Quantizing the Eisenhart lift

Kieran Finn^{1,*}, Sotirios Karamitsos^{2,†} and Apostolos Pilaftsis^{1,‡}

¹*Department of Physics and Astronomy, University of Manchester, Manchester M13 9PL, United Kingdom*

²*Dipartimento di Fisica dell' Università di Pisa, Largo Bruno Pontecorvo, 3, 56127 Pisa PI, Italy*

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The classical Eisenhart lift is a method by which the dynamics of a classical system subject to a potential can be recreated by means of a free system evolving in a higher-dimensional curved manifold, known as the lifted manifold. We extend the formulation of the Eisenhart lift to quantum systems, and show that the lifted manifold recreates not only the classical effects of the potential but also its quantum mechanical effects. In particular, we find that the solutions of the Schrödinger equations of the lifted system reduce to those of the original system after projecting out the new degrees of freedom. In this context, we identify a conserved quantum number, which corresponds to the lifted momentum of the classical system. We further apply the Eisenhart lift to quantum field theory (QFT). We show that a lifted field space manifold is able to recreate both the classical and quantum effects of a scalar field potential. We find that, in the case of QFT, the analog of the lifted momentum is a quantum charge that is conserved not only in time but also in space. The different possible values for this charge label an ensemble of Fock spaces that are all disjoint from one another. The relevance of these extended Fock spaces to the cosmological constant and gauge hierarchy problems is considered.

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I. INTRODUCTION

Although it is not often emphasized in the literature, fictitious or emergent forces underpin a significant part of modern physics. Fictitious forces arise in Newtonian mechanics in noninertial frames of reference such as the surface of Earth, but they are also the theoretical underpinning of general relativity (GR). As demonstrated by Einstein [1], the effects of gravity emerge from the curvature of the spacetime manifold. Particles living in anything other than a flat spacetime can never be in an inertial frame of reference, and, therefore, if we ignore the curvature of spacetime, gravity must be introduced as a fictitious force to explain the motion of these particles. In a similar vein, soon after the development of GR, Kaluza-Klein theory [2–4] was proposed as a way to explain electromagnetism as a fictitious force. By endowing spacetime with a compactified fifth dimension, it is possible to show that the additional degrees of freedom in the metric results in Maxwell's equations.

There is clearly precedent for using the geometrical properties of a higher-dimensional manifold in order to explain the origin of a force. As a result, we may naturally wonder whether geometrization is possible for all such forces. This question was answered by Eisenhart, who showed that the effects of *any* conservative force on a particle can be captured by embedding the particle in a higher dimensional space with *no potential* and an appropriate metric function [5]. This formalism is known as the *Eisenhart lift*.

Historically, the Eisenhart lift was the catalyst for the development of the Bargmann structure framework [6], which lifts a Euclidean manifold with the addition of two extra dimensions such that it is endowed with a Lorentz metric. In this context, the formalism is also known as the *Eisenhart–Duval lift*, and has seen applications in analytical mechanics, particularly in the context of integrable systems [7,8]; see [9] for a pedagogical overview.

Recently, we showed that the Eisenhart lift can also be applied to classical field theories [10]. In this case, the field theory is “lifted,” not by the addition of a spatial dimension but rather the addition of a new field. This has allowed for a geometric understanding of multifield inflation [11–13], as well as the initial conditions of the Universe [14,15].

Thus far, all work done on the Eisenhart lift has been restricted to the classical level. The aim of this paper is, therefore, to consider the quantum effects of the Eisenhart lift. We will apply the Eisenhart lift to quantum systems in the contexts of both quantum mechanics (QM) and

*kieran.finn@manchester.ac.uk

†sotirios.karamitsos@df.unipi.it

‡apostolos.pilaftsis@manchester.ac.uk

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quantum field theory (QFT) and compare the original and lifted systems at the quantum level.

The layout of this paper is as follows: in Sec. II, we provide a brief overview of the classical Eisenhart lift, including both its original Lagrangian formulation as well as how it appears in the Hamiltonian formalism. In Sec. III, we apply the Eisenhart lift to QM, demonstrating that solutions to the lifted Schrödinger equation correctly reproduce all the usual observables of the original system. As an illustrative example, we consider the quantum simple harmonic oscillator and construct its lifted formulation, as well as the associated lifted Hilbert space. In Sec. IV, we extend the Eisenhart lift to QFTs. We demonstrate that the lifted QFT contains a conserved quantum charge that labels an ensemble of disjoint Fock spaces. The value of this charge sets the scale of the energy quanta of the lifted creation and annihilation operators in the associated Fock space. We show that for an appropriate choice of the Fock vacuum, the original space can be embedded in the lifted space in a similar way to QM. Finally, we discuss how the Eisenhart lift may help answer the gauge hierarchy and cosmological constant problems in Sec. V, before summarizing our findings and discussing possible further applications in Sec. VI.

II. THE CLASSICAL EISENHART LIFT

In this section, we give an overview of the Eisenhart lift as applied to classical particles. First, we briefly summarize the Eisenhart lift, as it was originally developed in the Lagrangian formalism. We then examine how it can be rewritten in the Hamiltonian formalism, since this will be more convenient for quantization.

A. Lagrangian formalism

To start with, let us consider a particle of mass m moving in d dimensions. The dynamics of such a particle is described by the Lagrangian

$$L = \frac{1}{2}m \sum_{i=1}^d \dot{x}_i^2 - V(\mathbf{x}), \quad (2.1)$$

where

$$\mathbf{x} = (x_1, \dots, x_d). \quad (2.2)$$

As expected, the evolution of this system is governed by Newton's second law of motion,

$$m\ddot{x}_i = -\frac{dV(\mathbf{x})}{dx_i}. \quad (2.3)$$

However, as discussed in the Introduction, it is possible to reproduce these equations of motion by means of a *free* particle living within a higher-dimensional curved

manifold. To do this, we introduce a new “fictitious” coordinate y , and write the Lagrangian of the lifted system as

$$L_{\text{lift}} = \frac{1}{2}m \sum_{i=1}^d \dot{x}_i^2 + \frac{1}{2}\frac{M^2}{V(\mathbf{x})}\dot{y}^2. \quad (2.4)$$

Here we have introduced an arbitrary mass scale M in order to keep dimensions consistent.

Note that in the free limit, $V \rightarrow 0$, the lifted Lagrangian appears to be singular. However, as we will soon see, $\dot{y}^2/V$ vanishes in this limit, preventing L_{lift} from diverging.

The Lagrangian given in (2.4) has no potential term, and so the particle is indeed free and will, thus, move along geodesics of this “lifted” $(d+1)$ -dimensional manifold. As we shall see, these geodesics will recreate the equations of motion (2.3).

By varying (2.4), we see that the equations of motion for the lifted system are

$$m\ddot{x}_i = -\frac{M^2}{2}\frac{V_{,i}}{V^2}\dot{y}^2, \quad \frac{d}{dt}\left(\frac{\dot{y}}{V}\right) = 0. \quad (2.5)$$

The solution to the second equation in (2.5) is

$$\frac{\dot{y}M}{V(\mathbf{x})} = A, \quad (2.6)$$

where A is a dimensionless constant that depends on the initial conditions.

If we substitute (2.6) into the first equation of (2.5), we arrive at

$$m\ddot{x}_i = -\frac{A^2}{2}V_{,i}. \quad (2.7)$$

We see that if the constant A satisfies the Eisenhart condition

$$A^2 = 2 \quad (2.8)$$

then (2.7) reduces identically to Newton's second law (2.3). Thus, once projected down to the original x coordinates, the geodesics of this lifted manifold reproduce the motion of the particle described by the original Lagrangian with a potential (2.1).

We observe that the equations of motion of the lifted system (2.4) are invariant under rescalings of time t . This arises because we have the freedom to affinely reparametrize the geodesics of the lifted manifold. We can always use this freedom to satisfy (2.8) and, hence, recreate the equations of motion of the original system (2.3). If we do not, then the lifted system will still emulate the original system, but will appear to evolve either in “slow motion” ($A < \sqrt{2}$) or “fast forward” ($A > \sqrt{2}$).

Finally, we observe that (2.6) and (2.8) imply that $\dot{y} \rightarrow 0$ as $V \rightarrow 0$ in such a way that the second term in (2.4) vanishes in this limit, as promised.

B. Geometric interpretation of the Eisenhart lift

As mentioned in the Introduction, it is illuminating to view the Eisenhart lift from a geometrical point of view. The Lagrangian (2.1) describes a particle which lives in Euclidean space $\mathbb{R}^d$ with line element

$$ds^2 = \delta_{ij} dx_i dx_j. \quad (2.9)$$

This particle feels an *external* force that is not geometrical in origin but rather due to a potential that permeates the manifold.

On the other hand, the lifted Lagrangian (2.4) corresponds to the motion of a *free* particle on a curved manifold, equipped with a metric

$$G_{AB} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \frac{M^2}{mV} \end{pmatrix} \quad (2.10)$$

and *generalized coordinates* q^A , where $A \in \{i, y\}$ and $q^i = x^i$, $q^y = y$. The corresponding line element is therefore

$$ds^2 = G_{AB} dq^A dq^B = \delta_{ij} dx^i dx^j + \frac{M^2}{mV} dy^2. \quad (2.11)$$

We note that lowercase indices (corresponding to the original space) are raised and lowered using the Kronecker delta δ_{ij} . The vertical position of these indices is, therefore, irrelevant. Uppercase indices (corresponding to the lifted system) are, on the other hand, raised and lowered by the metric G_{AB} and its inverse G^{AB} . The vertical position of these indices is, therefore, extremely important.

Because the lifted system does not contain a potential, the trajectory of the particle will correspond to a geodesic of this manifold. These geodesics can be derived by extremizing the length of the trajectory in the lifted manifold

$$\Delta s_{\text{lift}} = \int ds = \int d\lambda \sqrt{G_{AB} \frac{dq^A}{d\lambda} \frac{dq^B}{d\lambda}}, \quad (2.12)$$

where λ is an affine parameter.

This leads us to the geodesic equation

$$\frac{d^2 q^A}{d\lambda^2} + \Gamma_{BC}^A \frac{dq^B}{d\lambda} \frac{dq^C}{d\lambda} = 0, \quad (2.13)$$

where the Christoffel symbols are given, as usual, by

$$\Gamma_{BC}^A = \frac{1}{2} G^{AD} (G_{CD,B} + G_{BD,C} - G_{CB,D}). \quad (2.14)$$

Explicitly calculating the Christoffel symbols and substituting them back in the geodesic equation (2.13) returns the lifted equations of motion (2.5), as expected. As we demonstrated in the previous section, these equations recover the original equations of motion when the fictitious coordinate is projected out.

It is also possible to write the geodesic equation in terms of covariant derivatives. These are defined with the help of Christoffel symbols to be

$$\nabla_C \dot{q}^A = \partial_C \dot{q}^A + \Gamma_{CB}^A \dot{q}^B. \quad (2.15)$$

We may also define the covariant time derivative by multiplying (2.15) by $\dot{q}^C$ and formally using the chain rule. This gives us

$$D_t \dot{q}^A \equiv \dot{q}^A + \Gamma_{CB}^A \dot{q}^C \dot{q}^B. \quad (2.16)$$

A similar derivation applies for the covariant derivative D_λ of any parameter λ belonging to the same affine class. We may therefore write the geodesic equation as

$$D_\lambda \frac{dq^A}{d\lambda} = 0. \quad (2.17)$$

Bear in mind that applying a covariant derivative on q^A is meaningless, as it is not a tensor. The appropriate covariant vector is $\dot{q}^A$.

C. Hamiltonian formalism

The Eisenhart lift was originally formulated in the Lagrangian formalism. However, since our plan is to study its quantum analog, it behooves us to examine it in the Hamiltonian formalism as well. To this end, we proceed as usual and derive the Hamiltonian H of a system by means of the standard Legendre transform,

$$H = p_A \dot{q}^A - L, \quad (2.18)$$

where p_A is the conjugate momentum defined as

$$p_A \equiv \frac{\partial L}{\partial \dot{q}^A}. \quad (2.19)$$

Therefore, the Hamiltonian corresponding to (2.1) is

$$H = \sum_{i=1}^d \frac{p_i^2}{2m} + V(x), \quad (2.20)$$

while the one pertinent to the lifted Lagrangian (2.4) is

$$H_{\text{lift}} = \sum_{i=1}^d \frac{p_i^2}{2m} + \frac{V(x)}{M^2} \frac{p_y^2}{2}. \quad (2.21)$$

Here, $p_i = m\dot{x}_i$ and $p_y = M^2\dot{y}/V$. Note that when $V = 0$, the original and lifted Hamiltonians coincide, which makes the free limit of Eisenhart lift more apparent in the Hamiltonian rather than in the Lagrangian formalism.

We now use Hamilton's equations, which may be written in a manifestly covariant manner as follows:

$$\dot{q}^A = \frac{\partial H}{\partial p_A}, \quad D_t p_A \equiv \dot{p}_A - \Gamma_{AB}^C \dot{q}^B p_C = -\frac{\partial U}{\partial q^A}, \quad (2.22)$$

where

$$U = U(q^A) = H|_{p^A=0} \quad (2.23)$$

is the generalized potential of the system.

To derive the second of Hamilton's equation in (2.22), we have employed a covariant expression for the Euler-Lagrange equation of motion, i.e.,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = D_t \left(\frac{\partial L}{\partial \dot{q}^A} \right) + \frac{\partial U}{\partial q^A} = 0, \quad (2.24)$$

as well as taking into account the metric compatibility of the covariant time derivative, i.e., $D_t G_{AB} = 0$. Applied to the Hamiltonian (2.20) we find

$$\dot{x}_i = p_i/m, \quad \dot{p}_i = -mV_{,i}, \quad (2.25)$$

where the subscript “ i ” implies differentiation with respect to x_i .

For the lifted Hamiltonian (2.21), on the other hand, Hamilton's equations read

$$\dot{x}_i = p_i/m, \quad \dot{p}_i + M^2 m V_{,i} \frac{p_y^2}{2} = 0, \quad (2.26)$$

$$\dot{y} = p_y V/M^2, \quad \dot{p}_y = 0, \quad (2.27)$$

with $i \neq y$. Note that for the lifted system, $U = 0$, since the dynamics are entirely captured by the kinetic term.

We observe that the lifted momentum $p_y = AM$ is a constant of motion. Physically, this is because there is no effective force in the y direction, since the Hamiltonian has no explicit y dependence. As before, if we choose initial conditions such that the constant A satisfies (2.8), then the lifted Hamiltonian equations of motion in (2.26) reduce completely to the original ones stated in (2.25).

It is worth noting that substituting the equation of motion for the fictitious coordinate in the lifted Hamiltonian (2.21) yields the original Hamiltonian (2.20). This is in contradistinction to the equivalent procedure for the original and lifted Lagrangians (2.1) and (2.4).

III. THE EISENHART LIFT IN QUANTUM MECHANICS

Now that we have an understanding of how the Eisenhart lift works in classical mechanics, let us turn our attention to its quantum aspects.

We have been calling the additional dimension in the lifted space *fictitious*. This term suggests that it is impossible to probe this dimension by any measurement. Indeed, if we live in a subspace of a higher-dimensional manifold, the only way that a theory and its lifted counterpart are physically equivalent is if we cannot observe motion in the y direction.

In the classical realm, this condition must be imposed “by hand.” After all, there is no inherent reason why y and p_y are less “physical” than x_i and p_i are.

In QM, however, it is possible to distinguish between compatible and incompatible observables, i.e., between observables whose measurements will or will not affect each other. Therefore, if we can show that the fictitious position and momentum are compatible with all nonlifted observables, we may then be assured that it is not possible to directly probe them.

A. Poisson brackets and commutator algebra

Before quantizing the lifted theory, we need to examine the Poisson brackets of the classical theory.

Consider a general theory of a system whose field space is parametrized by the canonical coordinates (q^A, p_A) , where $q^A = q^A(t)$ and $p_A = p_A(t)$. The dynamics of the system is governed by the Hamiltonian

$$H(q, p) = \frac{1}{2} G^{AB}(q) p_A p_B + V(q), \quad (3.1)$$

where the metric and potential have a general dependence on $q = (q_1, q_2, \dots)$. Note that by the definition (2.19), p_A is a proper vector. However q^A is *not* a covariant vector and should instead be treated as a coordinate. In general, the (inverse) field space metric G^{AB} can be explicitly deduced from the Hamiltonian as follows:

$$G^{AB} = \frac{\partial^2 H}{\partial p_A \partial p_B}. \quad (3.2)$$

Note that this definition of G^{AB} enables one to identify metrics for Hamiltonians that are not necessarily quadratic in p_A .

Let us write down the Poisson brackets for the system

$$\{f, g\} = \frac{\partial f}{\partial q^A} \frac{\partial g}{\partial p_A} - \frac{\partial f}{\partial p_A} \frac{\partial g}{\partial q^A}. \quad (3.3)$$

It is important to remark here that the usual Poisson brackets as defined in (3.3) are not covariant. If f and g are

tensors in the field space, their transformation properties will not be inherited by the Poisson bracket $\{f, g\}$. This means that after quantization, the operators and their commutators will also be noncovariant.

However, this will not affect any physical observables, since these are independent of parametrization (covariant or not) [16]. Therefore, for the sake of convenience, we will adopt the standard, chart-dependent formalism for the rest of the paper even though, in doing so, we will unavoidably lose manifest covariance. A discussion of how the Poisson brackets may be covariantized is presented in the Appendix.

We now turn our attention to the classical system with Hamiltonian (2.20). In order to simplify the discussion, we shall consider the case $d = 1$, so that the original system contains only one coordinate $x_1 \equiv x$ and one momentum $p_1 \equiv p_x$. After applying the Eisenhart lift, these are extended by a new fictitious coordinate y and momentum p_y .

We can apply (3.3) to the lifted system (2.21) to find the following Poisson brackets:

$$\begin{aligned} \{x, x\} &= \{y, y\} = \{x, y\} = 0, \\ \{p_x, p_x\} &= \{p_y, p_y\} = \{p_x, p_y\} = 0, \\ \{x, p_x\} &= \{y, p_y\} = 1, \\ \{x, p_y\} &= \{y, p_x\} = 0. \end{aligned} \quad (3.4)$$

In order to quantize the theory, we proceed as usual and promote the coordinates x, y, p_x , and p_y to operators $\hat{x}, \hat{y}, \hat{p}_x$, and $\hat{p}_y$. Thus, the original Hamiltonian (2.20) and the lifted Hamiltonian (2.21) become the operator-valued expressions

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \hat{V}, \quad (3.5)$$

$$\hat{H}_{\text{lift}} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2M^2} \hat{V}, \quad (3.6)$$

where the hats denote operators and $\hat{V} \equiv V(\hat{x})$.

Following the canonical quantization procedure, we promote the Poisson brackets (3.4), yielding

$$[\hat{x}, \hat{p}_x] = i, \quad [\hat{y}, \hat{p}_y] = i, \quad (3.7)$$

with all other commutators vanishing.

We notice that any operator constructed out of $\hat{x}$ and $\hat{p}_x$ commutes with any operator constructed out of $\hat{y}$ and $\hat{p}_y$. This indicates that fictitious and nonfictitious observables are completely compatible, and the fictitious subspace of a lifted manifold cannot be probed by observers living on the ordinary (nonfictitious) subspace.

Finally, we note that we do not run into any ordering issues with $\hat{V}$ and $\hat{p}_y$, since $[\hat{V}, \hat{p}_y] = 0$. This holds for any potential operator $V(\hat{x})$. Thus, no matter what functional

form the potential term assumes, there will be no kinetic mixing between $\hat{p}_x$ and $\hat{p}_y$.

B. The lifted Schrödinger equation

We now turn our attention to the time-independent Schrödinger equation (TISE). For the lifted system described by the Hamiltonian (3.6), this is given by

$$-\frac{\Psi_{,xx}}{2m} - \frac{V(x)\Psi_{,yy}}{2M^2} = E\Psi, \quad (3.8)$$

where $\Psi = \Psi(x, y)$ is the two-dimensional lifted wave function defined as $\Psi = \langle x, y | \Psi \rangle$ and the subscripts “ $,x$ ” and “ $,y$ ” indicate partial differentiation with respect to x and y , respectively. In writing down the TISE (3.8), we have used the fact that

$$\langle x | \hat{p}_y V(\hat{x}) | \Psi \rangle = \langle x | V(\hat{x}) \hat{p}_y | \Psi \rangle = -iV(x)\Psi_{,y}, \quad (3.9)$$

for the state vector $|\Psi\rangle$.

In order to solve (3.8) for the lifted wave function, we make the ansatz

$$\Psi(x, y) = \psi(x)\chi(y), \quad (3.10)$$

giving rise to a set of equations that are fully separable,

$$\left(\frac{\psi_{,xx}}{2m} + E\psi \right) \frac{M^2}{V(x)\psi} = -\frac{\chi_{,yy}}{2\chi} = P^2, \quad (3.11)$$

where $P = AM$ is a constant with units of momentum.

We immediately see that $\chi(y)$ is a pure momentum mode, as expected due to the shift symmetry of $\hat{H}_{\text{lift}}$ in y . It can therefore be written as

$$\chi(y) = e^{iPy}, \quad (3.12)$$

where we have absorbed an overall constant into the normalization for $\psi(x)$. This shows us that the constant P is, in fact, the momentum in the fictitious direction.

We can substitute this result to (3.11) and we find

$$-\frac{\psi_{,xx}}{2m} + \frac{P^2}{2M^2} V(x)\psi = E\psi. \quad (3.13)$$

Interestingly enough, if we choose a state satisfying (2.8) so that $P = \sqrt{2}M$, (3.13) becomes precisely the Schrödinger equation for the original system. In this case, the state ψ will represent an energy eigenstate of the original Hamiltonian (2.20).

The lifted wave function Ψ should be normalized so that

$$\int dx dy \Psi^* \Psi = \int dy \int dx \psi^* \psi = 1. \quad (3.14)$$

However, this equation cannot be satisfied if the fictitious coordinate is allowed to run over the entire real line. We therefore choose to compactify the fictitious direction y , restricting it to lie in the range $y \in [0, \ell)$, for some compactification length ℓ . This has the advantage of restricting the possibilities for the fictitious momentum to a set of discrete values

$$P = \frac{2\pi k}{\ell}, \quad (3.15)$$

where $k = 0, \pm 1, \pm 2, \dots$ is an additional quantum number.

We notice that, for the special case $k = 0$, the effect of the potential $V(x)$ in (3.13) vanishes. In this case, the solutions for $\psi(x)$ are just waves in the observable x direction. Instead, if $V(x)$ is nonzero, we must insist that $k \neq 0$, so the lowest energy state will be obtained for $k = \pm 1$. Indeed, without loss of generality, we can always choose our compactification scale ℓ , such that $k = \pm 1$ corresponds to $P^2 = 2M^2$ and so satisfies the Eisenhart condition (2.8).

In summary, solutions to the lifted TISE (3.13) are described by two quantum numbers: (i) the quantum number k of the conserved momentum p_y , and (ii) the quantum number n of a system with quantized energy eigenstates $E_{k,n}$ that result from a rescaled potential term, $V_k(x) \equiv k^2 V(x)$. Consequently, the states that satisfy the lifted TISE (3.13) with nonvanishing potential $V(x)$ can be written as

$$\Psi_{k,n}(x, y) = e^{2\pi i k y / \ell} \psi_{k,n}(x), \quad (3.16)$$

where

$$\psi_{k,n}(x) = \psi_n\left(\frac{x}{k}\right) \quad (3.17)$$

is the n th energy eigenstate resulting from the rescaled potential $V_k(x)$, and ψ_n is the n th energy eigenstate of the original system. Notice that, as explained above, one has $k \neq 0$, and so the latter expression is well defined.

C. The lifted harmonic oscillator

Let us examine closer the spectrum of the harmonic oscillator, which is described by the lifted Hamiltonian

$$\hat{H}_{\text{lift}} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{x}^2 \hat{p}_y^2}{2\mu}, \quad (3.18)$$

where $\mu = 2M^2/(m\omega^2)$. Solving the lifted TISE (3.8) for this Hamiltonian, we see that the energy spectrum of the lifted harmonic oscillator is

$$E_{k,n} = \frac{2\pi k \omega}{M\ell} \left(n + \frac{1}{2} \right), \quad (3.19)$$

with the restriction of energy positivity: $k > 0$. As discussed above, the compactification length ℓ can be chosen to be $\ell = 2\pi/M$, so that for $k = 1$ we have the standard spectrum of the harmonic oscillator.

Let us now analyze the lifted ladder operators, which will prove instructive when we generalize to QFT. The ladder operator approach is a particularly intuitive way of deriving the spectrum of a quadratic theory, and it will be helpful to see how it extends to the lifted case. We remind the reader that in the standard case, the ladder operators for the 1D quantum simple harmonic oscillator are given by

$$a \equiv \sqrt{\frac{m\omega}{2}} \left(\hat{x} + \frac{i\hat{p}_x}{m\omega} \right), \quad (3.20)$$

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2}} \left(\hat{x} - \frac{i\hat{p}_x}{m\omega} \right), \quad (3.21)$$

where we have suppressed the hats for a and $a^\dagger$.

A feature of the ladder operators is that the Hamiltonian takes on the simple quadratic form: $\hat{H} = \omega(a^\dagger a + \frac{1}{2})$. By analogy, we expect for the lifted ladder operators α and $\alpha^\dagger$ to have a similar relationship with $\hat{H}_{\text{lift}}$. This consideration prompts us to define the lifted ladders operators as follows:

$$\alpha \equiv \sqrt{\frac{m\omega}{2}} \left(\frac{\hat{x}\hat{p}_y}{\sqrt{2}M} + \frac{i\hat{p}_x}{m\omega} \right), \quad (3.22)$$

$$\alpha^\dagger \equiv \sqrt{\frac{m\omega}{2}} \left(\frac{\hat{x}\hat{p}_y}{\sqrt{2}M} - \frac{i\hat{p}_x}{m\omega} \right), \quad (3.23)$$

for which the lifted Hamiltonian can be written as

$$\hat{H}_{\text{lift}} = \omega \left(\alpha^\dagger \alpha + \frac{\hat{p}_y}{2\sqrt{2}M} \right). \quad (3.24)$$

The lifted ladder operators obey the following commutation relation:

$$[\alpha, \alpha^\dagger] = \frac{\hat{p}_y}{\sqrt{2}M}. \quad (3.25)$$

We notice that, when acting on an eigenstate of $\hat{p}_y$, the ladder operators will assume their usual role with $\alpha^\dagger$ and α adding and subtracting a quantum of energy to the system, respectively. Looking at the lifted Hamiltonian (3.24), we observe that the eigenvalue of the fictitious momentum operator $\hat{p}_y$ sets the ground state energy for the simple harmonic oscillator. In addition, the commutation relation (3.25) tells us that this eigenvalue also sets the separation between excited states of the harmonic oscillator.

If we apply these operators to an eigenstate of $\hat{p}_y$ with eigenvalue $p_y = \sqrt{2}M$, i.e., a $k = 1$ state, we will recover the standard relations for the quantum harmonic oscillator.

Any other eigenstate will shift the ground state energy, as well as the energies of all excited states, by a fixed amount.

Let us understand what this scaling of energy means, by considering the time dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle = \sum_i E_i \alpha_i |\Psi_i(t)\rangle. \quad (3.26)$$

Here, $|\Psi(t)\rangle$ is a general state that has been expanded in the energy eigenbasis

$$|\Psi(t)\rangle = \sum_i \alpha_i |\Psi_i(t)\rangle, \quad (3.27)$$

where Ψ_i is an energy eigenstate with eigenvalue E_i and α_i is a coefficient.

We see from (3.26) that we can compensate for a rescaling of energy by rescaling the time coordinate t . Therefore, just as in the classical case, a state not satisfying (2.8) corresponds to a system evolving in fast forward or slow motion.

Notice that

$$[\hat{H}_{\text{lift}}, \hat{p}_y] = 0, \quad (3.28)$$

and, hence the eigenvalue p_y is a good quantum number. Therefore, a system prepared in an eigenstate of $\hat{p}_y$ will remain in that state. This is important, because it means that there is no danger of tunneling to a state with $A \neq \sqrt{2}$ once the time parameter has been rescaled appropriately. Hence, the lifted system will evolve identically to the original system at both the classical and quantum levels.

We are also able to introduce ladder operators for the fictitious y momentum, which raise or lower the quantum number k . These are defined

$$\mathfrak{b} = e^{-2\pi i \hat{y}/\ell} \frac{\hat{p}_y}{\sqrt{2M}} = \frac{1}{\sqrt{2M}} \left(\frac{2\pi}{\ell} + \hat{p}_y \right) e^{-2\pi i \hat{y}/\ell}, \quad (3.29)$$

$$\mathfrak{b}^\dagger = \frac{\hat{p}_y}{\sqrt{2M}} e^{2\pi i \hat{y}/\ell} = \frac{1}{\sqrt{2M}} e^{2\pi i \hat{y}/\ell} \left(\frac{2\pi}{\ell} + \hat{p}_y \right), \quad (3.30)$$

where we note that $e^{2\pi i \hat{y}/\ell}$ is an operator-valued expression. The remaining commutators are

$$[\mathfrak{a}, \mathfrak{b}^\dagger] = -[\mathfrak{a}^\dagger, \mathfrak{b}] = \frac{\pi}{M^2 \ell} \sqrt{\frac{m\omega}{2}} \hat{x} \hat{p}_y e^{2\pi i \hat{y}/\ell}, \quad (3.31)$$

$$[\mathfrak{a}^\dagger, \mathfrak{b}^\dagger] = [\mathfrak{a}, \mathfrak{b}] = \frac{\pi}{M^2 \ell} \sqrt{\frac{m\omega}{2}} \hat{x} \hat{p}_y e^{-2\pi i \hat{y}/\ell}, \quad (3.32)$$

$$[\mathfrak{b}, \mathfrak{b}^\dagger] = \frac{\pi}{\ell M^2} \left(\frac{2\pi}{\ell} + \hat{p}_y \right). \quad (3.33)$$

We may now construct the lifted Hilbert space in which the states of the lifted harmonic oscillator live. We index states using their eigenvalues k and n as

$$|k, n\rangle = |k\rangle \otimes |n\rangle. \quad (3.34)$$

The lifted ladder operators $\mathfrak{a}^\dagger$, $\mathfrak{a}$ and $\mathfrak{b}^\dagger$, $\mathfrak{b}$, as defined above, act as follows on the states:

$$\mathfrak{a}^\dagger |k\rangle \otimes |n\rangle = k\sqrt{n+1} |k\rangle \otimes |n+1\rangle, \quad (3.35)$$

$$\mathfrak{a} |k\rangle \otimes |k\rangle = k\sqrt{n} |k\rangle \otimes |n-1\rangle, \quad (3.36)$$

and

$$\mathfrak{b}^\dagger |k\rangle \otimes |n\rangle = (k+1) |k+1\rangle \otimes |n\rangle, \quad (3.37)$$

$$\mathfrak{b} |k\rangle \otimes |n\rangle = k |k-1\rangle \otimes |n\rangle. \quad (3.38)$$

With the help of these relations, we may express arbitrary eigenstates in terms of the ground state by successively applying their creation operators as usual.

We note that the original ground state $n=0$ corresponds to a *class* of states $|k\rangle \otimes |0\rangle$, leading to an ensemble of Hilbert spaces indexed by different vacua. The most general state may be written as

$$|k, n\rangle \equiv |k\rangle \otimes |n\rangle = \frac{1}{k^n} \frac{(\mathfrak{a}^\dagger)^n (\mathfrak{b}^\dagger)^k}{\sqrt{n!} k!} |0\rangle \otimes |0\rangle, \quad (3.39)$$

where the vacua are defined as

$$\mathfrak{a} |k, 0\rangle = 0, \quad \mathfrak{b} |0, n\rangle = 0. \quad (3.40)$$

Note that our definition of Hilbert states in (3.39) ensures their orthonormality, i.e.,

$$\langle k', n' | k, n \rangle = \delta_{k'k} \delta_{n'n}. \quad (3.41)$$

This completes our treatment of the lifted harmonic oscillator. In the next section, we will demonstrate how the Eisenhart lift can be extended to QFTs.

IV. THE EISENHART LIFT IN QUANTUM FIELD THEORY

So far, we have discussed classical and quantum aspects of the Eisenhart lift for particles living in spacetime. However, as first shown in [10], the Eisenhart lift can also be applied to field theories. In this case it is not the spacetime manifold that is lifted, but rather the *field space*. Hence, the theory is lifted by introducing a new fictitious field (2.6).

Below we review how this is achieved for classical field theories before discussing the extension to QFT.

A. Classical field theory

As an illustrative archetype, consider the following Lagrangian (density):

$$\mathcal{L} = \frac{\partial_\mu \phi \partial^\mu \phi}{2} - V(\phi), \quad (4.1)$$

where all Lorentz indices are contracted with the Minkowski metric $\eta_{\mu\nu}$. The equation of motion for this theory is simply the generalized Klein-Gordon equation

$$\nabla^2 \phi + V'(\phi) = 0, \quad (4.2)$$

with $V'(\phi) \equiv dV(\phi)/d\phi$.

We can apply the Eisenhart lift as such a theory by introducing a new vector field B^μ . The lifted Lagrangian in this case is given by

$$\mathcal{L}_{\text{lift}} = \frac{\partial_\mu \phi \partial^\mu \phi}{2} + \frac{M^4}{V(\phi)} \frac{\partial_\mu B^\mu \partial_\nu B^\nu}{2}, \quad (4.3)$$

where once again the M^4 factor is used to keep dimensions consistent.

The equations of motion for the lifted Lagrangian (4.3) are

$$\partial^2 \phi = -\frac{M^4}{2} \frac{V'}{V^2} (\partial_\mu B^\mu)^2, \quad \partial_\mu \left(\frac{M^4}{V} \partial_\nu B^\nu \right) = 0. \quad (4.4)$$

We see that the second equation implies that

$$\frac{M^2 \partial_\mu B^\mu}{V(\phi)} = A, \quad (4.5)$$

where A is some (dimensionless) constant.

We note that when the fields depend on D spacetime dimensions, we require D conditions to ensure that A is truly a constant in both time and space. These D conditions are provided by the equations of motion for the D components of B^μ as shown in the second equation of (4.4). This tells us why B^μ must be a vector field, since it must contain D degrees of freedom.

Substituting (4.5) back into the first equation of (4.4) gives us

$$\partial^2 \phi + \frac{A^2}{2} V' = 0. \quad (4.6)$$

If we choose $A = \sqrt{2}$, we recover the original Klein-Gordon equation (4.2), exactly as we did in QM. Again, we may always satisfy this condition by rescaling the space-time coordinates. However, in the case of field theory, we must rescale not only time, but space as well. This rescaling leaves the lifted equations of motion (4.4) unchanged.

Let us now examine the above derivation in the Hamiltonian formalism. We note that the Hamiltonian approach to field theory necessarily breaks manifest covariance by singling out time. Thus, we should not be surprised that the following derivation is not covariant.

Nonetheless, we can be assured that it captures the same physics.

Our first step is to evaluate the canonical momenta for the fields through the standard relations,

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \quad \pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{B}^\mu}. \quad (4.7)$$

Normally, we would expect five canonical momenta in total, four of which correspond to the fictitious fields. However, in the case of the system described by the Lagrangian in (4.3), the three spatial components of B^μ are auxiliary fields with no kinetic term and so they have no canonical momentum. Indeed, substituting (4.3) into (4.7), we find

$$\pi_\phi = \dot{\phi}, \quad \pi_0 = \frac{M^4}{V(\phi)} \partial_\mu B^\mu, \quad \pi_i = 0, \quad (4.8)$$

with only one nonvanishing fictitious momentum. Nonetheless, in order to reproduce the correct equations of motion, we must refrain from setting $\pi_i = 0$ too early in our calculations, but only after we have derived Hamilton's equations.

Our next step is to determine the Hamiltonian (density) of the system by means of a Legendre transform: $\mathcal{H} = \pi_A \dot{\phi}^A - \mathcal{L}$. Using (4.8) in order to eliminate all time derivatives of ϕ and B^0 , we may write the Hamiltonian for the lifted system as

$$\mathcal{H}_{\text{lift}} = \frac{\pi_\phi^2 + (\nabla \phi)^2}{2} + \frac{V(\phi)}{M^4} \pi_0^2 + \pi_i \dot{B}^i - \pi_0 \partial_i B^i \quad (4.9)$$

where ∇ denotes the spatial three-derivative. Note that the term involving $\dot{B}^i$ cannot be eliminated, since it does not appear in the definition of any conjugate momenta. Moreover, even though we know that this term will vanish due to the condition $\pi_i = 0$, we must not remove it until after we apply Hamilton's equations.

The equations of motion can be derived from Hamilton's equations,

$$\dot{\phi}^A = \frac{\delta \mathcal{H}}{\delta \pi_A}, \quad D_i \pi_A = -\frac{\delta \mathcal{H}}{\delta \phi^A}, \quad (4.10)$$

which we have written in a manifestly covariant manner similar to (2.22) by now identifying the generalized potential *density* as

$$\mathcal{U}(q^A) = \mathcal{H}|_{\pi_A=0}, \quad (4.11)$$

which is an expression analogous to (2.23). The derivation of the second equation of (4.10) is similar to that of (2.24); namely, from the Euler-Lagrange equations, we can derive

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^A} \right) - \nabla_i \left(\frac{\partial \mathcal{L}}{\partial (\nabla_i \phi^A)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^A} \\ = D_t \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^A} \right) + \frac{\delta \mathcal{U}}{\delta \phi^A} = 0, \end{aligned} \quad (4.12)$$

where we must now take the functional derivative of the generalized potential so as to incorporate the gradient terms.

Using the equations of motion for the Hamiltonian (4.9), we find

$$\dot{\phi} = \pi_\phi, \quad \dot{\pi}_\phi + \frac{\pi_0^2 V'(\phi)}{2M^4} = \nabla^2 \phi, \quad (4.13)$$

$$\dot{B}^0 = \frac{V(\phi)}{M^4} \pi_0^2 - \partial_i B^i, \quad \dot{\pi}_0 = 0, \quad (4.14)$$

$$\dot{B}^i = \dot{B}^i, \quad \dot{\pi}_i + \partial_i \pi_0 = \dot{\pi}_i. \quad (4.15)$$

These equations can be rearranged into a more familiar form by substituting the first equation of (4.13) into the second one, giving

$$\partial^2 \phi = -\frac{1}{2} \pi_0^2 V'(\phi) = -\frac{1}{2} \left[\frac{\partial_\mu B^\mu}{V(\phi)} \right]^2 V'(\phi), \quad (4.16)$$

which is identical to (4.6). Rearranging the first equation of (4.14), we obtain

$$\pi_0 = \frac{M^4 \partial_\mu B^\mu}{V(\phi)}, \quad (4.17)$$

as expected. Finally, combining the second equation of (4.14) with the second one of (4.15) yields

$$\partial_\mu \pi_0 = 0. \quad (4.18)$$

Thus, we have

$$\pi_0 = \frac{M^4 \partial_\mu B^\mu}{V(\phi)} = AM^2, \quad (4.19)$$

where A is a constant in agreement with (4.5). Substituting this into (4.16) gives the equations of motion of the original system, provided $A^2 = 2$, exactly as we found before in the Lagrangian formalism.

We have seen that Hamilton's equations also reduce to the generalized Klein-Gordon equation. However, this derivation gives us more insight into the constant A . We see that, up to a rescaling, it is π_0 : the conjugate momentum of the fictitious degree of freedom B^0 .

B. Quantum field theory

Let us now quantize the above lifted theory in the canonical way, by promoting the fields and conjugate momenta to operators. These operators satisfy the following canonical (equal-time) commutation relations:

$$\begin{aligned} [\hat{\phi}(x), \hat{\pi}_\phi(y)] &= i\delta^{(3)}(x-y), \\ [\hat{B}^\mu(x), \hat{\pi}_\nu(y)] &= i\delta_\nu^\mu \delta^{(3)}(x-y), \\ [\hat{\phi}(x), \hat{\pi}_\mu(y)] &= [\hat{B}^\mu(x), \hat{\pi}_\phi(y)] = 0, \end{aligned} \quad (4.20)$$

with all other commutators vanishing.

For illustration, let us consider a free theory with a Hamiltonian,

$$H = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2. \quad (4.21)$$

The equations of motion for this theory are given by

$$\partial^2 \phi + m^2 \phi = 0. \quad (4.22)$$

Therefore, this QFT can be viewed as a simple harmonic oscillator at every point in space.

To highlight this interpretation, we perform the Fourier transform of the above operators

$$\hat{\phi}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\phi}_{\mathbf{k}}. \quad (4.23)$$

Here, we work in the Heisenberg picture in which the Fourier components have no time dependence. Substituting these into the Hamiltonian (4.21) gives

$$\hat{H} = \frac{1}{2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [\dot{\hat{\phi}}_{\mathbf{k}}^2 + (\mathbf{k}^2 + m^2) \hat{\phi}_{\mathbf{k}}^2]. \quad (4.24)$$

The equations of motion for (4.24) give us the dispersion relation

$$\ddot{\hat{\phi}}_{\mathbf{k}} + (\mathbf{k}^2 + m^2) \hat{\phi}_{\mathbf{k}} = 0. \quad (4.25)$$

Thus, we see that each Fourier mode oscillates as a simple harmonic oscillator with unit mass and frequency $\omega_{\mathbf{k}} = \mathbf{k}^2 + m^2$.

Proceeding as in QM, we can define creation and annihilation operators

$$a_{\mathbf{k}} = \int d^3 \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \left[\sqrt{\frac{\omega_{\mathbf{k}}}{2}} \hat{\phi}(\mathbf{x}) + \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} \hat{\pi}(\mathbf{x}) \right], \quad (4.26)$$

$$a_{\mathbf{k}}^\dagger = \int d^3 \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \left[\sqrt{\frac{\omega_{\mathbf{k}}}{2}} \hat{\phi}(\mathbf{x}) - \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} \hat{\pi}(\mathbf{x}) \right], \quad (4.27)$$

where we suppress hats for $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$. The canonical commutation relations for $\hat{\phi}$ and $\hat{\pi}$ imply that the creation

and annihilation operators obey the following commutation relations:

$$[a_{\mathbf{k}}, a_{\mathbf{q}}] = [a_{\mathbf{k}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \quad (4.28)$$

$$[a_{\mathbf{k}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{q}). \quad (4.29)$$

With $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ thus defined, we can rewrite the Hamiltonian in the form

$$\hat{H} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} [a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}]. \quad (4.30)$$

Note that the usual divergence term $\delta^{(3)}(0)$ resulting from the commutator (4.29) can be taken care of with the standard normal ordering.

We are now in the position to apply the Eisenhart lift to this free theory. We see from (4.9) that, classically, this gives us the lifted Hamiltonian

$$\mathcal{H}_{\text{lift}} = \frac{\pi_\phi^2 + (\nabla \phi)^2}{2} + \frac{m^2 \phi^2}{2M^4} \pi_0^2 + \pi_i \dot{B}^i - \pi_0 \partial_i B^i. \quad (4.31)$$

Let us now quantize this theory using the framework of canonical quantisation. Taking our inspiration from the lifted quantum harmonic oscillator in Sec. III, we define the lifted creation and annihilation operators

$$\mathbf{a}_{\mathbf{k}} = \int d^3 \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \left[\frac{\hat{\pi}_0(\mathbf{x})}{\sqrt{2M^2}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \hat{\phi}(\mathbf{x}) + \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} \hat{\pi}_\phi(\mathbf{x}) \right], \quad (4.32)$$

$$\mathbf{a}_{\mathbf{k}}^\dagger = \int d^3 \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \left[\frac{\hat{\pi}_0(\mathbf{x})}{\sqrt{2M^2}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \hat{\phi}(\mathbf{x}) - \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} \hat{\pi}_\phi(\mathbf{x}) \right]. \quad (4.33)$$

This allows us to rewrite the lifted Hamiltonian in the form

$$\hat{H}_{\text{lift}} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} [\mathbf{a}_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger + \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}}]. \quad (4.34)$$

Observe that the form for $\hat{H}_{\text{lift}}$ is analogous to (3.24) in QM. We note that the last two terms in (4.31) do not appear because they vanish once integrated over.

Upon normal ordering of the Hamiltonian where all creation operators occur before annihilation operators, we have

$$:\hat{H}_{\text{lift}} := \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}}. \quad (4.35)$$

The following commutation relations may then be derived:

$$[\mathbf{a}_{\mathbf{k}}, \mathbf{a}_{\mathbf{q}}^\dagger] = \frac{\hat{\pi}_{0\mathbf{k}}}{\sqrt{2M^2}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{q}), \quad (4.36)$$

$$[:\hat{H}_{\text{lift}} :, \mathbf{a}_{\mathbf{k}}^\dagger] = \frac{\hat{\pi}_{0\mathbf{k}}}{\sqrt{2M^2}} \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger, \quad (4.37)$$

$$[:\hat{H}_{\text{lift}} :, \mathbf{a}_{\mathbf{k}}] = -\frac{\hat{\pi}_{0\mathbf{k}}}{\sqrt{2M^2}} \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}. \quad (4.38)$$

We note that if we are in an eigenstate of $\hat{\pi}_0$ with the same eigenvalue $\pi_0 = \sqrt{2M^2}$ for all wave numbers, these commutation relations reduce to their standard expressions (4.28) and (4.29).

To fully understand the states of the lifted system, we calculate the four-momentum operator $\hat{P}^\mu$. To this end, we first derive the classical four-momentum P^μ from the canonical energy-momentum tensor,

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)^A} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}. \quad (4.39)$$

Spacetime translational invariance of the original Lagrangian implies that the four-momentum,

$$P^\mu = \int d^3 x T^{0\mu}, \quad (4.40)$$

is a conserved Noether's charge.

For the lifted system (4.31), the energy-momentum tensor is found to be

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + \frac{1}{2} \frac{2M^4}{m^2 \phi^2} (\partial_\rho B^\rho) (\partial^\mu B^\nu + \partial^\nu B^\mu) - \eta^{\mu\nu} \left[\frac{(\partial \phi)^2}{2} + \frac{2M^4}{m^2 \phi^2} \frac{(\partial_\rho B^\rho)^2}{2} \right]. \quad (4.41)$$

In particular, the elements that contribute to the four momentum are

$$T^{00} = \frac{\dot{\phi}^2 + (\nabla \phi)^2}{2} + \frac{2M^4}{m^2 \phi^2} (\partial_\rho B^\rho)^2, \quad (4.42)$$

$$T^{0i} = \dot{\phi} \partial^i \phi + \frac{1}{2} \frac{2M^4}{m^2 \phi^2} (\partial_\rho B^\rho) (\dot{B}^i + \partial^i B^0). \quad (4.43)$$

After promoting the fields to operators, these become

$$\hat{T}^{00} = \frac{\hat{\pi}_\phi^2 + (\nabla \hat{\phi})^2}{2} + \frac{m^2 \hat{\phi}^2}{2M^4} \hat{\pi}_0^2, \quad (4.44)$$

$$\hat{T}^{0i} = \hat{\pi}_\phi \partial^i \hat{\phi} + \frac{1}{2} \hat{\pi}_0 (\hat{\pi}_i + \partial^i \hat{B}^0). \quad (4.45)$$

Integrating these expressions over the spatial volume, we obtain the Noether-charge operators:

$$\hat{P}^0 = \hat{H} = \int d^3x \hat{H}, \quad (4.46)$$

$$\hat{P}^i = \int d^3x \left[\hat{\pi}_\phi \partial^i \phi + \frac{1}{2} \hat{\pi}_0 (\hat{\pi}^i + \partial^i \hat{B}^0) \right]. \quad (4.47)$$

By employing the commutation relation

$$[\hat{\pi}_0, \partial^i \hat{B}^0] = \partial^i [\hat{\pi}_0, \hat{B}^0] - [\partial^i \hat{\pi}_0, \hat{B}^0] = 0, \quad (4.48)$$

which can be deduced thanks to (4.18), we find that

$$[\hat{H}, \hat{\pi}_0] = 0, \quad (4.49)$$

$$[\hat{P}^i, \hat{\pi}_0] = 0. \quad (4.50)$$

The vanishing of the commutator in (4.49) implies that the eigenstates of the operator $\hat{\pi}_0$ are compatible with those of the Hamiltonian, and so its eigenvalue π_0 represents good quantum number, which is conserved in time.

In addition, the vanishing of the commutator (4.50) implies that this quantum number is also the same throughout all space. Thus, the eigenvalue π_0 , represents a global quantum number that is the same everywhere in the Universe.

This conserved charge can be interpreted in two ways: either as the definition of our system of (energy) units if we choose to assume a fixed particle mass, or as the mass of the particle if we assume a fixed system of units. In the latter case, jumping to an excited state of $\hat{\pi}_0$ would be interpreted as a change in the mass of the ϕ particle. However (4.49) and (4.50) ensure that once π_0 is set by some initial conditions, its value will remain constant throughout space and time. Therefore, the spectrum of the theory will not suddenly change as the system evolves.

We can now examine the Fock space of the lifted theory. In the standard case, the Fock space is the direct sum of tensor products of copies of the same Hilbert space. A general state in the original Fock space can be written in slightly expanded notation as

$$|\psi\rangle = \sum_{\pi_0} |(n_1)_{\mathbf{k}_1}, (n_2)_{\mathbf{k}_2}, \dots\rangle, \quad (4.51)$$

denoting n_i quanta of ϕ all with momentum $\mathbf{k}_i$.

Once the system is lifted, however, the Fock space of the theory is now extended to an ensemble of Fock spaces, indexed by the value of π_0 . This is similar to the case we saw in QM. Explicitly, an eigenstate of the system can be written

$$|\Psi\rangle_{\pi_0} = |\pi_0\rangle \otimes |(n_1)_{\mathbf{k}_1}, (n_2)_{\mathbf{k}_2}, \dots\rangle, \quad (4.52)$$

where

$$|\pi_0\rangle \otimes |(n_1)_{\mathbf{k}_1}, (n_2)_{\mathbf{k}_2}, \dots\rangle = \frac{(\mathbf{a}_{\mathbf{k}_1}^\dagger)^{n_{\mathbf{k}_1}} (\mathbf{a}_{\mathbf{k}_2}^\dagger)^{n_{\mathbf{k}_2}}}{\sqrt{n_{\mathbf{k}_1}!} \sqrt{n_{\mathbf{k}_2}!}} \dots \times |\pi_0\rangle \otimes |0_{\mathbf{k}_1}, 0_{\mathbf{k}_2}, \dots\rangle, \quad (4.53)$$

and the vacua are defined so that

$$a_{\mathbf{k}} |\pi_0\rangle \otimes |0_{\mathbf{k}_1}, 0_{\mathbf{k}_2}, \dots\rangle = 0, \quad \forall \mathbf{k}. \quad (4.54)$$

These vacua are orthogonal, since the ladder operators commute with $\hat{\pi}_0$. Thus, the eigenvalue π_0 labels an ensemble of independent Fock spaces that are completely separate from one another. Note that

$$\hat{\pi}_0 |\Psi\rangle_{\pi_0} = \pi_0 |\Psi\rangle_{\pi_0} \quad (4.55)$$

and so $|\Psi\rangle_{\pi_0}$ is an eigenstate of the operator $\hat{\pi}_0$.

The most general state can be written as a superposition of states in different Fock spaces:

$$|\Psi\rangle = \sum_{\pi_0} \alpha_{\pi_0} |\Psi\rangle_{\pi_0}. \quad (4.56)$$

Thanks to the orthogonality of the Fock spaces, any observer will only have access to a single slice of this state, corresponding to a single value of π_0 . Different observers will measure different values of π_0 and, hence, different values for the mass of ϕ , but these observers will never be able to communicate with one another.

Therefore, we can interpret the state (4.56) as a superposition of different “universes” with different masses for the particles, all of which evolve completely independently. We happen to live within one such universe, and will forever stay within it, since the states are orthogonal.

V. DYNAMICAL GENERATION OF HIERARCHIES VIA THE EISENHART LIFT

The gauge hierarchy problem [17,18] and the cosmological constant problem [19] are two of the most puzzling aspects of modern particle physics and indicate an extreme level of *fine-tuning* in the Standard Model. In absence of a mechanism that can explain how they arise, a very delicate balance must be struck such that radiative corrections cancel out to match measurements. This is compounded by the fact that such cancellations are sensitive to short-scale physics that remain unprobed. These issues do not necessarily indicate an underlying problem with our current theories, but rather are signs pointing to the existence of a more complete theory.

There have been many attempts to resolve the gauge hierarchy and the cosmological constant problems. The most prominent of these involve theories beyond the Standard Model such as supersymmetry [20]. However, recent observational bounds on supersymmetric parameters

lead to a similar level of fine-tuning, at least as far as the cosmological constant problem is concerned [21].

A possible way to evade the cosmological constant problem is by dynamical generation of the cosmological constant, such as via quintessence models, in which a scalar field takes on the role of dark energy [22–24]. In these models, the density of the field eventually dominates the expansion rate of the Universe by rolling down a potential [25]. Even then, such models cannot fully evade fine-tuning: careful parameter matching is required in order to generate the low scale of the vacuum energy observed today [26].

Another method to resolve fine-tuning issues comes from the anthropic principle. Fine-tuning was originally identified by Dicke in relation to the age of the Universe [27], who noted that living observers must necessarily observe the age of the universe to within a window of opportunity that is neither too young nor too old. This argument was codified in the so-called *anthropic principle* [28] and used as a heuristic to argue that physical and cosmological quantities are necessarily biased towards values that can support intelligent observers. This provides an explanation for the apparent “cosmic coincidences” of our universe (such as the present gauge hierarchy and the small value of the cosmological constant); if these coincidences did not exist, we would not be here to observe them.

While the anthropic principle has often been criticized as a truism, it becomes much more useful when coupled with a landscape of theories, some of which can support observers and some of which cannot, and from which a particular universe can be anthropically selected. An example of this is the *multiverse* [29,30], which provides such a landscape. Even then, anthropic arguments remain controversial, since they offer no new physical insights [31] and because assigning statistical weights to different universes can be an ambiguous procedure [32,33].

The Eisenhart lift may provide an alternative solution to both the gauge hierarchy and the cosmological constant problems without reference to short-scale physics. The gauge parameters, instead of being imposed *ad hoc*, can now be thought of as part of the geometry of the field space of our theory. By absorbing the offending quadratic divergence terms into the metric, it becomes possible to tame them by choosing an appropriate subspace. This completely eliminates this kind of fine-tuning since, as discussed in the previous subsection, the extended Fock space ensemble incorporates all possible particle masses.

The extended Fock space can incorporate all possible spectra of different particles simply by adding more fictitious fields. As a result, the observed hierarchy depends wholly on which slice we live on. This solves the issue of finely tuned parameters in dynamical models.

We can also describe a truly constant vacuum energy using the Eisenhart lift, resulting in a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{2M_P^2} \nabla_\mu B^\mu \nabla_\nu B^\nu, \quad (5.1)$$

where $\mathcal{L}_{\text{SM}}$ is the Lagrangian of the Standard Model (minus the cosmological constant term), M_P is the Planck mass, and B^μ is the fictitious field. The cosmological constant problem would then be solved if we lived on a slice with $\pi_0 \sim 10^{-60} M_P$.

Alternatively, we may live in a slice where $\pi_0 = 0$, leading to an exactly vanishing cosmological constant. The small observed value could then be generated by suppressed anomalous tunneling effects between slices.

The Eisenhart lift resolves fine-tuning issues in a way that combines both anthropic considerations and the geometry of the field space. In fact, the Fock space ensemble is closely analogous to the multiverse, in that they both allow for the realization of a landscape of theories. The crucial difference is that the multiverse spans the entirety of spacetime, while the ensemble spans the entirety of possible QFTs. As with any anthropic argument, we must necessarily contend with the difficulties of assigning an indifferent prior to an infinite landscape. However, even in the face of this issue, the Eisenhart lift provides a novel avenue for dealing with problems of fine-tuning.

VI. DISCUSSION

In this work, we have studied the Eisenhart lift and its applications to quantum theory. By reformulating the lift in the Hamiltonian formalism, we have been able to apply it to QM and QFT. We have shown that the introduction of additional, “fictitious” degrees of freedom, which replicate the effects of a conservative force in a purely geometric manner in the classical case, also replicate the observables of the corresponding quantum theory. This applies both when quantizing particles in QM and when quantizing fields in QFT.

In order for the Eisenhart lift to fully reproduce the equations of motion of the original theory, we must impose a particular value for the fictitious momentum. This degree of freedom stems from the simple fact that the lifted manifold is higher dimensional. However, it does not spoil the correspondence between a system and its lifted counterpart. This is because geodesics in the lifted manifold are affinely reparametrizable, and so different choices of the fictitious momentum still lead to the same trajectories in the original manifold, with the only difference being a different scaling of time or, more generally, choice of units. This is reflected in the lifted quantum mechanical system (2.4), where different fictitious momenta correspond to different masses for the particle. Therefore, an Eisenhart-lifted system will always correspond to an *equivalence class* of systems.

In QM, the additional degrees of freedom introduced by the Eisenhart lift is reflected in the choice of the vacuum. Quantizing the lifted theory results in the same results as the original theory (with the same probability distributions

and spectrum) only if the vacuum is appropriately selected. However, different eigenstates of the fictitious momentum evolve separately, which means that once again, the lifted manifold gives rise to an equivalence class of Hilbert spaces that differ in their zero-point energy. Similarly, in QFT, the lifted manifold corresponds to an equivalence class of Fock spaces, with different values for the mass of the original particle. States that belong in different subspaces of the equivalence class evolve separately. Therefore, observers in the real world effectively live in one of the “slices” of the lifted Hilbert space.

We have focused on the simple harmonic oscillator in QM and free particles in QFT, since the ladder operator formalism makes their study somewhat easier. However, the Eisenhart lift is not restricted to particular potentials. In the spirit of perturbation theory, we may use the free theory as a starting point and slightly deform it by a small potential term to find the relevant observables. Iterating this procedure will allow us to approximate any potential we like. Using this technique, it is in principle possible to derive expressions for the ladder operators of any potential [34,35].

With the development of the quantum formulation of the Eisenhart lift, the scope of possible applications is considerably expanded. We have seen how the dynamic generation of hierarchies offered by the Eisenhart lift may provide a novel way to tackle the gauge hierarchy and cosmological constant problems. It would then be interesting to examine how symmetries and gauge fields translate to a lifted theory, paving the way to a complete geometrical interpretation of the Standard Model.

Finally, our formulation has focused on scalar fields, but we expect the Eisenhart lift to be applicable to tensors [36] as well as spinors [37], possibly paving the way to study not only the Standard Model through the lens of geometry, but also providing a fundamentally different insight to our attempts to quantize gravity.

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APPENDIX: COVARIANT POISSON BRACKETS

In order to arrive at a covariant definition for Poisson brackets in curved space we try promoting the ordinary derivatives with respect to the coordinates to covariant derivatives. This leads us to define

$$\{f, g\}_G = \nabla_A f \frac{\partial g}{\partial p_A} - \frac{\partial f}{\partial p_A} \nabla_A g. \quad (\text{A1})$$

We have used the subscript G to remind us that this definition of the Poisson bracket depends on the metric G_{AB} . We can easily see that this definition respects the tensor properties of f and g . In addition, for flat space the covariant derivatives reduce to partial derivatives and we recover the usual result.

However, defining the covariant Poisson brackets does not fully resolve the issue of noncovariance. While we may be tempted to find the covariant commutator $\{q^A, p_B\}_G$, we must remember that q^A is not a field space vector, but rather a field space coordinate belonging to a particular chart. Therefore, $\{q^A, p_B\}_G$ is a noncovariant quantity despite the use of (A1). In fact, as noted in Sec. II.2, $\nabla_B q^A$ is a meaningless expression. This means that in order to write down Poisson brackets in a covariant manner, we must use a vector in place of q^A .

It is possible to promote q^A to a vector in a manner similar to that used in the (famously covariant) Vilkovisky–De Witt formalism [38–40]. We can define the *tangent vector* $\sigma^A(q_*, q)$ by considering an arbitrary base point q_* .

The tangent vector $\sigma^A(q_*, q)$ is a scalar with respect to its first argument and a vector with respect to its second argument, and can be expanded as

$$\sigma^A(q_*, q) = -(q_*^A - q^A) - \frac{1}{2} \Gamma_{BC}^A(q_*)(q_*^B - q^B)(q_*^C - q^C) + \mathcal{O}[(q_* - q)^3]. \quad (\text{A2})$$

We then see that $\{\sigma^A(q_*, q), p_B\}_G$ is a tensor due to the fact that q_* does not depend on q or p . Taking into account the curvature of the space, we may write the following tensor expressions for the Poisson brackets, suppressing the arguments for $\sigma^A(q_*, q)$:

$$\{\sigma^A, \sigma^B\}_G = \{p_A, p_B\}_G = 0, \quad (\text{A3})$$

$$\{\sigma^A, p_B\}_G = \delta_B^A - \frac{1}{3} R_{CBD}^A \sigma^C(q_*, q) \sigma^D(q_*, q) + \dots \quad (\text{A4})$$

We observe that the commutator in (A4) is not δ_B^A as we expect, but is modified by curvature terms induced by G_{AB} .

Once we know the Poisson structure of the theory, we can canonically quantize by using

$$\{f, g\}_G \rightarrow -i[\hat{f}, \hat{g}]_G, \quad (\text{A5})$$

where $[\hat{f}, \hat{g}]_G$ is defined such that it reduces to the standard commutator $[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}$ for a flat space.

We also need to find covariant representations for the operators $\hat{p}_A$ and $\hat{\sigma}^A$. We suspect that the momentum operator will be represented by $\hat{p}_A = -i\nabla_A$, but the representation of σ^A is not as readily apparent. We will not address here the covariant quantization of the quantum operator e.g., the mapping $p_A \rightarrow \hat{p}_A = -i\nabla_B$ or the role of σ^A , but shall instead leave it for future work.

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