From integrability to the Galois coaction on Feynman periods

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We argue that the description of Feynman loop integrals as integrable systems is intimately connected with their motivic properties and the action of the Cosmic Galois Group. We show how in the case of a family of fishnet graphs, coaction relations between them follow directly from iterative constructions of Q -functions in the quantum spectral curve formalism. Using this observation we conjecture a “differential equation for numbers” that enter these periods.

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I. INTRODUCTION

One of the lessons from recent developments in perturbative quantum field theory is that the most efficient way to deal with Feynman loop integrals is to avoid them where possible. This point of view is prevailingly supported by the achievements of the cluster bootstrap programme for scattering amplitudes in maximally supersymmetric Yang-Mills theory (MSYM), in which loop integration is traded with linear constraints on a linear space of functions conjectured to contain the amplitude [1]. These methods rely heavily on the coalgebra structure exhibited by Feynman integrals as iterated period integrals. In particular the symbol, the central ingredient of cluster bootstrap, is essentially an iteration of the Galois coaction on multiple polylogarithms [2,3].

A number of powerful conjectures on periods in QFT are expressed in terms of a motivic Galois coaction once they are lifted to motivic periods [4–6]. The (motivic) periods in ϕ^4 theory are expected to be closed under a “cosmic” Galois group that acts on these periods, and this was the guiding principle of studies of these periods through 11 loops [7]. Dimensionally-regulated one- and two-loop integrals observe this principle order-by-order in the expansion around an integer dimension, where the coaction is expressed in terms of the original graph with cut and contracted edges [8]. Furthermore, evidence for the Galois coaction relations have been observed in six-particle amplitudes in MSYM evaluated on special kinematical

datapoints [1] and in results for the electron anomalous magnetic moment. And most recently, the coaction on general one-loop graphs in four dimensions has been studied in [9]. All these studies that reveal fascinating properties of Feynman periods employ their often-complicated integral representations, and the underlying geometry determined by the integrands. More often than not, this complexity is in contrast with the simplicity of the emergent structures.

On the other hand, the description of certain Feynman integrals as integrable systems paints an entirely different picture [10–13]. This is for example the case in the fishnet model [10], which is an *integrable* QFT of two complex scalars ϕ_1 and ϕ_2 interacting with a particular quartic coupling of the form $\text{Tr}\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger$. The virtue of this coupling is that it only allows up to a single Feynman diagram to contribute to several quantities at each loop order, and this provides an opportunity to realise such descriptions and the possibility of computing the periods with no mention of loop integration whatsoever. The state-of-the-art approach on integrability in QFT is formulated in terms of the quantum spectral curve [14,15] and it has been successfully adapted to describe observables in the fishnet model [16].

The purpose of this paper is to argue that there is a close link between the integrability of the fishnet model and the coaction properties of its periods: We will present a problem in which *coaction relations can be straightforwardly derived from the properties of the quantum spectral curve*.

To this end, we will focus on the periods of wheel graphs with three spokes in the planar limit of the fishnet model. These graphs enter the perturbative expansion of the correlator

$$\langle \text{Tr}\phi_1^3(0)\text{Tr}\bar{\phi}_1^3(x) \rangle = [x^2]^{-D}, \quad D = 3 - \xi^3\delta \quad (1)$$

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FIG. 1. (a) A Feynman diagram that contributes to the operator (1) and (b) a corresponding Feynman graph.

where D is the quantum scaling dimension of the operator $\text{Tr}\phi_1^3$. The anomalous dimension δ has a perturbative expansion in the cube of the fishnet coupling ξ :

$$\delta = \sum_{m=0}^{\infty} \delta_m \xi^{3m}. \quad (2)$$

This expansion is justified from the diagrams that contribute to these correlators [depicted in Fig. 1(a)] by noting that the number of interactions is always a multiple of three. In the Feynman diagram, the solid and dashed lines represent propagators of complex scalars ϕ_1 and ϕ_2 , respectively. One of the operators in (1) is placed at the origin while the other is imagined to reside at infinity. Up to a normalization the numbers δ_m are identified [17] with the periods of the corresponding “wheel” Feynman graph in Fig. 1(b).

Direct calculations are only available for δ_0 [19] and for δ_1 [7], and in general it is proven that they are linear combinations of multiple zeta values (MZVs) [6,18]. Owing to the integrability of the fishnet model, powerful techniques of integrability can be employed to algorithmically determine further anomalous dimensions δ_m [16] in terms of these numbers.

II. SOLVING THE $L=3$ BAXTER EQUATION RECURSIVELY

As detailed in [16], in the QSC setup, the wheel periods are determined through a quantization condition

$$q_2(0, \mu)q_4(0, -\mu) + q_2(0, -\mu)q_4(0, \mu) = 0, \quad (3)$$

where the Q-functions $q_2(u, \mu)$ and $q_4(u, \mu)$ satisfy the Baxter equation,

$$\square q_\alpha = \frac{\mu}{u^3} q_\alpha + \frac{(D-3)(D-1)}{4u^2} q_\alpha \quad \alpha = 2, 4 \quad (4)$$

and are distinguished by their asymptotics:

$$\begin{aligned} q_2(u, \mu) &= u^{+D/2-1/2} + \mathcal{O}(u^{-1}) \\ q_4(u, \mu) &= u^{-D/2+3/2} + \mathcal{O}(u^{-1}). \end{aligned} \quad (5)$$

The $\square$ denotes the second-order difference operator: $\square f(u) = f(u+i) - 2f(u) + f(u-i)$. The parameter μ is related to the fishnet coupling as $\mu^2 = \xi^3$ [20].

To compute the anomalous dimensions, we consider the expansions of q_α in δ and μ in with coefficients $q_\alpha^{i,j}(u)$:

$$q_\alpha(u, \mu) = \sum_{i=-1}^{\infty} \sum_{j=-1}^{\lfloor i/2 \rfloor} q_\alpha^{i,j}(u) \mu^i \delta^j, \quad (6)$$

with

$$\begin{aligned} q_2^{i,j} &= 0 \quad \text{for } i < -1 \quad \text{or } j < -1, \\ q_4^{i,j} &= 0 \quad \text{for } i < 0 \quad \text{or } j < 0, \end{aligned} \quad (7)$$

while keeping in mind that δ also has a μ expansion as in Eq. (2). For a similar perturbative analysis of the Konishi anomalous dimensions in MSYM in the context of QSC see [21].

The Baxter equation (4) implies the following equation for the components of q_2 and q_4 :

$$\square q_\alpha^{i,j} = \frac{1}{2u^2} q_\alpha^{i-2,j-1} - \frac{1}{4u^2} q_\alpha^{i-4,j-2} + \frac{1}{2u^3} q_\alpha^{i-1,j}, \quad (8)$$

which can be algorithmically solved as a linear combination [22] of multiple Hurwitz zeta functions (MHZFs):

$$\eta_{s_1, \dots, s_k}(u) = \sum_{n_1 > \dots > n_k \geq 0} \frac{1}{(u + in_1)^{s_1} \dots (u + in_k)^{s_k}} \quad (9)$$

using well-known relations that they satisfy [14–16]. Importantly, when evaluated at $u = \sqrt{-1}$, they reduce to multiple zeta values (MZVs):

$$\eta_{s_1, \dots, s_k}(\sqrt{-1}) = (-1)^{\frac{1}{2} \sum_{i=1}^k s_i} \zeta_{s_1, \dots, s_k}. \quad (10)$$

III. PATTERNS IN NESTED WHEELS

A. General structure

The poles in u in the rhs of Eq. (8) imply that δ_m are linear combinations of MZVs of weight up to $4m+3$ and indices $\{1, 2, 3\}$ [23]. The first few numbers, computed in an “ f -alphabet” [4] using the Maple package HyperlogProcedures [24] are presented in Table I. [25] The expressions for $\delta_{\leq 3}$ match the those provided in [16] in terms of MZVs, and we obtained all periods through $\delta_{\leq 7}$. Their lengthy expressions, as well as representations as MZVs, can be found in the ancillary file `wheels.m` [26].

The main point of rewriting the MZVs in an f -alphabet is that it is a basis where the Galois coaction is simply deconcatenation [4]:

TABLE I. First few wheel periods in an f -alphabet.

m	δ_m
0	$-12f_3$
1	$-189f_7 + 288f_{3,3}$
2	$+\frac{2222721}{40}f_{11} + 6804f_{3,7} + 6804f_{7,3} - 18144f_{3,3,3} - 2160f_{3,3,5} - 2160f_{3,5,3} + 1728f_{5,3,3}$
3	$-\frac{140451186537}{30800}f_{15} - \frac{13336326}{5}f_{3,11} + 142884f_{7,7} - \frac{13336326}{5}f_{11,3}$ $-489888f_{3,3,7} - 67320f_{3,3,9} - 34020f_{3,5,7} - 489888f_{3,7,3} - 34020f_{3,7,5} - 67320f_{3,9,3} + 47628f_{5,3,7}$ $+47628f_{5,7,3} - 489888f_{7,3,3} - 34020f_{7,3,5} - 34020f_{7,5,3} - 13752f_{9,3,3}$ $+1866240f_{3,3,3,3} + 311040f_{3,3,3,5} + 311040f_{3,3,5,3} + 124416f_{3,5,3,3} - 248832f_{5,3,3,3} + 124416f_{3,3,3,3,3}$

$$\Delta f_w = \sum_{w_1 w_2 = w} f_{w_1} \otimes f_{w_2}, \quad (11)$$

provided that w only contains odd letters, while multiplication of such words is the shuffle product

$$f_{w_1} f_{w_2} = \sum_{w \in w_1 \Delta w_2} f_w. \quad (12)$$

Multiplication of a word by $\zeta_2 = f_2$ works by appending the letter 2 to it:

$$f_2 f_w = f_{w2}. \quad (13)$$

One can also define derivatives that act on the first entry of a word in an f -alphabet in the following way:

$$\partial_{2n+1} f_{aw} = \begin{cases} f_w & a = 2n+1 \\ 0 & a \neq 2n+1. \end{cases} \quad (14)$$

Consequently this framework allows one to consider “differential equations” for MZVs, which we will use to relate various numbers that appear in wheel periods.

We first note some glaring properties of these numbers: Unlike wheels with two frames [10], the depths of MZVs that appear in δ_m increase with m and use all available depths, cf. Table II.

The f basis is also convenient for implementing the “sv” map, that maps MZVs to a single-valued subspace. It can thus be easily checked that all δ_m we consider are single-valued MZVs.

Moreover, they include only odd letters and are symmetric in the last two letters, which is a property shared by other wheel families in the fishnet model [27] but not generic ϕ^4 periods [7], still begs for a good explanation.

TABLE II. Maximum depth of MZVs in f -basis that appear in each δ_m .

m	0	1	2	3	4	5	6	7	...
Maximum depth in δ_m	1	2	3	5	6	7	9	10	...

B. Wheel MZVs and their coaction

The quantization condition (3) leads to a natural decomposition of a given δ_m as a polynomial in $\delta_{m' < m}$ with MZV coefficients. To see this we first observe that using

$$q_2^{i,-1} q_4^{j+1,0} = q_2^{j,-1} q_4^{i+1,0} \quad \forall i, j \quad (15)$$

the condition (3) can be rewritten in the following form:

$$W(\delta) := \sum_{i=0}^{\infty} W_i(\mu) \delta^i = 0. \quad (16)$$

The coefficients $W_i(\mu)$ are generating functions

$$W_i(\mu) := \sum_{n=0}^{\infty} W_{i,n} \mu^{2n} \quad (17)$$

of “wheel MZVs” $W_{i,n}$ that are defined as bilinear combinations of $q_2^{i,j}$ and $q_4^{i,j}$, evaluated at $u = 0$:

$$W_{m,n} = 4 \sum_{i,j} (-1)^{i-\frac{1}{2}} q_4^{i,j}(0) q_2^{2m+2-i,n-j}(0), \quad (18)$$

where the sum is over all pairs i, j that give a nonzero contribution according to Eq. (7).

It is now easy to see the structure of the polynomials one obtains by expanding (16) in μ :

$$\begin{aligned} W = & W_{0,0} + \mu^2(W_{0,1} + W_{1,1}\delta_0) \\ & + \mu^4(W_{0,2} + W_{1,2}\delta_0 + W_{2,2}\delta_0^2 + W_{1,1}\delta_1) \\ & + \mu^6(W_{0,3} + W_{1,3}\delta_0 + W_{2,3}\delta_0^2 + W_{3,3}\delta_0^3 \\ & + W_{1,2}\delta_1 + 2W_{2,2}\delta_0\delta_1 + W_{1,1}\delta_2) \\ & + \mathcal{O}(\mu^8). \end{aligned} \quad (19)$$

TABLE III. First few coefficients in the μ expansion of W_0 .

m	$W_{0,m}$
1	$-12f_3$
2	$-189f_7 + 144f_{3,2^2}$
3	$\frac{2222721}{40}f_{11} + \frac{342}{5}f_{3,2^4} + 2268f_{7,2^2} - 1944f_{3,5,3}$
4	$-\frac{140451186537}{30800}f_{15} + \frac{682046379}{3455}f_{3,2^6} + \frac{10773}{10}f_{7,2^4} - \frac{6668163}{10}f_{11,2^2} + 13392f_{3,3,9} - 51030f_{3,5,7}$ $-20088f_{3,9,3} - 30618f_{7,5,3} - 51840f_{3,3,3,2^3} + 31104f_{3,3,5,2^2} + 38880f_{3,5,3,2^2} + 46656f_{3,3,3,3,3}$

Requiring the vanishing of (19) at each order, and noting that $W_{1,1} = -1$ allows us to express any δ_m as a polynomial in $\delta_{m' < m}$ with MZV coefficients.

As a first iterative pattern, we note that the only new numbers that enter a period δ_m are the coefficients of the powers of δ_0 , while coefficients of other monomials will have occurred as the coefficient of some power of δ_0 in a smaller wheel.

We will now show that the W_i have curious properties under the coaction that they inherit from the solutions of the Baxter equation. As an explicit illustration of these, we will focus on the sequence $W_{0,m}$. First few elements of this sequence are presented in Table III.

Conjecture 1. The components $q_2^{i-1} = q_4^{i,0}$ of solutions to the system of equations (4) & (5) satisfy

$$c_k^{-1} \partial_k q_4^{i,0}(0) = \begin{cases} c_{k-4}^{-1} \partial_{k-4} q_4^{i-2,0}(0) & k \equiv 3 \pmod{4} \\ 0 & k \equiv 1 \pmod{4} \end{cases} \quad (20)$$

with the same rational constant c_k for all $i \geq 2$, with the first few numbers c_k :

$$(c_3, c_7, c_{11}, \dots) = \left(1, \frac{63}{4}, -\frac{740907}{160}, \frac{46817062179}{123200}, \frac{2748660980272441072644717}{94776262380800}, \dots\right). \quad (21)$$

The relation (20) together with the fact that $q_2^{i-1} = q_4^{i,0}$ immediately lead to the differential equations that hold for any $k \equiv 3 \pmod{4}$:

$$\mathcal{O}_k W_0 := (c_k^{-1} \partial_k - c_{k-4}^{-1} \partial_{k-4}) W_0 = 0 \quad (22)$$

while any other derivative ∂_k with $k \not\equiv 3 \pmod{4}$ annihilates W_0 .

In other words, the coaction relations between wheel MZVs (22) rely on a conjectural identity for the Q-functions, which we have verified for $m \leq 7$, ie up to weight 31.

C. A differential equation for wheel MZVs

Equation (22) in fact follows from a remarkable relation satisfied by W_0 . We first note that the differential equation (22) implies that W_0 has the form

$$W_0 = Z \cdot S, \quad (23)$$

where $\cdot$ denotes concatenation that is distributive over addition, Z has a Taylor expansion in μ that involves only *single* zeta values:

$$Z(\mu) = \sum_{m=1}^{\infty} c_{4m-1} f_{4m-1} \mu^m, \quad (24)$$

and S is a function that can be worked out *a posteriori*. Upon defining

$$\tilde{Z}(\mu) = \sum_{m=1}^{\infty} \tilde{c}_{4m+1} f_{4m+1} \mu^m \quad (25)$$

with

$$(\tilde{c}_5, \tilde{c}_9, \tilde{c}_{13}, \dots) = \left(1, \frac{31}{3}, -\frac{718191}{200}, \frac{7408590567067}{484000}, -\frac{1276765401753338789096939955193067}{2169303723306720000}, \dots\right), \quad (26)$$

and solving for the function S , we observe that W_0 obeys (up to the available order) an almost homogeneous differential equation of the form:

$$W_0 - \frac{1}{\mu^4} Z \cdot (Z \cdot \partial_3 \partial_3 + \tilde{Z} \cdot \partial_5 \partial_3) W_0 = Z \cdot P, \quad (27)$$

where $P = P(\mu)$ is a function whose expansion only contains powers of π :

$$P(\mu) = 12 + 144\mu^2 \zeta_4 + \frac{342}{5} \mu^8 \zeta_8 + \dots \quad (28)$$

Since the powers of ζ_2 cancel out in δ_m , we can ignore the $\mathcal{O}(\mu)$ terms in P .

To emphasize the predictivity of (27) we remind that both Z and $\tilde{Z}$ are $\mathcal{O}(\mu)$. In summary, *the coactions on all $W_{k,0}$ are encoded in a pair of functions whose expansion contains only single zeta values.* This is reminiscent of an observation made about string-theory amplitudes considered in [28]

While we focused on the simplest case of $W_0(\mu)$ in this work, similar (and more involved) constructions are in principle possible for other sequences of wheel MZVs. For example the analogue of Eq. (22) for W_1 is

$$(\mathcal{O}_k - \mu^2 \mathcal{O}_{k-4}) W_1 = 0, \quad (29)$$

and this relation relies on a relation satisfied by “higher” components of the Q-functions:

$$\begin{aligned} (\mathcal{O}_k - 2\mu^2 \mathcal{O}_{k-4} + \mu^4 \mathcal{O}_{k-8}) q_4^{i,1} &= 0 \\ (\mathcal{O}_k - 2\mu^2 \mathcal{O}_{k-4} + \mu^4 \mathcal{O}_{k-8}) q_2^{i,0} &= 0, \end{aligned} \quad (30)$$

for all $k = 3 \bmod 4$ in all the available data.

We conclude the discussion with a comment on the prospects of generalizations of (22) and (29). One may be tempted to speculate equations for W_i using operators $\mathcal{O}_k^{(i)}$ that are defined recursively as

$$\mathcal{O}_k^{(i)} := \mathcal{O}_k^{(i-1)} - \mu^2 \mathcal{O}_{k-4}^{(i-1)}, \quad \mathcal{O}_k^{(0)} = \mathcal{O}_k. \quad (31)$$

However, given the range of depths that enter these equations being comparable to the amount of data available up to weight 31 does not allow for convincing checks. Therefore, it will be crucial to prove Conjecture 1, and its generalizations to make further statements on the coaction on wheel periods.

D. Subleading-weight pieces as polynomials in δ_m

Separating the anomalous dimensions into the leading- and subleading-weight parts:

$$\delta_m = \bar{\delta}_m + (-2)^m \tilde{\delta}_m \quad (32)$$

where $\tilde{\delta}_m$ contains all terms of weight *less than* $4m - 3$, gives further insight to patterns that relate various anomalous dimensions to each other.

We observe that all known $\tilde{\delta}_m$, can be expressed in terms of $\delta_{m'}$ with $m' < m$:

$$\begin{aligned} \tilde{\delta}_1 &= -\delta_0^2 \\ \tilde{\delta}_2 &= -8\delta_1\delta_0 + \frac{1}{2}(5\delta_0^2 - 4\delta_1)\delta_0 \\ \tilde{\delta}_3 &= -16\delta_2\delta_0 + 4(7\delta_0^2 - 4\delta_1)\delta_1 - 7\delta_0^4 \\ \tilde{\delta}_4 &= -40\delta_3\delta_0 + 10(9\delta_0^2 - 4\delta_1)\delta_2 \\ &\quad - 30(4\delta_0^2 - 3\delta_1)\delta_1\delta_0 + 21\delta_0^5 \\ \tilde{\delta}_5 &= -96\delta_4\delta_0 + 24(11\delta_0^2 - 4\delta_1)\delta_3 - 88(5\delta_0^2 - 6\delta_1)\delta_0\delta_2 \\ &\quad - 44(15\delta_0^2 - 2\delta_1)\delta_1^2 - 48\delta_2^2 - 33(2\delta_0^2 - 15\delta_1)\delta_0^4 \\ &\quad \dots \end{aligned} \quad (33)$$

This motivates the following *all-loop* conjecture:

Conjecture 2. The subleading-weight parts of the anomalous dimensions $\tilde{\delta}_m$ are always polynomials in $\tilde{\delta}_{m' < m}$, and are given by the following formula:

$$\tilde{\delta}_m = (-1)^m (m+1) \sum_{\mathbf{a}} \frac{(-2)^{2-A} (2m+1)!}{(2m+3-A)!} \delta^{\mathbf{a}} \quad (34)$$

The sum is over all multi-plets $\mathbf{a} = (a_1, \dots, a_{|\mathbf{a}|})$ of *positive* integers subject to the conditions $\sum_{i=1}^{|\mathbf{a}|} ia_i = m+1$ and we used the shorthand using words $\mathbf{a} = (a_1, a_2, \dots)$:

$$\delta^{\mathbf{a}} = \prod_{i=1}^{|\mathbf{a}|} \frac{(\delta_{i-1})^{a_i}}{a_i!}, \quad (35)$$

where $|\mathbf{a}|$ denotes the length of $\mathbf{a}$, and $A = \sum_{i=1}^{|\mathbf{a}|} a_i$ the sum of its elements, i.e., the degree of $\delta^{\mathbf{a}}$. We verified Conjecture 2 through $\tilde{\delta}_7$, i.e., up to weight 30.

IV. DISCUSSION

In this paper we demonstrated that the integrability of fishnet theory has consequences on the properties of its periods under the Galois coaction. We identified a coaction principle that controls a sequence of numbers that enter the anomalous dimensions of the operator $\text{Tr}\phi_1^3$ and therefore a family of nested wheel diagrams. Using these observations we constructed a powerful equation satisfied by these numbers, showing that complicated MZV expressions are in fact encoded in sequence of single Riemann zeta values. Furthermore we identified other recursive features of these numbers: We presented a structure that shows that numbers associated with smaller wheels recur naturally in the results for larger ones. We also made a conjecture determining the parts of anomalous dimensions of subleading transcendental weight as polynomials in those of lower loop order.

While we analysed the anomalous dimensions order by order in the perturbative expansion of the anomalous dimension δ , it would be very much in the spirit of integrability to make nonperturbative statements about its coaction. In particular the coaction on the resummed

quantity $W_0(\mu)$ is reminiscent of that of hypergeometric functions, and we also remark that a nonphysical toy model of (4) in [16] is solved in terms of these functions, whose coaction has been recently considered in [8,29]. The relationship between the difference equations that these functions satisfy, and their Galois coaction is not yet clear.

We would like to point out that the cancellation of the terms involving powers of π in W_i in Eq. (16) relies on conspiracies between the periods δ_m and on combinatorial identities for MZVs in $\{1, 2, 3\}$. These would be generalizations of identities considered in number theory literature [30,31], and their proofs would be a key to advancing the observations presented here.

An obvious question, and one of the main motivations of this paper, is on the possibility of reversing the logic presented: The Galois coaction on periods is believed to be a universal property of QFTs, and coactions on Feynman amplitudes are expected to satisfy relations that may be expressed as differential equations. It would be an exciting

prospect to recast such relations into a difference equation such as (8), which could in turn imply previously unknown integrable structures in QFTs.

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- [1] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Phys. Rev. Lett.* **105**, 151605 (2010); L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou, and M. Spradlin, *J. High Energy Phys.* **02** (2017) 137; L. J. Dixon, J. M. Drummond, and J. M. Henn, *J. High Energy Phys.* **11** (2011) 023; J. Drummond, J. Foster, Ö. Gürdoğan, and G. Papathanasiou, *J. High Energy Phys.* **03** (2019) 087; J. M. Drummond, G. Papathanasiou, and M. Spradlin, *J. High Energy Phys.* **03** (2015) 072; S. Caron-Huot, L. J. Dixon, F. Dulat, M. Von Hippel, A. J. McLeod, and G. Papathanasiou, *J. High Energy Phys.* **09** (2019) 061;
 - [2] F. Brown, *arXiv:1102.1312*.
 - [3] A. Goncharov, *Duke Math. J.* **128**, 209 (2002).
 - [4] F. Brown, *arXiv:1102.1310*.
 - [5] F. Brown, *Commun. Numer. Theor. Phys.* **11**, 453 (2017).
 - [6] F. Brown and O. Schnetz, *Duke Math. J.* **161**, 1817 (2012).
 - [7] E. Panzer and O. Schnetz, *Commun. Numer. Theor. Phys.* **11**, 657 (2017).
 - [8] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, *Proc. Sci. RADCOR2019* (2019) 067 [*arXiv:1912.03205*]; *LL2018* (2018) 047 [*arXiv:1808.00069*]; *J. High Energy Phys.* **12** (2017) 090; *Phys. Rev. Lett.* **119**, 051601 (2017).
 - [9] M. Tapušković, *arXiv:1911.01540*.
 - [10] Ö. Gürdoğan and V. Kazakov, *Phys. Rev. Lett.* **117**, 201602 (2016); **117**, 259903(A) (2016).
 - [11] A. Zamolodchikov, *Phys. Lett.* **97B**, 63 (1980).
 - [12] A. Isaev, *Nucl. Phys.* **B662**, 461 (2003).
 - [13] F. Loebbert, J. Miczajka, D. Müller, and H. Münkler, *Phys. Rev. Lett.* **125**, 091602 (2020).
 - [14] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, *Phys. Rev. Lett.* **115**, 251601 (2015).
 - [15] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Phys. Rev. Lett.* **112**, 011602 (2014).
 - [16] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro, and G. Sizov, *J. High Energy Phys.* **01** (2018) 095.
 - [17] Cfr equation (2.3.9) of [18] where it is apparent that the definition of periods is identical to that of anomalous dimensions.
 - [18] E. Panzer, Ph.D. thesis, Humboldt University, 2015.
 - [19] D. J. Broadhurst, *Phys. Lett.* **164B**, 356 (1985).
 - [20] In contrast with [16], we use μ to denote the parameter m therein.
 - [21] S. Leurent and D. Volin, *Nucl. Phys.* **B875**, 757 (2013).
 - [22] With coefficients at most linear in u .
 - [23] It is straightforward to write $\eta_w(u)$ in terms of other MHZFs with argument $u + i$, and this does not change the alphabet of their indices.
 - [24] O. Schnetz, *Hyperlogprocedures* (2020), <https://www.math.fau.de/person/oliver-schnetz/>.
 - [25] We should note that “the f -alphabet” satisfying (11) is not unique.
 - [26] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevD.103.L081703>, for Computer-readable lists of wheel periods and wheel MZVs.
 - [27] We defer their study to a future work.
 - [28] O. Schlotterer and S. Stieberger, *J. Phys. A* **46**, 475401 (2013); J. Drummond and E. Ragoucy, *J. High Energy Phys.* **08** (2013) 135.
 - [29] F. Brown and C. Dupont, *arXiv:1907.06603*.
 - [30] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, *arXiv:math/9812020*.
 - [31] S. Charlton, *J. Number Theory* **148**, 463 (2015).