mathbf{10}_L(T_7^+) + 3 \cdot \bar{\mathbf{5}}_L(T_7^-). \quad (45)$$

These spectra in Eqs. (44) and (45) do not lead to non-Abelian gauge anomalies. For $\bar{\mathbf{5}}_L$ in Eq. (45), in fact it is a linear combination of $\bar{\mathbf{5}}_L(T_7^-)$ ($\bar{F}$ in Table XIX) and $\bar{\mathbf{5}}_L(T_2^-)$ (η in Table XIX). In addition, note that there appear 72 pairs of $\mathbf{10} \oplus \overline{\mathbf{10}}$ from T_6 , which are needed for breaking the anti-SU(5) GUT.

C. Singlets, quintets, and Higgs fields

We obtained the rank 16 group $\text{SU}(5) \times \text{SU}(9)' \times \text{U}(1)^4$. The four $\text{U}(1)$ charges are also shown in the following tables, where Q_X is the $\text{U}(1)_X$ charge of anti-SU(5) GUT. The four $\text{U}(1)$ charges in the gauge group $\text{SU}(9)' \times \text{SU}(5) \times \text{U}(1)^4$ are defined by

$$Q_1 = \frac{1}{9}(-2, -2, -2, -2, -2, -2, -2, -2, -2; 0, 0; 0^5) \quad (46)$$

$$Q_2 = (0^9; -2, 0; 0^5) \quad (47)$$

$$Q_3 = (0^9; 0, -2; 0^5) \quad (48)$$

$$Q_X = (0^9; 0, 0; -2, -2, -2, -2, -2). \quad (49)$$

We summarize the $\text{SU}(5) \times \text{SU}(9)'$ singlets in the following Tables XVI–XVIII, with the format of Sec. III B. In Table XVIII, we tabulate the four $\text{U}(1)$ charges given in Eqs. (46)–(49). For the $\text{SU}(5)$ singlets, the electromagnetic charge is

TABLE XVI. Entries in calculating the multiplicities in the following Table XVII.

Model($T_k^{0,+,-}$)	$-\tilde{s} \cdot \phi_s$	$-kP_{\text{vec}}^{\text{th}} \cdot \phi_s$	$kP_{\text{Group}} \cdot V_{0,+,-}$	$(k/2)\phi_s^2$	$-(k/2)V_{0,+,-}^2$	Δ_{Group}^N	$-\delta_{\text{Group}}^N$	Θ_{Group}
$A(T_9^0)$	$\frac{+5}{12}$	$\frac{-3}{12}$	$\frac{0}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-4}{12}$
$B(T_9^+)$	$\frac{+5}{12}$	$\frac{-3}{12}$	$\frac{0}{12}$	$\frac{189}{144}$	$\frac{-405}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{+2}{12}$
$C(T_9^-)$	$\frac{+5}{12}$	$\frac{-3}{12}$	$\frac{0}{12}$	$\frac{189}{144}$	$\frac{-837}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{+2}{12}$
$D(T_4^+)$	$\frac{+5}{12}$	$\frac{+8}{12}$	$\frac{+4}{12}$	$\frac{84}{144}$	$\frac{-180}{144}$	$\frac{+8}{12}$	$\frac{0}{12}$	$\frac{+5}{12}$
$E(T_4^-)$	$\frac{+5}{12}$	$\frac{+8}{12}$	$\frac{+4}{12}$	$\frac{84}{144}$	$\frac{-372}{144}$	$\frac{+8}{12}$	$\frac{0}{12}$	$\frac{+1}{12}$
$F(T_1^-)$	$\frac{+5}{12}$	$\frac{+5}{12}$	$\frac{-5}{12}$	$\frac{21}{144}$	$\frac{-93}{144}$	$\frac{0}{12}$	$\frac{-2}{12}$	$\frac{+9}{12}$
$G(T_2^0)$	$\frac{+5}{12}$	$\frac{+10}{12}$	$\frac{-6}{12}$	$\frac{42}{144}$	$\frac{+54}{144}$	$\frac{+8}{12}$	$\frac{0}{12}$	$\frac{+1}{12}$
$H(T_2^-)$	$\frac{+5}{12}$	$\frac{+10}{12}$	$\frac{-2}{12}$	$\frac{42}{144}$	$\frac{-186}{144}$	$\frac{0}{12}$	$\frac{0}{12}$	$\frac{+1}{12}$
$I(T_7^0)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{-6}{12}$	$\frac{147}{144}$	$\frac{+189}{144}$	$\frac{+10}{12}$	$\frac{-2}{12}$	$\frac{0}{12}$
$J(T_7^-)$	$\frac{+5}{12}$	$\frac{+1}{12}$	$\frac{-1}{12}$	$\frac{147}{144}$	$\frac{-651}{144}$	$\frac{0}{12}$	$\frac{-2}{12}$	$\frac{+9}{12}$

TABLE XVII. Summary of chiral singlets, $\mathbf{1}$'s. The value for $-\hat{s} \cdot \phi_s$ is for $(\ominus; - - -)$, i.e., the ones in the top rows in the tables. We also listed the vectorlike $(\mathbf{10} \oplus \overline{\mathbf{10}}, \mathbf{1})_R$ obtained previously in Table XV.

Twisted Sector	$P; P + kV_i$	$-\hat{s} \cdot \phi_s$	Θ_k	Mult. of singlets (L or R)
$A(T_9^0)$	$(+^9; +, -, -^5)$	$\frac{5}{12}$	$\frac{-4}{12}$	1(L)
$B(T_9^+)$	$(-^9; -, -, +^5)$	$\frac{5}{12}$	$\frac{+2}{12}$	1(R)
$C(T_9^-)$	$(-^9; -, +, +^5)$	$\frac{5}{12}$	$\frac{+2}{12}$	1(R)
$D(T_4^+)$	$(-^9; +, \pm; +^5)$	$\frac{5}{12}$	$\frac{+5}{12}$	3(L) + 3(L), 3(R) + 3(R)
$E(T_4^-)$	$(-^9; -, \mp^3 \text{ or } \mp^5; -^5)$	$\frac{5}{12}$	$\frac{+1}{12}$	7(R) + 7(R)
$F(T_1^-)$	$(0^9; -1, -1; 0^5)$	$\frac{5}{12}$	$\frac{+9}{12}$	3(L), neutral
$G(T_2^0)$	$(0^9; +1, +1; 0^5)$	$\frac{5}{12}$	$\frac{+1}{12}$	3(R), neutral
$H(T_2^-)$	$(0^9; +1, +1; 0^5)$	$\frac{5}{12}$	$\frac{+1}{12}$	3(R), neutral
$I(T_7^0)$	$(+^9; \mp^3, -, -^5)$	$\frac{5}{12}$	$\frac{0}{12}$	3(L)
$J(T_7^-)$	$(-^9; +, -, +^5)$	$\frac{5}{12}$	$\frac{+9}{12}$	3(L)
$K_1(T_6^0)$	$(+^9; \mp^3, \mp^5; +^5)$	$\frac{5}{12}$	$\frac{+5}{12}$	$3 \times 7(L), 3 \times 7(R)$
$K_2(T_6^0)$	$(+^9; \mp^3, \mp^7; +^5)$	$\frac{5}{12}$	$\frac{+5}{12}$	$3 \times 7(L), 3 \times 7(R)$
$K_3(T_6^0)$	$(+^9; \mp^3, \mp^5; -^5)$	$\frac{5}{12}$	$\frac{+5}{12}$	$3 \times 7(L), 3 \times 7(R)$
$K_4(T_6^0)$	$(+^9; \mp^3, \mp^7; -^5)$	$\frac{5}{12}$	$\frac{+5}{12}$	$3 \times 7(L), 3 \times 7(R)$
$T_6(\mathbf{10})$	$(+^9; \mp^3, \mp^5 \text{ or } \mp^7; \pm \pm - - -)$	$\frac{5}{12}$	$\frac{+5}{12}$	36(L), 36(R)
$T_6(\overline{\mathbf{10}})$	$(+^9; \mp^3, \mp^5 \text{ or } \mp^7; \pm \pm + - -)$	$\frac{5}{12}$	$\frac{+5}{12}$	36(L), 36(R)

TABLE XVIII. U(1) charges of $\mathbf{1}$'s and Higgs $\mathbf{10}$ and $\overline{\mathbf{10}}$. Here, Q_R is calculated from Eq. (52).

Twisted Sector	$P; P + kV_i$	Q_1	Q_2	Q_3	Q_X	Q_R	Mult. of singlets (L or R)
$A(T_9^0)$	$(+^9; +, -, -^5)$	-1	-1	+1	+5	0	1(L)
$B(T_9^+)$	$(-^9; -, -, +^5)$	+1	+1	+1	-5	-3	1(R)
$C(T_9^-)$	$(-^9; -, +, +^5)$	+1	+1	-1	-5	0	1(R)
$D(T_4^+)$	$(-^9; +, \pm; +^5)$	+1	-1	∓ 1	-5	(+2, -1)	3(L)+3(L), 3(R)+3(R)
$E(T_4^-)$	$(-^9; -, \mp^3 \text{ or } \mp^5; -^5)$	+1	-1	+3/+5	+5	(+6, +3)	7(R)+7(R)
$F(T_1^-)$	$(0^9; -1, -1; 0^5)$	0	+2	+2	0	-5	3(L), neutral
$G(T_2^0)$	$(0^9; +1, +1; 0^5)$	0	-2	-2	0	+5	3(R), neutral
$H(T_2^-)$	$(0^9; +1, +1; 0^5)$	0	-2	-2	0	+5	3(R), neutral
$I(T_7^0)$	$(+^9; \mp^3, -, -^5)$	-1	+3	+1	+5	-4	3(L)
$J(T_7^-)$	$(-^9; +, -, +^5)$	+1	-1	+1	-5	-1	3(L)
$K_1(T_6^0)$	$(+^9; \mp^3, \mp^5; +^5)$	-1	+3	+5	-5	-20	$3 \times 7(L), 3 \times 7(R)$
$K_2(T_6^0)$	$(+^9; \mp^3, \mp^7; +^5)$	-1	+3	+7	-5	-23	$3 \times 7(L), 3 \times 7(R)$
$K_3(T_6^0)$	$(+^9; \mp^3, \mp^5; -^5)$	-1	+3	+5	+5	-10	$3 \times 7(L), 3 \times 7(R)$
$K_4(T_6^0)$	$(+^9; \mp^3, \mp^7; -^5)$	-1	+3	+7	+5	-13	$3 \times 7(L), 3 \times 7(R)$
$T_6(\mathbf{10})$	$(+^9; \mp^3, \mp^5 \text{ or } \mp^7; \pm \pm - - -)$	-1	+3	+5/+7	+1	(-14, -17)	36(L), 36(R)
$T_6(\overline{\mathbf{10}})$	$(+^9; \mp^3, \mp^5 \text{ or } \mp^7; \pm \pm + - -)$	-1	+3	+5/+7	-1	(-14, -17)	36(L), 36(R)

$$Q_{\text{em}} = \frac{1}{5} Q_X. \quad (50)$$

In Table XIX, we collect all $\mathbf{5}$'s, $\overline{\mathbf{5}}$'s, $\mathbf{10}$'s, and $\overline{\mathbf{10}}$'s. A set of anti-SU(5) GUT representation, which is of course

anomaly-free chiral set, is $\overline{\mathbf{10}}_{-1} \oplus \mathbf{5}_{+3} \oplus \overline{\mathbf{1}}_{-5}$. Indeed, three $\overline{\mathbf{1}}_{-5}$'s (A , B , and C) in Table XVIII belong to the anti-SU(5) families. The remaining $Q_X = \pm 5$ singlets form vectorlike representations under anti-SU(5). The neutral singlets, F , G , and H are the heavy neutrinos.

TABLE XIX. U(1) charges of $\mathbf{5}$, $\bar{\mathbf{5}}$, $\mathbf{10}$ and $\bar{\mathbf{10}}$, where Q_R is given in Eq. (52). The L fields are changed to R fields in the first column by taking charge conjugated quantum numbers presented in the text.

States	$P; P + kV_i$	Q_1	Q_2	Q_3	Q_X	Multiplicities of $\mathbf{5}$, $\bar{\mathbf{5}}$, $\mathbf{10}$ and $\bar{\mathbf{10}}$	Q_R
$\rho(\mathbf{5}_R(T_1^+))$	$(0^9; +1, 0; -1\ 0^4)$	0	-2	0	+2	3(R)	+4
$\sigma(\bar{\mathbf{5}}_R(T_1^+))$	$(0^9; 0, +1; +1\ 0^4)$	0	0	-2	-2	3(L)	+1
$\xi(\bar{\mathbf{5}}_R(T_2^-))$	$(-^9; +, +; - + + +)$	+1	-1	-1	-3	3(R)	+4
$\eta(\mathbf{5}_R(T_2^-))$	$(+^9; -, -; + - - -)$	-1	+1	+1	+3	3(L)	-4
$\alpha(\mathbf{5}_R(T_1^-))$	$(0^9; -1, 0; +1\ 0^4)$	0	+2	0	-2	3(R), Higgs $\mathbf{5}_{hR}$ containing H_u	-4
$\beta(\bar{\mathbf{5}}_R(T_1^-))$	$(0^9; +1, 0; -1\ 0^4)$	0	-2	0	+2	3(L), Higgs $\bar{\mathbf{5}}_{hR}$ containing H_d	+4
$\bar{F}(\mathbf{5}_R(T_7^-))$	$(+^9; ++; + - - -)$	-1	-1	-1	+3	3(L)	+1
$T(\bar{\mathbf{10}}_R(T_7^-))$	$(-^9; -, -; + + + -)$	+1	+1	+1	-1	3(R)	+1
$H(\mathbf{10}_R(T_6))$	$(-^9; -, -; + + - -)$	+1	+1	+1	+1	36(L), 36(R), $VEV = 0$	+3
$\bar{H}(\bar{\mathbf{10}}_R(T_6))$	$(-^9; -, +; + + + -)$	+1	+1	-1	-1	36(L), 36(R), $VEV \neq 0$	+4

From Table XIX, the vector-like pairs $\{\rho, \sigma\}$ and $\{\xi, \eta\}$ are removed. The remaining ones constitute the anti-SU(5) spectra $T, \bar{F}, A, B, C$ and the MSSM Higgs fields α and β , containing H_u and H_d . The vectorlike pairs H and $\bar{H}$ are needed to break the anti-SU(5) down to the SM.

IV. ASSIGNMENT OF R PARITY

For a $\mathbf{Z}_{4R}$ symmetry for guaranteeing proton longevity, we may assign the following U(1)_R charges to the MSSM plus σ fields [50], represented as name (R) [Q_R],³

$$q(T(\bar{\mathbf{10}}_{-1,R})) [+1], \quad \ell(\eta(\mathbf{5}_{+3,R})) [+1], \quad \tau^+(B(\mathbf{1}_{-5,R})) [+1], \\ H_u(\alpha(\mathbf{5}_{-2,R})) [-4], \quad H_d(\beta(\bar{\mathbf{5}}_{+2,R})) [+4], \quad \sigma [+1]. \quad (51)$$

With the four gauged U(1) charges, let us choose Q_R as

$$Q_R = \frac{9}{2}Q_1 - Q_2 - \frac{3}{2}Q_3 + Q_X \quad (52)$$

which is tabulated in Table XIX and also in Table XVIII. To break U(1)_R down to $\mathbf{Z}_{4R}$, we need the GUT breaking VEVs, $\langle \bar{H}(\bar{\mathbf{10}}_R(T_6)) \rangle \neq 0$, but we demand $\langle H(\mathbf{10}_R(T_6)) \rangle$ should not develop a VEV such that $\mathbf{Z}_{4R}$ is not broken further by $\langle H(\mathbf{10}_R(T_6)) \rangle$. With $\mathbf{Z}_{4R}$, the Yukawa couplings are required to carry the Q_R charge 2 modulo 4. With (52), the Higgs quintets carry $Q_R = 0$ modulo 4 and the Yukawa couplings $T\bar{F}\alpha$, $TT\beta$ and $\bar{F}B\beta$ are obtained. So, B is τ^+ .

³The $\mathbf{Z}_{4R}$ parity has been extensively discussed in the literature, taking into account the Green-Schwartz mechanism. For an earlier reference, see [51].

A. One pair of Higgs quintets

The Higgs quintets are α and β . Twisted sector T_1^- of α and β gives multiplicity 3. Thus, three objects at these fixed points must have permutation symmetry S_3 , the largest discrete symmetry of three objects. From three permuting elements of S_3 , $\{x_1, x_2, x_3\}$, the S_3 singlet is formed as

$$(x_1 + x_2 + x_3)/\sqrt{3}, \quad (53)$$

and the S_3 doublet becomes

$$\begin{pmatrix} (x_1 + \omega x_2 + \omega^2 x_3)/\sqrt{3} \\ (x_1 + \omega^2 x_2 + \omega x_3)/\sqrt{3} \end{pmatrix} \quad (54)$$

where ω is the complex cube root of unity. The tensor product of two S_3 doublets, (x_1, x_2) and (y_1, y_2) , is [52],

$$\mathbf{2} \otimes \mathbf{2} \rightarrow \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2} \quad (55)$$

where

$$\begin{aligned} \mathbf{1} &= x_1 y_1 + x_2 y_2, & \mathbf{1}' &= x_1 y_2 - x_2 y_1, \\ \mathbf{2} &= \begin{pmatrix} x_2 y_2 - x_1 y_1 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}. \end{aligned} \quad (56)$$

The Higgsino mass terms, constructed with the α and β must be an S_3 singlet $\mathbf{1}$, but not $\mathbf{1}'$. With this condition, we try to estimate the unmatched pairs of $\mathbf{5}_{hR}$ and $\bar{\mathbf{5}}_{hR}$, i.e., the number of massless Higgsino pairs. Note that SU(9)' confines at a high energy scale, and we consider the condensation $\bar{\mathbf{36}}' \cdot \mathbf{9}' \cdot \mathbf{9}'$. The Q_R charges of the hidden sector fields given by Eq. (52) are

$$\overline{\mathbf{36}}'(T_9^0) [+18], \quad \mathbf{9}'(T_4^0) [+19], \quad \mathbf{9}'(T_1^-) [+7]. \quad (57)$$

The condensation we consider is the scalar component of $\overline{\mathbf{36}}'$ and the fermion components of $\mathbf{9}'$. Then, the Q_R charge of $C \propto \overline{\mathbf{36}}'_{g^0} \cdot \mathbf{9}'_{g^1} \cdot \mathbf{9}'_{g^1}$ are either 54 (both $\mathbf{9}'$ from T_4^0), 30 (both $\mathbf{9}'$ from T_1^-), or 42 (one $\mathbf{9}'$ from T_4^0 and the other from T_1^-), i.e., $Q_R = 2$ modulo 4. Thus, the coupling $\alpha_1 \beta_1 C$ is allowed and α_1 and β_1 are removed at the scale where $SU(9)'$ confines, $\langle C \rangle$. Note, however, that $\mathbf{Z}_{4R}$ is broken to $\mathbf{Z}_{2R}$ at the scale $\langle C \rangle$.

Next, let us consider doublets. Two doublets, one each from the α and β sets, i.e., α_2 and β_2 [the form given in Eq. (54)], combine to form 1: $\beta_2 \xi_2$, as noted in Eq. (55). As above, the coupling $\alpha_2 \beta_2 C$ is allowed and one of α_2 and one of β_2 are removed at the scale where $SU(9)'$ confines, $\langle C \rangle$, and there remains only one pair of $\mathbf{5}_{hR}$ and $\bar{\mathbf{5}}_{hR}$. This is the vector-like pair of MSSM, $\mathbf{5}'_{hR}$ and $\bar{\mathbf{5}}'_{hR}$. Summarizing,

$$\mathbf{5}_{hR} \oplus \bar{\mathbf{5}}_{hR}: \quad \text{one MSSM pair, two pairs at the scale } \langle C \rangle. \quad (58)$$

From two permuting elements x_2 and x_3 , the S_2 singlet is $(x_2 + x_3)/\sqrt{2}$, and the orthogonal component to that is $(x_2 - x_3)/\sqrt{2}$. Thus, the MSSM doublets are

$$\frac{1}{\sqrt{2}}(\mathbf{5}_{hR}(T_1^-)_2 - \mathbf{5}_{hR}(T_1^-)_3), \quad \frac{1}{\sqrt{2}}(\bar{\mathbf{5}}_{hR}(T_1^-)_2 - \bar{\mathbf{5}}_{hR}(T_1^-)_3). \quad (59)$$

B. The third family members

It is natural to choose the singlet combination $(x_1 + x_2 + x_3)/\sqrt{3}$ as the third family member, and the doublet as the light family members. The quark and lepton mass matrix can be constructed based on these bases.

V. CONCLUSION

We derived a $\mathbf{Z}_{4R}$ parity in a family unification model with a GUT anti- $SU(5)$ possessing three families and one pair of Higgs quintets. Spontaneous symmetry breaking of anti- $SU(5)$ GUT is achieved by $\langle \mathbf{10}_{+1} \rangle \oplus \langle \bar{\mathbf{10}}_{-1} \rangle$. The $\mathbf{Z}_{4R}$ parity together with the permutation symmetry S_3 are useful to select the third family members and one MSSM pair of the Higgs quintets.

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TABLE XX. $\tilde{\chi}(k, l)$ in the $\mathbf{Z}_{12-I}$ orbifold [53].

k	$l =$										
	0	1	2	3	4	5	6	7	8	9	10
1	3	3	3	3	3	3	3	3	3	3	3
2	3	3	3	3	3	3	3	3	3	3	3
3	4	1	1	4	1	1	4	1	1	4	1
4	9	1	1	1	9	1	1	1	9	1	1
5	3	3	3	3	3	3	3	3	3	3	3
6	16	1	1	4	1	1	16	1	1	4	1

APPENDIX A: FIXED POINTS IN $\mathbf{Z}_{12-I}$

In the k th twisted sector of $\mathbf{Z}_N$ orbifold, multiplicities $\mathcal{P}_k$ is

$$\mathcal{P}_k = \frac{1}{N} \sum_{l=0}^N \tilde{\chi}(k, l) e^{i2\pi l \Theta_k}, \quad (A1)$$

where $\tilde{\chi}(k, l)$ in the $\mathbf{Z}_{12-I}$ orbifold are listed in Table XX. In the calculation of multiplicity the information on Θ is the key. The RHS mover is denoted by the spinor

$$s = (s_0; \tilde{s}) = (\ominus \text{ or } \oplus; \pm, \pm, \pm), \quad (A2)$$

where s_0 corresponds to L- or R- movers. In this paper, + or - in the spinor form as in Eqs. (27) and (A2), denote $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The phase Θ_k in the k th twisted sector in is

$$\begin{aligned} \Theta_k &= \sum_j (N_L^j - N_R^j) \hat{\phi}^j - \frac{k}{2} (V_a^2 - \phi_s^2) + (P + kV_a) \cdot V_a \\ &\quad - (\tilde{s} + k\phi_s) \cdot \phi_s + \text{integer}, \\ &= -\tilde{s} \cdot \phi_s + \Delta_k, \end{aligned} \quad (A3)$$

where Δ_k is

$$\Delta_k = (P + kV_a) \cdot V_a - \frac{k}{2} (V_a^2 - \phi_s^2) + \sum_j (N_L^j - N_R^j) \hat{\phi}^j \quad (A4)$$

$$\equiv \Delta_k^0 + \Delta_k^N. \quad (A5)$$

V_a is the shift vector V distinguished by Wilson lines a , $V_{0,+,-}$, and

$$\Delta_k^0 = P \cdot V_a + \frac{k}{2} (-V_a^2 + \phi_s^2), \quad (A6)$$

$$\Delta_k^N = \sum_j (N_L^j - N_R^j) \hat{\phi}^j, \quad (A7)$$

TABLE XXI. Multiplicities in the k th twisted sectors of $\mathbf{Z}_{12-I}$. $\mathcal{P}_k(\text{angle})$ is calculated with $\text{angle} = \frac{2\pi}{12} \cdot I$ in Eq. (A1).

i	Multiplicity				
	$\mathcal{P}_k(0)$	$\mathcal{P}_k(\pm \frac{\pi}{3})$	$\mathcal{P}_k(\pm \frac{2\pi}{3})$	$\mathcal{P}_k(\pm \frac{\pi}{2})$	$\mathcal{P}_k(\pi)$
1	3	0	0	0	0
2	3	0	0	0	0
3	2	0	1	0	0
4	3	0	0	2	2
5	3	0	0	0	0
6	4	2	3	0	2

TABLE XXII. H momenta, p_{orb} , in the twisted sectors of $\mathbf{Z}_{12-I}$, Table 10.1 of [53]. Requiring $(p_{\text{vec}} + k\hat{\phi})^2 =$ [2nd line in Eq. (A9)], we have p_{vec} in the 4th column. In the last column, δ_k is shown, from which we have the energy contribution from right movers $2\delta_k \geq 0$.

Orbifold	Twisted Sector	$k\hat{\phi}$	p_{vec}	p_{orb}	$-kp_{\text{vec}}^{\text{th}} \cdot \phi_s$	δ_k
$\mathbf{Z}_{12-I}$	T_1	$(\frac{5}{12}, \frac{4}{12}, \frac{1}{12})$	$(-1, 0, 0)$	$(\frac{-7}{12}, \frac{+4}{12}, \frac{+1}{12})$	$\frac{5}{12}$	$\frac{1}{12}$
	T_2	$(\frac{5}{6}, \frac{4}{6}, \frac{1}{6})$	$(-1, 0, 0)$	$(\frac{-1}{6}, \frac{+4}{6}, \frac{+1}{6})$	$\frac{10}{12}$	0
	T_3	$(\frac{5}{4}, \frac{4}{4}, \frac{1}{4})$	$(-1, -1, -1)$	$(\frac{1}{4}, \frac{0}{4}, \frac{-3}{4})$	$\frac{6}{12}$	$\frac{3}{12}$
	T_4	$(\frac{5}{3}, \frac{4}{3}, \frac{1}{3})$	$(-2, -1, 0)$	$(\frac{-1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{8}{12}$	0
	T_5	$(\frac{25}{12}, \frac{20}{12}, \frac{5}{12})$	$(-2, -2, -1)$	$(\frac{1}{12}, \frac{-4}{12}, \frac{-7}{12})$	$\times$	$\times$
	T_6	$(\frac{5}{2}, \frac{4}{2}, \frac{1}{2})$	$(-2, -2, 0)$	$(\frac{1}{2}, \frac{0}{2}, \frac{1}{2})$	0	0
	T_7	$(\frac{35}{12}, \frac{28}{12}, \frac{7}{12})$	$(-3, -1, 0)$	$(\frac{-1}{12}, \frac{4}{12}, \frac{7}{12})$	$\frac{1}{12}$	$\frac{1}{12}$
	T_8	$(\frac{40}{12}, \frac{32}{12}, \frac{8}{12})$	$(-3, -3, -1)$	$(\frac{1}{3}, \frac{-1}{3}, \frac{-1}{3})$	$\frac{8}{12}$	0
	T_9	$(\frac{45}{12}, \frac{36}{12}, \frac{9}{12})$	$(-4, -3, -1)$	$(\frac{-3}{12}, \frac{0}{12}, \frac{-3}{12})$	$\frac{-3}{12}$	$\frac{3}{12}$

and $\hat{\phi}^j$ is in the range $0 < \hat{\phi}^j \leq 1 \bmod \text{integer}$. The oscillator contributions in Δ_k^N is for $N_{L,R} \geq 0$. We satisfy the massless spectrum condition for gauge sectors by

$$(P + kV_a)^2 + 2 \sum_j N_L^j \hat{\phi}^j = 2\tilde{c}_k, \quad (\text{A8})$$

where $2\tilde{c}_k$ of $\mathbf{Z}_{12-I}$ orbifold is

$$2\tilde{c}_k: \frac{210}{144}(k=1), \quad \frac{216}{144}(k=2), \quad \frac{234}{144}(k=3), \quad \frac{192}{144}(k=4), \quad \frac{210}{144}(k=5), \quad \frac{216}{144}(k=6), \quad (\text{A9})$$

which will lead to $\mathcal{N} = 1$ supersymmetry in 4D.

In the twisted sectors with $3a_3 = 0 \bmod \text{integer}$, i.e., at $T_{3,6,9}$ and also U [37],⁴ the gauge invariance condition has to be taken into account explicitly,

$$(P + kV_0) \cdot a_3 = 0. \quad (\text{A10})$$

We distinguish the twisted sector T_k by 0, + and -. The phase Δ_k^0 of Eq. (A7) contains an extra $\frac{k}{2}$ factor, but at T_6 the factor $\frac{k}{2}$ is an integer. At $T_{1,2,4,5}$, we distinguish three fixed points just by $V_{0,+,-}$.

APPENDIX B: USEFUL TABLES FOR MODEL BUILDING

Here we present multiplicities in Table XXI and H -momenta in Table XXII [53].

⁴ T_9 contains the CTP conjugate states of T_3 .

TABLE XXIII. The multiplicities for each $\hat{s}$ calculated with the row $i = 3$ in Table XXI.

Chirality	$\tilde{s}$	$-\tilde{s} \cdot \phi_s$	$-k p_{\text{vec}}^{k\text{th}} \cdot \phi_s$	$k P_9 \cdot V_0$	$(k/2)\phi_s^2$	$-(k/2)V_0^2$	Δ_9^N	$-\delta_9^N$	Θ_9	Mult. of SU(9)
$\ominus = L$	$(- - -)$	$\frac{+5}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{+2}{12} + \frac{-2}{12}$	2
$\ominus = L$	$(- + +)$	$\frac{0}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-5}{12}$	0
$\ominus = L$	$(+ - +)$	$\frac{-1}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-6}{12}$	0
$\ominus = L$	$(+ + -)$	$\frac{-4}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-9}{12}$	0
$\oplus = R$	$(+ + +)$	$\frac{-5}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-10}{12}$	0
$\oplus = L$	$(+ - -)$	$\frac{0}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-5}{12}$	0
$\oplus = L$	$(- + -)$	$\frac{+1}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-4}{12}$	1
$\oplus = L$	$(- - +)$	$\frac{+4}{12}$	$\frac{-3}{12}$	$\frac{+6}{12}$	$\frac{189}{144}$	$\frac{+243}{144}$	$\frac{0}{12}$	$\frac{-6}{12}$	$\frac{-1}{12}$	0

APPENDIX C: MULTIPLICITIES FOR EACH $\hat{s}$

The multiplicities are calculated for each $\hat{s}$. In Table XXIII, we present one example for a complete list of the rhs spin vectors $\hat{s}$, deriving 36 of SU(9)' in T_9^0 .

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