Alternative to collective coordinates

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(Received 8 December 2020; accepted 19 January 2021; published 12 February 2021)

Collective coordinates provide a powerful tool for separating collective and elementary excitations, allowing both to be treated in the full quantum theory. The price is a canonical transformation which leads to a complicated starting point for subsequent calculations. Sometimes the collective behavior of a soliton is simple but nontrivial, and one is interested in the elementary excitations. We show that in this case an alternative prescription suffices, in which the canonical transformation is not necessary. The use of a nonperturbative operator which creates a soliton state allows the theory to be constructed perturbatively in terms of the soliton normal modes. We show how translation invariance may be perturbatively imposed. We apply this to construct the two-loop ground state of an arbitrary scalar kink.

DOI: [10.1103/PhysRevD.103.L041701](https://doi.org/10.1103/PhysRevD.103.L041701)

I. INTRODUCTION

When a theory is reformulated in terms of collective coordinates, some phenomena involving large numbers of elementary quanta, such as plasma waves, can be treated in perturbation theory [1]. Two groups applied collective coordinates to quantum solitons in the fateful summer of 1975, allowing a treatment of the scattering of quantum solitons. In [2], following the spirit of [1], the collective coordinates are related to the elementary fields by a canonical transformation. This transformation allows a straightforward quantization of the system. However it comes with a price, the theory becomes rather complicated and power counting renormalizability is lost. Nevertheless, the authors are still able to study a soliton in motion and in Refs. [3,4] the two-loop correction to a soliton energy is reproduced. A functional integral formulation is employed to avoid the complications of quantum states.

In [5] collective coordinates are introduced without the canonical transformation. Following [1], the formulation was Hamiltonian. As no transformation was used, the authors are forced to quantize the theory without collective coordinates to determine the operator ordering in the theory with collective coordinates. The theory is still “considerably more complex than that usually encountered in quantum field theory” but is now simple enough that the authors can treat two soliton scattering. Due to the complexity of both approaches, quantum states were never

considered beyond one loop, where the theories are sums of uncoupled quantum harmonic oscillators.

Sometimes one is not interested in the collective excitations. For example, one may be interested in the quantum structure of a soliton in its rest frame. After all, intuition from large N [6] suggests that hadrons are quantum solitons and so their masses, form factors, and general matrix elements may be calculated by solving for the corresponding quantum state. In this case, we will propose a much simpler alternative to collective coordinates which allows one to pass to higher numbers of loops using reasonably elementary computations.

II. REVIEW

For concreteness we will describe our formalism [7,8] in the case of a real scalar field theory in $1 + 1$ dimensions, described by the Hamiltonian

$$H = \int dx \mathcal{H}(x),$$

$$\mathcal{H}(x) = \frac{1}{2} : \pi(x) \pi(x) :_a + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) :_a$$

$$+ \frac{1}{g^2} : V[g\phi(x)] :_a \quad (1)$$

where $::_a$ is the normal ordering defined below. Let

$$\phi(x, t) = f(x) \quad (2)$$

be a kink solution to the classical equations of motion. We will always work in the Schrodinger picture.

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We assume that $V''[gf(-\infty)] = V''[gf(\infty)]$ and define $M^2/2$ to be equal to this value. Here the prime denotes a functional derivative of V with respect to its argument.

As (2) is a solution of the classical equations of motion, we might be tempted to expand the quantum field as $\phi(x) = f(x) + \eta(x)$. Then $\phi \rightarrow \eta = \phi - f$ would be a passive transformation of the fields. After this transformation, the quadratic part of the Hamiltonian $H[\eta]$ would describe small perturbations about the classical kink, and one could proceed perturbatively.

Instead of this passive transformation of the fields, following the standard approach [9,10], we will consider an active transformation of the functionals acting on the fields. In particular, we transform the Hamiltonian

$$H[\phi, \pi] \rightarrow H'[\phi, \pi] = H[f + \phi, \pi]. \quad (3)$$

Below we will perform the same transformation on the momentum operator P . The new observation that lies behind our approach is that H' and H are unitarily equivalent, because

$$H' = \mathcal{D}_f^\dagger H \mathcal{D}_f \quad (4)$$

where we have defined the translation operator

$$\mathcal{D}_f = \exp\left(-i \int dx f(x) \pi(x)\right). \quad (5)$$

In general Eq. (4) will be applied to the regularized and renormalized H and will be our definition of the regularized and renormalized H' . This eliminates the need to separately regularize H' and then to guess the correct regulator matching condition to apply when both regulators are taken to infinity. It has long been known [11] that the dependence on the unknown matching condition leads to wrong answers in otherwise correct calculations. In (1) all UV divergences are removed by the normal ordering, but this choice was not necessary for our approach.

The unitary equivalence (4) implies that H and H' have the same spectrum. Therefore the vacuum and the kink ground state are eigenstates of both Hamiltonians, with the same eigenvalues. We are then free to use the vacuum Hamiltonian H to calculate the vacuum energy and the kink Hamiltonian H' to calculate the kink ground state energy. We will argue that this choice allows both calculations to be performed in perturbation theory.

This procedure will give us not only the energies of the kink states, but also the kink states themselves. Once an eigenstate of H' is found, one need only apply $\mathcal{D}_f$ to arrive at the corresponding H eigenstate. For example, if $|0\rangle$ is the eigenstate of H' corresponding to the kink ground state, then $\mathcal{D}_f|0\rangle$ is the corresponding eigenstate $|K\rangle$ of H .

This correspondence works already at tree level. Let $|\Omega\rangle$ be a free vacuum of H that satisfies [12]

$$\langle\Omega|\phi(x)|\Omega\rangle = 0. \quad (6)$$

Then $\mathcal{D}_f^\dagger|\Omega\rangle$ is the free vacuum as an eigenstate of H' .

On the other hand, $|\Omega\rangle$ is not an eigenstate of H' , or even of its free part. However it has a vanishing form factor (6) which one may expect for a tree-level vacuum. The corresponding state $\mathcal{D}_f|\Omega\rangle$ in the eigenbasis of H is obtained via the unitary transformation. As a result of (6) it has a form factor which reproduces the classical kink profile

$$\langle\Omega|\mathcal{D}_f^\dagger\phi(x)\mathcal{D}_f|\Omega\rangle = f(x). \quad (7)$$

The state $\mathcal{D}_f|\Omega\rangle$ is not the kink ground state $|K\rangle$, indeed it is not even an eigenstate of H just as $|\Omega\rangle$ is not an eigenstate of H' . However it has the correct form factor (7), leading one to suspect that the difference between the two can be calculated in perturbation theory as we now describe.

We have argued that the eigenstate $|0\rangle$ of H' corresponding to the kink ground state is close to $|\Omega\rangle$. Our goal in this article will be to obtain a procedure which provides successively better approximations to $|0\rangle$.

The corresponding eigenstate of H will be

$$|K\rangle = \mathcal{D}_f|0\rangle. \quad (8)$$

The eigenvalue equation

$$H'|0\rangle = Q|0\rangle \quad (9)$$

is easily solved at leading order as it reduces to a free theory and subleading orders can be solved by simply fixing higher order coefficients, and so it is in principle possible to find an all-orders solution for $|0\rangle$. To obtain the correct eigenstate, we fix the leading order energy to be minimal among eigenstates of the free part of H' . Had we not performed the unitary transformation, this program would have failed already at the leading order, due to the inverse coupling appearing in the leading term in the soliton mass.

To perform this perturbative calculation, we first expand H' in powers of the coupling

$$\begin{aligned} H' &= \mathcal{D}_f^\dagger H \mathcal{D}_f = Q_0 + \sum_{n=2}^{\infty} H_n, \\ H_2 &= \frac{1}{2} \int dx [:\pi^2(x):_a + :(\partial_x \phi(x))^2:_a \\ &\quad + V''[gf(x)]:\phi^2(x):_a]. \end{aligned} \quad (10)$$

Q_0 is the classical kink mass and H_n is order g^{n-2} .

At one loop, only H_2 is relevant. The constant frequency ω solutions of its classical equations of motion are continuum normal modes $g_k(x)$ with $\omega_k = \sqrt{M^2 + k^2}$, discrete breathers and a Goldstone mode $g_B(x) = f'(x)/\sqrt{Q_0}$. Note that the definition of ω_k fixes the parametrization of k up to a

sign. For brevity of notation, we will not distinguish between continuum solutions and breathers, and so it will be implicit that integrals over the continuous variable k include a sum over the breathers, and 2π times a Dirac delta function of continuum k should be understood as a Kronecker delta of breathers.

We choose the normalization conditions

$$\int dx g_{k_1}(x) g_{k_2}^*(x) = 2\pi \delta(k_1 - k_2), \quad \int dx |g_B(x)|^2 = 1 \quad (11)$$

and conventions

$$g_k(-x) = g_k^*(x) = g_{-k}(x), \quad \tilde{g}(p) = \int dx g(x) e^{ipx} \quad (12)$$

leading to the completeness relations

$$g_B(x) g_B(y) + \int \frac{dk}{2\pi} g_k(x) g_k^*(y) = \delta(x - y). \quad (13)$$

As it is independent of time, the Schrodinger picture field $\phi(x)$ may be expanded in any basis of functions even in the full, interacting theory. We will expand it in terms of plane waves

$$\begin{aligned} \phi(x) &= \int \frac{dp}{2\pi} \left(A_p^\dagger + \frac{A_{-p}}{2\omega_p} \right) e^{-ipx}, \\ \pi(x) &= i \int \frac{dp}{2\pi} \left(\omega_p A_p^\dagger - \frac{A_{-p}}{2} \right) e^{-ipx}, \end{aligned} \quad (14)$$

and also normal modes [13]

$$\begin{aligned} \phi(x) &= \phi_0 g_B(x) + \int \frac{dk}{2\pi} \left(B_k^\dagger + \frac{B_{-k}}{2\omega_k} \right) g_k(x), \\ \pi(x) &= \pi_0 g_B(x) + i \int \frac{dk}{2\pi} \left(\omega_k B_k^\dagger - \frac{B_{-k}}{2} \right) g_k(x). \end{aligned} \quad (15)$$

Define the plane wave (normal mode) normal ordering $::_a$ ($::_b$) by moving all $A^\dagger$ (all ϕ_0 and $B^\dagger$) to the left. The canonical algebra obeyed by $\phi(x)$ and $\pi(x)$ then implies

$$\begin{aligned} [A_p, A_q^\dagger] &= 2\pi \delta(p - q), \\ [\phi_0, \pi_0] &= i, \quad [B_{k_1}, B_{k_2}^\dagger] = 2\pi \delta(k_1 - k_2). \end{aligned} \quad (16)$$

Decomposing fields in terms of the plane wave operators, Bogoliubov transforming to the normal mode operators and then normal mode normal ordering one finds that the one-loop Hamiltonian is a sum of quantum harmonic oscillators plus a free quantum mechanical particle for the center of mass

$$\begin{aligned} H_2 &= Q_1 + \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi} \omega_k B_k^\dagger B_k, \\ Q_1 &= -\frac{1}{4} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{(\omega_p - \omega_k)^2}{\omega_p} \tilde{g}_k^2(p) \\ &\quad - \frac{1}{4} \int \frac{dp}{2\pi} \omega_p \tilde{g}_B(p) \tilde{g}_B(p), \end{aligned} \quad (17)$$

where Q_1 is the one-loop kink mass. The one-loop kink ground state $|0\rangle_0$ is therefore the solution of

$$\pi_0 |0\rangle_0 = B_k |0\rangle_0 = 0. \quad (18)$$

The whole spectrum may be obtained exactly at one-loop by creating normal modes with $B_k^\dagger$ and boosting with $e^{i\phi_0 k}$. The state $|0\rangle_0$ is the first term in the semiclassical expansion in powers of $\sqrt{\hbar}$

$$|0\rangle = \sum_{i=0}^{\infty} |0\rangle_i \quad (19)$$

where the n -loop ground state is the sum up to $i = 2n - 2$.

III. IMPOSING TRANSLATION INVARIANCE

We will now consider the construction of the ground state $|K\rangle$ at higher orders. As the one-loop spectrum is known exactly, the generalization of what follows to other states is trivial. Recall that, using (8), it is sufficient to construct $|0\rangle$. $|K\rangle$ is annihilated by the momentum operator

$$P = - \int dx \pi(x) \partial_x \phi(x). \quad (20)$$

Therefore $|0\rangle$ is annihilated by its unitary transform

$$P' = \mathcal{D}_f^\dagger P \mathcal{D}_f = P - \sqrt{Q_0} \pi_0. \quad (21)$$

As g has dimensions of $[\text{action}]^{-1/2}$, the quantity $g\hbar^{1/2}$ is dimensionless. Setting $\hbar$ to unity, the semiclassical expansion in $\hbar$ is therefore equivalent to an expansion in g . While P and π_0 are independent of g , $\sqrt{Q_0}$ is proportional to g^{-2} and so $\hbar^{-1}$. Thus the action of P preserves the order in the semiclassical expansion while $\sqrt{Q_0} \pi_0$ reduces the order by one. Therefore

$$(P - \sqrt{Q_0} \pi_0) |0\rangle = 0 \quad (22)$$

implies the recursion relation

$$P |0\rangle_i = \sqrt{Q_0} \pi_0 |0\rangle_{i+1}. \quad (23)$$

Up to the kernel of π_0 , this determines order $i + 1$ states from order i states.

We can now state the critical difference between our approach and the collective coordinate approach. Whereas the collective coordinate approach imposes translation invariance exactly, we only solve the recursion relation (23) up to the order at which we intend to find the state. As a result, no nonlinear canonical transformation is required, only the linear Bogoliubov transformation that relates the A_p and B_k . Thus we do not arrive at a complicated Hamiltonian. *On the contrary, perturbation theory is greatly simplified as we only need to solve for components in the kernel of π_0 , the rest of the state is fixed by the recursion relation.*

The momentum operator (20) is

$$P = \int \frac{dk}{2\pi} \Delta_{kB} \left[i\phi_0 \left(-\omega_k B_k^\dagger + \frac{B_{-k}}{2} \right) + \pi_0 \left(B_k^\dagger + \frac{B_{-k}}{2\omega_k} \right) \right] + i \int \frac{d^2k}{(2\pi)^2} \Delta_{k_1 k_2} \left(-\omega_{k_1} B_{k_1}^\dagger B_{k_2}^\dagger + \frac{B_{-k_1} B_{-k_2}}{4\omega_{k_2}} - \frac{1}{2} \left(1 + \frac{\omega_{k_1}}{\omega_{k_2}} \right) B_{k_1}^\dagger B_{-k_2} \right) \quad (24)$$

where we have defined the matrix

$$\Delta_{ij} = \int dx g_i(x) g_j'(x). \quad (25)$$

Integration by parts, using the fact that all $g_i(x)$ vanish asymptotically, exchanges the indices and introduces a minus sign, so Δ_{ij} is antisymmetric. We can expand the i th order kink ground state as

$$|0\rangle_i = Q_0^{-i/2} \sum_{m,n=0}^{\infty} \int \frac{d^n k}{(2\pi)^n} \gamma_i^{mn}(k_1 \cdots k_n) \times \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \quad (26)$$

Then the recursion relation becomes

$$\begin{aligned} \gamma_{i+1}^{mn}(k_1 \cdots k_n) = & \Delta_{kB} \left(\gamma_i^{m,n-1}(k_1 \cdots k_{n-1}) + \frac{\omega_{k_n}}{m} \gamma_i^{m-2,n-1}(k_1 \cdots k_{n-1}) \right) + (n+1) \int \frac{dk'}{2\pi} \Delta_{-k'B} \left(\frac{\gamma_i^{m,n+1}(k_1 \cdots k_n, k')}{2\omega_{k'}} \right. \\ & \left. - \frac{\gamma_i^{m-2,n+1}(k_1 \cdots k_n, k')}{2m} \right) + \frac{\omega_{k_{n-1}} \Delta_{k_{n-1} k_n}}{m} \gamma_i^{m-1,n-2}(k_1 \cdots k_{n-2}) \\ & + \frac{n}{2m} \int \frac{dk'}{2\pi} \Delta_{k_n, -k'} \left(1 + \frac{\omega_{k_n}}{\omega_{k'}} \right) \gamma_i^{m-1,n}(k_1 \cdots k_{n-1}, k') \\ & - \frac{(n+2)(n+1)}{2m} \int \frac{d^2 k'}{(2\pi)^2} \frac{\Delta_{-k'_1, -k'_2}}{2\omega_{k'_2}} \gamma_i^{m-1,n+2}(k_1 \cdots k_n, k'_1, k'_2). \end{aligned} \quad (27)$$

We have assumed here that γ_i is symmetric under a permutation of the k_j , but (27) yields a γ_{i+1} which is not symmetric. Therefore, before each successive application of the recursion relation, it is necessary to symmetrize γ_{i+1} . The definition of the state (26) is invariant under this symmetrization.

As is, the recursion relation applies to any kink state whose center of mass is at rest. To restrict to the ground state, we need only impose the initial condition

$$\gamma_0^{mn} = \delta_{m0} \delta_{n0} \gamma_0^{00}. \quad (28)$$

One recursion yields

$$\begin{aligned} \gamma_1^{12}(k_1, k_2) &= \frac{(\omega_{k_1} - \omega_{k_2}) \Delta_{k_1 k_2}}{2} \gamma_0^{00}, \\ \gamma_1^{21}(k_1) &= \frac{\omega_{k_1} \Delta_{k_1 B}}{2} \gamma_0^{00}. \end{aligned} \quad (29)$$

Two yield the two-loop state up to the kernel of π_0 , corresponding to γ_2^{0n} . These are reported in Ref. [14].

IV. OLD-FASHIONED PERTURBATION THEORY

The terms γ_2^{0n} , which are in the kernel of π_0 , can be found using ordinary perturbation theory as follows.

First define Γ to be any solution of

$$\begin{aligned} & \sum_{j=0}^i (H_{i+2-j} - Q_{i-j+1}) |0\rangle_j \\ &= \sum_{mn} \int \frac{d^n k}{(2\pi)^n} \Gamma_i^{mn}(k_1 \cdots k_n) \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0 \end{aligned} \quad (30)$$

where Γ_i is of order $O(g^i)$. Recall that $|0\rangle_j$ is determined by γ_j and so Γ is a function of γ . Then observe that the Schrodinger equation

$$(H - Q)|0\rangle = 0 \quad (31)$$

is solved by any γ such that

$$\Gamma_i^{mn} = 0. \quad (32)$$

Recall that only the γ_i^{0n} need be determined perturbatively, as only they lie in the kernel of π_0 . The other components were already fixed by the recursion relation (23).

To solve (30) we first note that

$$H_n = \frac{1}{n!} \int dx V^{(n)}[gf(x)] : \phi^n(x) :_a \quad (33)$$

where $V^{(n)}[gf(x)]$ is the n th derivative of $g^{n-2}V[g\phi(x)]$ evaluated at $\phi(x) = f(x)$. These are converted into normal mode normal ordered expressions using the Wick's theorem stated and proved in [15]. As normal mode normal ordered expressions act simply on $|0\rangle_0$, one can easily use Eq. (30) to write Γ in terms of γ . At each new order i , the γ_i appear linearly and so the condition that $\Gamma_i = 0$ in (32) is uniquely solved for γ_i .

V. CONCLUSIONS

The usual IR problems associated with perturbation theory in the presence of a continuous spectrum are resolved here by the momentum constraint (22), as they are resolved in the case of the collective coordinate approach. As this perturbative calculation is standard, it is reported in the companion paper [14]. It yields a general formula valid for the energy of any scalar kink at two loops

$$\begin{aligned} Q_2 = & \frac{V_{II}}{8} - \frac{1}{8} \int \frac{dk'}{2\pi} \frac{|V_{Ik'}|^2}{\omega_k^2} \\ & - \frac{1}{48} \int \frac{d^3k'}{(2\pi)^3} \frac{|V_{k'_1 k'_2 k'_3}|^2}{\omega_{k'_1} \omega_{k'_2} \omega_{k'_3} (\omega_{k'_1} + \omega_{k'_2} + \omega_{k'_3})} \\ & + \frac{1}{16Q_0} \int \frac{d^2k'}{(2\pi)^2} \frac{|(\omega_{k'_1} - \omega_{k'_2}) \Delta_{k'_1 k'_2}|^2}{\omega_{k'_1} \omega_{k'_2}} \\ & - \frac{1}{8Q_0} \int \frac{dk'}{2\pi} |f''(x)|^2 \end{aligned}$$

where

$$V_{\mathcal{I}^m \mathcal{I}, \alpha_1 \dots \alpha_n} = \int dx V^{(2m+n)}[gf(x)] \mathcal{I}^m(x) g_{\alpha_1}(x) \dots g_{\alpha_n}(x). \quad (34)$$

Here we have introduced the contraction factor $\mathcal{I}(x)$ determined by [15]

$$\partial_x \mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{1}{2\omega_k} \partial_x |g_k(x)|^2 \quad (35)$$

and the condition that it vanish at infinity.

The two-loop scalar kink mass was previously only known in the Sine-Gordon case [3,4]. There it was derived from 13 UV divergent diagrams, which can be combined into five finite combinations. Our terms are always each UV finite, as we have normal-ordered from the beginning. In the Sine-Gordon case the terms in our energy formula are these five finite combinations. Our formula on the other hand also applies to kinks in many other models, such as ϕ^{2n} models.

However, by finding the two-loop state, and not just the mass, one can do much more. For example, it would be straightforward to calculate form factors [16] and matrix elements. This would allow, for the first time, a truly quantum approach to meson-kink scattering [17–19], breather excitation, acceleration [20,21] and more.

J.E. is supported by the CAS Key Research Program of Frontier Sciences grant QYZDY-SSW-SLH006 and the NSFC MianShang Grants No. 11875296 and No. 11675223. J.E. also thanks the Recruitment Program of High-end Foreign Experts for support.

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