Iterated integrals over letters induced by quadratic forms

J. Ablinger,¹ J. Blümlein,² and C. Schneider¹

¹*Research Institute for Symbolic Computation (RISC), Johannes Kepler University,
Altenbergerstraße 69, A-4040 Linz, Austria*

²*Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, D-15738 Zeuthen, Germany*

(Received 13 March 2021; accepted 9 April 2021; published 25 May 2021)

An automated treatment of iterated integrals based on letters induced by real-valued quadratic forms and Kummer–Poincaré letters is presented. These quantities emerge in analytic single and multiscale Feynman diagram calculations. To compactify representations, one wishes to apply general properties of these quantities in computer-algebraic implementations. We provide the reduction to basis representations, expansions, analytic continuation and numerical evaluation of these quantities.

DOI: [10.1103/PhysRevD.103.096025](https://doi.org/10.1103/PhysRevD.103.096025)

I. INTRODUCTION

In analytic calculations of single and multiscale Feynman integrals different principal structures have been revealed in particular during the last 30 years.¹ Beyond the multiple zeta values [2] and other special numbers for zero-scale quantities, there are the spaces of harmonic sums [3,4], harmonic polylogarithms [5], generalized harmonic sums [6,7] and Kummer–Poincaré iterated integrals [6–8], cyclotomic harmonic sums and iterated integrals [9], finite and infinite binomial sums and inverse binomial sums and the associated root-letter integrals [10,11], and iterative non-iterative integrals [12], including those containing complete elliptic integrals [12,13]. This list is expected still to extend in analytic calculations at even higher loops and for more contributing scales in the future.

In decomposing Feynman parameter representations in the general case, cf., e.g., [14], often real polynomials of higher degree have to be factored. According to the fundamental theorem of algebra [15] this leads to either linear and quadratic factors in real representations or to linear complex-valued factors with conjugated pairs. Real representations have often advantages in calculations. This is the main reason to extend the class of Kummer–Poincaré iterated integrals based on the alphabet

$$\mathfrak{A}_{KP} = \left\{ \frac{1}{x - c_i} \middle| c_i \in \mathbb{C} \right\} \quad (1.1)$$

¹For a survey see [1].

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

into²

$$\mathfrak{A}_R = \left\{ \frac{1}{x - a_i}, \frac{1}{x^2 + b_i x + c_i}, \frac{x}{x^2 + b_i x + c_i} \middle| a_i, b_i, c_i \in \mathbb{R}, 4c_i \geq b_i^2 \right\}. \quad (1.2)$$

There is an overlap with the cyclotomic iterated integrals [9] with respect to the letters $1/\Phi_k(x)$, $k = 3, 4, 6$. The associated iterative integrals are given by

$$H_{b,\vec{a}}(z) = \int_0^z dx f_b(x) H_{\vec{a}}(x), \quad H_\emptyset = 1, \quad f_l(x) \in \mathfrak{A}_R. \quad (1.3)$$

Iterated integrals do also obey the differential property

$$\frac{d}{dz} H_{b,\vec{a}}(z) = f_b(z) H_{\vec{a}}(z), \quad (1.4)$$

which is instrumental for building the corresponding algebra, to be closed under differentiation. Sometimes more general iterated integrals are used, cf. e.g., [16], with letters outside $\mathfrak{A}_R$. As we will show below this class of integrals can be cast into the class generated by $\mathfrak{A}_R$.

Using the alphabet (1.2) has the advantage to stay inside a real representation in calculating a real quantity. Complex decompositions [17] request the thorough observation of the pairing of complex conjugated letters. Both approaches can handle poles inside the integration region.

We will devise an algorithm to transform the mentioned formal iterated integrals into real ones, also referring to one

²Here the condition in (1.2) implies that the polynomial $x^2 + b_i x + c_i$ is irreducible over $\mathbb{R}$.

simple main variable. This has the advantage that the corresponding results can be iterated over in further integrations, which will be necessary for the use in a higher order calculation.

Using iterative integrals in the description of physical quantities it is required to give them a clear definition. In some cases it is possible that a first definition has singularities in subintegrals, which have to be dealt with to obtain a measurable quantity. Furthermore, the real and imaginary parts of the respective integrals have to be separated from the beginning, because they have a different physical meaning and it then allows to deal with real integrals only. One has also to observe that certain transformations in the main argument may effect the position of cuts chosen. In the case of singularities of the real integrals we will apply Cauchy's principal value for definiteness, as

it is also the case in amplitudes referring to the Källén–Lehmann representation [18].

In Sec. II we will describe the different operations for the iterated integrals induced by quadratic forms in the package HARMONICSUMS [3–5,7,9,10,19–21] and provide test examples. Section III deals with integrals of a recent physical application [16] which we reconsider in the present formalism, and Sec. IV contains the conclusion.

II. OPERATIONS FOR THE ITERATED INTEGRALS

In the following we describe a series of operations which allow to deal with iterated integrals containing letters of the alphabet $\mathfrak{A}_R$.

The statement

$$\text{QL}[\{\{\{a_1, b_1, c_1\}, d_1\}, \{\{a_2, b_2, c_2\}, d_2\}, \{\{a_3, b_3, c_3\}, d_3\}\}, z] \quad (2.1)$$

represents the integral

$$\begin{aligned} & H_{((a_1, b_1, c_1), d_1), ((a_2, b_2, c_2), d_2), ((a_3, b_3, c_3), d_3)}(z) \\ &= \int_0^z \int_0^{t_1} \int_0^{t_2} \frac{t_1^{d_1} t_2^{d_2} t_3^{d_3}}{(a_1 + b_1 t_1 + c_1 t_1^2)(a_2 + b_2 t_2 + c_2 t_2^2)(a_3 + b_3 t_3 + c_3 t_3^2)}, \\ & \text{with } d_i \in \{0, 1\}, a_i, b_i, c_i \in \mathbb{R}, \end{aligned} \quad (2.2)$$

covering a number of iterated letters out of $\mathfrak{A}_R$. The command `ToHarmonicSumsIntegrate` reveals the integral structure in explicit form. One may convert these integrals into GL-functions, cf. [10], by `QLToGL` and GL-functions with letters out of $\mathfrak{A}_R$ to QL-functions by `GLToQL`.

It is allowed that the Kummer–Poincaré letters in $\mathfrak{A}_{KP}$ have poles in the integration region. The iterative integral is then defined taking Cauchy's principal value. However, the quadratic denominators are assumed to not factorize in real numbers.

The numerical evaluation of QL-functions is performed as in the following example

$$\begin{aligned} r &= \text{ToHarmonicSumsIntegrate} \left[\text{QL}[\{\{\{1, 1, 2\}, 3\}, \{\{1, -1, 1\}, 1\}, \{\{-1, 1, 1\}, 1\}\}, \frac{1}{2}], \right. \\ & \quad \left. \text{NIntegrate} \rightarrow \text{True} \right] // \text{ReleaseHold} \end{aligned} \quad (2.3)$$

$$r \approx -0.0000649218. \quad (2.4)$$

Here not all letters corresponding to quadratic forms are yet in the standard form. There is no singularity, however, in the integration region since $(\sqrt{5} - 1)/2 > 1/2$.

To be able to deal with properly defined letters, the mapping `QLToStandardForm` is used. A typical example is

$$\text{QL}[\{\{\{1, 1, 2\}, 2\}\}, z] = \int_0^z dt \frac{t^2}{1 + t + 2t^2} \quad (2.5)$$

$$\text{QLToStandardForm}[\text{QL}[\{\{\{1, 1, 2\}, 2\}\}, z]] \Rightarrow \frac{z}{2} - \frac{1}{4} \int_0^z dt \frac{1}{\frac{1}{2} + \frac{t}{2} + t^2} - \frac{1}{4} \int_0^z dt \frac{t}{\frac{1}{2} + \frac{t}{2} + t^2}. \quad (2.6)$$

The shuffle operation is

$$H_{a_1, \dots, a_m}(z) \cdot H_{b_1, \dots, b_n}(z) = H_{a_1, \dots, a_n}(z) \sqcup H_{b_1, \dots, b_n}(z) = \sum_{c_i \in \{a_1, \dots, a_m \sqcup b_1, \dots, b_n\}} H_{c_i}(z), \quad (2.7)$$

where all combinations of the two index sets are allowed, which preserve the ordering in these two sets. Here H labels a general iterated integral. The corresponding command is `LinearHEExpand`.

One obtains

$$\begin{aligned} & \text{LinearHEExpand} \left[\text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\}, \frac{1}{2} \right\}^2 \text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 2 \right\}, 0 \right\}, \frac{1}{2} \right\} \right] \right. \right. \\ & \Rightarrow 2 \left\{ \text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 2 \right\}, 0 \right\} \right\}, \frac{1}{2} \right] \right. \\ & \quad + \text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 2 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, \frac{1}{2} \right] \\ & \quad \left. + \text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 2 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, \frac{1}{2} \right] \right\}. \end{aligned} \quad (2.8)$$

The algebraic reduction with respect to shuffle relations of a given expression is performed by the command `ReduceToQLBasis`, which will also transform the shuffled expression (2.8) into the corresponding product expression.

Often one would like to remove trailing indices or leading indices of QL-functions. This is done by the commands `RemoveTrailing0`, `RemoveTrailingIndex` or `RemoveLeading1`, `RemoveLeadingIndex`. Here `Trailing0` refers to the letter $1/x$ and `Leading1` to $1/(1-x)$. Examples are

$$\begin{aligned} & \text{RemoveTrailingIndex}[\text{QL}[\{\{1, 1, 1\}, 0\}, \{-1, 1, 0\}, 0\}, \{\{1, 1, 1\}, 0\}, \{\{1, 1, 1\}, 0\}, z], \{\{1, 1, 1\}, 0\}] \\ & = \frac{1}{2} \text{QL}[\{\{1, 1, 1\}, 0\}, z]^2 \text{QL}[\{\{1, 1, 1\}, 0\}, \{-1, 1, 0\}, 0\}, z] \\ & \quad - 2 \text{QL}[\{\{1, 1, 1\}, 0\}, z] \text{QL}[\{\{1, 1, 1\}, 0\}, \{\{1, 1, 1\}, 0\}, \{-1, 1, 0\}, 0\}, z] \\ & \quad + 3 \text{QL}[\{\{1, 1, 1\}, 0\}, \{\{1, 1, 1\}, 0\}, \{\{1, 1, 1\}, 0\}, \{-1, 1, 0\}, 0\}, z], \end{aligned} \quad (2.9)$$

$$\begin{aligned} & \text{RemoveLeadingIndex}[\text{QL}[\{\{1, 1, 1\}, 0\}, \{-1, 1, 0\}, 0\}, \{\{1, 1, 1\}, 0\}, z], \{\{1, 1, 1\}, 0\}] \\ & = \text{QL}[\{\{1, 1, 1\}, 0\}, z] \text{QL}[\{\{-1, 1, 0\}, 0\}, \{\{1, 1, 1\}, 0\}, z] \\ & \quad - 2 \text{QL}[\{\{-1, 1, 0\}, 0\}, \{\{1, 1, 1\}, 0\}, \{\{1, 1, 1\}, 0\}, z]. \end{aligned} \quad (2.10)$$

The series expansion of the QL-function about $z = 0$ is obtained by the command `QLSeries`, e.g.,

$$\begin{aligned} & \text{QLSeries} \left[\text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 1 \right\}, \{3, 1, 1\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, -1, 1 \right\}, 1 \right\} \right\}, z \right], z, 10 \right] \\ & = \frac{z^5}{45} - \frac{z^6}{216} - \frac{11z^7}{1260} + \frac{13z^8}{1620} - \frac{7z^9}{972} - \frac{4793z^{10}}{680400} \end{aligned} \quad (2.11)$$

or

$$\begin{aligned} & \text{QLSeries}[\text{QL}[\{\{1, 1, 1\}, 1\}, \{3, 1, 1\}, 0\}, \{\{0, 1, 0\}, 0\}, z], z, 5] \\ & = -\frac{4z^3}{27} + \frac{11z^4}{96} - \frac{5z^5}{972} + \left[\frac{z^3}{9} - \frac{7z^4}{72} + \frac{z^5}{162} \right] \ln(z). \end{aligned} \quad (2.12)$$

Often a change in the argument of the QL-functions is desirable. The following transformations are implemented and are carried out by the command `TransformQL`

$$k \cdot z \rightarrow z, \quad k \in \mathbb{Q}, \quad 1 - z \rightarrow z, \quad \frac{1}{z} \rightarrow z, \quad \frac{1-z}{1+z} \rightarrow z. \quad (2.13)$$

Examples are

$$\text{TransformQL}[\text{QL}[\{\{\{1, 1, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, 2z], z] \Rightarrow \frac{1}{2} \text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{4}, \frac{1}{2}, 1 \right\}, 1 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, x \right], \quad (2.14)$$

$$\begin{aligned} & \text{TransformQL}[\text{QL}[\{\{\{1, 1, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, 1-z], z] \\ & \Rightarrow \text{QL}[\{\{\{2, 1, 1\}, 0\}\}, 1](-\text{QL}[\{\{\{3, -3, 1\}, 0\}\}, z] + \text{QL}[\{\{\{3, -3, 1\}, 1\}\}, z]) \\ & + \text{QL}[\{\{\{1, 1, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, 1] + \text{QL}[\{\{\{3, -3, 1\}, 0\}, \{\{4, -3, 1\}, 0\}\}, z] \\ & - \text{QL}[\{\{\{3, -3, 1\}, 1\}, \{\{4, -3, 1\}, 0\}\}, z], \end{aligned} \quad (2.15)$$

$$\begin{aligned} & \text{TransformQL} \left[\text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 1 \right\}, \left\{ \{2, 1, 1\}, 0 \right\} \right\}, \frac{1}{z} \right], z \right] \\ & \Rightarrow \left[-\text{QL}[\{\{\{0, 1, 0\}, 0\}\}, z] - \text{QL}[\{\{\{1, 1, 1\}, 0\}\}, 1] + \text{QL}[\{\{\{1, 1, 1\}, 0\}\}, z] \right. \\ & - \text{QL}[\{\{\{1, 1, 1\}, 1\}\}, 1] + \text{QL}[\{\{\{1, 1, 1\}, 1\}\}, z] \left. + \frac{1}{2} \left[\text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, 1 \right] \right. \right. \\ & + 2\text{QL}[\{\{\{2, 1, 1\}, 0\}\}, 1] - \text{QL} \left[\left\{ \left\{ \{0, 1, 0\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, 1 \right] \\ & + \text{QL} \left[\left\{ \left\{ \{0, 1, 0\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, z \right] + \text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, 1 \right] \right. \\ & - \text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 0 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, z \right] + \text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 1 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, 1 \right] \\ & \left. - \text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 1 \right\}, \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, z \right] + 2\text{QL}[\{\{\{1, 1, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, 1] \right], \end{aligned} \quad (2.16)$$

$$\begin{aligned} & \text{TransformQL} \left[\text{QL} \left[\left\{ \left\{ \{1, 1, 1\}, 1 \right\}, \left\{ \{2, 1, 1\}, 0 \right\} \right\}, \frac{1-z}{1+z} \right], z \right] \\ & \Rightarrow \text{QL}[\{\{\{2, 1, 1\}, 0\}\}, 1](\text{QL}[\{\{\{1, 1, 0\}, 0\}\}, 1] - \text{QL}[\{\{\{1, 1, 0\}, 0\}\}, z]) \\ & - \text{QL}[\{\{\{3, 0, 1\}, 0\}\}, 1] + \text{QL}[\{\{\{3, 0, 1\}, 0\}\}, z] - \text{QL}[\{\{\{3, 0, 1\}, 1\}\}, 1] \\ & + \text{QL}[\{\{\{3, 0, 1\}, 1\}\}, z] - \text{QL}[\{\{\{1, 1, 0\}, 0\}, \{\{2, 1, 1\}, 0\}\}, 1] \\ & + \text{QL}[\{\{\{1, 1, 0\}, 0\}, \{\{2, 1, 1\}, 0\}\}, z] + \text{QL}[\{\{\{3, 0, 1\}, 0\}, \{\{2, 1, 1\}, 0\}\}, 1] \\ & - \text{QL}[\{\{\{3, 0, 1\}, 0\}, \{\{2, 1, 1\}, 0\}\}, z] + \text{QL}[\{\{\{3, 0, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, 1] \\ & - \text{QL}[\{\{\{3, 0, 1\}, 1\}, \{\{2, 1, 1\}, 0\}\}, z]. \end{aligned} \quad (2.17)$$

In these transformations new special numbers are occurring beyond the multiple zeta values. In particular one obtains the values

$$\text{QL}[\{\{\{3, 0, 1\}, 1\}\}, 1] = \ln(2) - \frac{1}{2} \ln(3) \quad (2.18)$$

$$\text{QL} \left[\left\{ \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, 1 \right\}, 0 \right\} \right\}, 1 \right] = \frac{4}{\sqrt{7}} \arctan \left(\frac{\sqrt{7}}{3} \right) \quad (2.19)$$

$$\text{QL}[\{\{\{2, 1, 1\}, 0\}\}, 1] = \frac{2}{\sqrt{7}} \arctan\left(\frac{\sqrt{7}}{5}\right). \quad (2.20)$$

More involved constants are dilogarithms of arguments, e.g., of the type $\sqrt{3}(1 + i\sqrt{3})/(\sqrt{7} + \sqrt{3})$, etc. and further special constants.

It is furthermore useful to transform QL-functions in a representation in which they have a convergent Taylor series representation, which is also applied in their numerical evaluation. This is provided by the command `QLToConvergentRegion`. An example is

$$\begin{aligned} & \text{QLToConvergentRegion}[\text{QL}[\{\{\{1, 1, 0\}, 0\}, \{\{-1, 1, 0\}, 0\}, \{\{2, 1, 1\}, 1\}\}, 4]] \\ &= \text{QL}[\{\{\{0, 1, 0\}, 0\}\}, 3](\text{QL}[\{\{\{1, 1, 0\}, 0\}\}, 1]\text{QL}[\{\{\{2, 1, 1\}, 1\}\}, 1] + \text{QL}[\{\{\{2, 1, 1\}, 1\}\}, 1] \\ & \quad \text{QL}[\{\{\{4, 1, 0\}, 0\}\}, 1]) + \text{QL}[\{\{\{2, 1, 1\}, 1\}\}, 1](\text{QL}[\{\{\{0, 1, 0\}, 0\}\}, \{\{1, 1, 0\}, 0\}\}, 1] \\ & \quad - \text{QL}[\{\{\{2, 1, 0\}, 0\}, \{\{4, 1, 0\}, 0\}\}, 1]) + \text{QL}[\{\{\{1, 1, 0\}, 0\}\}, 1](\text{QL}[\{\{\{2, 1, 0\}, 0\}\}, 1] \\ & \quad \text{QL}[\{\{\{2, 1, 1\}, 1\}\}, 1] - \text{QL}[\{\{\{2, 1, 1\}, 1\}, \{\{-1, 1, 0\}, 0\}\}, 1]) + \text{QL}[\{\{\{4, 1, 0\}, 0\}\}, 1] \\ & \quad \times \left(\frac{1}{2} \text{QL}\left[\left\{\left\{\{0, 1, 0\}, 0\right\}, \left\{\left\{1, \frac{3}{2}, 1\right\}, 0\right\}\right\}, 1\right] + \text{QL}\left[\left\{\left\{\{0, 1, 0\}, 0\right\}, \left\{\left\{1, \frac{3}{2}, 1\right\}, 1\right\}\right\}, 1\right] \right. \\ & \quad \left. - \text{QL}[\{\{\{2, 1, 1\}, 1\}, \{\{-1, 1, 0\}, 0\}\}, 1]\right) + \frac{1}{2} \text{QL}\left[\left\{\left\{\left\{1, \frac{3}{2}, 1\right\}, 0\right\}\right\}, 1\right] \\ & \quad \times \text{QL}[\{\{\{4, 1, 0\}, 0\}, \{\{2, 1, 0\}, 0\}\}, 1] + \text{QL}[\{\{\{4, 1, 0\}, 0\}, \{\{2, 1, 0\}, 0\}\}, 1] \\ & \quad \times \text{QL}\left[\left\{\left\{\left\{1, \frac{3}{2}, 1\right\}, 1\right\}\right\}, 1\right] + \text{QL}[\{\{\{1, 1, 0\}, 0\}, \{\{-1, 1, 0\}, 0\}, \{\{2, 1, 1\}, 1\}\}, 1] \\ & \quad + \frac{1}{2} \text{QL}\left[\left\{\left\{\{1, 1, 0\}, 0\right\}, \{\{0, 1, 0\}, 0\}, \left\{\left\{1, \frac{3}{2}, 1\right\}, 0\right\}\right\}, 1\right] \\ & \quad + \text{QL}\left[\left\{\left\{\{1, 1, 0\}, 0\right\}, \{\{0, 1, 0\}, 0\}, \left\{\left\{1, \frac{3}{2}, 1\right\}, 1\right\}\right\}, 1\right] \\ & \quad + 3 \text{QL}[\{\{\{4, 1, 0\}, 0\}, \{\{2, 1, 0\}, 0\}, \{\{14, 7, 1\}, 0\}\}, 1] \\ & \quad + \text{QL}[\{\{\{4, 1, 0\}, 0\}, \{\{2, 1, 0\}, 0\}, \{\{14, 7, 1\}, 1\}\}, 1]. \end{aligned} \quad (2.21)$$

For removing poles in the integration domain we rely on the formula

$$\text{H}_{a_1, a_2, \dots, a_m}(z) = \sum_{i=0}^m \text{H}_{b_1, b_2, \dots, b_i}(z - p) \text{H}_{a_{i+1}, a_{i+2}, \dots, a_m}(p) \quad (2.22)$$

where $f_{b_j}(x) := f_{a_j}(x + p)$.

The following example illustrates the use of this formula to remove poles. Consider $\text{H}_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(z)$. It has a pole at 1 due to the letter

$$\frac{1}{-1+x} = ((-1, 1, 0), 0). \quad (2.23)$$

In order to remove this pole we use (2.22) which yields

$$\begin{aligned} \text{H}_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(z) &= \text{H}_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(1) + \text{H}_{((1,1,0),0)}(z-1) \text{H}_{((-1,1,0),0),((1,0,1),0)}(1) \\ & \quad + \text{H}_{((1,1,0),0),((0,1,0),0)}(z-1) \text{H}_{((1,0,1),0)}(1) + \text{H}_{((1,1,0),0),((0,1,0),0),((2,2,1),0)}(z-1). \end{aligned}$$

Remove trailing zeros, i.e., the letter $((0, 1, 0), 0)$ from the functions with argument $x - 1$, yields

$$\begin{aligned} & H_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(1) + H_{((1,1,0),0)}(z-1)H_{((-1,1,0),0),((1,0,1),0)}(1) \\ & + [H_{((-1,1,0),0)}(z-1)H_{((0,1,0),0)}(z-1) - H_{((0,1,0),0),((1,1,0),0)}(z-1)]H_{((1,0,1),0)}(1) + H_{((1,1,0),0),((0,1,0),0),((2,2,1),0)}(z-1). \end{aligned}$$

Now we replace

$$H_{((0,1,0),0)}(z-1) = H_{((0,1,0),0)}\left(\frac{z-1}{p}\right) - H_{((-1,1,0),0)}(1) + v \cdot i\pi \quad (2.24)$$

and finally we remove leading ones, i.e., the letter $((-1, 1, 0), 0)$, which results in

$$\begin{aligned} H_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(x) &= H_{((0,1,0),0)}(x-1)H_{((1,0,1),0)}(1)H_{((1,1,0),0)}(z-1) - H_{((1,0,1),0)}(1)H_{((0,1,0),0),((1,1,0),0)}(z-1) \\ &- H_{((1,1,0),0)}(z-1)H_{((1,0,1),0),((-1,1,0),0)}(1) + H_{((0,1,0),0),((-1,1,0),0),((1,0,1),0)}(1) \\ &+ H_{((1,1,0),0),((0,1,0),0),((2,2,1),0)}(z-1) + v \cdot i\pi H_{((1,0,1),0)}(1)H_{((1,1,0),0)}(z-1), \end{aligned}$$

where we set

$$v = \begin{cases} 1, & \text{for adding } +i\varepsilon \text{ to a singular letter} \\ 0, & \text{to compute Cauchy's principal value} \\ -1, & \text{for adding } -i\varepsilon \text{ to a singular letter.} \end{cases} \quad (2.25)$$

Note that it might be necessary to perform the above steps several times to different poles.

The series expansion of a QL-function $H_{\vec{m}}(z)$ with a letter

$$\frac{1}{x+a_i} \quad (2.26)$$

for $a_i \in \mathbb{R} \setminus \{0\}$ is convergent for $|z| < |a_i|$. Hence given a QL-function $H_{\vec{m}}(z)$ with letters from $\mathfrak{A}_{KP}$, i.e., with letters of the form (2.26), we remove all poles p at the real axis for

which $|p| < |z|$ using the strategy mentioned above. Now expanding the resulting functions about $z = 0$ will lead to a convergent series at z .

In the case that there are also quadratic forms present i.e., letters of the form

$$\frac{1}{a_i + b_i x + x^2}, \quad (2.27)$$

we have to additionally treat the $\sqrt{|a_i|}$ as poles and apply the method mentioned above also with $p = \pm\sqrt{|a_i|}$ for $\sqrt{|a_i|} < |z|$. This is due to the fact that the series expansion of a QL-function with a letter of the type (2.27) is convergent for $|z| < \sqrt{|a_i|}$.

The option `PrincipalValue → False` in `QLEvaluate` allows us to evaluate integrals sub-integrals containing poles yielding a complex result.

$$\begin{aligned} & \int_0^1 dx \int_0^x dy \frac{1}{[\frac{3}{8}-x]} \frac{1}{[\frac{15}{8}-y]} \rightarrow \lim_{\varepsilon \rightarrow 0^+} \int_0^1 dx \int_0^x dy \frac{1}{[\frac{3}{8}-x+i\varepsilon]} \frac{1}{[\frac{15}{8}-y]} \\ &= -\zeta_2 + 4\ln^2(2) + 2\ln(2)\ln(3) + \ln^2(3) - \ln^2(5) - [2\ln(2) + \ln(3) - \ln(5)]\ln(7) \\ &+ \text{Li}_2\left(-\frac{1}{4}\right) + \text{Li}_2\left(\frac{7}{12}\right) + i\pi[2\ln(2) - \ln(5)] \\ &= \text{QLEvaluate}\left[\text{QL}\left[\left\{\left\{\left\{\frac{3}{8}, -1, 0\right\}, 0\right\}, \left\{\left\{\frac{15}{8}, -1, 0\right\}, 0\right\}\right\}, 1\right], 25, \text{PrincipalValue} \rightarrow \text{False}\right] \\ &\approx -0.8205920210842043836307006 - 0.7010261415046584209879799i. \end{aligned} \quad (2.28)$$

Here the regularization is performed in adding $+i\varepsilon$ to the denominator of the singular letters.

The QL-functions in the variable $z \in [0, 1]$ can be Mellin transformed by

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^N f(x). \quad (2.29)$$

One obtains representations by harmonic sums [3,4] and (generalized) S-sums [7] at complex weights

$$S_{b,\vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, b, \quad a_i \in \mathbb{Z} \setminus \{0\} \quad (2.30)$$

$$S_{b,\vec{a}}(c, \vec{d}, N) = \sum_{k=1}^N \frac{b^c}{k^c} S_{\{\vec{a}\}}(\vec{d}, k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{C}, \quad c, d_i \in \mathbb{N} \setminus \{0\}. \quad (2.31)$$

Instead of working with differential equations one may work with difference equations, which are solved by the package SIGMA [22,23]. The ground field of the package needs then to contain the corresponding special constants arising in $\{\vec{c}\}$.

An example is the Mellin transformation of

$$F_1(x) = \text{QL} \left[\left\{ \left\{ \{0, 1, 0\}, 0 \right\}, \left\{ \left\{ \frac{1}{3}, 0, 1 \right\}, 1 \right\}, \left\{ \left\{ \frac{1}{3}, 0, 1 \right\}, 1 \right\} \right\}, x \right], \quad (2.32)$$

which is given by

$$\begin{aligned} \mathbf{M}[F_1(x)](N) = & -\frac{1}{(1+N)^4} - \frac{9T_1^2}{2(1+N)^2} - \frac{9T_3}{1+N} - \frac{9T_4}{1+N} + T_1 \left(\frac{3}{(1+N)^3} + \frac{9T_2}{1+N} \right) \\ & + \left\{ \left[\frac{3T_5}{2(1+N)^2} + T_0 \left(\frac{1}{2(1+N)^3} - \frac{3T_1}{2(1+N)^2} + \frac{S_1}{4(1+N)^2} - \frac{S_{-1}}{4(1+N)^2} \right) \right. \right. \\ & + i\sqrt{3}T_1 \left(-\frac{1}{2(1+N)^3} - \frac{S_1}{4(1+N)^2} - \frac{S_{-1}}{4(1+N)^2} + \frac{S_1(-i\sqrt{3})}{2(1+N)^2} \right) \\ & + \frac{3iT_1^2\sqrt{3}}{4(1+N)^2} - \frac{i}{\sqrt{3}} \left(\frac{S_1(-i\sqrt{3})}{2(1+N)^3} + \frac{S_{1,1}(1, -i\sqrt{3})}{4(1+N)^2} + \frac{S_{1,1}(-1, i\sqrt{3})}{4(1+N)^2} \right) \Big] i^N \\ & + \left[\frac{3T_5}{2(1+N)^2} + T_0 \left(\frac{1}{2(1+N)^3} - \frac{3T_1}{2(1+N)^2} + \frac{S_1}{4(1+N)^2} - \frac{S_{-1}}{4(1+N)^2} \right) \right. \\ & + i\sqrt{3}T_1 \left(\frac{1}{2(1+N)^3} + \frac{S_1}{4(1+N)^2} + \frac{S_{-1}}{4(1+N)^2} - \frac{S_1(i\sqrt{3})}{2(1+N)^2} \right) \\ & \left. \left. - \frac{3iT_1^2\sqrt{3}}{4(1+N)^2} + \frac{i}{\sqrt{3}} \left(\frac{S_1(i\sqrt{3})}{2(1+N)^3} + \frac{S_{1,1}(1, i\sqrt{3})}{4(1+N)^2} + \frac{S_{1,1}(-1, -i\sqrt{3})}{4(1+N)^2} \right) \right] \times i^{3N} \right\} 3^{-N/2}, \quad (2.33) \end{aligned}$$

where we dropped the argument N both for the harmonic and generalized harmonic sums. The constants T_i are iterated integrals at $x = 1$ and are given by

$$T_0 = \frac{\pi}{3\sqrt{3}} \quad (2.34)$$

$$T_1 = \frac{\ln(2)}{3} \quad (2.35)$$

$$T_2 = \frac{\zeta_2}{12} + \frac{\ln^2(3)}{24} + \frac{1}{12} \text{Li}_2\left(-\frac{1}{3}\right) \quad (2.36)$$

$$\begin{aligned} T_3 = & \frac{1}{72} \zeta_3 - \frac{1}{27} \ln^3(2) + \frac{1}{36} \ln^2(2) \ln(3) \\ & - \frac{1}{36} \ln(2) \text{Li}_2\left(\frac{1}{4}\right) - \frac{1}{72} \text{Li}_3\left(\frac{1}{4}\right) \end{aligned} \quad (2.37)$$

$$\begin{aligned} T_4 = & \frac{1}{36} \zeta_2 \ln(2) - \frac{1}{36} \zeta_3 + \frac{1}{54} \ln^3(2) \\ & + \frac{1}{36} \ln(2) \text{Li}_2\left(\frac{1}{4}\right) + \frac{1}{36} \text{Li}_3\left(\frac{1}{4}\right) \end{aligned} \quad (2.38)$$

$$\begin{aligned} T_5 = & -\frac{1}{6\sqrt{3}} \text{Re} \left[i \text{Li}_2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \right] \\ = & 2\zeta_2 - \frac{86}{33} \ln^2(3) + \frac{5}{33} \text{Li}_2\left(-\frac{1}{3}\right), \end{aligned} \quad (2.39)$$

where $\text{Li}_n(x)$ denotes the classical polylogarithm [24] with the representation

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}, \quad x \in [-1, 1], \quad n \in \mathbb{N}. \quad (2.40)$$

Not all of the above polylogarithms are independent, cf. [25,26], and the following relations hold,

$$\text{Li}_2\left(\frac{1}{4}\right) = -\text{Li}_2\left(-\frac{1}{3}\right) - 2\ln^2(2) + 2\ln(2)\ln(3) - \frac{1}{2}\ln^2(3) \quad (2.41)$$

$$= 2\text{Li}_2\left(-\frac{1}{2}\right) + \zeta_2 - \ln^2(2) \quad (2.42)$$

$$\text{Li}_3\left(\frac{1}{4}\right) = \frac{7}{2}\zeta_3 - 2\zeta_2\ln(2) + \frac{2}{3}\ln^3(2) + 4\text{Li}_3\left(-\frac{1}{2}\right). \quad (2.43)$$

The basis of contributing constants is here

$$\left\{ \ln(2), \zeta_2, \zeta_3, \pi, \ln(3), \text{Li}_2\left(-\frac{1}{2}\right), \text{Li}_3\left(-\frac{1}{2}\right) \right\}. \quad (2.44)$$

The constants at position 1–3 are multiple zeta values [2], at position 4 and 5 they are cyclotomic [9], and the ones at position 6 and 7 occurred already for square-root valued iterated integrals [26]. The Mellin transformation (2.33) is real.

III. PHYSICS EXAMPLES

As an application of the operations given in Sec. II we calculate a few examples of particular iterated integrals, which have emerged in the calculation of inclusive Compton scattering cross sections at next-to-leading order (NLO) [16] recently. The corresponding integrals were defined by

$$G_{b,\vec{a}}(x) = \int_0^x dw_b(x') G_{\vec{a}}(x'), \quad (3.1)$$

with

$$dw_y(x) = \frac{ydx}{x}, \quad dw_a(x) = \frac{dx}{x-a}, \quad a \in \mathbb{R}, \quad (3.2)$$

where $y = \sqrt{x/(4+x)}$, $x = s/m^2 - 1$, s the cms energy and m a mass, implying $y \in [0, 1]$, and $y = y(x)$.

Iterative integrals obey shuffle algebras [27–29] and are first reduced to a corresponding basis representation by exploiting the relations implied by (2.7). The formal iterated integrals (3.1) are given by

$$G_{0,y,-1}(x), \quad G_{y,-1,0}(x), \quad G_{y,0,-1}(x), \quad G_{y,-1,-1}(x), \\ G_{y,-2,-1}(x), \quad G_{-4,y,-1}(x), G_{y,-1}(x), \quad G_{y,y,-1}(x), \quad (3.3)$$

and $x = 4y^2/(1-y^2)$.

We now cast these integrals into a root-free form. Furthermore, we choose a simple main argument, to allow further iterated integration, needed in potential higher order calculations. For the following representations we choose the variable

$$z = \frac{1-y}{1+y} \quad (3.4)$$

for the main argument and set $H_{\vec{a}}(z) \equiv H_{\vec{a}}$ for the harmonic polylogarithms and cyclotomic harmonic polylogarithms.

We now transform the integrals (3.3) into iterative integrals, see also [20,21], containing also letters generated by quadratic forms.

$$G_{0,y,-1}(x) = -\frac{4}{3}\zeta_3 - \frac{1}{6}H^3 + \frac{2}{3}\zeta_2H_1 + H_0^2H_1 \\ - \sqrt{3}\pi H_{\{3,0\},0}(1) - 2H_{0,0,1} - 2H_{0,0,\{6,0\}} \\ + 2H_{0,0,\{6,1\}} - 2H_{1,0,\{6,0\}} + 4H_{1,0,\{6,1\}} \\ + \frac{4z}{(1-z^2)^2} (H_{-1}(x) + H_{\{6,0\}} - 2H_{\{6,1\}}) \\ + \left[\frac{4z}{(1-z^2)^2} + \frac{1}{3}\zeta_2 + 2H_{0,1} \right] H_0. \quad (3.5)$$

Some of the iterated integrals in the right-hand side (rhs_ in (3.5) are obtained at main argument x , like $H_{-1}(x)$ and $H_{0,-1}(x)$. It appears to be useful to use the following relations

$$H_{-1}(x) = -H_0 - H_{\{6,0\}} + 2H_{\{6,1\}}. \quad (3.6)$$

$$H_{-2,-1}(x) = \frac{1}{2}H_0^2 + H_{0,\{6,0\}} - 2H_{0,\{6,1\}} - 2H_{\{4,1\},0} \\ - 2H_{\{4,1\},\{6,0\}} + 4H_{\{4,1\},\{6,1\}} - \frac{1}{2}\zeta_2 \quad (3.7)$$

$$H_{0,-1}(x) = \zeta_2 + \frac{1}{2}H_0^2 + 2H_0H_1 - 2H_{0,1} + H_{0,\{6,0\}} \\ - 2H_{0,\{6,1\}} + 2H_{1,\{6,0\}} - 4H_{1,\{6,1\}}. \quad (3.8)$$

which yields

$$G_{0,y,-1}(x) = -\frac{4}{3}\zeta_3 - \frac{1}{6}H^3 + \frac{2}{3}\zeta_2H_1 + H_0^2H_1 \\ - \sqrt{3}\pi H_{\{3,0\},0}(1) - 2H_{0,0,1} - 2H_{0,0,\{6,0\}} \\ + 2H_{0,0,\{6,1\}} - 2H_{1,0,\{6,0\}} + 4H_{1,0,\{6,1\}} \\ + \left[\frac{1}{3}\zeta_2 + 2H_{0,1} \right] H_0. \quad (3.9)$$

For the other integrals the following results are obtained,

$$G_{y,-1,0}(x) = \frac{2}{3}\zeta_3 + \zeta_2 H_0 - \frac{1}{6}H_0^3 - 2H_{0,0,1} - H_{0,\{6,0\},0} - 2H_{0,\{6,0\},1} + 2H_{0,\{6,1\},0} + 4H_{0,\{6,1\},1} \quad (3.10)$$

$$G_{y,0,-1}(x) = \frac{2}{3}\zeta_3 - \zeta_2 H_0 - \frac{1}{6}H_0^3 - 2H_0 H_{0,1} + \sqrt{3}\pi H_{\{3,0\},0}(1) + 4H_{0,0,1} - H_{0,0,\{6,0\}} + 2H_{0,0,\{6,1\}} - 2H_{0,1,\{6,0\}} + 4H_{0,1,\{6,1\}} \quad (3.11)$$

$$G_{y,-1,-1}(x) = \frac{5}{9}\zeta_3 - \frac{1}{6}H_0^3 + \frac{\pi}{\sqrt{3}}H_{\{3,0\},0}(1) - H_{0,0,\{6,0\}} + 2H_{0,0,\{6,1\}} - H_{0,\{6,0\},0} - H_{0,\{6,0\},\{6,0\}} + 2H_{0,\{6,0\},\{6,1\}} + 2H_{0,\{6,1\},0} + 2H_{0,\{6,1\},\{6,0\}} - 4H_{0,\{6,1\},\{6,1\}} \quad (3.12)$$

$$G_{y,-2,-1}(x) = \frac{1}{12}\zeta_3 + \frac{1}{2}\zeta_2 H_0 - \frac{1}{6}H_0^3 - \frac{1}{4}\sqrt{3}\pi H_{\{3,0\},0}(1) - H_{0,0,\{6,0\}} + 2H_{0,0,\{6,1\}} + 2H_{0,\{4,1\},0} + 2H_{0,\{4,1\},\{6,0\}} - 4H_{0,\{4,1\},\{6,1\}} \quad (3.13)$$

$$G_{-4,y,-1}(x) = \frac{16}{9}\zeta_3 - \frac{1}{6}H_0^3 + H_{-1}\left(-\frac{2}{3}\zeta_2 + H_0^2\right) + H_0\left(\frac{1}{2}\zeta_2 - 2H_{0,-1}\right) + \frac{2\pi}{\sqrt{3}}H_{\{3,0\},0}(1) + 2H_{-1,0,\{6,0\}} - 4H_{-1,0,\{6,1\}} + 2H_{0,0,-1} - H_{0,0,\{6,0\}} + 2H_{0,0,\{6,1\}} \quad (3.14)$$

$$G_{y,-1}(x) = -\frac{1}{3}\zeta_2 + \frac{1}{2}H_0^2 + H_{0,\{6,0\}} - 2H_{0,\{6,1\}} \quad (3.15)$$

$$G_{y,y,-1}(x) = \frac{2}{3}\zeta_3 + \frac{1}{3}\zeta_2 H_0 - \frac{1}{6}H_0^3 - H_{0,0,\{6,0\}} + 2H_{0,0,\{6,1\}}, \quad (3.16)$$

where

$$H_{\{3,0\},0}(1) = -\frac{2}{\sqrt{3}}\text{Cl}_2\left(\frac{2}{3}\pi\right) = \frac{2}{9}\left[4\zeta_2 - \psi'\left(\frac{1}{3}\right)\right] \quad (3.17)$$

is another cyclotomic constant and Cl_2 denotes a Clausen functions with

$$\text{Cl}_2(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2} \quad (3.18)$$

and ψ' denotes the first derivative of the digamma function.

The above functions contain letters of the usual harmonic polylogarithms and those of cyclotomy $\mathbf{c} = 3, 4$ and 6 [9], to which the notation in (3.9)–(3.16) corresponds. This is the case because of the choice of the variable z , which is possible in the univariate case, although one does not know the optimal choice *a priori*. The cyclotomic letters are characterized by two indices [9] as, e.g.,

$$f_{\{5,3\}}(x) = \frac{x^3}{1+x+x^2+x^3+x^4}, \quad (3.19)$$

where the second index is smaller than the degree of the denominator polynomial.

Choosing the variable y instead as the main argument, one transforms the above expressions using Eq. (2.17). As a consequence the additional letters

$$\left\{ \frac{1}{1+3x^2}, \frac{x}{1+3x^2} \right\} \quad (3.20)$$

are needed, which are not cyclotomic. As an example one obtains

$$G_{y,-1}(x) = -\frac{1}{2}H_{-1}^2(y) - H_{-1}(y)H_1(y) + \frac{1}{2}H_1^2(y) + 2H_{-1,1}(y) \\ - 2\text{QL}\left[\left\{\left\{-1, 1, 0\right\}, 0\right\}, \left\{\left\{\frac{1}{3}, 0, 1\right\}, 1\right\}\right], y\right] + 2\text{QL}\left[\left\{\left\{1, 1, 0\right\}, 0\right\}, \left\{\left\{\frac{1}{3}, 0, 1\right\}, 1\right\}\right], y\right], \quad (3.21)$$

and similar in the case of the other integrals. In general also QL-functions at argument $x = 1$ will appear, inducing new constants. This set can be reduced to the algebraic basis by shuffle and stuffle relations [29]. Depending on the defining constants of the contributing quadratic forms, further sets of relations may be present. Often particular algebraic numbers occur in this context.

IV. CONCLUSIONS

In the analytic calculation of Feynman diagrams hierarchies of function spaces and algebras emerge both in the z space and N space representation consisting out of iterative integrals or nested sums of different kind. The simplest structures are harmonic polylogarithms, followed by Kummer–Poincaré iterated integrals and cyclotomic integrals, and also iterated integrals over square-root valued letters and the associated sums and special constants. Here we consider a real extension of the Kummer–Poincaré iterated integrals, allowing also for letters generated by general real quadratic forms without real factorization. Here the range of constants is not limited to $c_i \in \mathbb{Q}$, but general real numbers are allowed, which are usually implied by the values of different masses and virtualities in the processes to be considered. Quantities of this kind appear in higher order and multileg calculations. Since real representations have sometimes advantages compared to complex representations, we provide algorithms to build the associated algebra to a set of these letters, their basis

representation, different mappings of the main argument, including analytic continuation in the case of the presence of cuts. Finally, also the expressions can be evaluated numerically. The different commands in `HarmonicSums` to provide these operations are described and illustrated by examples. We have applied the corresponding mappings to a class of functions which have emerged recently in the NLO calculation of the inclusive Compton cross section. In viewing physics results within this class, it is for structural reasons also interesting to see whether the result can be expressed by functions out of a particular function space. In the case of Ref. [16] it turns out to be the space of cyclotomic harmonic polylogarithms if the final expression is written using the variable z , (3.4).

The different commands to treat iterated integrals of the QL-type are implemented in the package `HarmonicSums`, which is available from <https://risc.jku.at/sw/harmonicsums/>. As well we attach the Supplemental Material `Quadratic-Letters.nb` to this paper [30].

ACKNOWLEDGMENTS

We thank S. Weinzierl for a discussion. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 764850, SAGEX and from the Austrian Science Fund (FWF) Grant No. SFB F50 (F5009-N15).

-
- [1] J. Blümlein and C. Schneider, *Int. J. Mod. Phys. A* **33**, 1830015 (2018).
  - [2] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren, *Comput. Phys. Commun.* **181**, 582 (2010).
  - [3] J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **14**, 2037 (1999).
  - [4] J. Blümlein and S. Kurth, *Phys. Rev. D* **60**, 014018 (1999).
  - [5] E. Remiddi and J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **15**, 725 (2000).
  - [6] S. Moch, P. Uwer, and S. Weinzierl, *J. Math. Phys. (N.Y.)* **43**, 3363 (2002).
  - [7] J. Ablinger, J. Blümlein, and C. Schneider, *J. Math. Phys. (N.Y.)* **54**, 082301 (2013).
  - [8] E. E. Kummer and J. Reine, *Angew. Math. (Crelle)* **21**, 74 (1840); H. Poincaré, *Acta Math.* **4**, 201 (1884); J. A. Lappo-Danilevsky, *Mémoires sur la Théorie des Systèmes Différentielles Linéaires* (Chelsea Publishing Company, New York, 1953); K. T. Chen, *Trans. Am. Math. Soc.* **156**, 359 (1971); A. B. Goncharov, *Math. Res. Lett.* **5**, 497 (1998).
  - [9] J. Ablinger, J. Blümlein, and C. Schneider, *J. Math. Phys. (N.Y.)* **52**, 102301 (2011).
  - [10] J. Ablinger, J. Blümlein, C. G. Raab, and C. Schneider, *J. Math. Phys. (N.Y.)* **55**, 112301 (2014).
  - [11] A. I. Davydychev and M. Yu. Kalmykov, *Nucl. Phys.* **B699**, 3 (2004); S. Weinzierl, *J. Math. Phys. (N.Y.)* **45**, 2656 (2004).
  - [12] J. Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C. G. Raab, C. S. Radu, and C. Schneider, *J. Math. Phys. (N.Y.)* **59**, 062305 (2018).

- [13] A. Sabry, *Nucl. Phys.* **33**, 401 (1962); D. J. Broadhurst, [arXiv:0801.4813](#); L. Adams, C. Bogner, and S. Weinzierl, *J. Math. Phys. (N.Y.)* **54**, 052303 (2013); **55**, 102301 (2014); **56**, 072303 (2015); **57**, 032304 (2016); E. Remiddi and L. Tancredi, *Nucl. Phys.* **B907**, 400 (2016); L. Adams, C. Bogner, A. Schweitzer, and S. Weinzierl, *J. Math. Phys. (N.Y.)* **57**, 122302 (2016); G. Passarino, *Eur. Phys. J. C* **77**, 77 (2017); E. Remiddi and L. Tancredi, *Nucl. Phys.* **B925**, 212 (2017); L. Adams and S. Weinzierl, *Phys. Lett. B* **781**, 270 (2018); D. J. Broadhurst, η quotients, Eichler integrals and L-series, in Proceedings at HMI Bonn, 2018; L. Adams and S. Weinzierl, *Commun. Num. Theor. Phys.* **12**, 193 (2018); C. Bogner, A. Schweitzer, and S. Weinzierl, *Nucl. Phys.* **B922**, 528 (2017); J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, *J. High Energy Phys.* **05** (2019) 120; **01** (2019) 023; J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, *J. High Energy Phys.* **08** (2018) 014; J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, *J. High Energy Phys.* **05** (2018) 093; *Phys. Rev. D* **97**, 116009 (2018).
- [14] C. Bogner and S. Weinzierl, *Int. J. Mod. Phys. A* **25**, 2585 (2010).
- [15] R. Argand, *Essai sur une Manière de Représenter les Quantités Imaginaires Dans Les Constructions Géométriques, 1806, 1813*, 2nd ed. (Gauthier–Villars, Paris, 1874).
- [16] R. N. Lee, M. D. Schwartz, and X. Zhang, [arXiv:2102.06718](#).
- [17] J. Vollinga and S. Weinzierl, *Comput. Phys. Commun.* **167**, 177 (2005).
- [18] A. Cauchy, in *Sur un Nouveau Genre de calcul Analogue au Calcul Infinitésimal. Exercices de mathématiques 1826. Reprinted in Oeuvres complètes, Ser. 2*, 6 (Gauthier–Villars, Paris, 1882–1974), pp. 23–37; G. Källén, *Helv. Phys. Acta* **25**, 417 (1952); H. Lehmann, *Nuovo Cimento* **11**, 342 (1954).
- [19] J. Ablinger, J. Blümlein, and C. Schneider, *J. Phys. Conf. Ser.* **523**, 012060 (2014); J. Ablinger, *Proc. Sci. LL2014* (2014) 019 [[arXiv:1407.6180](#)]; Diploma thesis, JKU Linz, 2009 [[arXiv:1011.1176](#)]; Ph.D. thesis, Linz U., 2012 [[arXiv:1305.0687](#)]; *Proc. Sci. LL2016* (2016) 067; *Exp. Math.* **26** (2017); *Proc. Sci. RADCOR2017* (2018) 001 [[arXiv:1801.01039](#)]; *Proc. Sci. LL2018* (2018) 063; J. Blümlein, *Comput. Phys. Commun.* **180**, 2218 (2009).
- [20] J. Ablinger, *Exp. Math.* **26** (2017).
- [21] J. Ablinger, *Exp. Math.* **26**, 62 (2016).
- [22] C. Schneider, *Sém. Lothar. Combin.* **56**, 1 (2007).
- [23] C. Schneider, in *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts and Monographs in Symbolic Computation, edited by C. Schneider and J. Blümlein (Springer, Wien, 2013), pp. 325–360.
- [24] L. Lewin, *Polylogarithms and Associated Functions* (North Holland, New York, 1981).
- [25] J. Blümlein, *Integrals in Diagram Calculations* (unpublished).
- [26] J. Ablinger, J. Blümlein, C. Raab, C. Schneider, and F. Wißbrock, *Nucl. Phys.* **B885**, 409 (2014).
- [27] J. Blümlein, *Comput. Phys. Commun.* **159**, 19 (2004).
- [28] C. Reutenauer, *Free Lie Algebras* (Clarendon Press, Oxford, 1993).
- [29] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, *Trans. Am. Math. Soc.* **353**, 907 (2001).
- [30] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevD.103.096025> for details.