Schur correlation functions from q -deformed Yang-Mills theory

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We construct the wave functions in the q -deformed 2D Yang-Mills theory that compute torus correlation functions of affine currents in the vertex operator algebra associated to a class of 4D $\mathcal{N} = 2$ superconformal field theory. These wave functions are then shown to reduce to the topological correlators of a set of Coulomb branch operators in the $T[SU(N)]$ theory, from which those correlators in the 3D mirror dual of the 4D T_N theories can be computed.

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I. INTRODUCTION

Four dimensional $\mathcal{N} = 2$ superconformal field theory (SCFTs) are found to contain a privileged class of operators, referred to as the “Schur operators,” endowed with a vertex operator algebra (VOA) structure [1]. The $\mathcal{R}$ symmetry of the 4D theory guarantees the presence of a Virasoro subalgebra, while any flavor symmetry $\mathfrak{g}$ enhances to an affine algebra $\hat{\mathfrak{g}}_k$, whose generators descend from the flavor moment maps in 4D. The full VOAs of a variety of 4D SCFTs have been identified and analyzed in great depth, see for example [2–5]. The underlying topological quantum field theory that captures the Schur index $\mathcal{I}$ of a 4D $\mathcal{N} = 2$ SCFT is identified with the q -deformed 2D Yang-Mills theory, which allows one to identify the index $\mathcal{I}$ as a correlator in the latter 2D theory [6].

Recently, correlation functions of Schur operators on the torus are computed via supersymmetric localization [7,8] and are further mapped to the deformation quantization of the Higgs branch of vacua [9–14] of the circle-reduced theory in 3D.

Hence, the aim of this article is to propose a parallel approach to computing some of these correlators, by constructing the wave functions corresponding to the affine current operators for a class of 4D SCFTs of class $\mathcal{S}$ of type $\mathfrak{g}$, and show that they are determined by the correlators of critical current algebras. Moreover, we will relate the dimensionally reduced wave functions for the T_N theories with the Coulomb branch correlators in the 3D mirror dual [15,16], namely, the star-shaped quiver gauge theories

with three $T[SU(N)]$ tails. We also discuss the possible generalization to other $T[G]$ theories.

II. CURRENT CORRELATION FUNCTIONS

In this section we first focus on class $\mathcal{S}$ theories associated to Riemann surfaces with only regular punctures. We will derive the q -Yang-Mills formula for the correlation functions of affine currents associated to a maximal puncture, by combining the known formula for the Schur index and Zhu’s recursion formula. Then we generalize the computation to a class of Argyres-Douglas theories containing a full puncture.

Let us begin with a quick review of class $\mathcal{S}$ theories with only regular punctures. First we recall that each full puncture of such a theory $\mathcal{T}$ is associated to a factor of flavor symmetry $\mathfrak{g}$ with flavor central charge $k^{4D} = 2h^\vee$. Under the map from Schur operators in 4D to the vertex operators in the associated VOA $V \equiv \text{VOA}[\mathcal{T}]$, the moment map operators of $\mathfrak{g}$ descend to the affine current generators of a current subalgebra $\hat{\mathfrak{g}}_{k=-h^\vee}$ at the critical level.

The Schur index $\mathcal{I}(\mathbf{x}, q)$ of a theory associated to a $(s+1)$ -punctured Riemann surface, which is equivalent to the vacuum character ch_V of $V \equiv \text{VOA}[\mathcal{T}]$, can be written as a 2D q -deformed Yang-Mills correlator [2,6]

$$\begin{aligned} \text{ch}_V(\mathbf{x}, \mathbf{y}; q) &\equiv \text{tr}_V q^{L_0 - \frac{c}{24}} \mathbf{x}^{\mathbf{h}} \mathbf{y}^{\mathbf{h}'} \\ &= q^{-\frac{c}{24}} \sum_{\lambda} C_{\lambda}(q)^{2g-2+s} q^{(\lambda, \rho)} \text{ch}_{\lambda}^{\text{crit}}(\mathbf{x}, q) \\ &\quad \times \prod_{i=1}^s \psi_{\lambda}^{(i)}(\mathbf{y}, q). \end{aligned} \quad (1)$$

Here, the sum over λ exhausts all integral dominant weights λ of the finite Lie algebra $\mathfrak{g}$, and $\text{ch}_{\lambda}^{\text{crit}}(\mathbf{x}, q)$ denotes the $\hat{\mathfrak{g}}_{k=-h^\vee}$ character of a module whose highest weight subspace is given by the $\mathfrak{g}$ -irreducible representation $\mathcal{R}_{\lambda}$ with

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the highest weight λ [17]. As in the usual practice, c denotes the central charge of V , and g the genus of the Riemann surface used to engineer the 4D theory. Weight ρ is the Weyl vector of $\mathfrak{g}$ with the standard inner product $(\cdot, \cdot)$ defined by the Killing form, and $\mathbf{h}, \mathbf{h}'$ collectively denote the cartans of the flavor symmetry group associated with punctures, and we have singled out one particular full puncture whose flavor fugacity is $\mathbf{x}$; the remaining flavor fugacities are collectively written as $\mathbf{y}$. The appearance of the critical character on the right is due to its simple relation with the full puncture wave function ψ^{full} , $\text{ch}_\lambda^{\text{crit}}(\mathbf{x}, q) = q^{-(\lambda, \rho)} C_\lambda(q) \psi_\lambda^{\text{full}}(\mathbf{x}, q)$, where [3]

$$C_\lambda(q) \equiv \frac{1}{\text{PE}[\sum_{j=1}^r \frac{q^{d_j}}{1-q}] \dim_q \mathcal{R}_\lambda} \quad (2)$$

$$\psi_\lambda^{\text{full}} \equiv \text{PE} \left[\frac{1}{1-q} q \chi_{\text{adj}}^{\mathfrak{g}} \right] \chi_\lambda^{\mathfrak{g}}. \quad (3)$$

Here, PE denotes the plethystic exponential defined as $\text{PE}[f(x)] = \exp(\sum_{i=1}^\infty \frac{f(x^i)}{i})$. The sum in (2) is over all the degree- d_j invariants (Casimirs) of Lie algebra $\mathfrak{g}$. We refer $\chi_\lambda^{\mathfrak{g}}$ to the character of $\mathcal{R}_\lambda$, with $\chi_{\text{adj}}^{\mathfrak{g}}$ that of the adjoint representation, and $\dim_q \mathcal{R}_\lambda$ denotes the q -deformed dimension of the irreducible representation $\mathcal{R}_\lambda$,

$$\dim_q \mathcal{R}_\lambda \equiv \prod_{\alpha \in \Delta_+} \frac{q^{\frac{1}{2}(\lambda+\rho, \alpha^\vee)} - q^{-\frac{1}{2}(\lambda+\rho, \alpha^\vee)}}{q^{\frac{1}{2}(\rho, \alpha^\vee)} - q^{-\frac{1}{2}(\rho, \alpha^\vee)}}. \quad (4)$$

The remaining $\psi_\lambda^{(i)}$ in the formula denotes the wave functions of the other regular punctures.

Our goal is to derive the torus correlation functions (in the unflavoring limit $\mathbf{x} \rightarrow 1$) of the currents J^a associated to a full puncture. Obviously J^a have vanishing one-point functions.

To compute the two-point functions, we recall Zhu's recursion relations [18], now applied to any module M of the algebra V and the generators J^a of the vertex subalgebra $\hat{\mathfrak{g}}_k$ associated to a full puncture (see Appendix B below for a quick review of the notations on vertex operator algebras, and more on the special functions P_m can be found in Appendix A),

$$\begin{aligned} & \text{tr}_M J^a[z_1] J^b[z_2] q^{L_0} \\ &= \text{tr}_M o(J^a) o(J^b) q^{L_0} \\ & - \sum_{m \geq 0} (-1)^m P_{m+1}(z_1 - z_2) \text{tr}_M o(J_{[m]}^a J^b) q^{L_0}. \end{aligned} \quad (5)$$

Here, for any state $a \in V$, the state-operator correspondence is given by $a[z] \equiv Y(e^{2\pi i z L_0} a, e^{2\pi i z}) = e^{2\pi i \text{wt}(a)z} a(e^{2\pi i z})$, which is appropriate for insertions on a torus. In the second

term of (5), only the summand with $m = 1$ contributes thanks to the commutation relations for $\hat{\mathfrak{g}}_k$. Therefore,

$$\begin{aligned} & \text{tr}_M J^a[z_1] J^b[z_2] q^{L_0} \\ &= \text{tr}_M o(J^a) o(J^b) q^{L_0} + \frac{k K^{ab}}{(2\pi i)^2} P_2(z_1 - z_2) \text{tr}_M q^{L_0}. \end{aligned} \quad (6)$$

Here K denotes the Killing form. Group theory ensures that the first term on the right can be written as $K^{ab} \mathcal{J}_M(q)$, with $\mathcal{J}_M(q)$ a function to be determined later. In particular, with a basis of Cartan generators h^i for the finite part $\mathfrak{g}$ of $\hat{\mathfrak{g}}_k$, we have

$$K^{jj} \mathcal{J}_M(q) = \left(x_j \frac{d}{dx_j} \right)^2 \Big|_{x \rightarrow 1} \text{tr}_M q^{L_0} \prod_{i=1}^r x_i^{h^i}. \quad (7)$$

Therefore, we conclude that the current two-point functions in any module M can be obtained from the associated character by acting with a uniform differential operator, $\text{tr}_M J^a[z_1] J^b[z_2] q^{L_0 - c/24} = \mathcal{D}^{ab}(z_1, z_2) \text{ch}_M(q)$,

$$\mathcal{D}^{ab}(z_i) \equiv K^{ab} \left[\frac{1}{K^{jj}} \left(x_j \frac{d}{dx_j} \right)^2 \Big|_{x \rightarrow 1} + k P_2(z_1 - z_2) \right].$$

Now, we restrict the module M to be V itself, and the relevant character is simply the Schur index. The action of this differential operator on the Schur index leads easily to

$$\begin{aligned} & \text{tr}_V J^a[z_1] J^b[z_2] q^{L_0} \\ &= \sum_\lambda C_\lambda^{2g-2+s} q^{(\lambda, \rho)} \text{tr}_\lambda J^a[z_1] J^b[z_2] q^{L_0^{\text{crit}}} \prod_{i=1}^s \psi^{(i)}. \end{aligned} \quad (8)$$

Said differently, the wave function $\Psi_\lambda^{ab}(z_1, z_2)$ in the q -deformed Yang-Mills theory corresponding to the two-point current insertions in V is simply the (normalized) critical current two-point function

$$\Psi_\lambda^{ab}(z_1, z_2) = \langle J^a[z_1] J^b[z_2] \rangle_\lambda^{\text{crit}} \equiv \frac{\text{tr}_\lambda J^a[z_1] J^b[z_2] q^{L_0^{\text{crit}}}}{\text{tr}_\lambda q^{L_0^{\text{crit}}}}.$$

In fact, one can do better by observing, for critical currents algebras, that if we write

$$\text{tr}_\lambda o(J^a) o(J^b) q^{L_0^{\text{crit}}} = K^{ab} \mathcal{J}_\lambda(q), \quad (9)$$

we have

$$\mathcal{J}_\lambda(q) = \left[\frac{|\lambda + \rho|^2}{\dim \mathfrak{g}} + h^\vee E_2(\tau) \right] \text{ch}_\lambda^{\text{crit}}(q). \quad (10)$$

Here $E_2(\tau)$ denotes the second Eisenstein series; see also Appendix A. To prove this observation, it is convenient to rewrite the formal character $\text{ch}_\lambda^{\text{crit}}$ evaluated at $2\pi i \tau$ as

$$\text{ch}_\lambda^{\text{crit}}(2\pi i t \rho) = f(q) \prod_{\alpha \in \Delta_+} \frac{e^{\pi i t(\lambda + \rho, \alpha)} - e^{-\pi i t(\lambda + \rho, \alpha)}}{\vartheta_1(t(\rho, \alpha) | \tau)}, \quad (11)$$

where $f(q)$ denotes some functions of q whose form is irrelevant for this proof, and the Jacobi-theta function $\vartheta_1(z | \tau)$ is defined by

$$i\vartheta_1(z | \tau) \equiv e^{\pi i z} q^{\frac{1}{8}} \prod_{n=0}^{+\infty} (1 - q^{n+1})(1 - e^{2\pi i z} q^{n+1})(1 - e^{-2\pi i z} q^n). \quad (12)$$

Now we can apply on both sides of the relation $\partial_t^2|_{t=0}$ (and proper normalized by the Killing form [Eq. (11)]). Using the identity for any weights μ, ν

$$\sum_{\alpha \in \Delta_+} (\mu, \alpha)(\alpha, \nu) = h^\vee(\mu, \nu), \quad (13)$$

and the relation between (A4) $E_2(\tau)$ with the Jacobi-theta functions ϑ_1

$$E_2(\tau) = \frac{1}{12\pi^2} \frac{\vartheta_1'''(0 | \tau)}{\vartheta_1'(0 | \tau)}, \quad (14)$$

one easily proves the statement. Explicitly, we now have the wave function

$$\Psi_\lambda^{ab}(z_i) = -\frac{h^\vee K^{ab}}{(2\pi i)^2} \wp_0(z_1 - z_2, \tau) + K^{ab} \frac{|\lambda + \rho|^2}{\dim \mathfrak{g}}. \quad (15)$$

Here we made use of the relation between P_2 and the Weierstrass $\wp$ function; see Appendix A for their definitions.

Next we turn to the three-point functions. To proceed, we recall Zhu's general recursion relations, proposition 4.3.4 of [18]. Again, substituting in $a = J^a$, $a^{(i)} = J^{a_i}$ we have

$$\begin{aligned} \text{tr}_M J^a[w] J^{a_1}[z_1] J^{a_2}[0] q^{L_0} \\ = \text{tr}_M (J_{[-1]}^a J^{a_1})[z_1] J^{a_2}[0] q^{L_0} + \frac{1}{2} \text{tr}_M (J_0^a J^b)[z_1] J^c[0] q^{L_0} \\ + \frac{i}{2\pi} [(P_1(-z_1) - P_1(-w)) \text{tr}_M J^{a_1}[z_1] (J_0^a J^{a_2})[0] q^{L_0} \\ - P_1(z_1 - w) \text{tr}_M (J_0^a J^{a_1})[z_1] J^{a_2}[0] q^{L_0}]. \end{aligned}$$

Let us first focus on Lie algebras $\mathfrak{g}$ with a vanishing cubic Casimir. Combining with $J_0^a J^b = i f^{ab}_c J^c$, where $f^{aa_1}_b$ denotes the structure constant of Lie algebra $\mathfrak{g}$, and the Ward identity

$$\begin{aligned} \text{tr}_M (J_{[-1]}^a J^{a_1})[z_1] J^{a_2}[0] q^{L_0} \\ = -\frac{1}{2} f^{aa_1}_b \partial_{z_1} \text{tr}_M J^b[z_1] J^{a_2}[0] q^{L_0}, \quad (16) \end{aligned}$$

we arrive at a simple recursion relation

$$\text{tr} J^a[w] \prod_{i=1}^2 J^{a_i}[z_i] q^{L_0} = \mathcal{D}^{aa_1}_b \text{tr} J^b[z_1] J^{a_2}[z_2] q^{L_0}$$

in terms of a uniform differential operator

$$\begin{aligned} \mathcal{D}^{aa_1}_b \equiv \frac{1}{2} f^{aa_1}_b \left[-\partial_{z_1} + 1 \right. \\ \left. + \frac{1}{\pi i} (P_1(z_2 - z_1) - P_1(z_2 - w) + P_1(z_1 - w)) \right]. \quad (17) \end{aligned}$$

We will also use the notation $\mathcal{D}^{a_1 a_2 a_3}$ with a raised last index using the Killing form. Acting with the differential operator $\mathcal{D}^{a_1 a_2}_b$ on the two-point correlators of V , we therefore conclude that the wave function corresponding to three-point current insertions is simply the normalized three-point current correlator,

$$\Psi_\lambda^{a_1 a_2 a_3}(z_1, z_2, z_3) = \langle J^{a_1}[z_1] J^{a_2}[z_2] J^{a_3}[z_3] \rangle_\lambda^{\text{crit}}. \quad (18)$$

For $\mathfrak{g} = \mathfrak{su}(n)$ with $n \geq 3$, an additional term proportional to the cubic Casimir $d^{a_1 a_2 a_3}$ appears [19]. This term is independent of z_i and can be obtained by computing $\text{tr}_V o(J^{a_1}) o(J^{a_2}) o(J^{a_3}) q^{L_0}$ with a_i symmetrized. As shown in [19], the term can also be written as a uniform differential operator acting on the character. Therefore the above argument goes through and the result (18) holds. It will be important later to observe that (after symmetrizing a_i)

$$\text{tr}_\lambda \prod_{i=1}^3 o(J^{a_i}) q^{L_0^{\text{crit}}} = \mathfrak{d}(\lambda) d^{a_1 a_2 a_3} \text{ch}_\lambda^{\text{crit}}(q), \quad (19)$$

where $\mathfrak{d}$ is a function of weight λ which has a nontrivial limit $\mathfrak{d}_{3D}(\lambda) \equiv \lim_{\beta \rightarrow +0} \beta^3 \mathfrak{d}(\beta^{-1} \lambda)$ as an odd function of λ .

Continuing with the same logic, one can apply the general Zhu's recursion formula to derive uniform differential operators that extract higher-point current correlation functions, and we conjecture the general wave function to be given simply by the normalized critical current correlators,

$$\Psi_\lambda^{a_1 \dots a_n}(z_i) = \langle J^{a_1}[z_1] \dots J^{a_n}[z_n] \rangle_\lambda^{\text{crit}}. \quad (20)$$

In the case of $\mathfrak{g} = \mathfrak{su}(N)$, the 4D SCFTs admit surface defects engineered by the renormalization group flow initialized by some position dependent vacuum expectation value of baryonic Higgs branch operator [20]. As shown in [21], these defects are labeled by the dominant integral weights λ' of $\mathfrak{g}$, and the associated wave functions are given by $S_{\lambda\lambda'}/S_{\lambda 0}$. Here $S_{\lambda\lambda'}$ denotes the (analytically continued) modular S matrix for $SU(N)$. Explicitly, $S_{\lambda\lambda'} = S_{00\lambda}(q^\rho) \chi_{\lambda'}(q^{\rho+\kappa})$ with $\rho = (\frac{N-1}{2}, \dots, -\frac{N-1}{2})$, and κ is a set of integers linearly dependent on the Dynkin labels of λ .

Hence, in the presence of p such defects, the n -point correlation functions of currents J^a are simply given by

$$\begin{aligned} \text{tr}_V J^{a_1}[z_1] \dots J^{a_n}[z_n] & \left(\prod_{\alpha=1}^p \mathfrak{D}_{\lambda'_\alpha} \right) q^{L_0} \\ &= \sum_{\lambda} C_{\lambda}(q)^{2g-2-s} \Psi_{a_1 \dots a_n}(z_i) \prod_{\alpha=1}^p \frac{S_{\lambda \lambda'_\alpha}}{S_{\lambda 0}} \prod_{i=1}^s \psi^{(i)}(q). \end{aligned} \quad (21)$$

Here $\mathfrak{D}_{\lambda'_\alpha}$ symbolically denotes insertion of the α th defect labeled by a weight λ'_α , and are not to be confused with the differential operators $\mathcal{D}^{a_1 \dots}$ that recursively generate current correlation functions.

Next we turn to a class of Argyres-Douglas theories denoted as $(J^b[\kappa], F)$ which admit class- $\mathcal{S}$ construction in terms of an irregular puncture and a full puncture [4,5,22]. In this notation, b, κ are integers characterizing the pole $\sim z^{-2-\frac{\kappa}{b}}$ associated to the irregular puncture. Their associated VOAs are either given by, or contain as a vertex subalgebra, the affine algebra $V = \hat{J}_{k=\frac{b}{b+\kappa}-h^\vee}$. For a class of such Argyres-Douglas theories, the Schur index can be computed from the 2D q -deformed Yang-Mills theory as [4,23,24]

$$\text{ch}_V(\mathbf{x}, q) = \sum_{\lambda} q^{-\frac{\kappa}{24}} C_{\lambda}(q) q^{(\lambda, \rho)} \psi_{\lambda}^{b, \kappa}(\mathbf{a}, q) \text{ch}_{\lambda}^{\text{crit}}(\mathbf{x}, q).$$

Here $\psi_{\lambda}^{b, \kappa}$ is the wave function associated to the irregular puncture with $\mathbf{a}$ the potential $U(1)^n$ flavor fugacities. The critical character is again associated to the full puncture with $\mathbf{g}$ flavor fugacities $\mathbf{x}$.

Similar to the previous subsection, we can study the correlation functions of the currents employing Zhu's recursion formula. We begin by recalling the recursion formula (6) for two-point functions. Note that the level k is no longer critical. Treating the right-hand side again as the differential operator $\mathcal{D}^{ab}(z_1, z_2)$ acting on the index, which only acts on the fugacities of the full puncture but ignore the irregular puncture, we obtain the wave function

$$\begin{aligned} \Psi_{\lambda}^{ab}(z_1, z_2) &= K^{ab} \frac{|\lambda + \rho|^2}{\dim \mathfrak{g}} \\ &+ K^{ab} \left[(k + h^\vee) E_2(\tau) + \frac{k g_2(z, \tau)}{(2\pi i)^2} \right]. \end{aligned} \quad (22)$$

Note that the operator $\mathcal{D}^{ab}(z_1, z_2)$ actually depends on the level k , and the $k + h^\vee$ term is the consequence of the mismatch between the critical level of the full puncture and the actual level k of the full VOA.

To obtain the wave function of the three-point function, we simply apply the uniform differential operator $\mathcal{D}^{a_1 a_2}_b(z_1, z_2, z_3)$ on the two-point wave function $\Psi^{ba_3}(z_2, z_3)$. Note that $\mathcal{D}^{a_1 a_2}_b(z_1, z_2, z_3)$ is independent of level k , and therefore the three-point wave function is simply

$$\begin{aligned} \Psi_{\lambda}^{a_1 a_2 a_3}(z_i) &= \langle J^{a_1}[z_1] \dots J^{a_3}[z_3] \rangle_{\lambda}^{\text{crit}} + d^{a_1 a_2 a_3} \text{ term} \\ &+ \frac{b}{b + \kappa} \mathcal{D}^{a_1 a_2 a_3}(z_1, z_2, z_3) \left(\frac{P_2(z_2 - z_3)}{(2\pi i)^2} \right). \end{aligned}$$

Before moving on, let us comment on stress tensor insertions. In general, the stress tensor of a theory of class $\mathcal{S}$ is not associated with any single puncture. Therefore, there seems to be no notion of an associated wave function. This also manifests in the fact that the summand ψ_{λ} of a general Schur index as a function of weight λ is not an eigenfunction of $q\partial q$ which inserts T . However, it is easy to show that for $(J^{h^\vee}[\kappa], F)$ the full summand ψ_{λ} with a given λ is in fact an eigenfunction of $q\partial q$,

$$\left(q \frac{d}{dq} - \frac{\dim \mathfrak{g}}{2} E_2(\tau) - \frac{|\lambda + \rho|^2}{2(k + h^\vee)} \right) \psi_{\lambda} = 0. \quad (23)$$

Therefore one may loosely associate the eigenvalue $\Psi_T \equiv \frac{\dim \mathfrak{g}}{2} E_2(\tau) - \frac{|\lambda + \rho|^2}{2(k + h^\vee)}$ with the “wave function” for T insertions in these theories.

III. COULOMB BRANCH CORRELATION FUNCTIONS IN STAR-SHAPED QUIVER

In any three dimensions $\mathcal{N} = 4$ SCFT $\mathcal{T}^{3D}$, one can extract two protected subsectors forming two associative topological algebras, one in the Higgs branch and the other in the Coulomb branch. These algebras have been shown to be the deformation quantization of hyperkähler cones describing the two branches of vacua [9].

One can derive such an algebra associated to the Higgs branch by performing suitable “circle reduction” or “high-temperature limit” of the VOA $V[\mathcal{T}^{4D}]$ where $\mathcal{T}^{4D}$ reduces to $\mathcal{T}^{3D}$ via the said reduction [11,12]. More precisely, one puts $\mathcal{T}^{4D}$ on a twisted product $S^3 \times_q S^1$, where $q = e^{2\pi i \beta}$ with β parametrizing the size of the circle factor. The $S^3 \times_q S^1$ -partition function of such a theory then computes the Schur index [7,8], while the correlation functions of suitably defined “Schur operators” inserted along a torus $T_q^2 \subset S^3 \times_q S^1$ compute the torus correlation functions of the corresponding vertex operators in the VOA $V[\mathcal{T}^{4D}]$.

The proper circle reduction sends $\beta \rightarrow 0$ while rescaling all Schur operators $\mathcal{O} \rightarrow \beta^h \mathcal{O}$ according to their conformal weights h . During such a reduction, only the 4D Schur operators belonging to the Higgs branch chiral ring survive, while others decouple in the sense that any correlation function involving these non-Higgs branch operators vanishes in the limit. The prime examples that survive are the moment map operators of the 4D flavor symmetry.

The circle reduction $\mathcal{T}^{3D}$ of a $\mathcal{T}^{4D}$ is not necessary Lagrangian, while it always has a Lagrangian 3D mirror dual $\check{\mathcal{T}}^{3D}$. The Higgs branch correlation functions $\langle \mathcal{O}_H(x) \dots \rangle$ of $\mathcal{T}^{3D}$ are mapped to those of $\check{\mathcal{T}}^{3D}$ in the Coulomb branch by the 3D mirror symmetry. It is therefore

FIG. 1. The 3D mirror of a T_N theory is a star-shaped quiver theory with three $T[SU(N)]$ theories fused at the central $SU(N)$ node. When we start with a 4D T_N theory of type D, the gauge nodes in the 3D mirror are of SO and Sp types [15,28].

natural to expect the high-temperature limit of the current correlation functions of $\mathcal{T}^{4D}$ to reproduce a set of Coulomb branch correlators $\langle \mathcal{O}_C(x) \dots \rangle$ in $\check{\mathcal{T}}^{3D}$.

In the following we shall use the wave functions in the q -deformed Yang-Mills theory we have previously constructed to compute a class of topological Coulomb branch correlators for the 3D mirror dual of the T_N theories.

We begin with the T_2 theory whose Schur index is given by

$$\mathcal{I} = q^{-\frac{c}{24}} \sum_{\lambda} C_{\lambda}(q)^{-2} \prod_{i=1}^3 q^{(\lambda, \rho)} \text{ch}_{\lambda}^{\text{crit}}(\mathbf{x}^{(i)}, q). \quad (24)$$

The naive limit of sending $\beta \rightarrow 0$ with fugacity identification $x = e^{2\pi i \xi}$ does not lead to a sensible outcome. Instead, following [25], one should also rescale the Dynkin labels $\lambda_i \rightarrow \beta^{-1} \lambda_i$, originally denoting integral weights of the underlying algebra $\mathfrak{g}$, and make it continuous by replacing the sum $\sum_{\lambda}$ with an integral $\int \prod_{i=1}^r d\lambda_i$. For the case at hand, $r = 1$, and each critical character reduces to (up to overall factors involving only on β)

$$q^{(\lambda, \rho)} \text{ch}_{\beta^{-1}\lambda}^{\text{crit}}(e^{2\pi i \xi}, e^{2\pi i \beta}) \xrightarrow{\beta \rightarrow 0} \frac{\sin 4\pi \lambda \xi}{\sinh(2\pi \lambda) \sinh(2\pi \xi)},$$

while

$$C_{\beta^{-1}\lambda}(e^{2\pi i \beta})^{-2} \rightarrow \frac{1}{2} \sinh^2(2\pi \lambda) = \Delta_{SU(2)}(\lambda)$$

reduces to the $SU(2)$ Haar measure factor. One immediately recognizes that the reduction of the critical character is nothing but the S^3 partition function of $T[SU(2)]$ theory with mass λ and Fayet-Iliopoulos (FI) parameters ξ . The full index now becomes the matrix integral that computes the S^3 -partition function of the star-shaped quiver theory which fuses three $T[SU(2)]$ theories [25,26], with the λ integral integrating over the vacuum expectation value of the vector multiplet scalar which gauges the manifest

diagonal $SU(2) \subset SU(2)^3$ flavor symmetry of three $T[SU(2)]$ theories.

Now we extend the above computation to include rescaled current operators $\beta J^a[z]$ associated with one full puncture of T_2 . First, we reduce the wave function $\beta^2 \Psi_{\lambda}^{ab}(z, w)$, assuming $\text{Re}(z - w) > 0$,

$$\beta^2 \Psi_{\beta^{-1}\lambda}^{ab}(z, w) \xrightarrow{\beta \rightarrow 0} \frac{K^{ab}}{6} (1 + 4\lambda_1^2). \quad (25)$$

We recognize this result to be the two-point function of a pair of magnetic monopoles, or two vector multiplet scalars in the massive $T[SU(2)]$ (identifying $m_1 = -m_2 = \lambda_1$ while turning off the FI parameters) [16],

$$\begin{aligned} \langle \mathcal{M}_+ \mathcal{M}_- \rangle^{S^3} &= \frac{1}{Z_{T[SU(2)]}^{S^3}} \int d\sigma \prod_{I=1}^2 \frac{i(\sigma + m_I) - 1/2}{2 \cosh \pi(\sigma + m_I)} \\ &= \frac{1}{6} (1 + 4\lambda_1^2). \end{aligned} \quad (26)$$

The three-point wave function also has a nontrivial limit independent of the insertion coordinate,

$$\beta^3 \Psi_{\beta^{-1}\lambda}^{abc}(z_i) \xrightarrow{\beta \rightarrow 0} -\frac{if^{abc}}{12} (1 + 4\lambda_1^2), \quad (27)$$

which we can identify with the three-point function in the $T[SU(2)]$,

$$\begin{aligned} \langle \Phi \mathcal{M}_+ \mathcal{M}_- \rangle^{S^3} &= \frac{1}{Z^{S^3}} \int d\sigma d\sigma' \prod_{I=1}^2 \frac{i(\sigma + m_I) - 1/2}{2 \cosh \pi(\sigma + m_I)} \\ &= -\frac{i}{12} (1 + 4\lambda_1^2). \end{aligned} \quad (28)$$

Finally, we identify the reduction of the current correlation function with the full correlation functions of said Coulomb branch operators in the star-shaped quiver

$$\begin{aligned} &\text{tr}_V J^a[z] \dots J^c[0] q^{L_0 - \frac{c}{24}} \\ &\rightarrow \int d\lambda_1 \Delta_{SU(2)} \langle \mathcal{O}^a \dots \mathcal{O}^c \rangle_{T[SU(2)]}^{S^3} (Z_{T[SU(2)]}^{S^3})^3. \end{aligned} \quad (29)$$

The above result immediately generalizes to T_N theories. See Fig. 1 for the 3D mirror dual of a T_N theory, where three $T[SU(N)]$ theories are glued at the center. First of all, the critical character reduces to the S^3 -partition function of $T[SU(N)]$. This can be seen by first observing that the critical character can be rewritten as

$$\text{ch}_{\lambda}^{\text{crit}} = \frac{(\prod_{\alpha \in \Delta_+} (q; q) e^{\frac{\alpha}{2}} q^{\frac{1}{2}}) \sum_w \epsilon(w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - q^{(\lambda + \rho, \alpha^{\vee})}) i \vartheta_1(\frac{\alpha}{2\pi i} | \tau)}. \quad (30)$$

Here we have used that for any function f

$$\prod_{\hat{\alpha} \in \hat{\Delta}_+^{\text{re}}} f(e^{-\hat{\alpha}}) = \prod_{\alpha \in \Delta_+} \prod_{n=0}^{+\infty} f(e^{+\alpha} q^{n+1}) f(e^{-\alpha} q^n), \quad (31)$$

and the Jacobi-theta function $\vartheta_1(z|\tau)$ is defined in (12).

After this is evaluated on a weight $2\pi i\xi$, and setting $q = e^{-2\pi\beta}$, $\lambda \rightarrow \beta^{-1}\lambda$, $\xi \rightarrow \beta\xi$, one can perform the high-temperature limit $\beta \rightarrow +0$ and get (where we have dropped some unimportant factors)

$$q^{(\lambda, \rho)} \text{ch}_\lambda^{\text{crit}}(\xi) \rightarrow \frac{\sum_w e(w) e^{2\pi i(w(\lambda), \xi)}}{\prod_{\alpha \in \Delta_+} \sinh \pi(\alpha, \lambda) \sinh \pi(\alpha, \xi)}.$$

This is simply the S^3 partition function of the $T[SU(N)]$ theory [26,27], where the now-continuous Dynkin labels λ play the role of the $SU(N)$ mass parameters. Apparently this computation is also valid for other A-type, D-type and E-type (ADE) Lie algebras, and indeed it has been proposed that the above expression gives the S^3 partition function of the $T[G]$ theory [27,28], and our result for general ADE $\mathfrak{g}$ is another piece of evidence supporting the proposal.

In a similar fashion, the correlation functions of the $SU(N)$ currents are expected to reduce to the topological correlation functions $\langle \mathcal{O}^a \mathcal{O}^b \dots \rangle$ of a set of Coulomb branch operators $\mathcal{O}^a$ in the $T[SU(N)]$ theory (again with vanishing FI-parameter ξ but nonzero masses λ_i). For example, we have

$$\begin{aligned} q^{(\lambda, \rho)} \text{ch}_\lambda^{\text{crit}}(\xi) &\rightarrow Z_{T[SU(N)]}^{S^3}(\lambda, \xi), \\ \langle J^a[z] J^b[w] \rangle_\lambda^{\text{crit}} &\rightarrow \langle \mathcal{O}^a \mathcal{O}^b \rangle = K^{ab} \left(\frac{h^\vee}{12} + \frac{|\lambda|^2}{\dim \mathfrak{g}} \right), \\ \langle J^a[z] J^b[w] J^c[0] \rangle_\lambda^{\text{crit}} &\rightarrow \langle \mathcal{O}^a \mathcal{O}^b \mathcal{O}^c \rangle \\ &= -\frac{i}{2} f^{abc} \left(\frac{h^\vee}{12} + \frac{|\lambda|^2}{\dim \mathfrak{g}} \right). \end{aligned} \quad (32)$$

Here $|\lambda|^2$ is the length squared of the weight λ with continuous Dynkin labels, now identified with the masses of the $T[SU(N)]$ theories [29]. The odd function $\mathfrak{d}_{3d}(\lambda)$ which captures the high-temperature limit of the d^{abc} -term in the VOA current correlator would vanish when we glue three $T[SU(N)]$ theories by an integral corresponding to the central gauge node in the star-shaped quiver. In particular, the structure constant $C_\lambda(q)^{-2}$ has a simple high-temperature limit (up to some irrelevant factors)

$$C_\lambda(q)^{-2} \rightarrow \left[\prod_{\alpha \in \Delta_+} e^{-\pi(\alpha, \lambda)} - e^{\pi(\alpha, \lambda)} \right]^2, \quad (33)$$

which conveniently provides the vector multiplet contribution from the central $SU(N)$ (or the proper gauge group for other ADE algebra $\mathfrak{g}$) gauge node.

In [27], the partition function and Coulomb branch correlation functions of $T[SU(N)]$ are decomposed into weighted traces over Verma modules of the Coulomb branch operator algebra $\mathcal{A}_C$. This structure is naturally visible in the critical affine character (see Appendix C for more detail)

$$\text{ch}_\lambda^{\text{crit}} = \prod_{\alpha \in \Delta_+} \frac{1}{1 - q^{(\lambda + \rho, \alpha)}} \sum_{w \in W} \frac{e^{w \cdot \lambda}}{\prod_{\hat{\alpha} \in \hat{\Delta}_+^{\text{re}}} (1 - q^{-\hat{\alpha}})}. \quad (34)$$

The sum over the Weyl group can be viewed as a sum over reducible modules of $\hat{\mathfrak{g}}_{-h^\vee}$ with finite highest weights $w \cdot \lambda$ for all $w \in W$. The high-temperature limit sends each term in the above sum to a Coulomb branch twisted character χ_i^C in [27], while the overall factor $\prod_{\alpha \in \Delta_+} (1 - q^{(\lambda + \rho, \alpha)})^{-1}$ becomes the factors f_i^H there. Similarly, the critical current correlation functions inherit such sum structure, since the overall factor $\prod_{\alpha \in \Delta_+} (1 - q^{(\lambda + \rho, \alpha)})$ is independent of flavor fugacities. Consequently, the critical current correlation functions can be written as a sum over those evaluated in the above reducible modules, which then reduce in the high-temperature limit to the weighted sum of traces of the Coulomb branch operators.

The above computation is naturally compatible with the inclusion of surface defects. As discussed above, such a defect introduces factors of modular S matrices into the q -deformed Yang-Mills correlators. In performing the high-temperature limit, we send $\lambda_i \rightarrow \beta^{-1}\lambda_i$, meaning that the κ 's are also rescaled $\kappa_A \rightarrow \beta^{-1}\kappa_A$ for $A = 1, \dots, N$. As a result, one simply has

$$\frac{S_{\lambda\lambda'}}{S_{\lambda 0}} \rightarrow \chi_{\lambda'}(e^{-2\pi\kappa_1}, \dots, e^{-2\pi\kappa_N}). \quad (35)$$

This factor captures a Wilson loop in the central $SU(N)$ gauge node of the star-shaped quiver. Geometrically, the Wilson loop links but does not intersect with the great circle where the Coulomb branch operators are inserted. Alternatively, this Wilson loop can be viewed as inserting a line operator into one of the $T[SU(N)]$ by acting on the $Z_{T[SU(N)]}^{S^3}$ with a difference operator $\mathfrak{D}_{\lambda'}$ labeled by the representation λ' [21,30]. In particular, the line operator can be inserted into the same $T[SU(N)]$ theory in which the Coulomb branch operators are defined, since insertion of the Coulomb branch operators can be realized computationally by differentiating with respect to the FI parameter (which is sent to zero in the end), and such differentiation commutes with the difference operator $\mathfrak{D}_{\lambda'}$ [16].

In this article we proposed a new way of computing current correlation functions on the torus via the q -deformed 2D Yang-Mills theory, and they are further identified with Coulomb branch correlation functions in the 3D mirror dual. The discussions in this article can be generalized to moment map operators of generic regular

punctures by following carefully the correlation functions along the quantum Drinfeld-Sokolov reduction [2]. Furthermore, it should be straightforward to derive the flavored correlation functions by a refined version of recursion relations [31]. It will be interesting to further investigate the relation between the Verma modules of the 3D Coulomb branch/Higgs branch algebra and the modules of the associated VOAs in 4D, and compute from four dimensions the correlators of a large class of Coulomb branch operators in the 3D mirror dual. On S^4 , a similar localization computation adapted to Schur-like operators can be performed [32], which prompts the question that whether the correlation functions of those Schur-like operators have interpretation on the Liouville/Toda theory side. A positive result would help further bridge the Alday-Gaiotto-Tachikawa duality [33] and the SCFT/VOA correspondence.

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APPENDIX A: SPECIAL FUNCTIONS

Here we recall some definitions and properties of the various functions in [18]. The Eisenstein series $G_{2k}(\tau)$ is defined by

$$G_{2k}(\tau) \equiv 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \frac{n^{2k-1} q^n}{1-q^n}, \quad (\text{A1})$$

where ζ denotes the well-known Riemann ζ function. In the main text, we use also the Eisenstein series with a different normalization

$$E_{2k}(\tau) \equiv (2\pi i)^{-2k} G_{2k}(\tau). \quad (\text{A2})$$

With G_{2k} , a series of “Weierstrass functions” can be defined to be

$$P_1(z, \tau) \equiv -\frac{1}{z} + \sum_{n \geq 1} G_{2n+2}(\tau) z^{2n+1} + G_2(\tau) z - \pi i,$$

$$P_{k \geq 2}(z, \tau) \equiv \frac{(-1)^k}{z^k} + \sum_{n \geq 1} \binom{2n+1}{k-1} G_{2n+2}(\tau) z^{2n+2-k}.$$

For $k \geq 2$, $P_k(z, \tau)$ are periodic in $z \sim z + m + n\tau$, while $P_1(z, \tau)$ is only periodic in $z \sim z + m$ and

$P_1(z + \tau, \tau) = P_1(z) + 2\pi i$. Note that $P_2(z, \tau)$ is related to the standard Weierstrass $\wp$ function by $P_2(z, \tau) = \wp_0(z, \tau) + G_2(\tau)$ with

$$\wp_0(z, \tau) \equiv \frac{1}{z^2} + \sum'_{m,n} \frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}. \quad (\text{A3})$$

To obtain the high-temperature limit of the above functions, it is most convenient to observe that they are related to the Jacobi-theta functions, for example,

$$G_2(\tau) = -\frac{\vartheta_1'''(0, \tau)}{3\vartheta_1'(0, \tau)}$$

$$P_1(z, \tau) = -\frac{\vartheta_1'(z|\tau)}{\vartheta_1(z|\tau)} - \pi i$$

$$P_{k+1}(z, \tau) = \frac{1}{k} \frac{d}{dz} P_k(z, \tau). \quad (\text{A4})$$

One has (assuming $\text{Re } z > 0$)

$$G_2(\tau) \sim -\frac{\pi^2}{3\beta^2}, \quad P_1(z, \tau) \sim \frac{\pi}{\beta}(2z-1), \quad (\text{A5})$$

$$P_2(z, \tau) \sim \frac{2\pi}{\beta}, \quad \wp_0(z, \tau) \sim \frac{\pi^2}{3\beta^2}. \quad (\text{A6})$$

Note that in making P_2 , the β^{-2} asymptotic behavior in $\wp$ and G_2 cancel, leaving behind a $2\pi\beta^{-1}$ singularity.

APPENDIX B: VERTEX OPERATOR ALGEBRA

Here we recall some definitions and results widely used in the community of vertex operator algebra, following mainly [18].

A vertex operator algebra V is characterized by a state space V and a state-operator map $Y: V \times \mathbb{C} \rightarrow \text{End}(V)$, where, in physics literature, one usually denotes simply $a(z) = Y(a, z) = \sum_n a_n z^{-n-1}$. An important state ω in V corresponds to the familiar stress tensor T whose modes are L_n forming the Virasoro algebra. In particular, the eigenvalue of L_0 on an eigenstate a is referred to as $\text{wt}(a)$. One also defines the zero mode $o(a) \equiv a_{\text{wt}(a)-1}$.

For operators on a torus, it is common to consider $Y(e^{2\pi i L_0} a, e^{2\pi i z})$, which we sometimes write compactly as $a[z] = \sum_n a_{[n]} z^{-n-1}$, where the square modes $a_{[n]}$ are defined by the expansion. Mode $a_{[m]}$ can be computed explicitly as

$$a_{[m]} \equiv (2\pi i)^{-m-1} \sum_{i \geq m} c(m, i, \text{wt}(a)) a_i, \quad (\text{B1})$$

with $c(m, i, h)$ defined via $[\ln(1+z)]^m (1+z)^{\text{wt } a-1} = \sum_{i \geq m} c(m, i, \text{wt } a) z^i$.

APPENDIX C: CRITICAL AFFINE CHARACTER

The characters of integrable representations of the affine Kac-Moody algebras are well known; however, general representations at nonintegral levels are less understood. In [17] a character formula at level $\hat{\mathfrak{g}}_{k=-h^\vee}$ is proposed,

$$\text{ch}_\lambda^{\text{crit}} = \frac{\sum_{w \in W} e^{w \cdot \lambda}}{\prod_{\hat{\alpha} \in \hat{\Delta}_+^{\text{re}}} (1 - e^{-\hat{\alpha}})} \frac{1}{\prod_{\alpha \in \Delta_+} 1 - q^{(\lambda + \rho, \alpha^\vee)}}, \quad (\text{C1})$$

where λ is the integral dominant with respect to $\mathfrak{g}$ and is the finite part of the affine highest weight of the $\hat{\mathfrak{g}}_{-h^\vee}$ module, $q \equiv e^{-\delta}$ with δ being the imaginary root, $\hat{\Delta}_+^{\text{re}}$ denotes the set of real positive affine roots

$$\hat{\Delta}_+^{\text{re}} \equiv \{\alpha + n\delta | n > 0, \alpha \in \Delta\} \cup \{\alpha | \alpha \in \Delta_+\}, \quad (\text{C2})$$

and finally Δ_+ is the set of all positive roots of the finite Lie algebra $\mathfrak{g}$. Sometimes we also denote $\mathcal{R}_\lambda$ as the horizontal subspace of the module, which itself is a finite dimensional irreducible representation of $\mathfrak{g}$. In [3], the critical character is shown to be proportional to the full puncture wave function,

$$\text{ch}_\lambda^{\text{crit}} = q^{-(\lambda, \rho)} \frac{\text{P.E.}[-\sum_{j=1}^r \frac{q^{d_j}}{1-q^j}]}{\dim_q \mathcal{R}_\lambda} \psi_\lambda^{\text{full}}, \quad (\text{C3})$$

where d_j are the degrees of the Casimirs of $\mathfrak{g}$.

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