Relation between instant and light-front formulations of quantum field theory

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(Received 16 February 2021; accepted 27 April 2021; published 19 May 2021)

The scattering equivalence of quantum field theories formulated with light-front and instant-form kinematic subgroups is established using nonperturbative methods. The difficulty with field theoretic formulations of Dirac's forms of dynamics is that the free and interacting unitary representations of the Poincaré group are defined on inequivalent representations of the Hilbert space, which means that the concept of kinematic transformations must be modified on the Hilbert space of the field theory. This work addresses this problem by assuming the existence of a field theory with the expected properties and constructs equivalent representations with instant and front-form kinematic subgroups. The underlying field theory is not initially associated with an instant form or light-front form of the dynamics. In this construction the existence of a vacuum and one-particle mass eigenstates is assumed and both the light-front and instant-form representations are constructed to share the same vacuum and one-particle states. If there is spontaneous symmetry breaking there will be a 0 mass particle in the mass spectrum (assuming no Higgs mechanism). The free field Fock space plays no role. There is no “quantization” of a classical theory. The property that survives from the perturbative approach is the notion of a kinematic subgroup, which means kinematic Poincaré transformations can be trivially implemented by acting on suitable basis vectors. This nonperturbative approach avoids dealing with issues that arise in perturbative treatments where it is necessary to have a consistent treatment of renormalization, rotational covariance, and the structure of the light-front vacuum. While addressing these issues in a computational framework is the most important unanswered question for applications, this work may provide some insight into the nature of the expected resolution and identifies the origin of some of the differences between the perturbative and nonperturbative approaches.

DOI: [10.1103/PhysRevD.103.105017](https://doi.org/10.1103/PhysRevD.103.105017)

I. INTRODUCTION

This paper discusses the relation between light-front and instant-form formulations of quantum field theory. The identification of different forms of dynamics is due to Dirac [1]. In a relativistic quantum theory the invariance of quantum probabilities in different inertial coordinate systems requires that equivalent states in different inertial coordinate systems are related by a unitary ray representation of the subgroup of the Poincaré group continuously connected to the identity [2]. The Poincaré Lie algebra has 3 independent commutators involving rotationless boost and translation generators that have the Hamiltonian on the right

$$[K^i, P^j] = i\delta_{ij}H. \quad (1)$$

If $H = H_0 + V$ for some interaction V then the operators on the left side of each commutator must also know about the interaction. Dirac identified three representations of the Poincaré Lie algebra with the minimum number (3–4) of interaction-dependent Poincaré generators. He called these the instant, point and front-forms of the dynamics. In the instant form the generators of space translations and rotations are free of interactions, in the point form the generators of Lorentz transformations are free of interactions and in the front form the generators of transformations that leave a hyperplane tangent to the light cone invariant are free of interactions.

The equivalence of these different representations of relativistic quantum mechanics was settled by Sokolov and Shatnii [3,4]. In quantum field theory the problem is more complicated because the free and interacting dynamics are formulated on different inequivalent representations of the Hilbert space [5], so the decomposition of the Hamiltonian into the sum of a free Hamiltonian plus interaction is not defined on the Hilbert space of the field theory. Such a decomposition makes sense in perturbative quantum field theory with cutoffs, so the notion of instant- and light-front formulations [6–11] of quantum field theory make sense

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perturbatively. The price paid is that as the cutoffs are removed the theory has infinities that have to be renormalized. The relation between the different forms of dynamics depends on a consistent treatment of the renormalization. The light-front formulation of quantum field theory is of particular interest for applications. The most appealing property of the theory is the apparent triviality of the light-front vacuum, which reduces the solution of the field theory to linear algebra on a Hilbert space, like nonrelativistic quantum mechanics. In addition to the computational challenges of implementing this program in a theory with an infinite number of degrees of freedom, there are a number of puzzles that appear in comparing the two approaches. These include:

- (i) Is the light-front vacuum the same as the Fock vacuum?
- (ii) How to understand $P^+ = 0$ (zero mode) singularities?
- (iii) How to understand spontaneous symmetry breaking in a light-front dynamics?
- (iv) How to renormalize the theory consistent with rotational covariance.
- (v) What is the relation between light-front and canonical quantization of a quantum field theory?
- (vi) Are both approaches equivalent?

While there is a general consensus that the two formulations are equivalent, the answers to the above questions are not as clean as desired. These issues have been discussed extensively in the literature [6,7,12,13,8–11,14–39].

Most of these discussions are based on perturbation theory; in particular on the assumption that the light-front dynamics can be formulated on the Fock space of a free field theory. The continued interest in these questions is because a compelling resolution of these questions is obscured by the need to perform a nonperturbative renormalization of the theory. In this work these complications are avoided by assuming the existence of the theory with the expected properties that are normally assumed in axiomatic formulations of quantum field theory [40–43]. One departure from both of Dirac's Hamiltonian approaches is that in axiomatic formulations of quantum field theory states are constructed by applying fields integrated against Schwartz test functions [44] in four space-time variables to the vacuum. These states can be evolved, but the initial data is not on a sharp time slice or light front. The existence of an asymptotically complete scattering theory is also assumed, based on the Haag-Ruelle formulation of scattering theory [45–48]. Haag-Ruelle scattering has the advantage that it can be formulated in terms of wave operators defined by strong limits. The strong limits justify using many of the methods that are used in nonrelativistic scattering theory and they are used in the construction that follows. The difference with the LSZ formulation of scattering is that the interpolating field operators must create one-particle states rather than just create states that have some overlap with one-particle states. The advantage

of the Haag-Ruelle formulation of scattering is that the weak limits that are used to define the scattering asymptotic conditions in the LSZ formulation of scattering can be replaced by strong limits.

A two-Hilbert space representation, which does not require the existence of a free dynamics on the Hilbert space of the theory, is used to formulate the scattering theory. This avoids any concerns about Haag's theorem [5,40], which implies that the interacting and free field theories are formulated on inequivalent Hilbert-space representations of the field algebra. The second Hilbert space, which is called the asymptotic Hilbert space, is the direct sum of tensor products of single-particle Hilbert spaces with physical masses. This representation is natural in nonrelativistic many-body scattering. For example, in lead-lead scattering the lead nuclei are treated as elementary particles in the asymptotic Hilbert space. Products of the lead state vectors, $|L_1, \mathbf{p}_1, \mu_1\rangle \times |L_2, \mathbf{p}_2, \mu_2\rangle$, define a mapping from the asymptotic Hilbert space of suitably symmetrized square integrable functions, $f(\mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2)$, to the Hilbert space of the theory. In the field theory case the lead wave functions are replaced by products of operators, $A_L^\dagger(\mathbf{p}, \mu)$, that create point spectrum lead mass eigenstates out of the vacuum, applied to the vacuum [49,50,51]. In the field theory case the operators $A_L^\dagger(\mathbf{p}, \mu)$ are not creation operators. They do not commute or anticommute with each other and do not satisfy canonical commutation relations. Inner products of states constructed by applying two or more of these operators to the vacuum can have intermediate states with different particle content.

In the simplest case the asymptotic Hilbert space is a Fock space with physical particle masses. The mapping from the second Hilbert space to the Hilbert space of the field theory puts in the internal structure of the physical particles, including the effects of self-interactions, when the particles are asymptotically separated.

There is a natural unitary representation of the Poincaré group on the asymptotic Hilbert space, which treats the particles as free particles with their observed masses. There is sufficient freedom to choose the mappings from the asymptotic Hilbert space to the Hilbert space of the field theory so the unitary representation of the instant or light-front kinematic subgroup of the Poincaré group on the asymptotic space is mapped to the dynamical unitary representation of the corresponding kinematic subgroup on the field-theory Hilbert space without changing the scattering operator.

A theorem due to Ekstein [52] gives necessary and sufficient conditions for unitary transformations on a Hilbert space to preserve the scattering operator. This result assumes the scattering is formulated in terms of asymptotically complete wave operators defined as strong limits. In the two-Hilbert space formulation Ekstein's condition is a constraint on the unitary transformation

relating two different Hamiltonians that does not require modifying the Hamiltonian in the asymptotic Hilbert space. Roughly speaking it means that the unitary operator is a short-range perturbation of the identity in a manner that Ekstein makes precise [see Eq. (34)]. This freedom can be exploited to construct the most general class of S -matrix preserving unitary transformations that are invariant under a kinematic subgroup of the dynamical representation of the Poincaré group. The proof uses the unitarity of the wave operators which holds provided the vacuum and one-body channels are included in the asymptotic Hilbert space. Unitary transformations must leave the one-particle masses and the mass gap unchanged. In this work these transformations are also constructed to leave the vacuum and one-particle states unchanged.

When these unitary transformations are applied to the unitary representation of the Poincaré group on the physical Hilbert space, they result in two new scattering-equivalent representations with light-front and instant-form kinematic subgroups. These representations have the property that the injection operators map the kinematic subgroup on the asymptotic Hilbert space to the corresponding subgroup of the unitary representation of the Poincaré group on the physical Hilbert space. Both representations are equivalent to the original “axiomatic” formulation of the theory in the sense that they are related by S -matrix preserving unitary transformations.

One thing that is missing in this construction is the existence of a noninteracting representation of the Poincaré group on the Hilbert space of the field theory. Haag’s theorem implies that free fields, even with experimental masses, are defined on an inequivalent Hilbert space representation of the field algebra, so there is no corresponding local free field theory on the Hilbert space of the field theory. In the event that the mapping from the asymptotic space to the field theory Hilbert space has dense range, and the unitarized form of this mapping satisfies Ekstein’s asymptotic condition, Eq. (34), then it can be used to map the unitary representation of the Poincaré group on the asymptotic Hilbert space to a unitary representation of the Poincaré group on the field theory Hilbert that plays the role of a free-particle dynamics. Since the two-Hilbert space injection operators are not unique and are not local operators, this will not be a local field theory.

The results can be summarized as follows. The construction assumes a unitary representation of the Poincaré group on the Hilbert space of the field theory and constructs scattering equivalent representations with light-front and instant-form kinematic symmetries. The representations are related by a unitary transformation that satisfies an asymptotic condition that preserves the scattering matrix elements. Because the one-body states are eigenstates of the mass operator, there is no mass renormalization. While the form of the interactions is representation dependent,

the absence of mass renormalization suggests that the simplest types of interactions that change particle number are short-range $2 \leftrightarrow 3$ -particle interactions rather than $1 \leftrightarrow 2$ vertex interactions.

This paper has five sections and two appendices. Section II summarizes the assumptions used in this work and discusses a two-Hilbert space formulation of Haag-Ruelle scattering that has kinematically invariant injection operators that map an asymptotic many-particle Hilbert space to the Hilbert space of the field theory. Haag-Ruelle scattering is the natural generalization of the usual formulation of time-dependent scattering theory. The construction of equivalent instant and light-front representations of the field theory is given in Sec. III. Section IV has a brief discussion of spontaneous symmetry breaking. The construction is compared to the corresponding construction in relativistic formulations of quantum mechanics of a finite number of degrees of freedom in Sec. V. Section VI has a summary of the results and conclusions. There are two appendices. The first one contains a discussion of Ekstein’s treatment of scattering equivalences [52] that is used in Sec. III. The second appendix discusses the construction of Haag-Ruelle injection operators with the kinematic symmetry properties that are used in Secs. II and III.

II. GENERAL CONSIDERATIONS

This paper examines the relation between light-front and instant-form formulations of quantum field theory from a more abstract perspective that does not assume that the Hamiltonian can be decomposed as the sum of free and interacting operators acting on one representation of the Hilbert space. The starting assumptions are typical assumptions about abstract properties of a quantum field theory. These include:

- (i) The Hilbert space of the field theory is generated by applying bounded functions, F , of field operators smeared over test functions to a unique vacuum vector, denoted by $|0\rangle$. The smearing is assumed to be over Schwartz test functions [44] in four space-time dimensions. It is not assumed that fields restricted to a light-front or fixed-time manifold make sense. A dense set of vectors, $|\psi\rangle$, can be taken to have the form:

$$|\psi\rangle = F|0\rangle \quad F = \sum_{n=1}^N c_n e^{i\phi_n(f_n)} \quad (2)$$

where the c_n are complex coefficients, the f_n are Schwartz functions, ϕ_n are field operators and N is finite.

- (ii) There is a unitary representation of the Poincaré group, $U(\Lambda, a)$, on this representation of the Hilbert

space which is given by covariance. In the spinless case the action of $U(\Lambda, a)$ on states of the form (2) is

$$U(\Lambda, a)|\psi\rangle = \sum_c c_n e^{i\phi(f'_n)}|0\rangle$$

$$f'_n(x) = f_n(\Lambda^{-1}(x - a)). \quad (3)$$

- (iii) The vacuum, $|0\rangle$, is Poincaré invariant and normalized to unity:

$$U(\Lambda, a)|0\rangle = |0\rangle \quad \langle 0|0\rangle = 1. \quad (4)$$

- (iv) The unitary representation of the Poincaré group acts like the identity on the vacuum subspace and can be decomposed into a direct integral of positive-mass positive-energy irreducible representations of the Poincaré group [2,53] on the orthogonal complement of the vacuum subspace. The mass Casimir operator is assumed to have point-spectrum eigenstates representing particles of the theory.
- (v) There is a complete set of asymptotically complete Haag-Ruelle [45–48] scattering states.

A characteristic element of Dirac's forms of dynamics is the notion of a *kinematic subgroup*. Kinematic subgroups leave a manifold in Minkowski space, that classically intersects every (massive) particle's world line once, invariant. In an instant-form dynamics the invariant manifold is a fixed-time plane, which is preserved under the group generated by rotations and space translations. In a light-front dynamics the manifold is a hyperplane that is tangent to the light cone. This is invariant under a seven-parameter subgroup of the Poincaré group. These manifolds are prominent in Dirac's work; in this work the focus will be on the subgroups of the Poincaré group that leave these manifolds invariant. These subgroups will be referred to as kinematic subgroups.

For quantum mechanical systems of a finite number of degrees of freedom kinematical and dynamical unitary representations of the Poincaré group exist in the same representation of the Hilbert space. In the field theoretic case the representation of the Hilbert space depends on the dynamics. While free fields with physical masses have the same mass spectrum, they act on an inequivalent Hilbert space representation of the field algebra by Haag's theorem, so *there is no free dynamics* on the Hilbert space of the interacting field theory. This is one of the distinctions with the perturbative approach, which attempts to define the interacting theory as a perturbation of a free field theory. The incompatibility of these formulations leads to the infinities that complicate the analysis of the relation between light-front and instant-form treatments of field theory.

The first step in the abstract formulation is to define what is meant by the kinematic subgroup in the absence of a noninteracting representation of the Poincaré group.

The unitary representation of the Poincaré group has a set of 10 infinitesimal generators that are self-adjoint operators on the Hilbert space of the field theory satisfying the Poincaré Lie algebra. This follows because each one is the generator of a unitary one-parameter group. The unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations. In this work a typical spectral condition is assumed, where the representations that appear in the direct integral are assumed to be positive-mass positive-energy representations, and the identity on the vacuum subspace.

The direct integral of irreducible representations of the Poincaré group results in a direct integral decomposition of the Hilbert space into irreducible subspaces or slices. The next step is to choose a basis on each irreducible subspace. Two choices will be considered in this work. The first basis consists of simultaneous eigenstates of the mass, spin, linear momentum, and the projection of the canonical spin on the z axis. These are all functions of the infinitesimal generators of the unitary representation of the Poincaré group. There will also be Poincaré invariant degeneracy parameters, which can be taken as discrete quantum numbers, since any continuous parameters can be replaced by a basis of square integrable functions of the continuous degeneracy parameters. The other basis on the same irreducible subspace consists of simultaneous eigenstates of the mass, spin, the three light-front components of the four momentum, and the projection of the light-front spin on the z axis. These are also functions of the infinitesimal generators with the same degeneracy parameters.

These bases can be formally expressed as

$$|0\rangle \cup \{|(m, s, d_n)\mathbf{p}, \mu_c\rangle\} \quad \text{and}$$

$$|0\rangle \cup \{|(m, s, d_n)\tilde{\mathbf{p}}, \mu_F\rangle\} \quad (5)$$

where the light-front components of the four momentum are defined by

$$\tilde{\mathbf{p}} = (\mathbf{p} \cdot \hat{\mathbf{x}}, \mathbf{p} \cdot \hat{\mathbf{y}}, p^0 + \mathbf{p} \cdot \hat{\mathbf{z}}) := (\mathbf{p}_\perp, p^+). \quad (6)$$

The variables (6) are eigenvalues of the generators of translations tangent to the light-front hyperplane. The first basis in (5) will be referred to as the “instant-form” basis and the second basis will be referred to as the “light-front” basis.

These basis states are related by a unitary change of basis. The change of basis is block diagonal in the direct integral. The different basis vectors are related by

$$|(m, s, d_n)\mathbf{p}, \mu_I\rangle = \sum_{\mu_F=-s}^s |(m, s, d_n)\tilde{\mathbf{p}}(p), \mu_F\rangle$$

$$\times \sqrt{\frac{p^+}{\omega_m(\mathbf{p})}} D_{\mu_F \mu_I}^s [B_F^{-1}(p/m) B_I(p/m)] \quad (7)$$

which assumes that both bases have delta function normalizations:

$$\begin{aligned} & \langle (m, s, d_n) \mathbf{p}, \mu_I | (m', s', d_k) \mathbf{p}', \mu'_I \rangle \\ & = \delta(\mathbf{p} - \mathbf{p}') \delta_{ss'} \delta_{\mu_I \mu'_I} \delta_{nk} \delta[m - m'] \end{aligned} \quad (8)$$

and

$$\begin{aligned} & \langle (m, s, d_n) \tilde{\mathbf{p}}, \mu_F | (m', s', d_k) \tilde{\mathbf{p}}', \mu'_F \rangle \\ & = \delta(p^+ - p'^+) \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp) \delta_{ss'} \delta_{\mu_F \mu'_F} \delta_{nk} \delta[m - m'] \end{aligned} \quad (9)$$

where $\delta[m - m']$ is either a Dirac or Kronecker delta function depending on whether m is in the point or continuous spectrum of the mass operator. $B_I(p/m)$ and $B_F(p/m)$ are $SL(2, C)$ matrices representing a rotationless boost and a light-front preserving boost from $(1, 0, 0, 0)$ to p/m . The combination $B_F^{-1}(p/m) B_I(p/m)$ is a $SU(2)$ representation of a Melosh rotation [54] that changes the light-front spin to the canonical spin. $D_{\mu\nu}^s[R] = \langle s, \mu | U(R) | s, \nu \rangle$ is the $2s + 1$ dimensional unitary representation of $SU(2)$ and $\omega_m(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$.

There are similar transformations relating light-front bases corresponding to different orientations of the light front (i.e., replacing $\hat{\mathbf{z}}$ by some other unit vector $\hat{\mathbf{n}}$). Like (7) they involve a variable change, the square root of a Jacobian and a momentum-dependent rotation matrix.

The action of $U(\Lambda, a)$ on each of these irreducible basis states is a consequence of the transformation properties of the Poincaré generators and the basis choice. For the “instant-form” basis Λ

$$U(\Lambda, a) |0\rangle = |0\rangle, \quad (10)$$

$$\begin{aligned} & U(\Lambda, a) | (m, s, d_n) \mathbf{p}, \mu \rangle \\ & = e^{ia \cdot \Lambda p} \sum_{\mu'=-s}^s | (m, s, d_n) \Lambda p, \mu' \rangle \\ & \times \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(\mathbf{p})}} D_{\mu' \mu}^s [B_I^{-1}(\Lambda(p/m)) \Lambda B_I(p/m)] \end{aligned} \quad (11)$$

and for the “light-front” basis,

$$U(\Lambda, a) |0\rangle = |0\rangle, \quad (12)$$

$$\begin{aligned} & U(\Lambda, a) | (m, s, d_n) \tilde{\mathbf{p}}, \mu \rangle \\ & = e^{ia \cdot \Lambda p} \sum_{\mu'=-s}^s | (m, s, d_n) \tilde{\Lambda p}, \mu' \rangle \\ & \times \sqrt{\frac{(\Lambda p)^+}{p^+}} D_{\mu' \mu}^s [B_F^{-1}(\Lambda(p/m)) \Lambda B_F(p/m)]. \end{aligned} \quad (13)$$

These equations define the dynamical unitary representation of the Poincaré group on these two irreducible bases.

The connection of these bases with kinematic subgroups is that the coefficients of the basis functions on the right-hand side of Eqs. (11) and (13) do not depend on the mass eigenvalue m when (Λ, a) is an element of the kinematic subgroup. For the basis (11) the kinematic subgroup is generated by rotations and spatial translations while for the basis (13) the kinematic subgroup is the subgroup that leaves the light front hyperplane $x^+ = x^0 + \mathbf{x} \cdot \hat{\mathbf{z}} = 0$ invariant.

The next step is to discuss the formulation of scattering theory. A two-Hilbert space representation [51,55], will be used for this purpose. The starting point is the assumption that the mass operator, $M = \sqrt{-p^2}$ has one-particle eigenstates. In this nonperturbative context one-particle means that the mass operator, $M = \sqrt{-p^2}$, has a nonempty point spectrum. The discrete mass eigenvalues are assumed to be strictly positive. There is no distinction between elementary and composite particles.

Normalizable one-particle states in the “instant form” basis can be constructed by integrating the irreducible basis functions with a wave packet:

$$|\psi_{Ig}\rangle := \int d\mathbf{p} \sum_{\mu=-s}^s | (m, s, d_n) \mathbf{p}, \mu \rangle g(\mathbf{p}, \mu) \quad (14)$$

where m is a discrete mass eigenvalue and $g(\mathbf{p}, \mu)$ is a square integrable wave packet. The corresponding construction in the “light-front” basis has the form

$$|\psi_{Fg}\rangle := \int d^2 \mathbf{p}_\perp \int_0^\infty dp^+ \sum_{\mu=-s}^s | (m, s, d_n) \tilde{\mathbf{p}}, \mu \rangle \tilde{g}(\tilde{\mathbf{p}}, \mu) \quad (15)$$

where in the light front case $\tilde{\mathbf{p}}$ should vanish as $p^+ \rightarrow 0$ in a manner that results in a square integrable function of $\mathbf{p}$ under the variable change (6) $\tilde{\mathbf{p}} \rightarrow \mathbf{p}$.

These normalizable vectors are linear in the functions $g(\mathbf{p}, \mu)$ and $\tilde{g}(\tilde{\mathbf{p}}, \mu)$. They can be represented by elements of the field algebra applied to the vacuum. This means that they can be expressed as

$$\begin{aligned} |\psi_{Ig}\rangle & = \sum_{\mu=-s}^s \int A_{m,s,d_n}^\dagger(p, \mu) |0\rangle d\mathbf{p} g(\mathbf{p}, \mu) \\ & = A^\dagger(g) |0\rangle \end{aligned} \quad (16)$$

and

$$\begin{aligned} |\psi_{Fg}\rangle & = \sum_{\mu=-s}^s \int \tilde{A}_{m,s,d_n}^\dagger(\tilde{p}, \mu) |0\rangle dp^+ d^2 \mathbf{p}_\perp \tilde{g}(\tilde{\mathbf{p}}, \mu) \\ & = \tilde{A}^\dagger(\tilde{g}) |0\rangle. \end{aligned} \quad (17)$$

These states will be equal if $g(\mathbf{p}, \mu) = \langle (m, s, d_n) \mathbf{p}, \mu_I | g \rangle$ and $\tilde{g}(\tilde{\mathbf{p}}, \mu) = \langle (m, s, d_n) \tilde{\mathbf{p}}, \mu_F | g \rangle$ are related by the change of basis (7).

The operators $A_{m,s,d_n}^\dagger(p, \mu)$ and $\tilde{A}_{m,s,d_n}^\dagger(\tilde{p}, \mu)$ are functions of the fields that create particles of mass m , spin s , momentum $\mathbf{p}$ and magnetic quantum number μ out of the vacuum. The construction of these operators starting from operators that couple the vacuum to the one-particle states of the theory is discussed in Appendix B.

Since $A_{m,s,d_n}^\dagger(p, \mu)$ and $\tilde{A}_{m,s,d_n}^\dagger(\tilde{p}, \mu)$ are in the field algebra (after smearing with test functions), they can be multiplied.

The following quantity

$$A_{m_1,s_1,d_{n_1}}^\dagger(\mathbf{p}_1, \mu_1) \cdots A_{m_k,s_k,d_{n_k}}^\dagger(\mathbf{p}_k, \mu_k) | 0 \rangle \quad (18)$$

can be considered as a mapping from a Hilbert space of square integrable suitably symmetrized functions of $\mathbf{p}_1, \mu_1 \cdots \mathbf{p}_k, \mu_k$ to the Hilbert space of the field theory. Note that while $A^\dagger(g) | 0 \rangle = \tilde{A}^\dagger(\tilde{g}) | 0 \rangle$ for g and $\tilde{g}$ related by (7), because both operators create the same single-particle states out of the vacuum, this identity is not true for products of these operators applied to the vacuum. From Eq. (B17) in Appendix B it follows that this mapping has the following properties

$$\begin{aligned} & \mathbf{p} A_{m_1,s_1,d_{n_1}}^\dagger(\mathbf{p}_1, \mu_1) \cdots A_{m_N,s_N,d_{n_N}}^\dagger(\mathbf{p}_N, \mu_N) | 0 \rangle \\ &= \sum_{n=1}^N \mathbf{p}_n A_{m_1,s_1,d_{n_1}}^\dagger(\mathbf{p}_1, \mu_1) \cdots A_{m_N,s_N,d_{n_N}}^\dagger(\mathbf{p}_N, \mu_N) | 0 \rangle \end{aligned} \quad (19)$$

and

$$\begin{aligned} & U(R, 0) A_{m_1,s_1,d_{n_1}}^\dagger(\mathbf{p}_1, \mu_1) \cdots A_{m_N,s_N,d_{n_N}}^\dagger(\mathbf{p}_1, \mu_N) | 0 \rangle \\ &= \sum A_{m_1,s_1,d_{n_1}}^\dagger(R\mathbf{p}_1, \nu_1) \cdots A_{m_N,s_N,d_{n_N}}^\dagger(R\mathbf{p}_N, \nu_N) | 0 \rangle | 0 \rangle \\ &\quad \times \prod_{n=1}^N D_{\nu_n \mu_n}^{s_n}(R). \end{aligned} \quad (20)$$

For the mapping in the “light-front” basis the corresponding relations are

$$\begin{aligned} & \tilde{\mathbf{p}} \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(\tilde{\mathbf{p}}_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(\tilde{\mathbf{p}}_N, \mu_N) | 0 \rangle \\ &= \sum_{n=1}^N \tilde{\mathbf{p}}_n \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(\tilde{\mathbf{p}}_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(\tilde{\mathbf{p}}_N, \mu_N) | 0 \rangle \end{aligned} \quad (21)$$

$$\begin{aligned} & U(B_f, 0) \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(\tilde{\mathbf{p}}_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(\tilde{\mathbf{p}}_N, \mu_N) | 0 \rangle \\ &= \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(\tilde{\mathbf{B}}_f p_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(\tilde{\mathbf{B}}_f p_N, \mu_N) | 0 \rangle \\ &\quad \times \prod_{n=1}^N \sqrt{\frac{B_f(p_n)^+}{p_n^+}}. \end{aligned} \quad (22)$$

$$\begin{aligned} & U(R_z(\phi), 0) \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(\tilde{\mathbf{p}}_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(\tilde{\mathbf{p}}_N, \mu_N) | 0 \rangle \\ &= \tilde{A}_{m_1,s_1,d_{n_1}}^\dagger(R_z(\phi) \tilde{\mathbf{p}}_1, \mu_1) \cdots \tilde{A}_{m_N,s_N,d_{n_N}}^\dagger(R(\phi) \tilde{\mathbf{p}}_N, \mu_N) | 0 \rangle \\ &\quad \times \prod_{n=1}^N e^{i\mu_n \phi}. \end{aligned} \quad (23)$$

If the Fourier transform of these states are integrated against wave packets that are localized in regions separated by large space-like separations and the field theory satisfies cluster properties (normally a consequence of uniqueness of the vacuum) then the resulting vectors look like states of N asymptotically separated particles when they are used in inner products. This property is used [48,50] to prove the existence of the strong limits that define wave operators in Haag-Ruelle scattering. The interpretation of the operators $A_{m,s,d_n}^\dagger$ and $\tilde{A}_{m,s,d_n}^\dagger$ is that they *asymptotically* behave like creation operators.

A scattering channel α is associated with a finite collection of particles asymptotically. It could correspond to the state of a target and an incoming projectile or a collection of particles that are detected in a scattering experiment. The set of all scattering channels of the theory is denoted by $\mathcal{A}$. In what follows both the vacuum and one-particle states are included in the collection of channels, $\mathcal{A}$.

Channel injection operators are defined as products of the operators (16) or (17), applied to the vacuum, one for each particle in a scattering channel α . In the canonical basis they are

$$\Phi_\alpha(\mathbf{p}_1, \mu_1 \cdots \mathbf{p}_N, \mu_N) := \prod_{k \in \alpha} A_{m_k,s_k,d_{n_k}}^\dagger(\mathbf{p}_k, \mu_k) | 0 \rangle \quad (24)$$

and in the light-front basis they are

$$\tilde{\Phi}_\alpha(\tilde{\mathbf{p}}_1, \mu_1 \cdots \tilde{\mathbf{p}}_N, \mu_N) := \prod_{k \in \alpha} \tilde{A}_{m_k,s_k,d_{n_k}}^\dagger(\tilde{\mathbf{p}}_k, \mu_k) | 0 \rangle. \quad (25)$$

The asymptotic channel Hilbert space $\mathcal{H}_\alpha$ is the space of square integrable functions of the variables $\mathbf{p}_k, \mu_k$ or $\tilde{\mathbf{p}}_k, \mu_k$ for $k \in \alpha$. It is interpreted as a space of N particles of mass m_k and spin s_k . If any of the particles in the channel α are identical then the functions representing identical bosons should be symmetrized and those representing identical fermions should anti-symmetrized. The order of the product in (25) does not matter for suitably symmetrized states in the asymptotic Hilbert space. The channel injection operators (24) and (25) are interpreted as mappings from the k -particle channel subspace of the asymptotic Hilbert space $\mathcal{H}_\alpha$ to the Hilbert space $\mathcal{H}$ of the field theory.

The asymptotic unitary representation of the Poincaré group on $\mathcal{H}_\alpha$ is defined by treating $\mathcal{H}_\alpha$ as a space of N mutually noninteracting particles of mass m_k and spin s_k for $k \in \alpha$. This representation is defined by the tensor

product of one-particle irreducible representations in terms of “instant-form” variables

$$\begin{aligned}
 U_\alpha(\Lambda, a)|\mathbf{p}_1, \mu_1, \dots, \mathbf{p}_k, \mu_k\rangle \\
 := \sum e^{ia \cdot \Lambda(\sum_n p_n)} |\Lambda p_1, \nu_1 \dots \Lambda p_k, \nu_k\rangle \\
 \times \prod_{n=1}^k \sqrt{\frac{\omega_{m_n}(\Lambda p_n)}{\omega_{m_n}(\mathbf{p}_n)}} D_{\nu_n \mu_n}^s [B_c^{-1}(\Lambda p_n/m_n) \Lambda B_c(p_n/m_n)].
 \end{aligned} \quad (26)$$

The corresponding expression in terms of “light-front” variables is

$$\begin{aligned}
 U_\alpha(\Lambda, a)|\tilde{\mathbf{p}}_1, \mu_1, \dots, \tilde{\mathbf{p}}_k, \mu_k\rangle \\
 := \sum e^{ia \cdot \Lambda(\sum_n p_n)} |\tilde{\Lambda} p_1, \nu_1 \dots \tilde{\Lambda} p_k, \nu_k\rangle \\
 \times \prod_{n=1}^k \sqrt{\frac{(\tilde{\Lambda} p_n)^+}{p_n^+}} D_{\nu_n \mu_n}^s [B_f^{-1}(\Lambda p_n/m_n) \Lambda B_f(p_n/m_n)].
 \end{aligned} \quad (27)$$

The states in (26) and (27) are basis vectors in $\mathcal{H}_\alpha$, not $\mathcal{H}$. Since the Hilbert space vectors in the range of Φ_α or $\tilde{\Phi}_\alpha$ are not necessarily N-particle states it follows that in general

$$\begin{aligned}
 U(\Lambda, a)\Phi_\alpha &\neq \Phi_\alpha U_\alpha(\Lambda, a). \\
 U(\Lambda, a)\tilde{\Phi}_\alpha &\neq \tilde{\Phi}_\alpha U_\alpha(\Lambda, a).
 \end{aligned} \quad (28)$$

This is because (B6) does not hold for P^0 due to the p^0 integrals in (B9) and (B6) does not hold for P^- due to the p^- integrals on (B9) (see Appendix B).

When Λ_K and a_K are elements of the instant-form or light-front kinematic subgroup, it follows from (19)–(23) that

$$\begin{aligned}
 U(\Lambda_K, a_K)\Phi_\alpha &= \Phi_\alpha U_\alpha(\Lambda_K, a_K) \\
 U(\Lambda_K, a_K)\tilde{\Phi}_\alpha &= \tilde{\Phi}_\alpha U_\alpha(\Lambda_K, a_K)
 \end{aligned} \quad (29)$$

because the coefficients of the kinematic transformations are mass independent. The operators Φ_α and $\tilde{\Phi}_\alpha$ that map the channel α Hilbert space $\mathcal{H}_\alpha$ into the field theory Hilbert space $\mathcal{H}$ are Haag-Ruelle [45–47] injection operators. The only difference with the standard choices of Haag Ruelle injection operators is that Φ_α and $\tilde{\Phi}_\alpha$ are designed to preserve the kinematic subgroup (29).

In the two-Hilbert space formulation [49,50] channel wave operators $\Omega_{\alpha\pm} = \Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha)$ are defined by the strong limits

$$\lim_{t \rightarrow \pm\infty} \|(\Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha) - e^{iHt}\Phi_\alpha e^{-iH_\alpha t})|\psi\rangle\| = 0 \quad (30)$$

or

$$\lim_{t \rightarrow \pm\infty} \|(\Omega_{\alpha\pm}(H, \tilde{\Phi}_\alpha, H_\alpha) - e^{iHt}\tilde{\Phi}_\alpha e^{-iH_\alpha t})|\psi\rangle\| = 0. \quad (31)$$

The two Hilbert space wave operators satisfy the Poincaré covariance condition

$$\begin{aligned}
 U(\Lambda, a)\Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha) &= \Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha)U_\alpha(\Lambda, a) \\
 U(\Lambda, a)\Omega_{\alpha\pm}(H, \tilde{\Phi}_\alpha, H_\alpha) &= \Omega_{\alpha\pm}(H, \tilde{\Phi}_\alpha, H_\alpha)U_\alpha(\Lambda, a)
 \end{aligned} \quad (32)$$

where, unlike (28), (Λ, a) are not restricted to the kinematic subgroup.

The purpose of the injection operators in Haag-Ruelle scattering is to define the asymptotic boundary conditions. They are constructed so they behave like creation operators in asymptotically separated regions. They replace the free-particle asymptotic states of ordinary scattering theory by states in the Hilbert spaces of the field theory that only behave like a system of noninteracting particles in the asymptotic region. The two injection operators defined in Appendix B both have this property. They differ in which variable enforces the one-particle mass-shell delta function when applied the vacuum. This means that as operators

$$\Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha) = \Omega_{\alpha\pm}(H, \tilde{\Phi}_\alpha, H_\alpha) \quad (33)$$

if they are evaluated in the same one-particle basis. This identity is equivalent to the requirement

$$\begin{aligned}
 0 &= \lim_{t \rightarrow \pm\infty} \|e^{iHt}(\Phi_\alpha - \tilde{\Phi}_\alpha)e^{-iH_\alpha t}|\psi_\alpha\rangle\| \\
 &= \lim_{t \rightarrow \pm\infty} \|(\Phi_\alpha - \tilde{\Phi}_\alpha)e^{-iH_\alpha t}|\psi_\alpha\rangle\|
 \end{aligned} \quad (34)$$

which is precisely the condition that the injection operators agree asymptotically.

The scattering operator for scattering from channel α to channel β is

$$S_{\beta,\alpha} : \mathcal{H}_\alpha \rightarrow \mathcal{H}_\beta := \Omega_{\beta+}^\dagger(H, \Phi_\beta, H_\beta)\Omega_{\alpha-}(H, \Phi_\alpha, H_\alpha). \quad (35)$$

It follows from the identity (34) that $S_{\alpha\beta}$ is independent of the choice of injection operator. It follows from (32) that

$$U_\beta(\Lambda, a)S_{\beta,\alpha} = S_{\beta,\alpha}U_\alpha(\Lambda, a). \quad (36)$$

This can be extended to all channels, $\mathcal{A}$, by defining the asymptotic Hilbert space as the direct sum of all channel Hilbert spaces $\mathcal{H}_\alpha$ for $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is defined to include the vacuum and one-particle channels as well as the scattering channels:

$$\mathcal{H}_\mathcal{A} := \bigoplus_{\alpha \in \mathcal{A}} \mathcal{H}_\alpha. \quad (37)$$

The asymptotic unitary representation of the Poincaré group is defined on $\mathcal{H}_\mathcal{A}$ by

$$U_A(\Lambda, a) = \bigoplus_{\alpha \in \mathcal{A}} U_\alpha(\Lambda, a). \quad (38)$$

Multi-channel injection operators are defined as the sum of all channel injection operators

$$\Phi_A := \sum_{\alpha \in \mathcal{A}} \Phi_\alpha \quad \text{or} \quad \tilde{\Phi}_A := \sum_{\alpha \in \mathcal{A}} \tilde{\Phi}_\alpha \quad (39)$$

where $\Phi_\alpha: \mathcal{H}_\alpha \rightarrow \mathcal{H}$. These can be used to define multi-channel wave operators

$$\Omega_{A\pm}(H, \Phi_A, H_A) = \tilde{\Omega}_{A\pm}(H, \tilde{\Phi}_A, H_A) \quad (40)$$

by the strong limits

$$\lim_{t \rightarrow \pm\infty} \|(\Omega_{A\pm}(H, \Phi_A, H_A) - e^{iHt} \Phi_A e^{-iH_A t} |\psi\rangle)\| = 0 \quad (41)$$

or

$$\lim_{t \rightarrow \pm\infty} \|(\Omega_{A\pm}(H, \tilde{\Phi}_A, H_A) - e^{iHt} \tilde{\Phi}_A e^{-iH_A t} |\psi\rangle)\| = 0 \quad (42)$$

where $H_A := \sum_{\alpha \in \mathcal{A}} H_\alpha \Pi_\alpha$ and Π_α is the projection on the subspace $\mathcal{H}_\alpha$ of $\mathcal{H}_A$. The assumed asymptotic completeness of the scattering theory means that the wave operators (including the vacuum and one-body channels) are unitary mappings from the asymptotic Hilbert space, $\mathcal{H}_A$, to the Hilbert, $\mathcal{H}$, space of the quantum field theory.

With this definition (32) becomes

$$\begin{aligned} U(\Lambda, a) \Omega_{A\pm}(H, \Phi_A, H_A) \\ = \Omega_{A\pm}(H, \Phi_A, H_A) U_A(\Lambda, a) \end{aligned} \quad (43)$$

which also holds with Φ_A replaced by $\tilde{\Phi}_A$.

The multi-channel scattering operator is

$$S(H, \Phi_A, \mathcal{H}_A) = \Omega_+^\dagger(H, \Phi_A, \mathcal{H}_A) \Omega_-(H, \Phi_A, \mathcal{H}_A) \quad (44)$$

where it was constructed to be the identity on the vacuum and one particle states.

Poincaré invariance of the multichannel scattering operator on $\mathcal{H}_A$ follows from the intertwining property of the wave operators (43) [49]

$$\begin{aligned} U_A(\Lambda, a) S(H, \Phi_A, \mathcal{H}_A) \\ = U_A(\Lambda, a) \Omega_+^\dagger(H, \Phi_A, \mathcal{H}_A) \Omega_-(H, \Phi_A, \mathcal{H}_A) \\ = \Omega_+^\dagger(H, \Phi_A, \mathcal{H}_A) U(\Lambda, a) \Omega_-(H, \Phi_A, \mathcal{H}_A) \\ = \Omega_+^\dagger(H, \Phi_A, \mathcal{H}_A) \Omega_-(H, \Phi_A, \mathcal{H}_A) U_A(\Lambda, a) \\ = S(H, \Phi_A, \mathcal{H}_A) U_A(\Lambda, a). \end{aligned} \quad (45)$$

At this point all that has been demonstrated is that the same scattering theory is obtained by using injection operators that agree asymptotically but commute with different

kinematic subgroups. This will be used in the next section to construct unitarily equivalent light-front and instant-form representations of the field theory that preserve the scattering matrix.

III. CONSTRUCTION

The next step is to use the kinematic symmetries of the injection operators to show that the Hamiltonian H in the expressions for the wave operators can be replaced by the light front Hamiltonian, P^- . The relations

$$H = P^- + P^3 \quad H = \frac{1}{2}(P^+ + P^-) \quad (46)$$

will be used in what follows. In the first case note because $P^3 \Phi_A = \Phi_A P_A^3$ it follows that

$$\begin{aligned} e^{iHt} \Phi_A e^{-iH_A t} &= e^{i(P^- + P^3)t} \Phi_A e^{-i(P_A^- + P_A^3)t} \\ &= e^{iP^- t} \Phi_A e^{-iP_A^- t}. \end{aligned} \quad (47)$$

Similarly since $P^+ \tilde{\Phi}_A = \tilde{\Phi}_A P_A^+$

$$\begin{aligned} e^{iHt} \tilde{\Phi}_A e^{-iH_A t} &= e^{i(P^- + P^+)t/2} \tilde{\Phi}_A e^{-i(P_A^- + P_A^+)t/2} \\ &= e^{iP^- t/2} \tilde{\Phi}_A e^{-iP_A^- t/2}. \end{aligned} \quad (48)$$

This means that the wave operators constructed using H are identical to the ones using P^- with both injection operators. Combining these results with (40) gives the following identifications

$$\begin{aligned} \Omega_\pm(H, \Phi_A, H_A) &= \Omega_\pm(H, \tilde{\Phi}_A, H_A) \\ &= \Omega_\pm(P^-, \Phi_A, P_A^-) \\ &= \Omega_\pm(P^-, \tilde{\Phi}_A, P_A^-). \end{aligned} \quad (49)$$

The final step in the construction is to introduce two unitary operators V_F and V_I that act on the field theory Hilbert space $\mathcal{H}$ with the following properties:

$$V_F |0\rangle = V_I |0\rangle = |0\rangle \quad (50)$$

$$[U(\Lambda_{K_I}, a_{K_I}), V_I] = 0 \quad [U(\Lambda_{K_F}, a_{K_F}), V_F] = 0 \quad (51)$$

$$\lim_{t \rightarrow \pm\infty} \|(\Phi_A - V_I \Phi_A) e^{-iH_A t} |\psi_\alpha\rangle\| = 0 \quad (52)$$

$$\lim_{t \rightarrow \pm\infty} \|(\tilde{\Phi}_A - V_F \tilde{\Phi}_A) e^{-iH_A t} |\psi_\alpha\rangle\| = 0 \quad (53)$$

where (Λ_{K_I}, a_{K_I}) and (Λ_{K_F}, a_{K_F}) are in the instant-form and light-front kinematic subgroups respectively. Note that for Eqs. (52) and (53) to hold the asymptotic Hamiltonian must have a nontrivial absolutely continuous mass spectrum. While this is true on the scattering channel subspaces, it is not true for the vacuum or one-particle states, where

$e^{-iH_A t}$ either factors out because the energy depends on the mass and kinematic variables or is 1 in the case of the vacuum. In these cases (53) becomes

$$\|(\tilde{\Phi}_A - V_F \tilde{\Phi}_A)|\psi_\alpha\rangle\| = 0. \quad (54)$$

In the construction of this section the operators V_I and V_F were chosen to leave the vacuum unchanged. While this is not necessary, this choice gives S-matrix equivalent light-front and instant-form field theories with the same vacuum vectors.

The next step is to define two new unitary representations of the Poincaré group on the Hilbert space $\mathcal{H}$ of the quantum field theory by:

$$\begin{aligned} U_F(\Lambda, a) &:= V_F U(\Lambda, a) V_F^\dagger \\ U_I(\Lambda, a) &:= V_I U(\Lambda, a) V_I^\dagger \end{aligned} \quad (55)$$

with the corresponding dynamical generators:

$$H_F := V_F H V_F^\dagger \quad H_I := V_I H V_I^\dagger \quad (56)$$

$$P_F^- := V_F P^- V_F^\dagger \quad P_I^- := V_I P^- V_I^\dagger. \quad (57)$$

It follows from (51) that

$$\begin{aligned} U_I(\Lambda_{K_I}, a_{K_I}) &= U(\Lambda_{K_I}, a_{K_I}) \\ U_F(\Lambda_{K_F}, a_{K_F}) &:= U(\Lambda_{K_F}, a_{K_F}) \end{aligned} \quad (58)$$

where (Λ_{K_I}, a_{K_I}) is an instant-form kinematic Poincaré transformation and (Λ_{K_F}, a_{K_F}) is a front-form kinematic transformation.

In Appendix A it is shown [52] that as a consequence of (52) and (53) that

$$\Omega_\pm(H_I, \Phi_A, H_A) = V_I \Omega_\pm(H, \Phi_A, H_A) \quad (59)$$

and

$$\Omega_\pm(H_F, \tilde{\Phi}_A, H_A) = V_F \Omega_\pm(H, \tilde{\Phi}_A, H_A). \quad (60)$$

Taken together with (49) gives:

$$\begin{aligned} \Omega_\pm(H_I, \Phi_A, H_A) &= V_I \Omega_\pm(H, \Phi_A, H_A) \\ &= V_I \Omega_\pm(H, \tilde{\Phi}_A, H_A) \\ &= V_I \Omega_\pm(P^-, \tilde{\Phi}_A, P_A^-) \\ &= V_I V_F^\dagger \Omega_\pm(P_F^-, \tilde{\Phi}_A, P_A^-). \end{aligned} \quad (61)$$

Since (61) holds for the same $V_I V_F^\dagger$ for both time limits it follows that

$$\begin{aligned} S(H_I, \Phi_A, H_A) &= \Omega_+^\dagger(H_I, \Phi_A, H_A) \Omega_-(H_I, \Phi_A, H_A) \\ &= \Omega_+^\dagger(H_F, \tilde{\Phi}_A, H_A) V_F V_I^\dagger V_I V_F^\dagger \Omega_-(H_F, \tilde{\Phi}_A, H_A) \\ &= \Omega_+^\dagger(P_F^-, \tilde{\Phi}_A, P_A^-) \Omega_-(P_F^-, \tilde{\Phi}_A, P_A^-) \\ &= S(P_F^-, \tilde{\Phi}_A, P_A^-). \end{aligned} \quad (62)$$

This means both representations of the dynamics result in the same scattering operators on the asymptotic Hilbert space. Next consider the two unitary representations of the Poincaré group, $U_I(\Lambda, a)$ and $U_F(\Lambda, a)$ defined above. For the first one the operators $\{\mathbf{P}, s, s_z\}$ are mutually commuting self-adjoint functions of the instant-form kinematic generators of the representation $U_I(\Lambda, a)$. For the second one the operators $\{\tilde{\mathbf{P}}, s_{z_F}\}$ are mutually commuting self-adjoint functions of the front-form kinematic generators of the representation $U_F(\Lambda, a)$.

It is possible to construct a basis for $U_I(\Lambda, a)$ consisting of the eigenvalues of $\{\mathbf{P}, s, s_{z_I}\}$ and some additional kinematically invariant commuting observables x_I . Similarly it is possible to construct a basis for $U_F(\Lambda, a)$ consisting of the eigenvalues of $\{\tilde{\mathbf{P}}, s_{z_F}\}$ and some additional kinematically invariant commuting observables x_F .

It follows that in these bases wave functions have the form

$$\langle \mathbf{p}, s, \mu_I, x_I | \psi \rangle \quad \text{or} \quad \langle \tilde{\mathbf{p}}, \mu_F, x_F | \psi \rangle. \quad (63)$$

Matrix elements of the dynamical operators are nontrivial matrices in the “ x ” variables:

$$\begin{aligned} \langle \mathbf{p}, s, \mu_I, x_I | H_I | \mathbf{p}', s', \mu'_I, x'_I \rangle \\ = \delta(\mathbf{p} - \mathbf{p}') \delta_{ss'} \delta_{\mu_I \mu'_I} \langle x_I | H_I(\mathbf{P}, s) | x'_I \rangle \end{aligned} \quad (64)$$

and

$$\begin{aligned} \langle \tilde{\mathbf{p}}, \mu_F, x_F | P_F^- | \tilde{\mathbf{p}}', \mu'_F, x'_F \rangle \\ = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') \delta_{\mu_F \mu'_F} \langle x_F | P_F^-(\tilde{\mathbf{P}}, \mu_F) | x'_F \rangle. \end{aligned} \quad (65)$$

The eigenvalue problems for the dynamical operators have the forms

$$\begin{aligned} \sum \int \langle x_I | H_I(\mathbf{P}, s) | x'_I \rangle dx'_I \langle \mathbf{p}, s, \mu_I, x_I | \psi \rangle \\ = E(\mathbf{P}, s) \langle \mathbf{p}, s, \mu_I, x_I | \psi \rangle \end{aligned} \quad (66)$$

and

$$\begin{aligned} \sum \int \langle x_F | P_F^-(\tilde{\mathbf{P}}, \mu_F) | x'_F \rangle dx'_F \langle \tilde{\mathbf{p}}, \mu, x_F | \psi \rangle \\ = P^-(\tilde{\mathbf{P}}, \mu_F) \langle \tilde{\mathbf{p}}, \mu_F, x_F | \psi \rangle. \end{aligned} \quad (67)$$

The matrices $\langle x_I \| H_I(\mathbf{P}, s) \| x'_I \rangle$ or $\langle x_F \| P_F^-(\tilde{\mathbf{P}}, \mu) \| x'_F \rangle$ must be diagonalized in order to compute dynamical Poincaré transformations. On the other hand kinematic transformations can be computed by applying the inverse kinematic transformation to basis vectors:

$$\begin{aligned} & \langle \mathbf{p}, s, \mu, x_I | U_I(R, \mathbf{a}) | \psi \rangle \\ &= \langle \psi | U_I^\dagger(R, \mathbf{a}) | \mathbf{p}, s, \mu, x_I \rangle^* \\ &= e^{i\mathbf{a}\cdot\mathbf{p}} \langle R^{-1}\mathbf{p}, s, \nu, x_I | \psi \rangle D_{\mu\nu}^s(R). \end{aligned} \quad (68)$$

Similarly in the light-front case for light-front preserving boosts and translations:

$$\begin{aligned} & \langle \tilde{\mathbf{p}}, \mu, x_F | U_F(\Lambda_{KF}, a_{KF}) | \psi \rangle \\ &= \langle \psi | U_F^\dagger(\Lambda_{KF}, a_{KF}) | \tilde{\mathbf{p}}, \mu, x_F \rangle^* \\ &= e^{i\tilde{\mathbf{a}}\cdot\tilde{\mathbf{p}}} \langle \tilde{\Lambda}_k^{-1} p, \mu, x_F | \psi \rangle \sqrt{\frac{(\Lambda_k^{-1} p)^+}{p^+}} \end{aligned} \quad (69)$$

and for rotations about the z axis

$$\begin{aligned} & \langle \tilde{\mathbf{p}}, \mu, x_F | U_F(R_z(\phi), 0) | \psi \rangle \\ &= \langle \psi | U_F^\dagger(R_z(\phi)) | \tilde{\mathbf{p}}, \mu, x_F \rangle^* \\ &= \langle \tilde{\mathbf{R}}_z(\phi)^{-1} p, \mu, x_F | \psi \rangle s^{i\mu\phi}. \end{aligned} \quad (70)$$

Equation (68) shows that $U_I(\Lambda, a)$ has an instant-form kinematic subgroup while Eqs. (69) and (70) show that $U_F(\Lambda, a)$ has a light-front kinematic subgroup. In addition the two representations are related by a unitary transformation:

$$U_F(\Lambda, a) = V_F V_I^\dagger U_I(\Lambda, a) V_I V_F^\dagger \quad (71)$$

$$|0\rangle_F = V_F V_I^\dagger |0\rangle_I = |0\rangle_I. \quad (72)$$

In this construction it was chosen so that it preserves the vacuum.

It follows from (62) that both representations give the same unitary scattering operators on the asymptotic Hilbert space $\mathcal{H}_A$. In addition because V_I and V_f are kinematically invariant (51), the injection operators satisfy

$$U_I(\Lambda_{KI}, a_{KI}) \Phi_A = \Phi_A U_A(\Lambda_{KI}, a_{KI}) \quad (73)$$

for the instant-form kinematic subgroup and

$$U_F(\Lambda_{KF}, a_{KF}) \tilde{\Phi}_A = \tilde{\Phi}_A U_A(\Lambda_{KF}, a_{KF}) \quad (74)$$

for the light-front kinematic subgroup.

The results can be summarized by noting that it is possible to choose bases in the field theory Hilbert space that transform covariantly under either kinematic subgroup. In both cases a free dynamics is not assumed. The two

representations are related by S -matrix preserving unitary transformations. Both representation have the same vacuum and one-particle subspaces. There are no bare particles in these representations.

This construction did not assume the existence of a free dynamics on the Hilbert space of the field theory. If the range of $\tilde{\Phi}$ is all of $\mathcal{H}$ and

$$W := (\tilde{\Phi}_A \tilde{\Phi}_A^\dagger)^{-1/2} \tilde{\Phi}_A \quad (75)$$

satisfies

$$\lim_{t \rightarrow \pm\infty} \|(W - \tilde{\Phi}_A) e^{-iH_A t} | \psi_\alpha \rangle\| = 0 \quad (76)$$

then W is a kinematically invariant unitary mapping from $\mathcal{H}_A$ to $\mathcal{H}$ and $U_0(\Lambda, a) = W U_A(\Lambda, a) W^\dagger$ defines a consistent “free dynamics” on $\mathcal{H}$ that satisfies

$$U_I(\Lambda_{KI}, a_{KI}) = U_0(\Lambda_{KI}, a_{KI}) \quad (77)$$

with a similar relation in the light-front case. If these representation exist, they are not unique since they depend on the choice of injection operator.

IV. SPONTANEOUS SYMMETRY BREAKING

One of the puzzling aspects of light-front quantum field theory is how spontaneous symmetry breaking is realized. The signal for spontaneous symmetry breaking is a conserved local current, $j^\mu(x)$, that arises from the symmetry and the presence of a 0 mass Goldstone boson in the spectrum. For a local scalar field theory the following condition [56]

$$\lim_{R \rightarrow \infty} \langle 0 | [Q_R, \phi(y)] | 0 \rangle \neq 0 \quad (78)$$

implies the existence of a 0 mass particle. Here

$$\langle 0 | [Q_R, \phi(y)] | 0 \rangle := \langle 0 | \left[\int d\mathbf{x} \chi_R(|\mathbf{x}|) j^0(\mathbf{x}, t), \phi(y) \right] | 0 \rangle \quad (79)$$

where $\chi_R(|\mathbf{x}|)$ is a smooth function that is 1 for $|\mathbf{x}| < R$ and 0 for $|\mathbf{x}| > R + \epsilon$ for some finite positive ϵ . The cutoff function χ_R ensures that the integral converges for large $|\mathbf{x}|$. The commutator vanishes for $x - y$ spacelike. For fixed y and t and sufficiently large R , if $|\mathbf{x}| > R$ then $x - y$ is spacelike and the commutator vanishes. In this case

$$\begin{aligned} \langle 0 | [Q_R, \phi(y)] | 0 \rangle &:= \langle 0 | \left[\int d\mathbf{x} \chi_R(|\mathbf{x}|) j^0(\mathbf{x}, t), \phi(y) \right] | 0 \rangle \\ &= \int d\mathbf{x} \langle 0 | [j^0(\mathbf{x}, t), \phi(y)] | 0 \rangle. \end{aligned} \quad (80)$$

This does not require that the charge operator exists.

Current conservation implies

$$\langle 0 | \left[\int d\mathbf{x} \partial_\mu j^\mu(\mathbf{x}, t), \phi(y) \right] | 0 \rangle = 0. \quad (81)$$

Inserting a complete set of intermediate states gives

$$\begin{aligned} 0 &= \sum \int d\mathbf{x} \left(\langle 0 | \partial_\mu j^\mu(\mathbf{x}, t) | p, n \rangle \frac{d\mathbf{p}}{2p_n^0} \langle p, n | \phi(y) | 0 \rangle - \langle 0 | \phi(y) | p, n \rangle \frac{d\mathbf{p}}{2p_n^0} \langle p, n | \partial_\mu j^\mu(\mathbf{x}, t) | 0 \rangle \right) \\ &= \sum \int d\mathbf{x} \left(\langle 0 | \partial_\mu j^\mu(\mathbf{0}, 0) | p_r, n \rangle \frac{d\mathbf{p}}{2p_n^0} \langle p_r, n | \phi(0) | 0 \rangle e^{ip \cdot (x-y)} - \langle 0 | \phi(0) | p_r, n \rangle \frac{d\mathbf{p}}{2p_n^0} \langle p_r, n | \partial_\mu j^\mu(\mathbf{0}, 0) | 0 \rangle e^{ip \cdot (y-x)} \right) \end{aligned} \quad (82)$$

where Poincaré covariance has been used to remove the nontrivial x , y and p dependence from the matrix elements. p_r is the constant rest four momentum for massive states and a constant light-like vector for massless states. For a scalar field theory the vacuum expectation value of the current vanishes so the vacuum does not appear as an intermediate state. The Lehmann weights that appear in this matrix element

$$\sigma(m_n) m_n^2 = \langle 0 | \partial_\mu j^\mu(\mathbf{0}, 0) | p_r, n \rangle \langle p_{nr}, n | \phi(0) | 0 \rangle \quad (83)$$

and

$$\sigma^*(m_n) m_n^2 = \langle 0 | \phi(0) | p_{nr}, n \rangle \langle p_{nr}, n | \partial_\mu j^\mu(\mathbf{0}, 0) | 0 \rangle \quad (84)$$

are functions of the invariant mass of the intermediate states. The factor $m_n^2 = -p_n^2$ is due to the $\partial_\mu j^\mu(\mathbf{0}, 0)$ assuming that the current is a local function of the scalar field.

Current conservation requires that this quantity vanishes which follows if either $\sigma(m_n) = 0$ or $m_n^2 = 0$. Inserting intermediate states in (79) gives a similar result with $\sigma(m_n) m_n^2$ replaced by $p_{nr}^0 \sigma(m_n)$ where p_{nr}^0 is a nonzero constant.

Since this does not include the m^2 factor, it vanishes by current conservation unless $\sigma(m_n)$ contains a $\delta(m_n)$. Thus for (78) to be nonvanishing the sum over intermediate states must have a contribution from a 0 mass particle. This is the nonperturbative form of Goldstone's theorem.

In [56] Coleman avoids directly computing the charge operator. The current is an operator valued distribution, so there is no reason to expect that integrating against the constant 1 will result in a well-defined operator. There is another reason for this. If the integrals that define the charge operator converge and there is no contribution to the current on the surface at infinity, then the charge operator will be translationally invariant and time independent. This means that it would commute with the four momentum operator. It follows that even if $Q_I | 0 \rangle$ is not 0, it is eigenstate of the four momentum, with 0 eigenvalue. While the Goldstone boson has 0 mass, its momentum

cannot be 0, which means that a translationally invariant time-independent charge operator cannot couple the vacuum to the Goldstone boson.

In the light-front representation it is still possible to define $Q_{IR}(t)$ and the analysis above still applies. The assumption that the current is a local function of the fields means that the light-front dynamics must be used to get its value on the fixed-time surface where the locality of the theory can be utilized to control the volume integral.

While the nonvanishing of (78) is a sign of spontaneous symmetry breaking, the argument used above does not work when it is applied to the light-front charge. The problem is that there is no compact region on the light front where outside of that region $(x - y)$ is always spacelike for fixed y and $x^+ = 0$. This is true even if the light-front charge is computed in the instant-form representation. If the integrals that define the light-front charge exist and there are no contributions from the divergence theorem, the situation is similar to the instant-form case—the charge operator will commute with the four momentum which means that the light-front charge applied to the vacuum is an eigenstate of the four momentum with 0 eigenvalue, which by the argument used above means that the charge operator cannot couple the vacuum to the Goldstone boson.

V. MISCELLANEOUS REMARKS

This section provides a brief discussion comparing the structure of the light-front and instant-form representations discussed in this work to the corresponding relation in relativistic quantum mechanics of a finite number of degrees of freedom.

The starting point of this work is the assumed existence of a local field theory with the expected properties. Normally in an instant or front-form representation of a quantum field theory “solving the field theory” means constructing a unitary transformation that decomposes the Hilbert space into a direct integral of irreducible representations. This is the relativistic analog of diagonalizing the Hamiltonian. This is also true in relativistic quantum theories of a finite number of degrees of freedom, except in that case the interacting and noninteracting theories are

formulated on the same representation of the Hilbert space. In the finite number of degree of freedom case different choices of independent variables in each irreducible subspace are related by a unitary change of variables. This defines a unitary, S -matrix preserving transformation relating the two forms of dynamics. It is nontrivial because there is a different (mass-dependent) variable change on each irreducible subspace. The light-front and instant-form bases have the property that matrix elements of the unitary representation of the kinematic subgroups in each case are mass independent.

In this work this process is reversed; the direct integral is assumed to be a property of a quantum field theory with all of the expected properties. The general class of S -matrix preserving unitary transformations is used to construct equivalent representations of the dynamics. The freedom to choose different S -matrix preserving unitary transformations was exploited to choose S -matrix preserving unitary transformations that commute with the instant form or light-front kinematic subgroups of the theory, which resulted in equivalent light-front or instant-form representations of the field theory.

The unitarity of these transformations means that they must preserve the spectrum of the mass operator. This means that point-spectrum eigenvalues of the mass operator must be the same in all three representations. The unitary transformation must also leave the vacuum energy or the mass gap unchanged. Preserving the S -matrix means that acceptable unitary transformations must be short-range perturbations of the identity in the sense that they do not require a modification of scattering asymptotic conditions. The unitary transformations that satisfy all of these conditions are not unique and are discussed in Appendix A.

The construction of equivalent instant-form and light-front unitary representations of the Poincaré group in this paper is essentially identical to the corresponding construction in the finite number of degree of freedom case [3,4,57,58]. In that case, when there is physical particle production, the particles all have their physical masses and there are no vertex interactions that can change masses [59]. In that case the simplest interactions that change particle number are $2 \leftrightarrow 3$ interactions.

Given the spectrum of one-particle states, the mass spectrum of the field theory Hamiltonian includes the one-particle masses and continuous eigenvalues starting at the two-particle thresholds. This is identical to the mass spectrum of a sum of free-field Hamiltonians with physical particle masses. The conclusion of Haag's theorem [40] is that a spectrally equivalent free field theories are defined on inequivalent representation of the Hilbert space. This why there are no free multiparticle states in the field theory and is the motivation for using the two-Hilbert space representation in order to formulate the scattering asymptotic conditions.

In the absence of free multi-particle states in the theory, the notion of many-particle interactions needs to be defined. One

possibility is to define interactions using matrix elements in states constructed by applying products of the operators that create one-particle states out of vacuum to the vacuum. Matrix elements of the Hamiltonian in these states have two properties, the first is that they become products of one-particle states as the particles are asymptotically separated. In this representation, because the one-particle states are point spectrum eigenstates of the Hamiltonian, the Hamiltonian cannot couple them to the continuum. This means that in this representation a “production vertex” cannot couple to a multiparticle scattering state. This is one way of saying that stable particle cannot decay, however particle production is possible since interactions that couple “two-particle states” to “three-particle states”, etc. are possible in this representation.

A final comment concerns vacuum polarization in this representation. The physical vacuum is orthogonal to all bound and scattering eigenstates of the Hamiltonian, but operators that couple the vacuum to other states in the Hilbert space are possible and are needed to describe local vacuum excitations.

VI. ANALYSIS AND CONCLUSIONS

This work provides one way to understand Dirac's forms of dynamics in a nonperturbative setting. This approach assumed the existence of a local quantum field theory with all of the expected properties. Using this assumption unitarily equivalent representations with the same scattering operators were constructed. Bases that transform covariantly with respect to light-front and instant-form kinematic subgroups were constructed for each of these equivalent representations of the dynamics. Kinematic Poincaré transformations on these wave functions could be computed without diagonalizing the dynamical operators. One aspect of the construction used in this work is that both representations of the dynamics leave the vacuum and one-particle states unchanged and both representations give the same Poincaré invariant scattering operator. While preserving the vacuum states is not required, the construction demonstrates that it is a consistent assumption.

A key difference with the perturbative approach is the nonexistence of an underlying free-field dynamics on the Hilbert space of the field theory. If the injection operators have full range on the Hilbert space of the field theory and satisfy (76) there is the analog of a free dynamics on the Hilbert of the field theory. If such a representation exists, it cannot be a local free field theory with physical masses.

While the representations discussed in this analysis are not useful for applications, the results support the conclusions of most of the quoted references that the two formulations of the theory are equivalent. The vacuum is the same up to a unitary transformation and can be taken as the same in both representations. While the light-front vacuum is kinematically invariant, it is also rotationally invariant, which is an additional dynamical constraint on the light-front vacuum that separates it from the free-field

Fock vacuum. By starting with the assumed solution to the field theory there are no issues with renormalization, rotational covariance or 0 modes. These issues need to be addressed in applications. If there is spontaneous symmetry breaking, the nonperturbative condition for the presence of a Goldstone boson in the mass spectrum is the same in both representations, but it does not couple to the vacuum with a space-time translationally invariant charge operator, whether it is on the fixed-time surface or the light-front hyperplane.

ACKNOWLEDGMENTS

The author would like to thank the referee for some useful suggestions. This work supported by the U.S. Department of Energy, Office of Science, Grant No. DE-SC0016457.

APPENDIX A: SCATTERING EQUIVALENCES

The two-Hilbert space multichannel wave operators are defined by the strong limits where the Φ_A are two-Hilbert space injection operators that map the asymptotic Hilbert space $\mathcal{H}_A$ to the Hilbert space $\mathcal{H}$ of the quantum field theory:

$$\Omega_{\pm}(H, \Phi_A, H_A) = s - \lim_{t \rightarrow \pm\infty} e^{iHt} \Phi_A e^{-iH_A t}. \quad (\text{A1})$$

The wave operators are assumed to exist and be Poincaré invariant in the sense

$$\begin{aligned} U(\Lambda, a) \Omega_{\pm}^{\dagger}(H, \Phi_A, H_A) \\ = \Omega_{\pm}^{\dagger}(H, \Phi_A, H_A) U_A(\Lambda, a). \end{aligned} \quad (\text{A2})$$

where $U_A(\Lambda, a)$ is the natural representation of the Poincaré group on $\mathcal{H}_A$ that has the form of a direct sum of tensor products of irreducible representations. The set of channels $\mathcal{A}$ is assumed to include the vacuum channel, and one-particle channels in addition to multiparticle scattering channels. The wave operators are assumed to be asymptotically complete unitary mappings from asymptotic Hilbert space $\mathcal{H}_A$ to the Hilbert space $\mathcal{H}$ of the field theory.

The scattering operator is defined by

$$S(H, \Phi, H_A) = \Omega_{+}^{\dagger}(H, \Phi_A, H_A) \Omega_{-}(H, \Phi_A, H_A). \quad (\text{A3})$$

Consider two identical scattering operators based on different Hamiltonians

$$S(H, \Phi_A, H_A) = S(H', \Phi'_A, H_A) \quad (\text{A4})$$

It follows from the definitions and unitarity of the wave operators that

$$\begin{aligned} W &:= \Omega_{+}(H', \Phi'_A, H_A) \Omega_{+}^{\dagger}(H, \Phi_A, H_A) \\ &= \Omega_{-}(H', \Phi'_A, H_A) \Omega_{-}^{\dagger}(H, \Phi_A, H_A) \end{aligned} \quad (\text{A5})$$

is a unitary operator on $\mathcal{H}$. The identification of the scattering operators (A4) means that W is the same for the incoming ($t \rightarrow +\infty$) or outgoing ($t \rightarrow -\infty$) multichannel wave operator. It follows that

$$W \Omega_{\pm}(H, \Phi_A, H_A) = \Omega_{\pm}(H', \Phi'_A, H_A). \quad (\text{A6})$$

In addition, the intertwining property of the wave operators [49],

$$\begin{aligned} WH' &= \Omega_{\pm}(H, \Phi_A, H_A) \Omega_{\pm}^{\dagger}(H', \Phi'_A, H_A) H' \\ &= \Omega_{\pm}(H, \Phi_A, H_A) H_A \Omega_{\pm}^{\dagger}(H', \Phi'_A, H_A) \\ &= H \Omega_{\pm}(H, \Phi_A, H_A) \Omega_{\pm}^{\dagger}(H', \Phi'_A, H_A) = HW \end{aligned} \quad (\text{A7})$$

means that the two Hamiltonians are related by the unitary transformation W .

It follows that

$$\begin{aligned} \Omega_{-}(H', \Phi'_A, H_A) &= W \Omega_{-}(H, \Phi_A, H_A) \\ &= \Omega_{-}(WHW^{\dagger}, W\Phi_A, H_A) \\ &= \Omega_{-}(H', W\Phi_A, H_A). \end{aligned} \quad (\text{A8})$$

Taking the difference of the right and left side of Eq. (A8) gives

$$0 = \lim_{t \rightarrow \pm\infty} \|e^{iH't} (\Phi'_A - W\Phi_A) e^{-iH_A t} |\psi\rangle\|. \quad (\text{A9})$$

The unitarity of $e^{iH't}$ gives

$$0 = \lim_{t \rightarrow \pm\infty} \|(\Phi'_A - W\Phi_A) e^{-iH_A t} |\psi\rangle\|. \quad (\text{A10})$$

That this condition holds for both time limits is important. For the vacuum and one-particle channels this condition requires that

$$\Phi'_A - W\Phi_A = 0. \quad (\text{A11})$$

Next consider the converse. Assume that W is a unitary operator satisfying $H' = WHW^{\dagger}$, and the scattering operators

$$S(H, \Phi, H_A) = \Omega_{+}^{\dagger}(H, \Phi_A, H_A) \Omega_{-}(H, \Phi_A, H_A) \quad (\text{A12})$$

$$S(H', \Phi'_A, H_A) = \Omega_{+}^{\dagger}(H', \Phi'_A, H_A) \Omega_{-}(H', \Phi'_A, H_A) \quad (\text{A13})$$

both exist. If W satisfies (A10) for both time limits then the S matrices are identical

$$S(H', \Phi'_A, H_A) = S(H, \Phi_A, H_A). \quad (\text{A14})$$

The proof follows from

$$\begin{aligned}
 \Omega_{\pm}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}}) &= \Omega_{\pm}(WHW^{\dagger}, \Phi'_{\mathcal{A}}, H_{\mathcal{A}}) \\
 &= W\Omega_{\pm}(H, W^{\dagger}\Phi'_{\mathcal{A}}, H_{\mathcal{A}}) \\
 &= W\Omega_{\pm}(H, W^{\dagger}(\underbrace{\Phi'_{\mathcal{A}} - W\Phi_{\mathcal{A}} + W\Phi_{\mathcal{A}}}_{\rightarrow 0}), H_{\mathcal{A}}) \\
 &= W\Omega_{\pm}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}). \tag{A15}
 \end{aligned}$$

It then follows that

$$\begin{aligned}
 S(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}}) &= \Omega_{+}^{\dagger}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}})\Omega_{-}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}}) \\
 &= \Omega_{+}^{\dagger}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}})W^{\dagger}W\Omega_{-}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) \\
 &= S(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}). \tag{A16}
 \end{aligned}$$

Note that unitary equivalence is not a sufficient condition for S -matrix equivalence. This is not hard to understand in the case of ordinary quantum mechanics, where two Hamiltonians with short-range repulsive interactions have the same spectrum (so they are unitarily equivalent) but generally have different S -matrix elements or phase shifts.

The conclusion of this section is that unitary operators W that satisfy the asymptotic condition (A10) can be used to relate Hamiltonians that have the same scattering matrix. This asymptotic condition means that W does not disturb the asymptotic structure of the states that define the asymptotic condition.

The above condition applies to both quantum mechanics and quantum field theory assuming that they have asymptotically complete scattering operators. The field theory generalization of multiparticle scattering is the Haag-Ruelle formulation of scattering which involves the strong limits used in this Appendix.

APPENDIX B: TWO HILBERT SPACE INJECTION OPERATORS

This Appendix discusses the construction of injection operators from operators in the field algebra. The starting point is to let B be a function of the fields that creates a one-body state out of the vacuum. This means that the completeness sum

$$\langle 0|B^{\dagger}B|0\rangle = \sum_n \langle 0|B^{\dagger}|n\rangle \langle n|B|0\rangle \tag{B1}$$

has one-body intermediate states (here one-body means states with discrete positive-mass eigenvalues, so there is no distinction between elementary and composite one-body states). For simplicity it is assumed that the one-body spectrum is nondegenerate, which means that each one-particle state has a different mass, and each one-body mass eigenstate has a given spin.

To isolate the operators that create the one-body states use space-time translations to define the operator valued distributions

$$B(x) := e^{-iP \cdot x} B e^{-iP \cdot x} = U^{\dagger}(I, x) B U(I, x) \tag{B2}$$

where P^{μ} is the four momentum operator. It follows from (B2) that

$$\frac{\partial B(x)}{\partial x_{\mu}} = -i[P^{\mu}, B(x)]. \tag{B3}$$

The next step is to compute the Fourier transform of the operator density $B(x)$:

$$\hat{B}(q) := \int \frac{d^4 x}{(2\pi)^2} e^{iq \cdot x} B(x). \tag{B4}$$

Multiplying both sides of (B3) by $\frac{e^{iq \cdot x}}{(2\pi)^2}$ and integrating over $d^4 x$ by parts, assuming that $B(x)$ is an operator valued distribution, gives

$$-i[P^{\mu}, \hat{B}(q)] = -iq^{\mu} \hat{B}(q) \tag{B5}$$

or

$$[P^{\mu}, \hat{B}(q)] = q^{\mu} \hat{B}(q). \tag{B6}$$

It follows from (B6) that $\hat{B}(q)|0\rangle$ is either 0 or an eigenstate of the four momentum with eigenvalue q^{μ} :

$$P^{\mu} \hat{B}(q)|0\rangle = \hat{B}(q) P^{\mu}|0\rangle = q^{\mu} \hat{B}(q)|0\rangle = q^{\mu} \hat{B}(q)|0\rangle. \tag{B7}$$

Let m be a discrete mass eigenvalue of $M = \sqrt{-P^2}$ and let s denote the spin of the particle of mass m . Next let $h(q)$ be a smooth Lorentz invariant function of compact support in q^2 that is 1 when $q^2 = -m^2$ and $q^0 > 0$, and identically 0 on the rest of the spectrum of intermediate states in (B1) and define

$$A^{\dagger}(q) := \hat{B}(q) h(q). \tag{B8}$$

Additional integrations are used to define the operators:

$$\begin{aligned}
 A^{\dagger}(\mathbf{q}) &:= \int A^{\dagger}(q) dq^0 \quad \text{and} \\
 \tilde{A}^{\dagger}(\tilde{\mathbf{q}}) &:= \int A^{\dagger}(q) dq^{-}/2. \tag{B9}
 \end{aligned}$$

Because $h(q)$ is localized, these operators are distributions in the three momentum or light-front components of the four momentum. When these operators are applied to the vacuum they create one-particle states of mass m . Since by assumption, there is a unique spin s associated with each

discrete mass eigenvalue, the particle created out of the vacuum also has spin s .

Because of the integrals over q^0 or q^- it follows that (B6) must be replaced by

$$[\mathbf{P}, A^\dagger(\mathbf{q})] = \mathbf{q}A^\dagger(\mathbf{q}). \quad [\tilde{\mathbf{P}}, \tilde{A}^\dagger(\tilde{\mathbf{q}})] = \tilde{\mathbf{q}}\tilde{A}^\dagger(\tilde{\mathbf{q}}) \quad (\text{B10})$$

which when applied to the vacuum becomes

$$\begin{aligned} \mathbf{P}A^\dagger(\mathbf{q})|0\rangle &= \mathbf{q}A^\dagger(\mathbf{q})|0\rangle \\ \tilde{\mathbf{P}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle &= \tilde{\mathbf{q}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle. \end{aligned} \quad (\text{B11})$$

This means that $A^\dagger(\mathbf{q})$ creates a one-particle state with momentum $\mathbf{q}$, mass m and spin s out of the vacuum. Similarly

$$\tilde{\mathbf{P}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle = \tilde{\mathbf{q}}A^\dagger(\tilde{\mathbf{q}})|0\rangle \quad (\text{B12})$$

creates a one-particle state with light-front momentum $\tilde{\mathbf{q}}$ and mass m and spin s out of the vacuum.

The next step is to construct operators that create simultaneous eigenstates of mass and spin and magnetic quantum numbers.

Define the operator

$$\begin{aligned} A_{m,s}^\dagger(\mathbf{p}, \mu) \\ := \int_{SU(2)} U(R, 0) A^\dagger(R\mathbf{p}) U^\dagger(R, 0) D_{\mu s}^{s*}(R) dR \end{aligned} \quad (\text{B13})$$

where the integral is over the $SU(2)$ Haar measure and $D_{\mu s}^{s*}(R)$ is the spin- s $SU(2)$ Wigner function

$$D_{\mu s}^s(R) = \langle s, \mu | U(R) | s, s \rangle. \quad (\text{B14})$$

It follows from (B13) that

$$\begin{aligned} U(R', 0) A_{m,s}^\dagger(\mathbf{p}, \mu) U^\dagger(R', 0) \\ = \int_{SU(2)} U(R'R, 0) A^\dagger(R'^{-1}\mathbf{p}) U^\dagger(R'R, 0) D_{\mu s}^{s*}(R) dR. \end{aligned} \quad (\text{B15})$$

Changing variables, $R'' = R'R$ using the invariance of the Haar measure $dR = dR''$ for fixed R' gives

$$\begin{aligned} (\text{B15}) &= \int_{SU(2)} U(R'', 0) A^\dagger(R''^{-1}R'\mathbf{p}) U^\dagger(R'', 0) dR'' D_{\mu s}^{s*}(R'^{-1}R'') \\ &= \int_{SU(2)} \sum_{\nu=-s}^s U(R'', 0) A^\dagger(R''^{-1}R'\mathbf{p}) U^\dagger(R'', 0) dR'' D_{\mu\nu}^{s*}(R'^{-1}) D_{\nu s}^{s*}(R'') \\ &= \int_{SU(2)} \sum_{\nu=-s}^s U(R'', 0) A^\dagger(R''^{-1}R'\mathbf{p}) U^\dagger(R'', 0) dR'' D_{\nu s}^{s*}(R'^{-1}R'') D_{\nu\mu}^s(R') \\ &= \sum_{\nu=-s}^s A_{(m,j)}^\dagger(R'\mathbf{p}, \nu) D_{\nu\mu}^s(R'). \end{aligned} \quad (\text{B16})$$

When applied to the vacuum (B16) gives

$$U(R, 0) A_{(m,s)}^\dagger(\mathbf{p}, \mu) |0\rangle = \sum_{\nu=-s}^s A_{(m,s)}^\dagger(R\mathbf{p}, \nu) |0\rangle D_{\nu\mu}^s(R) \quad (\text{B17})$$

which means that either this vanishes or it transforms like a particle of mass m , spin s , momentum $\mathbf{p}$ and magnetic quantum number μ . The spin in these states created out of the vacuum is the canonical spin. This will vanish if there are no one-particle intermediate states with mass m and spin s in (B1). While the notation is purposely suggestive, $A_{m,s}^\dagger(\mathbf{p}, \mu)$ is *not* a creation operator. In addition, p^0 is only equal to $\sqrt{\mathbf{p}^2 + m^2}$ when $A_{m,s}^\dagger(\mathbf{p}, \mu)$ is applied to the vacuum.

Because rotations do not change the time component, these operators also satisfy

$$\begin{aligned} [\mathbf{P}, A_{(m,s)}^\dagger(\mathbf{q}, \mu)] \\ &= \int_{SU(2)} dR [\mathbf{P}, U(R, 0) A^\dagger(R^{-1}\mathbf{q}) U^\dagger(R, 0)] D_{\mu s}^{s*}(R) \\ &= \int_{SU(2)} dR U(R, 0) [R\mathbf{P}, A^\dagger(R^{-1}\mathbf{q})] U^\dagger(R, 0) D_{\mu s}^{s*}(R) \\ &= RR^{-1}\mathbf{q} \int_{SU(2)} dR U(R, 0) A^\dagger(R^{-1}\mathbf{q}) U^\dagger(R, 0) D_{\mu s}^{s*}(R) \\ &= \mathbf{q} A_{(m,s)}^\dagger(\mathbf{q}, \mu). \end{aligned} \quad (\text{B18})$$

The normalization can be chosen so the states created out of the vacuum have the normalization (9).

The construction above cannot be used in the “light-front” case due to the integral over p^- in (B9). However in that case it is enough to project out the magnetic quantum number using a rotation $R_z(\phi)$ which leaves p^- in (B9)

unchanged. The spin is identified with the highest nonzero weight, $s = \mu_{\max}$. In this case Eq. (B13) is replaced by

$$\begin{aligned} \tilde{A}_{m,s}^\dagger(\tilde{\mathbf{p}}, \mu) &:= \int_0^{2\pi} U(R_z(\phi), 0) \tilde{A}^\dagger(R_z(\phi)^{-1} \mathbf{p}) U^\dagger(R_z(\phi), 0) e^{-i\mu\phi} \frac{d\phi}{2\pi}, \\ &\quad (B19) \end{aligned}$$

(B17) and (B18) are replaced by

$$[\tilde{\mathbf{P}}, \tilde{A}_{(m,s)}^\dagger(\tilde{\mathbf{q}}, \mu)] = \tilde{\mathbf{q}} \tilde{A}_{(m,s)}^\dagger(\tilde{\mathbf{q}}, \mu), \quad (B20)$$

$$U(\Lambda_K, 0) \tilde{A}_{(m,s)}^\dagger(\tilde{\mathbf{q}}, \mu) U^\dagger(\Lambda_K, 0) = \tilde{A}_{(m,s)}^\dagger(\tilde{\Lambda}_K \mathbf{q}, \mu), \quad (B21)$$

$$\begin{aligned} U(R_z(\phi), 0) \tilde{A}_{(m,s)}^\dagger(\tilde{\mathbf{q}}, \mu) U^\dagger(R_z(\phi), 0) \\ = \tilde{A}_{(m,s)}^\dagger(R_z(\phi) \tilde{\mathbf{q}}, \mu) e^{i\mu\phi}. \end{aligned} \quad (B22)$$

The proof (B20) is essentially the same as the proof of (B18). To show (B22) note that for rotations about the z axis

$$B_F(p/m) R_z = R_z B_F(R_z^{-1} p/m) \quad (B23)$$

where $B_F(p/m)$ is a light-front boost. It follows that

$$\begin{aligned} &U(B_F(p/m), 0) \tilde{A}_{m,s}^\dagger(\tilde{\mathbf{q}}, \mu) U(B_F(p/m), 0)^\dagger \\ &= \int_0^{2\pi} U(B_F(p/m) R_z(\phi), 0) \tilde{A}^\dagger(R_z(\phi)^{-1} \mathbf{q}) U^\dagger(B_F(p/m) R_z(\phi), 0) e^{-i\mu\phi} \\ &= \int_0^{2\pi} U(R_z(\phi) B_F(R_z^{-1}(\phi) p/m), 0) \tilde{A}^\dagger(R_z(\phi)^{-1} \mathbf{q}) U^\dagger(R_z(\phi) B_F(R_z^{-1}(\phi) p/m), 0) e^{-i\mu\phi} \\ &= \int_0^{2\pi} U(R_z(\phi), 0) U(B_F(R_z^{-1}(\phi) p/m), 0) \tilde{A}^\dagger(R_z(\phi)^{-1} \mathbf{q}) U(B_F(R_z^{-1}(\phi) p/m), 0)^\dagger U^\dagger(R_z(\phi), 0)^\dagger e^{-i\mu\phi} \\ &= \int_0^{2\pi} U(R_z(\phi), 0) U(, 0) \tilde{A}^\dagger(B_F(R_z^{-1}(\phi) p/m) R_z(\phi)^{-1} \mathbf{q}) U^\dagger(R_z(\phi), 0)^\dagger e^{-i\mu\phi} \\ &= \int_0^{2\pi} U(R_z(\phi), 0) U(, 0) \tilde{A}^\dagger(R_z(\phi)^{-1} B_F(p/m) \mathbf{q}) U^\dagger(R_z(\phi), 0)^\dagger e^{-i\mu\phi} \\ &= \tilde{A}_{m,s}^\dagger(\tilde{\mathbf{B}}_F(p/m) \mathbf{q}, \mu) \end{aligned} \quad (B24)$$

which proves (B21). For (B22) note that

$$U(R_z(\phi'), 0) \tilde{A}_{(m,s)}^\dagger(\tilde{\mathbf{q}}, \mu) U^\dagger(R_z(\phi'), 0) = \int_0^{2\pi} U(R_z(\phi + \phi'), 0) \tilde{A}^\dagger(R_z(\phi)^{-1} \mathbf{p}) U^\dagger(R_z(\phi + \phi'), 0) e^{-i\mu\phi} \frac{d\phi}{2\pi}. \quad (B25)$$

Let $\phi'' = \phi + \phi'$ so (B25) becomes

$$\begin{aligned} &\int_0^{2\pi} U(R_z(\phi''), 0) \tilde{A}^\dagger(R_z(\phi'' - \phi')^{-1} \mathbf{p}) U^\dagger(R_z(\phi''), 0) e^{-i\mu(\phi'' - \phi')} \frac{d\phi''}{2\pi} \\ &\int_0^{2\pi} U(R_z(\phi''), 0) \tilde{A}^\dagger(R_z(\phi'')^{-1} R_z(\phi') \mathbf{p}) U^\dagger(R_z(\phi''), 0) e^{-i\mu(\phi'' - \phi')} e^{i\mu\phi'} = \tilde{A}_{(m,s)}^\dagger(R_z(\phi') \tilde{\mathbf{q}}, \mu) e^{i\mu\phi'}. \end{aligned} \quad (B26)$$

The normalization can be chosen so these states created out of the vacuum have the normalization (9).

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