Confinement and graded partition functions for $\mathcal{N} = 4$ SYM

Aleksey Cherman^{*} and Aditya Dhumuntarao[†]

School of Physics and Astronomy, University of Minnesota Minneapolis, Minnesota 55455, USA

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Gauge theories with confining phases at low temperatures tend to deconfine at high temperatures. In some cases, for example in supersymmetric theories, confinement can persist for all temperatures provided the partition function includes a grading by $(-1)^F$. When it is possible to define partition functions that smoothly interpolate between no grading and $(-1)^F$ grading, it is natural to ask if there are other choices of grading that have the same effect as $(-1)^F$ on confinement. We explore how this works for $\mathcal{N} = 4$ SYM on $S^1 \times S^3$ in the large N limit at both small and large coupling. We find evidence for a continuous range of grading parameters that preserve confinement for all temperatures at large coupling, while at small coupling only a discrete set of gradings preserves confinement.

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I. INTRODUCTION

Gauge theories tend to deconfine at high temperatures. This statement can be made precise in theories that have a center symmetry, such as $SU(N)$ Yang-Mills (YM) theory, see, e.g., [1,2]. Studying the properties of a $4d$ theory on a spatial manifold M_3 at finite temperature is equivalent to asking about the behavior of the Euclidean path integral on $M_3 \times S^1_\beta$ with periodic boundary conditions for bosons and antiperiodic boundary conditions for fermions. This path integral gives the thermodynamic partition function

$$Z(\beta) = \text{tr} e^{-\beta H}, \quad (1)$$

where $\beta = 1/T$. Wilson loops that wind around the thermal circle, $\text{tr} \Omega = \mathcal{P} e^{i \int_{S^1} A}$, transform by a $\mathbb{Z}_N$ phase under center symmetry, $\text{tr} \Omega \rightarrow e^{2\pi i/N} \text{tr} \Omega$. In pure $SU(N)$ YM theory, it is known that $\langle \text{tr} \Omega \rangle = 0$ at low temperatures, see, e.g., [3], but it is nonzero at high temperatures [1]. This implies that the high and low temperature regimes are separated by a phase transition in the thermodynamic limit. Note however that there are actually two physically distinct ways to take the thermodynamic limit for $SU(N)$ YM theory. The standard way is take the volume of M_3 to infinity. Alternatively, one can take $N \rightarrow \infty$ with the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and other parameters such as the

spatial volume held fixed [4–6]. There is a temperature-driven deconfinement transition in both of these two thermodynamic limits. In this paper we will discuss the deconfinement phase transition in the large N thermodynamic limit.

The deconfinement phase transition is rather difficult to evade by, e.g., varying parameters controlling the matter content. Indeed, in the large N limit, the deconfinement transition is forced by the fact that the confining phase features a Hagedorn density of states. This means that the density of states, $\rho(E)$, grows exponentially with the energy $\rho(E) \sim e^{+\beta_H E}$, where it is assumed that $E \sim N^0$ is large compared to the characteristic mass scale of the theory, and $\beta_H > 0$, the Hagedorn temperature, depends on the matter content of the theory, but it can be estimated as $\beta_H \sim \min(\Lambda^{-1}, R)$ where Λ is the large scale and R is the characteristic size of the spatial box. Given this density of states, the thermodynamic partition function cannot be analytic at $\beta = \beta_H$. Thus the theory must have a deconfinement transition at some $\beta_c \geq \beta_H$ at large N .

However, the deconfinement phase transition can be evaded if one considers some *graded* partition functions. The most famous example is

$$\tilde{Z}(\beta) = \text{tr} (-1)^F e^{-\beta H}, \quad (2)$$

where F is fermion parity. This partition function is calculated by a path integral with periodic boundary conditions for all fields on S^1_β . If the distribution of bosonic and fermionic states are uncorrelated, a $(-1)^F$ -graded partition function will also exhibit phase transitions as a function of β in the thermodynamic limit. However, if the spectra of bosonic and fermionic states are related in a sufficiently precise way, there will be large cancellations

^{*}acherman@umn.edu
[†]dhumu002@umn.edu

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and the graded density of states $\tilde{\rho}(E)$ can have subexponential growth at large E at arbitrary β . Then $\tilde{Z}(\beta)$ can avoid the Hagedorn instability and the deconfinement phase transition.

This naturally occurs in supersymmetric gauge theories, where the bosonic and fermionic states with energies $E > 0$ necessarily come in degenerate multiplets in the infinite-volume limit. Surprisingly, deconfinement phase transitions of (2) can sometimes be avoided even without supersymmetry. Indeed, this is the case for adjoint QCD, YM theory coupled to $N_f > 1$ adjoint fermions [7–11], and even in some theories with fundamental fermions, which require more complicated gradings [12]. In such theories, $\tilde{Z}(\beta)$ is smooth, and hence the theory remains confining at all temperatures.

In this paper, we want to study what happens to deconfinement transition in theories where it is possible to smoothly interpolate between (1) and (2), both at large and small 't Hooft coupling. This is most easily done for the large N limit of $4d \mathcal{N} = 4$ super Yang-Mills (SYM) theory, formulated in Euclidean signature on $S_R^3 \times S_\beta^1$, where R is the radius of S^3 . Indeed, it is known that, without any grading, large N , $\mathcal{N} = 4$ SYM has a deconfinement as a function of β/R , and the critical value of β/R is known both at small [4–6] and large [13–16] 't Hooft coupling¹ due to the AdS/CFT correspondence. Supersymmetry suggests that the theory should be in a confined phase for all β/R if the partition function includes a grading by $(-1)^F$, and indeed this has been verified at large coupling in Ref. [16]. Finally, $\mathcal{N} = 4$ SYM has a continuous internal global symmetry, $SO(6)_R$, which, as we will show, can be used to define a grading that smoothly interpolates between (1) and (2). We define this grading below and study the resulting phase structure at small and large 't Hooft coupling.

II. GRADED PARTITION FUNCTIONS

In a conformal field theory (such as $\mathcal{N} = 4$ SYM) on $S_R^3 \times S_\beta^1$, the partition function can be viewed as $Z(\beta/R) = \sum_{\Delta} e^{-\Delta\beta/R}$, where the sum runs over the scaling dimensions of local operators, so that the Hamiltonian is related to the dilation operator $H = D/R$. In theories with global symmetries it is also possible to turn on chemical potentials μ_i associated to conserved charges Q_i , yielding

$$Z(\beta, \mu_i) = \text{tr} e^{\beta \sum_i \mu_i Q_i} e^{-\beta H}. \quad (3)$$

In the specific case of $\mathcal{N} = 4$ SYM, there is an $SO(6)_R$ symmetry, and so one can turn on three independent chemical potentials associated with the three Cartan generators of $SO(6)_R$. We can consider $\mathcal{N} = 4$ SYM as $\mathcal{N} = 1$

SYM coupled to three adjoint chiral supermultiplets, then the three complex scalar fields ϕ_1, ϕ_2 , in $\mathcal{N} = 4$ SYM are the lowest components of the three chiral supermultiplets. The three scalars transform in the **6** representation of $SO(6)_R$, and following the conventions of Ref. [19], we assemble the three scalars into a six-dimensional vector $(\phi_1, \phi_1^*, \phi_2, \phi_2^*, \phi_3, \phi_3^*)^T$, and write the Cartan generators in the **6** representation as

$$Q_1^6 = \text{diag}(+1, -1, 0, 0, 0, 0) \quad (4)$$

$$Q_2^6 = \text{diag}(0, 0, +1, -1, 0, 0) \quad (5)$$

$$Q_3^6 = \text{diag}(0, 0, 0, 0, +1, -1), \quad (6)$$

Then $2Q_i^6$ acts with charge **2** on ϕ_i , and the superconformal R -charge r can be written as $r = \frac{2}{3}(Q_1 + Q_2 + Q_3)$, see, e.g., [20]. We will set $\mu_1 = \mu_2 = \mu_3 = 2i\theta/\beta$ and study the partition function

$$Z(\beta, \theta) = \text{tr} e^{2i\theta \sum_{i=1}^3 Q_i} e^{-\beta H} = \text{tr} e^{3i\theta r} e^{-\beta H}. \quad (7)$$

This graded partition function is periodic in θ with period 2π . It implements a simple interpolation from (1) at $\theta = 0$ to (2) at $\theta = \pi$, thanks to the fact that in $\mathcal{N} = 4$ SYM there is a spin-charge relation $e^{3\pi i r} = (-1)^F$, see, e.g., [21]. Charge conjugation symmetry further relates r to $-r$, so that the θ periodicity is reduced to $\theta \simeq \theta + \pi$.

As we already mentioned, $\mathcal{N} = 4$ SYM confines for all β/R when $\theta = \pi$, which is natural since this corresponds to working with a $(-1)^F = e^{3\pi i r}$ graded partition function in a supersymmetric theory. However, it is also true that in any supermultiplet, the r charge of the bosons and fermions differs by 1. This means that grading by $e^{i\pi r}$ should produce cancellations that are as strong as with $(-1)^F$, and one should expect to find confinement for all β/R when $\theta = \pi/3, 5\pi/3$. The calculations below show that these expectations are satisfied.

III. ZERO COUPLING

Let us determine the β and θ dependence of Z in (7) in the zero coupling limit $\lambda \rightarrow 0$, at large N . The partition function is given by a matrix integral which can be interpreted as an integral over the holonomy Ω [5], which implements the Gauss law constraint [1]. The matrix integral can be reduced to an integral over the eigenvalues of the holonomy

$$Z(\beta/R, \{\mu_i\}) = \int \left(\prod_k d\lambda_k \right) e^{-\mathcal{S}_{\text{eff}}(\lambda_k; \frac{\beta}{R}, \{\mu_i\})}, \quad (8)$$

where λ_k are the eigenvalues, $\{\mu_i\}$ are the three independent chemical potentials for mutually commuting charges in $SO(6)_R$, and $\mathcal{S}_{\text{eff}}$ is an effective action. The holonomy

¹More generally, the critical value of β/R is bounded from below by the Hagedorn temperature, which is known for all λ [17,18].

eigenvalues take values on a circle, and the effective action can be written in terms of Fourier coefficients ρ_n of the eigenvalue distribution ρ

$$\mathcal{S}_{\text{eff}} = N_c^2 \sum_{n=1}^{\infty} V_n \rho_n^2, \quad (9)$$

where $\rho_n \equiv \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \cos(n\alpha)$, and the V_n coefficients are given by

$$V_n \equiv \frac{1}{n} (1 - z_B(x^n, \{\mu_i\}) + (-1)^n z_F(x^n, \{\mu_i\})). \quad (10)$$

The parameter $x = e^{-\beta/R}$, and the functions $z_B = z_V + z_S, z_F$ are related to the single-particle partition functions for the massless vector, scalar, and fermion fields of $\mathcal{N} = 4$ SYM on S^3 [19]:

$$z_V(x) = \frac{6x^2 - 2x^3}{(1-x)^3}, \quad (11)$$

$$z_S(x, \{\mu_i\}) = \frac{x + x^2}{(1-x)^3} \sum_{a=1}^3 (x^{R\mu_i} + x^{-R\mu_i}), \quad (12)$$

$$z_F(x, \{\mu_i\}) = \frac{2x^{3/2}}{(1-x)^3} \prod_{a=1}^3 (x^{\frac{1}{2}R\mu_i} + x^{-\frac{1}{2}R\mu_i}). \quad (13)$$

We now set $\mu_i = \frac{2i\theta}{\beta}$, so that

$$z_S = 6 \frac{x + x^2}{(1-x)^3} \cdot [\cos(2\theta)], \quad (14a)$$

$$z_F = 16 \frac{x^{3/2}}{(1-x)^3} \cdot [\cos(\theta)]^3. \quad (14b)$$

So long as all of the coefficients V_n are positive, the matrix integral is dominated by the center-symmetric $\rho_n = 0$ minimum. When there is value of n for which V_n becomes negative there is a phase transition. We can interpret these phase transitions as center-breaking phase transitions where $\mathbb{Z}_N$ is spontaneously broken to an (approximate) subgroup $\mathbb{Z}_n$. (For an early discussion of partial deconfinement see, e.g., [22].) For large β/R , corresponding to $x \ll 1$, it is easy to verify that $V_n > 0$ for all n , but for small β/R some V_n generically become negative. Figure 1 is a plot of the phase diagram as a function of θ and β/R . Center symmetry is preserved above the black curve, and is broken either completely or to a subgroup of $\mathbb{Z}_N$ below the black curve.

There are two interesting features in Fig. 1. First, the lack of deconfinement for any β at $\theta = \pi$ is consistent with our original expectations: it amounts to working with a $(-1)^F$ graded partition function. Second, the theory also remains confining for all β/R if $\theta = \pi/3, 5\pi/3$. In the region

FIG. 1. Large N phase diagram of $\mathcal{N} = 4$ SYM theory on S^3_R for $\lambda \rightarrow 0$ as a function of inverse temperature β and grading θ . The theory is in a confined phase in the white region, and is partially confined in the colored regions. In between $\theta = \pi/3$ and $\theta = 5\pi/3$ center symmetry is only partially broken, with the unbroken subgroup of $\mathbb{Z}_N$ indicated by the color scheme in the legend. The theory confines at all β/R when $\theta = \pi/3, \pi, 5\pi/3$.

$(\pi/3, \pi) \cup (\pi, 5\pi/3)$ the system is in a (partially) center-broken phase for sufficiently small values of β/R . The dependence of the deconfinement temperature on θ is highly nonmonotonic.

What should we expect as λ is increased from zero? At a heuristic level, increasing the coupling λ should increase the fluctuations of the Polyakov loop eigenvalues, and it is natural to expect such fluctuations to increase the range of β/R values where all traces of powers of the Polyakov loop vanish. It would be interesting but challenging to directly compute corrections to our results in powers of λ , see, e.g., [6,23] for the challenges that arise already in pure YM theory. However, one of the special features of $\mathcal{N} = 4$ SYM is that the $\lambda \rightarrow \infty, N \rightarrow \infty$ limit is just as calculable as the $\lambda \rightarrow 0, N \rightarrow \infty$ limit thanks to the AdS/CFT correspondence [13–16]. In what follows, we take advantage of AdS/CFT and study the θ dependence of the phase diagram at large 't Hooft coupling.

IV. INFINITE COUPLING

At large N and λ , $\mathcal{N} = 4$ $SU(N)$ SYM is believed to have a dual gravitational description [13–16]. The dependence of the theory on R -charge chemical potentials has been extensively explored in the literature, see, e.g., [15,16,19,24,25]. The AdS/CFT dictionary for conserved charges implies that to study $\mathcal{N} = 4$ SYM with chemical potentials for R symmetry, we should consider the truncation of type IIB supergravity on $\text{AdS}_5 \times S^5$ to a 5d Einstein gravity theory coupled to three $U(1)$ gauge fields (A_I), $I = 1, 2, 3$, associated with the $U(1)^3$ Cartan subgroup of $SO(6)_R$ [26–30]:

$$\mathcal{I} = -\frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left[R + \frac{12}{\ell^2} - \frac{\ell^2}{4} \sum_{I=1}^3 (F_I)^2 \right], \quad (15)$$

where $(F_I)_{MN} = \partial_M(A_I)_N - \partial_N(A_I)_M$ are the field strengths of the three gauge fields, $M, N = 1, \dots, 5$, κ is related to the 5d Newton constant G_5 via $\kappa = 8\pi G_5$, and the cosmological constant is $\Lambda = -6/\ell^2$.

The boundary values of bulk gauge fields act as sources for conserved currents in the boundary theory, and our goal is to turn on equal chemical potentials for each of the three Cartan charges. For this purpose we can simply set $A^I = A$ above, and then the bulk action reduces to an Einstein-Maxwell-anti-de Sitter (AdS)₅ theory

$$\mathcal{I} = -\frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left[R + \frac{12}{\ell^2} - \frac{3\ell^2}{4} F_{MN} F^{MN} \right]. \quad (16)$$

The Einstein-Maxwell-AdS₅ system has been extensively explored in [24,25], and has Reissner-Nordstrom-AdS₅ black hole solutions² with parameters (L_m, L_q) , which are related to the ADM mass and the charge density, respectively. The charged black hole solution reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \quad (17)$$

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{L_m^2}{r^2} + \frac{L_q^4}{r^4}, \quad (18)$$

$$A_t(r) = \left(\mu - \frac{L_q^2}{\ell r^2} \right). \quad (19)$$

This nonsupersymmetric family of nonrotating solutions is expected to correspond to spatially homogeneous equilibrium states of the dual field theory.

We briefly review how the thermodynamic parameters of the bulk solution, i.e., the inverse temperature and chemical potential, map to the corresponding field theory parameters. First, we recall that ℓ can be identified with the radius of the S^3 spatial manifold in the dual field theory [19]. To identify β , we pass to Euclidean signature $t \rightarrow i\tau$, and recall that the metric is free of conical singularities provided τ is periodic with period

$$\beta = \frac{4\pi}{\partial_r f(r)} \Big|_{r_+} = \frac{2\pi\ell^2 r_+}{\ell^2(1 - (L_q/r_+)^4) + 2r_+^2}, \quad (20)$$

where r_+ is the outer horizon corresponding to the largest positive root of $f(r_+) = 0$. The Hawking temperature $T = 1/\beta$ is then identified with the temperature of the dual field theory. Next, the chemical potential is determined by the asymptotic value of the field strength along the $\tau - r$ disk, D_2 :

²Our normalization convention is different from [24,25], so that $A_{\text{there}} = \sqrt{4/3}\ell^2 A_{\text{here}}$.

$$\int_{D_2} F = \int_0^\beta d\tau \int_{r_+}^\infty dr F_{r\tau} = i\beta\mu. \quad (21)$$

The gauge field must be regular at r_+ , meaning that $A_\tau(r_+) = 0$, which yields the identification

$$\mu = \frac{L_q^2}{\ell r_+^2}. \quad (22)$$

The confinement/deconfinement phase transition is mapped onto a gravitational phase transition in the following way. The solution with $L_m = L_q = 0$ should be thought of as thermal AdS₅, and is dual to the confining phase of the dual gauge theory, whereas solutions with $L_m^2 \in \mathbb{R}_+$ are dual to the deconfined phase [16]. The difference of the free energies of these two phases is [24,25,31]

$$\mathcal{F}_{\text{deconfined}} - \mathcal{F}_{\text{confined}} = \frac{\pi r_+^2}{8G_5} \left(1 - (\ell\mu)^2 - \frac{r_+^2}{\ell^2} \right). \quad (23)$$

When this quantity changes sign, there is a deconfinement phase transition from the point of view of the field theory [15,16], which is realized as a Hawking-page phase transition in the gravitational theory [32].

To map out the dependence of the phase transition temperature on the parameter θ in (7), we study how (23) behaves as a function of an appropriate imaginary chemical potential. We take $\mu \rightarrow i\tilde{\mu} = \frac{2i\theta}{\beta}$ while keeping r_+ , which requires sending $L_q^2 \rightarrow i\tilde{L}_q^2$. Then A_τ is real and continues to satisfy $A_\tau(r_+) = 0$. The on shell Euclidean action becomes

$$\mathcal{F}_{\text{deconfined}} - \mathcal{F}_{\text{confined}} = \frac{\pi r_+^2}{8G_5} \left(1 + \left(\frac{2\ell\theta}{\beta} \right)^2 - \frac{r_+^2}{\ell^2} \right). \quad (24)$$

This expression is manifestly invariant under $\theta \rightarrow -\theta$. In addition, we should understand the θ dependence in this formula and the ones that follow mod 2π . In field theory, this basic fact follows from charge quantization. To see how this is matched in the bulk, we recall that type IIB string theory has an NS-NS two-form gauge field B . One can turn on a flat B potential (that is, obeying $dB = 0$) in the solutions discussed above at no cost in energy, with an arbitrary value of the 2π periodic parameter α [16,33,34]:

$$\alpha = \int_{D_2} B. \quad (25)$$

The periodicity of the chemical potential under shifts of $2\pi i/\beta$ is ensured by the shift freedom of $\alpha \rightarrow \alpha + 2\pi$.

It is now straightforward to determine the phase transition temperature as a function of θ . The critical temperature, T_c , is determined by the solution of $\mathcal{F}_{\text{deconfined}} - \mathcal{F}_{\text{confined}} = 0$, which reduces to

FIG. 2. Large N phase diagram of $\mathcal{N} = 4$ SYM theory on S^3_R as a function of inverse temperature β and grading θ at zero and infinite 't Hooft coupling λ . The grading θ interpolates between the thermal and $(-1)^F$ graded partition functions. At $\lambda = 0$ the theory has a deconfinement phase transition along the black curve, and has at least partially broken center symmetry everywhere in the colored regions. The analogous phase transition curve due to known bulk solutions at large coupling $\lambda \rightarrow \infty$ is shown in blue, with center symmetry broken everywhere in the hatched region. Note that the deconfined region at large coupling is a subset of the deconfined region at zero coupling.

$$1 = \frac{\ell^2}{r_+^2} \left[1 + \left(\frac{2\ell\theta}{\beta_c} \right)^2 \right]. \quad (26)$$

Using (20), we may trade r_+ for β_c , replace ℓ by R , and find the transition temperature in field theory variables

$$\beta_c(\theta) = \frac{2\pi R}{3} \sqrt{1 - \left(\frac{\theta}{\pi/3} \right)^2}. \quad (27)$$

When $\theta = 0$, corresponding to conventional thermal field theory we recover the standard Hawking-Page phase transition for uncharged SAdS₅ black holes $\beta_c = 2\pi R/3$ [32]. As θ is increased the deconfinement temperature decreases. At $\theta = \pi/3 \bmod 2\pi$ the critical temperature becomes infinite, and there are no sensible Reissner-Nordstrom-AdS₅ black hole solutions because β_c becomes complex. The fact that $\mathcal{N} = 4$ SYM is confined for all β when $\theta = \pi$, corresponding to $(-1)^F$ graded partition function, has been known since Ref. [16]. But our analysis here suggests that there is actually a whole window of θ values, namely

$$\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3} \right], \quad (28)$$

where $\mathcal{N} = 4$ $SU(N)$ SYM apparently remains confined for all values of β/R at large λ . We compare the zero and large coupling phase diagrams in Fig. 2.

We should emphasize that to reach (27) we have only studied the known (analytically continued) solutions that correspond to homogeneous equilibrium states in the dual field theory. It would be very interesting to see whether

there might be some sort of previously unknown (perhaps multicenter) black objects that would be associated to the partially confined phases³ we saw at small coupling, see, e.g., [42] for a discussion in the context of the superconformal index.

V. CONCLUSIONS

In $\mathcal{N} = 4$ SYM theory, it is possible to smoothly interpolate from the standard thermodynamic partition function to a $(-1)^F$ -graded partition function by taking advantage of the $SO(6)$ R symmetry. We studied the simplest such generalized partition function, where the grading factor is $e^{3i\theta r}$ uses the superconformal R -charge r . We found that the large N theory is in the confined phase on $S^3_R \times S^1_\beta$ when $\theta = \pi/3, \pi, 5\pi/3$ both at small and large coupling. The result for $e^{3\pi i r} = (-1)^F$ was already known. The other two points with complete confinement for all temperatures at both small and large 't Hooft coupling correspond to the more complicated grading $(-1)^F e^{\pm 2\pi i r}$. The small coupling and large coupling corners of parameter space differ in the behavior for $\theta \in (\pi/3, \pi) \cup (\pi, 5\pi/3)$. At small coupling, the phase transition temperature depends nonmonotonically on θ in the region $\theta \in (\pi/3, \pi) \cup (\pi, 5\pi/3)$, with center symmetry partially broken for high enough temperatures, with the critical temperature diverging as $\theta \rightarrow \pi$. At large coupling, on the other hand, we found no evidence of such rich behavior: the theory appears to be in the confined phase for all temperatures for $\theta \in [\pi/3, 5\pi/3]$.

It is interesting to connect our results to some recent studies of deconfinement in the superconformal index of $\mathcal{N} = 4$ SYM. At the outset, however, we should stress that our graded partition function is quite different from the superconformal index of $\mathcal{N} = 4$ SYM [43], see, e.g., Refs. [44,45] for recent reviews. The partition function we focus on here receives contributions from all of the states in the theory and depends nontrivially on the 't Hooft coupling. On the other hand, the superconformal index is a special $S^3_R \times S^1_\beta$ partition function that is designed to only receive contributions from $1/4$ BPS states (in $\mathcal{N} = 1$ language). It is independent of the 't Hooft coupling, so that its exact form for all values of λ and N can be determined in terms of a matrix integral. The superconformal index involves two chemical potentials (and corresponding fugacities p, q) built out of appropriate combinations of Δ R -charge r and the left and right moving angular momenta j_1, j_2 . When the fugacities are real, the superconformal index $\mathcal{I}$ is always in the “confining phase,” meaning that $\log \mathcal{I} \sim \mathcal{O}(1)$ [43]. It was recently discovered that $\log \mathcal{I}$ can become $\mathcal{O}(N^2)$ if the fugacities are rotated into the complex plane, so that the superconformal index can be used to study deconfinement and black holes, see

³There is another notion of “partial deconfinement/submatrix deconfinement” recently discussed in Refs. [35–41].

Ref. [20,21,42,46–65]. While the superconformal index and our partition functions are distinct, they do share an interesting parallel: the grading $(-1)^F e^{\pm 2\pi i r}$ plays the same role in both cases as being on the boundary between deconfinement for some β/R and confinement for all β/R . This was recently shown in Ref. [21], which studied the superconformal index as a function of $p = q = ye^{i\psi}$, finding a deconfined region for $\psi \in (-4\pi, -3\pi) \cup (-3\pi, -2\pi)$, with, e.g., $\psi = -2\pi$ corresponding to a grading by $(-1)^F e^{-2\pi i r}$.

Our large-coupling results are less complete than our results at small coupling; this is because we do not know how to exclude the possibility that there are some presently unknown black objects which correspond to partially confined phases at large coupling. One might guess that increasing the 't Hooft coupling should increase eigenvalue fluctuations, and heuristically this could be expected to increase the deconfinement temperature and (naïvely) wipe out partially confined phases. This simple picture is

consistent with our results. However, Ref. [42] recently also found evidence for partially confined phases within the superconformal index, which is independent of the 't Hooft coupling. So, from the viewpoint of field theory, partially confined phases appear to be fairly ubiquitous. It would be very interesting to find new black objects dual to partially confined phases, or to prove that none exist at least within the supergravity approximation.

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