Thick brane in $f(T, B)$ gravity

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In this paper we explore the five-dimensional $f(T, B)$ teleparallel modified gravity with $f_1(T, B) = T + k_1 B^{n_1}$ and $f_2(T, B) = B + k_2 T^{n_2}$ in the brane scenario. Asymptotically, the bulk geometry converges to an AdS_5 spacetime whose cosmological constant is produced by parameters that control torsion and the boundary term. The analysis of the energy density condition reveals a splitting brane process satisfying the weak and strong energy conditions for some values of the parameters $n_{1,2}$ and $k_{1,2}$. In addition, we investigate the behavior of the gravitational perturbations in this scenario. In the bulk, the torsion keeps a gapless nonlocalizable and stable tower of massive modes. Inside the brane core, the torsion produces new barriers and potential wells leading to small amplitude massive modes and a massless mode localized for some values of the parameters $n_{1,2}$ and $k_{1,2}$.

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I. INTRODUCTION

Randall-Sundrum (RS) brane models [1,2] have attracted much attention since they were proposed because of their great success in solving the hierarchy problem [2] and in enabling new approaches to address outstanding issues such as the cosmological problem [3], the nature of dark matter [4], and dark energy. However, the thickness of the branes in the RS model is disappearing. A new approach is the so-called thick brane model, where the brane is constructed by the scalar field [5–9]. Various thick brane models have been investigated in Refs. [10–24]. All of these works only considered the contribution of spacetime curvature without torsion. However, torsion can also lead to an equivalent gravity theory known as the teleparallel equivalent of general relativity (TEGR) [25–28].

The TEGR is constructed using the curvatureless tensor named Weitzenböck instead of the torsionless Levi-Civita connection. Furthermore, the fundamental dynamical quantity of the theory is not the metric tensor but the so-called tetrad field. In such a formulation the gravitational Lagrangian results from contractions of the torsion tensor

and is called the torsion scalar T , similarly to the general relativity (GR) Lagrangian, i.e., the curvature scalar R , which is constructed by contractions of the curvature tensor. The torsion scalar and the curvature scalar are related by a boundary quantity, B , where $R = -T + B$.

Hence, similarly to the $f(R)$ extensions of GR [29,30], one can construct $f(T)$ extensions of TEGR [31,32]. The interesting feature in this extension is that $f(T)$ does not coincide with $f(R)$ gravity, despite the fact that TEGR coincides with GR. Additionally, the advantage of this theory is that the equations of motion are of second order, in contrast to the fourth-order equations of $f(R)$ gravity. Since it is a new gravitational modification class, various thick brane models have been investigated in Refs. [33–39].

New teleparallel gravity models have emerged, such as $f(T, T_G)$ gravity, where T_G is the torsion scalar equivalent of Gauss-Bonnet (GB) [40–42]; $f(T, \mathcal{T})$ gravity, where $\mathcal{T}$ is the trace of the stress-energy tensor [43]; and $f(T, B)$ gravity, where B is the boundary term [44–46]. Among these models, the gravitational model $f(T, B)$ is more interesting because its features have less mathematical complexity as well as good agreement with observational data to describe the accelerated expansion of the Universe [47,48]. In addition, significant results were obtained in cosmological perturbations and thermodynamics, dark energy, and gravitational waves [49–55]. This model provides equivalence between torsion and curvature, where one simultaneously recovers the models of both $f(T)$ gravity and $f(R) = f(-T + B)$ gravity.

We consider the increasing interest in modified teleparallel gravity models and in the significant results

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obtained in $f(T, B)$ gravity, which gives us the possibility to treat model alternatives to GR. We investigate the impact of torsion and the boundary term on the structure of branes and their influence on the localization of gravity on the branes.

Assuming only one scalar field as a source, we obtained a splitting in the thick brane. By varying the parameters that control the torsion and boundary term, the source undergoes a phase transition revealed by the energy density components. The gravitational perturbations form a gapless Kaluza-Klein (KK) spectrum whose interaction with the brane depends on the torsion and boundary term parameters.

The paper is organized as follows. In Sec. II we review the main definitions of the teleparallel theory; we introduce the $f(T, B)$ theory and then give the field equations for the five-dimensional braneworld. Furthermore, the exterior and interior solutions are found. We examine the energy density behavior in the brane. In Sec. III we derive the tensor perturbed equations and explore the gravitational KK modes. Finally, additional comments are presented in Sec. IV.

II. MODIFIED TELEPARALLEL BRANEWORLD

In this section, we present the main concepts of the modified teleparallel $f(T, B)$ gravity and obtain the modified gravitational equations for the braneworld scenario.

In teleparallel gravity, the dynamic variable is provided by the vielbeins, defined by $g_{MN} = \eta_{ab} h^a_M h^b_N$, where the capital Latin index $M = 0, \dots, D-1$ denotes the bulk coordinate indices and the Latin index $a = 0, \dots, D-1$ is a vielbein index.

In order to allow a distant parallelism, the teleparallel gravity assumes a curvature-free connection, known as the Weitzenböck connection, defined by [28]

$$\tilde{\Gamma}^P_{NM} = h_a^P \partial_M h^a_N. \quad (1)$$

The Weitzenböck connection has a nonvanishing torsion, defined as

$$T^P_{MN} = \tilde{\Gamma}^P_{NM} - \tilde{\Gamma}^P_{MN}. \quad (2)$$

The Weitzenböck and the Levi-Civita connections are related by $\tilde{\Gamma}^P_{NM} = \Gamma^P_{NM} + K^P_{NM}$ where

$$K^P_{NM} = \frac{1}{2}(T_N^P{}_M + T_M^P{}_N - T^P_{NM}) \quad (3)$$

is the contorsion tensor [28].

By defining the so-called superpotential torsion tensor as

$$S_P{}^{MN} = \frac{1}{2}(K^{MN}{}_P - \delta_P^N T^{QM}{}_Q + \delta_P^M T^{QN}{}_Q), \quad (4)$$

the torsion scalar is

$$\begin{aligned} T &= \frac{1}{2} T^P{}_{MN} T^P{}^{MN} + T^P{}_{MN} T^{NM}{}_P - 2 T^P{}_{MP} T^{NM}{}_N \\ &= T_{PMN} S^{PMN}. \end{aligned} \quad (5)$$

Therefore, the Lagrangian of the TEGR reads

$$\mathcal{L} = -hT/4\kappa_g, \quad (6)$$

where $h = \sqrt{g}$, with g the determinant of the metric, and $\kappa_g = 4\pi G/c^4$ is the gravitational constant [28]. Such a Lagrangian is equivalent to the usual Einstein-Hilbert action. Indeed, the Ricci scalar for the Weitzenböck is proportional to T by [28]

$$R = -T - 2\nabla^M T^N{}_{MN}, \quad (7)$$

and thus we can identify the boundary term

$$B \equiv -2\nabla^M T^N{}_{MN} = \frac{2}{h} \partial_M (hT^M), \quad (8)$$

in which T^M is the torsion tensor that can be define by $T_M = T^N{}_{MN}$. Hence, one can immediately see that GR and TEGR will lead to exactly the same equations. However, this will not be the case if one uses $f(R)$ or $f(T)$ as the Lagrangian of the theory, which therefore corresponds to different gravitational modifications [55].

A modified gravity theory can be accomplished by considering as the gravitational Lagrangian a function of T and B , leading to $f(T, B)$ gravity [55]. We assume a five-dimensional $f(T, B)$ gravity in the form

$$\mathcal{S} = -\frac{1}{4\kappa_g} \int h f(T, B) d^5x + \int (\Lambda + \mathcal{L}_m) d^5x, \quad (9)$$

where $\mathcal{L}_m$ is the matter Lagrangian. By varying the action with respect to the vielbein we obtain the following field equations [51,55]:

$$\begin{aligned} \frac{1}{h} f_T [\partial_Q (h S_N{}^{MQ}) - h \tilde{\Gamma}^R{}_{SN} S_R{}^{MS}] + \frac{1}{4} [f - B f_B] \delta_N^M \\ + [(\partial_Q f_T) + (\partial_Q f_B)] S_N{}^{MQ} + \frac{1}{2} [\nabla^M \nabla_N f_B - \delta_N^M \square f_B] \\ = -\kappa_g (\Lambda \delta_N^M + T_N^M), \end{aligned} \quad (10)$$

where $\square \equiv \nabla^M \nabla_M$, $f \equiv f(T, B)$, $f_T \equiv \partial f(T, B)/\partial T$, and $f_B \equiv \partial f(T, B)/\partial B$, and T_N^M is the stress-energy tensor, which in terms of the matter Lagrangian is given by $\mathcal{T}_a^M = -\delta \mathcal{L}_m / \delta h^a_M$.

We would like to consider the braneworld scenario, for which the metric ansatz reads as [33]

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (11)$$

where $\eta_{\mu\nu} = (-1, 1, 1, 1)$ is the four-dimensional Minkowski metric and $e^{A(y)}$ is the so-called warp factor. Accordingly, adopting the *sechsbeins* in the form $h^a_M = \text{diag}(e^A, e^A, e^A, 1)$, the torsion scalar and the boundary term are obtained, respectively, as

$$\begin{aligned} T &= -12A'^2, \\ B &= -8(A'' + 4A'^2), \end{aligned} \quad (12)$$

which together reproduce the Ricci scalar $R = -T + B = -4(5A'^2 + 2A'')$, where the prime ($'$) denotes differentiation with respect to y .

In our model, we take a Lagrangian given by

$$\mathcal{L}_m = -h \left[\frac{1}{2} \partial_M \phi \partial^M \phi + V(\phi) \right], \quad (13)$$

where $\phi \equiv \phi(y)$ is a background scalar field that generates the brane. The explicit field equations (10) and the equation of motion of the scalar field can be written as

$$\phi'' + 4A'\phi' = \frac{\partial V}{\partial \phi}, \quad (14)$$

$$\frac{1}{4} [f + 8(A'' + 4A'^2)f_B] + 6A'^2 f_T = -\kappa_g \left(\Lambda - \frac{1}{2} \phi'^2 + V \right), \quad (15)$$

$$\begin{aligned} &-12[A'(A''' + 8A'A'')](f_{BB} + f_{TB}) + 3A''A'^2(f_{TT} + f_{BT}) \\ &+ \frac{1}{2}(A'' + 4A'^2)(4f_B + 3f_T) + \frac{1}{4}f \\ &= -\kappa_g \left(\Lambda + \frac{1}{2} \phi'^2 + V \right). \end{aligned} \quad (16)$$

Equations (14)–(16) form a quite intricate system of coupled equations. Thus, we first analyze the solutions exterior to the brane, and then, we propose a possible solution for the brane core.

In this work we consider a power-law modified gravity in the form $f_1(T, B) = T + k_1 B^{n_1}$ and $f_2(T, B) = B + k_2 T^{n_2}$, where $k_{1,2}$ and $n_{1,2}$ are parameters controlling the deviation of the usual teleparallel theory.

A. Thin brane regime

Let us first explore the effects of the torsion on the vacuum region, exterior to the brane. In the vacuum, the stress-energy tensor vanishes $\mathcal{T}_N^M = 0$, and then, the geometry is governed by the bulk cosmological constant Λ .

Assuming a vacuum solution $A' = -c$ that establishes a relation between the bulk cosmological constant, the

torsion parameters, and c , the corresponding exterior metric takes the form

$$ds^2 = e^{-2c|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (17)$$

which is the thin brane solution in the RS model [1,2]. However, unlike the usual GR braneworld, $f(T, B)$ gravity enables possible new configurations. We give two examples below.

$$I. f_1(T, B) = T + k_1 B^{n_1}$$

The modified gravitational field equations (15) and (16) yield

$$-3c^2 + (-4)^{n_1-1} k_1 (n_1 - 1) (8c^2)^{n_1} = \kappa_g \Lambda. \quad (18)$$

First, let us seek solutions of Eq. (18) for $\Lambda = 0$. If $n_1 = 1$ the only solution is $c = 0$, which leads to a factorizable Kaluza-Klein model. Yet, for $n_1 = 2$ we obtain the solution

$$c = \pm \sqrt{\frac{3\kappa_g}{256k_1}}, \quad (19)$$

whereas for $n = 3$ we find

$$c = \pm \sqrt[4]{\frac{3\kappa_g}{8192k_1}}. \quad (20)$$

Accordingly, the $f_1(T, B)$ yields to a warped compactified spacetime even in the absence of a bulk cosmological constant.

Now let us consider a nonzero bulk cosmological constant. For $n_1 = 1$, Eq. (18) yields to

$$c = \pm \sqrt{\frac{\kappa_g(-\Lambda)}{3}}, \quad (21)$$

which are solutions that represent the usual thin brane with an AdS_5 bulk. For $n_1 = 2$, we obtain four solutions,

$$c = \pm \sqrt{\frac{\kappa_g(-\Lambda)}{3}} \quad \text{or} \quad c = \pm \sqrt{\frac{\kappa_g(\Lambda + 1)}{256k_1}}. \quad (22)$$

The last two solutions are generated by a positive cosmological constant, provided $k > 0$. For $n_1 = 3$ we obtain six solutions:

$$c = \pm \sqrt{\frac{\kappa_g(-\Lambda)}{3}} \quad \text{or} \quad c = \pm \sqrt[4]{\frac{\kappa_g(-\Lambda - 1)}{8192k_1}}. \quad (23)$$

The last solutions are generated by a positive cosmological constant, provided $k < 0$. Thus, for $n_1 = 3$, $n_1 = 2$, and

$n_1 = 1$, the exterior geometry is the same as in the thin RS model.

2. $f_2(T, B) = B + k_2 T^{n_2}$

The modified gravitational field equations (15) and (16) yield to

$$(-4)^{n_2-1} k_2 (2n_2 - 1) (3c^2)^{n_2} = \kappa_g \Lambda. \quad (24)$$

For any n_2 , the only solution of Eq. (24) with $\Lambda = 0$ is $c = 0$. Now let us consider a nonzero bulk cosmological constant. For $n_2 = 1$, Eq. (24) yields

$$c = \pm \sqrt{\frac{\kappa_g(-\Lambda)}{3k_2}}. \quad (25)$$

If $k_2 = 1$, these solutions represent the usual thin brane with an AdS_5 bulk. For $n_2 = 2$, we obtain the solutions

$$c = \pm \sqrt[4]{\frac{\kappa_g(\Lambda)}{108k_2}}, \quad (26)$$

which are generated by a positive cosmological constant, provided $k_2 > 0$. For $n_2 = 3$ we obtain six solutions:

$$c = \pm \sqrt[6]{\frac{\kappa_g(-\Lambda)}{2160k_2}}. \quad (27)$$

The last solutions are generated by a positive cosmological constant, provided $k_2 < 0$. The solutions for any n_2 reveal a striking effect of $f_2(T, B)$ upon the exterior geometry.

B. Thick brane regime

We can rewrite Eqs. (15) and (16) as

$$6A'^2 = -\frac{\kappa_g}{f_T}(\Lambda + P + P_{TB}), \quad (28)$$

$$3A'' + 12A'^2 = -\frac{2\kappa_g}{f_T}(\Lambda + \rho + \rho_{TB}), \quad (29)$$

where

$$\kappa_g P_{TB} = \frac{1}{4}[f + 8(A'' + 4A'^2)f_B], \quad (30)$$

$$\begin{aligned} \kappa_g \rho_{TB} = & -12[A'(A''' + 8A'A'')](f_{BB} + f_{TB}) \\ & + 3A''A'^2(f_{TT} + f_{BT}) \\ & + 2(A'' + 4A'^2)f_B + \frac{1}{4}f. \end{aligned} \quad (31)$$

Note that the left sides of Eqs. (28) and (29) are equivalent to that obtained in TERG. So we can state that modified

gravity equations of motion of the $f(T, B)$ gravities is similar to including an additional source with ρ_{TB} and P_{TB} .

Since we have studied the effects of the torsion on the region exterior to the brane, let us turn our attention to the brane core region. In order to do this, we propose a smooth, thick, warp factor ansatz of the form [6]

$$e^{2A(y)} = \cosh^{-2p}(\lambda y), \quad (32)$$

where the parameters p and λ determine, respectively, the amplitude and the width of the source. Now, we have two important issues, namely, the energy conditions and the field solutions.

1. Energy conditions

Let us now analyze the profile of the energy density which yields Eq. (32). Here, we will study how the $f(T, B)$ gravity leads to the brane splitting process.

When $y \rightarrow \pm\infty$ (vacuum), the stress-energy tensor vanishes, $T_N^M = 0$. So for $f_1(T, B)$, we have

$$\Lambda = -\frac{1}{\kappa_g} \{3(p\lambda)^2 - (-4)^{n_1-1} k_1 (n_1 - 1) [8(p\lambda)^2]^{n_1}\}. \quad (33)$$

For $f_2(T, B)$, we have

$$\Lambda = \frac{1}{\kappa_g} \{(-4)^{n_2-1} k_2 (2n_2 - 1) [3(p\lambda)^2]^{n_2}\}. \quad (34)$$

Comparing Eq. (18) with (33) and Eq. (24) with (34), we can say that $A(y \rightarrow \pm\infty) \rightarrow -p\lambda|y|$. Thus, the spacetime solution is asymptotically AdS_5 .

The energy densities for $f_1(T, B)$ are

$$\begin{aligned} \rho_1(y) = & -\Lambda - \frac{3}{\kappa_g} \left[(p\lambda)^2 - \frac{1}{2}(1 + 2p)p\lambda^2 \text{sech}^2(\lambda y) \right] \\ & + \frac{\alpha}{\kappa_g} \left[2^{3n_1-2} (n_1 - 1) k_1 \right. \\ & \left. - \frac{8^{n_1-1} (n_1 - 1) 3n_1 k_1 (1 + 4p) \sinh^2(\lambda y)}{\beta^2} \right], \end{aligned} \quad (35)$$

with pressure

$$P_1(y) = -\Lambda - \frac{1}{4\kappa_g} [12(p\lambda)^2 \tanh^2(\lambda y) - 8^{n_1} \alpha^{n_1} k_1 (n_1 - 1)], \quad (36)$$

where we defined the functions $\alpha \equiv p\lambda^2 [\text{sech}^2(\lambda y) - 4p \tanh^2(\lambda y)]$, $\beta \equiv 1 + 2[1 - \cosh(2\lambda y)]p$, and Λ is given in Eq. (33).

For $n_1 = 1$, the source exhibits a localized profile satisfying the dominant and strong energy conditions independent of the parameter k_1 .

FIG. 1. Plots of the energy density for $p = \lambda = 1$. (a) $n_1 = 2$. (b) $n_1 = 3$. Plots of the pressure $P_1(y)$. (c) $n_1 = 2$. (d) $n_1 = 3$.

In Fig. 1, we plot the energy densities $\rho_1(y)$ and pressure $P_1(y)$, varying the parameter k_1 . The $n_1 = 2$ configuration [Fig. 1(a)] includes a new peak. For the $n_1 = 3$ configuration, we can say that three valleys appear for $k_1 = -0.002$ [Fig. 1(b)]. This feature reflects the brane's internal structure, which tends to split the brane. A similar splitting process was obtained in Ref. [33].

A noteworthy feature is the violation of the dominant energy condition for $n_1 = 2$ with $k_1 > 0$ and $n_1 = 3$ with $k_1 < 0$, where the source presents a negative energy density phase. Therefore, $f_1(T, B)$ produces modifications in the

source equation of state that might lead to the brane splitting. For $n_1 = 3$ with $k_1 > 0$, the source has a positive energy density phase.

In Fig. 2, we plot the energy densities $\rho_1(y)$, varying the parameter p for $n_1 = 2$. Note that by varying the parameter p , we obtain the splitting of the brane.

The energy densities for $f_2(T, B)$ are

$$\rho_2(y) = -\Lambda - \frac{1}{\kappa_g} \{ 2^{2n_2-3} (-3)^{n_2} p^{-1} k_2 (2n_2 - 1) \beta \text{csch}^2(\lambda y) \times [p\lambda \tanh(\lambda y)]^{2n_2} \}, \quad (37)$$

FIG. 2. Plots of the energy density for $n_1 = 2$ with $\lambda = 1$ and $k_1 = -0.01$.

and the pressure is

$$P_2(y) = -\Lambda - \frac{1}{\kappa_g} \{(-3)^{n_2} 4^{n_2-1} k_2 (2n_2 - 1) \times [(p\lambda)^2 \tanh^2(\lambda y)]^{n_2}\}, \quad (38)$$

where Λ is given in Eq. (34).

For $n_2 = 1$, the source exhibits a localized profile satisfying the dominant and strong energy conditions, but when $k_2 < 0$, the source presents a negative energy density phase.

In Fig. 3, we plot the energy densities $\rho_2(y)$ and pressure $P_2(y)$, varying the parameter k_2 . For the $n_2 = 2$ and $n_2 = 3$ configurations, the energy densities include a new peak regardless of the k_2 parameter. Again, this feature reflects the brane's internal structure, which tends to split the brane.

As in $f_1(T, B)$, we noticed the violation of the dominant energy condition for $n_2 = 2$ with $k_2 > 0$ and $n_2 = 3$ with $k_2 < 0$, where the source presents a negative energy density phase. Therefore, $f_2(T, B)$ produces modifications in the source equation of state that might lead to the brane splitting. For $n_2 = 3$ with $k_2 > 0$, the source has a positive energy density phase.

In Fig. 4, we plot the energy densities $\rho_2(y)$, varying the parameter p for $n_2 = 2$. Note that for higher values of p , we undo the splitting in the brane.

2. Field solution

In this subsection, we obtain the configurations of the scalar field that lead to the thick brane (32).

We follow the approach carried out in Ref. [33], where by manipulating the modified Einstein equations, an equation relating the metric components and the scalar field was obtained. In this case we can write Eqs. (15) and (16) as

$$\phi'^2(y) = -\frac{3}{2} f_T A'' + 12A'[(8A'A'' + A''')(f_{BB} + f_{TB}) + 3A'A''(f_{TT} + f_{BT})], \quad (39)$$

$$V(\phi) = 6A'[(8A'A'' + A''')(f_{BB} + f_{TB}) + 3A'A''(f_{TT} + f_{BT})] - \frac{1}{4}[3(8A'^2 + A'')f_T + 8(4A'^2 + A'')f_B + f], \quad (40)$$

FIG. 3. Plots of the energy density for $p = \lambda = 1$. (a) $n_2 = 2$. (b) $n_2 = 3$. Plots of the pressure $P_2(y)$. (c) $n_2 = 2$. (d) $n_2 = 3$.

FIG. 4. Plots of the energy density for $n_2 = 2$ with $\lambda = 1$ and $k_2 = -0.00005$.

where we set the gravitational constant $\kappa_g = 1$ for simplicity. Equation (39) allows us to obtain the scalar field for a given geometric solution [33]. As we know, solutions $\phi = \phi(y)$ of Eq. (39) must go to some constant value ϕ_c asymptotically. Thus, if we want our model to make physical sense, the potential (40) should go to a vacuum value for ϕ_c .

For $f_1(T, B)$ Eqs. (39) and (40) take the form

$$\phi'^2(y) = \frac{3}{2} p \lambda^2 \text{sech}^2(\lambda y) - \frac{\alpha^{n_1}}{\beta^2} [8^{n_1-1} k_1 n_1 (n_1 - 1) (1 + 4p) \sinh^2(\lambda y)], \quad (41)$$

$$V(\phi(y)) = \frac{3}{4} \alpha + 2^{3n_1-4} k_1 (n_1 - 1) (p \lambda^2)^2 \alpha^{n_1-2} \{4 \text{sech}^4(\lambda y) + 64 p^2 \tanh^4(\lambda y) - [3n_1 + 4(8 + 3n_1)] \text{sech}^2(\lambda y) \tanh^2(\lambda y)\}. \quad (42)$$

In this case, with Eq. (33), the asymptotic values of the potential and its derivative with respect to the field are, respectively,

$$\Lambda \equiv V(\phi \rightarrow \pm \phi_{c_1}) = (-4)^{n_1-1} k_1 (n_1 - 1) [8(p \lambda)^2]^{n_1} - 3(p \lambda)^2, \quad (43)$$

and $\partial V(\phi \rightarrow \pm \phi_{c_1}) / \partial \phi = 0$. We can solve Eq. (41) to find a function $\phi = g(y)$, which may be inverted to give $y = g^{-1}(\phi)$, allowing us to write the potential in the usual way, $V = V(\phi)$. The thick brane solution for $n_1 = 1$ and the potential are

$$\phi(y) = \sqrt{6p} \arctan \left[\tanh \left(\frac{\lambda y}{2} \right) \right], \quad (44)$$

$$V(\phi) = \frac{3p\lambda^2}{4} \left[(1 + 4p) \cos^2 \left(\frac{2\phi}{\sqrt{6p}} \right) - 4p \right], \quad (45)$$

which are the same expressions as those in Refs. [6,17]. One point here is that the expressions in those works were obtained by using a superpotential approach where the gravity is described by general relativity. As a consequence, the solutions for this case are equivalent to those in the case $f_1(T, B) = T$. For $n_1 = 2$

$$\begin{aligned} \phi(y) = & \sqrt{3p\lambda} \coth(\lambda y) \sqrt{1 - 32[1 - \text{sech}^2(\lambda y)]} \left\{ \frac{1}{\varrho} \left[\tau E \left(\arcsin[\cosh(\lambda y)], 1 - \frac{1}{\tau} \right) \right. \right. \\ & \left. \left. - F \left(\arcsin[\cosh(\lambda y)], 1 - \frac{1}{\tau} \right) \right] + \frac{\text{sech}(\lambda y)}{[1 + \cos(2\lambda y)]} \right\}, \end{aligned} \quad (46)$$

where $\tau = 32k_1 p (1 + 4p) \lambda^2$ and $\varrho = 2\sqrt{2[1 + \cosh(2\lambda y)]} \sqrt{(\tau - 1) \cosh(2\lambda y) - \tau - 1}$, and $F(y, q)$ and $E(y, q)$ are the first and second kinds of elliptic integrals, respectively. Note that $k_1 \geq 1/32p(1 + 4p)\lambda^2$ for the second solution (46) in order to ensure that the scalar field $\phi(y)$ is real. In this case, the $V(\phi)$ solution cannot be found analytically.

For $f_2(T, B)$ Eqs. (39) and (40) take the form

$$\phi'^2(y) = -2^{2n_2-3} (-3)^{n_2} p^{-1} k_2 n_2 (2n_2 - 1) \text{csch}^2(\lambda y) [p \lambda \tanh(\lambda y)]^{2n_2}, \quad (47)$$

$$V(\phi(y)) = 4^{n_2-2} (-3)^{n_2} p^{-1} k_2 (2n_2 - 1) [4p - n_2 \text{csch}^2(\lambda y)] [p \lambda \tanh(\lambda y)]^{2n_2}. \quad (48)$$

In this case, with Eq. (34), the asymptotic value of the potential is

$$\Lambda \equiv V(\phi \rightarrow \pm\phi_{c_2}) = (-4)^{n_2-1} k_2 (2n_2 - 1) [3(p\lambda)^2]^{n_2}, \quad (49)$$

and its derivative in relation to the field is $\partial V(\phi \rightarrow \pm\phi_{c_2})/\partial\phi = 0$. The thick brane solution for $n_2 = 1$ and the potential are

$$\phi(y) = \sqrt{6pk_2} \arctan\left[\tanh\left(\frac{\lambda y}{2}\right)\right], \quad (50)$$

$$V(\phi) = \frac{3pk_2\lambda^2}{4} \left[(1 + 4pk_2) \cos^2\left(\frac{2\phi}{\sqrt{6pk_2}}\right) - 4pk_2 \right]. \quad (51)$$

For the case $n_2 = 2$, we have

$$\phi(y) = -6\sqrt{-3p^3k_2\lambda} \operatorname{sech}(\lambda y), \quad (52)$$

$$V(\phi) = 108k_2(p\lambda)^4 + \frac{\lambda^2}{2}(1 + 4p)\phi^2 + \frac{1 + 2p}{216k_2p^3}\phi^4, \quad (53)$$

which is not a good solution. The same goes for cases where $n_2 = 4, 6, \dots$ (even numbers). When $n_2 = 3$, we have

$$\phi(y) = 9(p\lambda)^2 \sqrt{10pk_2} \{ \arctan[\sinh(\lambda y)] - \operatorname{sech}(\lambda y) \tanh(\lambda y) \}. \quad (54)$$

In this case the $V(\phi)$ solution cannot be found analytically. We note that for odd n_2 , the parameter k_2 must be greater than zero to obtain real solutions of $\phi(y)$. For even n_2 , the

parameter must be negative ($k_2 < 0$) in order to obtain real solutions of $\phi(y)$.

III. GRAVITATIONAL TENSOR MODES

In this section we investigate the effects of torsion and the boundary term on the propagation of the linear perturbations in the brane system. We follow closely the analysis performed in Ref. [36]. For that, we use the *fünfbein* perturbation

$$h^a_M = \begin{pmatrix} e^{A(y)}(\delta^a_\mu + w^a_\mu) & 0 \\ 0 & 1 \end{pmatrix}, \quad (55)$$

where $w^a_\mu = w^a_\mu(x^\mu, y)$. The resulting metric perturbation takes the form $ds^2 = e^{A(y)}(\eta_{\mu\nu} + \gamma_{\mu\nu})dx^\mu dx^\nu + dy^2$, where the metric and the *fünfbein* perturbations are related by

$$\begin{aligned} \gamma_{\mu\nu} &= (\delta^a_\mu w^b_\nu + \delta^b_\nu w^a_\mu) \eta_{ab}, \\ \gamma^{\mu\nu} &= (\delta^a_\mu w^b_\nu + \delta^b_\nu w^a_\mu) \eta^{ab}. \end{aligned} \quad (56)$$

We assume the transverse-traceless (TT) tensor perturbation, which is related to the gravitational wave and four-dimensional gravitons. The TT tensor perturbation satisfies the following TT conditions, $\partial_\mu \gamma^{\mu\nu} = 0 = \eta^{\mu\nu} \gamma_{\mu\nu}$, which leads to the *fünfbein*

$$\partial_\mu (\delta^a_\mu w^b_\nu + \delta^b_\nu w^a_\mu) \eta^{ab} = 0, \quad (57)$$

$$\delta^a_\mu w^a_\mu = 0. \quad (58)$$

The nonvanishing components of the torsion tensor are [36]

$$\begin{aligned} T^\rho_{\mu y} &= -A' \delta^\rho_\mu - (\delta^a_\mu w^a_\rho - \delta^a_\rho w^a_\mu) A' - \delta^a_\mu w^a_\rho, \\ T^\rho_{\mu\nu} &= \delta^a_\mu (\partial_\nu w^a_\rho - \partial_\rho w^a_\nu), \end{aligned} \quad (59)$$

whereas the nonvanishing contorsion components are [36]

$$\begin{aligned} K^\rho_{\mu y} &= A' (\delta^a_\mu w^a_\rho - \delta^a_\rho w^a_\mu) + \frac{1}{2} (\delta^a_\mu w'^a_\rho - \delta^a_\rho w'^a_\mu), \\ K^\rho_{y\nu} &= -A' \delta^\rho_\nu - \frac{1}{2} (\delta^a_\nu w'^a_\rho - \delta^a_\rho w'^a_\nu), \\ K^y_{\mu\nu} &= e^{2A} \left(A' \eta_{\mu\nu} + A' \gamma_{\mu\nu} + \frac{1}{2} \gamma'_{\mu\nu} \right), \\ K^\rho_{\mu\nu} &= \frac{1}{2} [\delta^a_\mu (\partial^\rho w_{a\nu} - \partial_\nu w^a_\rho) + \delta^a_\nu (\partial^\rho w_{a\mu} - \partial_\mu w^a_\rho) - \delta^a_\rho (\partial_\mu w^a_\nu - \partial_\nu w^a_\mu)]. \end{aligned} \quad (60)$$

Accordingly, the nonvanishing components of the dual torsion tensor are [36]

$$\begin{aligned}
S_\rho &= \frac{1}{2} \left[(3A' + B') \delta_\rho^\mu - \frac{1}{2} (\delta_\rho^\mu w_a'^\mu + \delta_a^\mu w'^a_\rho) \right], \\
S_y &= \frac{1}{2} \left[A' (\delta_a^\mu w^{a\mu} - \delta_a^\nu w^{a\mu}) + \frac{1}{2} (\delta_a^\mu w'^{a\mu} - \delta_a^\nu w'^{a\mu}) \right] e^{-2A}, \\
S_\rho &= \frac{1}{4} [\delta_a^\nu (\partial^\mu w_a^\nu - \partial_\rho w^{a\mu}) - \delta_a^\mu (\partial^\nu w_a^\rho - \partial_\rho w^{a\mu})] e^{-2A} + \frac{1}{4} \delta_\mu^a (\partial^\mu w_a^\nu - \partial^\nu w_a^\mu) + \frac{1}{2} [\delta_\rho^\nu \delta_a^\lambda \partial_\lambda w^{a\mu} - \delta_\rho^\mu \delta_a^\lambda \partial_\lambda w^{a\mu}] e^{-2A}, \\
S_y^{\mu\nu} &= \frac{1}{2} (\delta_a^\rho \partial_\rho w^{a\mu}) e^{-2A}.
\end{aligned} \tag{61}$$

In this work, we always neglect the second-order terms for the disturbed quantities. Considering the traceless condition (58), we have [36]

$$\delta h = h h_a^M \delta h^a_M = h e^{-A} e^A \delta_a^\mu w_a^\mu = 0. \tag{62}$$

After a lengthy but simple algebra, it is straightforward to verify that $\delta T = 0$ and $\delta B = 0$ [36–39]. The perturbed modified Einstein equation (10) now has the form

$$\begin{aligned}
&\frac{1}{4} (B f_B - f) \delta g_{MN} - \frac{1}{h} f_T [\delta g_{PN} \partial_Q (h S_M^{PQ}) + g_{PN} \partial_Q (h \delta S_M^{PQ}) - h (\delta \tilde{\Gamma}^Q_{PM} S_{QN}^P + \tilde{\Gamma}^Q \delta S_{QN}^P)] \\
&- [(\partial_Q f_B) + (\partial_Q f_T)] \delta S_{MN}^Q + \frac{1}{2} (\delta g_{MN} \square f_B - \delta g_{PN} \nabla^P \nabla_M f_B) = \delta T_{MN},
\end{aligned} \tag{63}$$

which yields

$$\begin{aligned}
&\frac{1}{4} \{ [e^{-2A} \square^{(4)} \gamma_{\mu\nu} + 4A' \gamma'_{\mu\nu} + \gamma''_{\mu\nu} - 6(A'' + 4A'^2) \gamma_{\mu\nu}] f_T \\
&+ [8(A''' + 8A'A'')(f_{BB} + f_{TB}) + 24A'A''(f_{TT} + f_{BT})] (6A' \gamma_{\mu\nu} - \gamma'_{\mu\nu}) - [f + 8(A'' + 4A'^2) f_B] \gamma_{\mu\nu} \} e^{2A} = \delta T_{\mu\nu},
\end{aligned} \tag{64}$$

where $\square^{(4)} = \eta^{\mu\nu} \partial_\mu \partial_\nu$. Here, we note that the perturbations vanish in the extra dimension. The linearized stress-energy tensor is written

$$\delta T_{\mu\nu} = \delta(\mathcal{T}_\mu^\mu g_{\mu\nu}) = \delta(\mathcal{T}_\mu^\mu) \eta_{\mu\nu} e^{2A} + \mathcal{T}_\mu^\mu \gamma_{\mu\nu} e^{2A}. \tag{65}$$

The gravitational field equation (10) provides the condition

$$12[A'(A''' + 8A'A'')(f_{BB} + f_{TB}) + 3A''A''(f_{TT} + f_{BT})] - \frac{1}{2}(A'' + 4A'^2)(4f_B + 3f_T) - \frac{1}{4}f = \mathcal{T}_\mu^\mu. \tag{66}$$

By plugging Eq. (66) into Eq. (64), and considering the vanishing trace $\delta(\mathcal{T}_\mu^\mu)$, we obtain the following perturbation equation:

$$-8[(A''' + 8A'A'')(f_{BB} + f_{TB}) + 3A'A''(f_{TT} + f_{BT})] \gamma'_{\mu\nu} + (e^{-2A} \square^{(4)} \gamma_{\mu\nu} + 4A' \gamma'_{\mu\nu} + \gamma''_{\mu\nu}) f_T = 0. \tag{67}$$

Assuming the Kaluza-Klein decomposition $\gamma_{\mu\nu}(x^\rho, y) = \epsilon_{\mu\nu}(x^\rho) \chi(y)$ and a 4D plane wave satisfying $(\square^{(4)} - m^2) \epsilon_{\mu\nu} = 0$, the perturbed Einstein equation (67) yields

$$\chi'' + \left\{ 4A' - \frac{1}{f_T} [24A'A''(f_{TT} + f_{TB}) + 8(A''' + 8A'A'')(f_{BB} + f_{TB})] \right\} \chi' + e^{-2A} m^2 \chi = 0. \tag{68}$$

A. Massive modes

Let us first consider the effects of torsion in the region exterior to the brane. That limit can also be interpreted as representing a thin brane. In this regime $A' = -c$, and Eq. (68) takes the form

$$\chi'' - 4c\chi' + e^{2c|y|} m^2 \chi = 0. \tag{69}$$

Equation (69) is the same as the thin brane in the RS model [1], except that the coefficient of the first-derivative term is 2 instead of 4 and that c depends on the parameters that control the torsion and boundary term $n_{1,2}$ and $k_{1,2}$.

On the other hand, for the thick brane regime, we numerically solved Eq. (68) using the interpolation method, thereby obtaining the massive modes for $f_1(T, B)$ and $f_2(T, B)$, adopting the usual boundary condition $\chi'(-\infty) = \chi'(\infty) = 0$. As depicted in Figs. 5 and 6, the asymptotic divergence of the massive gravitational field shows that they form a tower of nonlocalized states.

For $f_1(T, B)$ and $n_1 = 1$, there is no dependency on k_1 for the behavior of massive modes, as is illustrated in Fig. 5(a). We note that for $n_1 = 2$ and 3, there is dependency on k_1 . It is important to remark that the amplitude of ripples increases for $n_1 = 2$ and an increasing value of k_1 as well as for $n_1 = 3$ and a decreasing value of k_1 . The increasing amplitude of the ripples makes them more intense and allows their presence within the brane, as illustrated in Figs. 5(c) and 5(d), where $n_1 = 3$ and we vary

the parameter k_1 . For $n_1 = 2$ the behavior of massive modes is illustrated in Fig. 5(b).

For $f_2(T, B)$ there is no dependency on k_2 in massive modes. For $n_2 = 1$ we obtain the same differential equation as $f_1(T, B)$ with $n_1 = 1$; thus, we have the same solution and behavior already illustrated in Fig. 5(a). For any value of n_2 there is an amplitude within the brane. For $n_2 = 2$ the behavior of the massive modes is illustrated in Fig. 6(a) and for $n_2 = 3$ in Fig. 6(b).

B. Massless modes

Employing the change to a conformal coordinate $z = \int e^{-A} dy$, the tensor perturbation given by Eq. (68) is transformed as

$$(\partial_z^2 + 2H\partial_z + m^2)\chi(z) = 0, \quad (70)$$

where

FIG. 5. Massive modes for $f_1(T, B)$ and $p = \lambda = 1$. (a) $n_1 = 1$. (b) $n_1 = 2$ and $k_1 = -0.05$. (c,d) $n_1 = 3$, with its first fixed mass eigenfunction.

$$H = \frac{3}{2}\dot{A} + \frac{4e^{-2A}}{f_T} [3(\dot{A}^3 - \dot{A}\ddot{A})(f_{TT} + f_{BT}) + (6\dot{A}^3 - 4\dot{A}\ddot{A} - \ddot{A}^2)(f_{BB} + f_{TB})], \quad (71)$$

and the dot denotes differentiation with respect to z . With the change on the wave function $\chi(z) = F(z)\Psi(z)$ in Eq. (70), it is possible to recast into a Schrödinger-like equation

$$[-\partial_z^2 + U(z)]\Psi(z) = m^2\Psi(z), \quad (72)$$

where the potential is defined by

$$U(z) = \dot{H} + H^2, \quad (73)$$

and

$$F(z) = e^{-\frac{3}{2}A + \int K(z)dz}, \quad (74)$$

with

$$K(z) = -\frac{4e^{-2A}}{f_T} [3(\dot{A}^3 - \dot{A}\ddot{A})(f_{TT} + f_{BT}) + (6\dot{A}^3 - 4\dot{A}\ddot{A} - \ddot{A}^2)(f_{BB} + f_{TB})]. \quad (75)$$

The Schrödinger-like equation (72) can be factorized as

$$(-\partial_z + H)(\partial_z + H)\Psi(z) = m^2\Psi(z), \quad (76)$$

which represents an equation of the so-called supersymmetric quantum mechanics. The superpotential H and the quantum mechanics supersymmetric form of the potential U ensure the absence of tachyonic KK gravitational modes.

Besides the spectrum stability, the potential in Eq. (73) allows a massless KK mode of the form

$$\Psi_0 = N_0 e^{\frac{3}{2}A - \int K(z)dz}, \quad (77)$$

where N_0 is a normalization constant. In order to recover the four-dimensional gravity, the zero mode should be localized on the brane. Let us now investigate the problem of localization of the massless mode.

For $f_1(T, B)$ the superpotential (71) has the form

$$H_1 = -\frac{3pz\lambda^2}{\xi} - 8^{n_1-1}k_1n_1(n_1-1)zp\lambda^4\xi^{p-3} \times [(p\lambda)^2(3z^2-1) + 2p\xi + 3] \times \{p\lambda^2\xi^{p-2}[1 - (1-4p)(z\lambda)^2]\}^{n_1-2}, \quad (78)$$

where $\xi \equiv 1 + (z\lambda)^2$. The expression of the potential is too lengthy to be written here. Instead, we plot the potential and the zero mode and explore some qualitative features. Figures 7 and 8 show us how the potential and the zero mode recognize the division of the brane by varying the parameters that control the boundary term of $f_1(T, B)$.

For $n_1 = 1$ the effective potential is independent of the parameter k_1 . For $n_1 = 2$ and $k < 0$ as we can see from Fig. 7(a), when k_1 decreases, the shape of the effective potential is volcanolike, and the zero mode wave function has only one peak that gets more localized as seen in Fig. 7(b). When $k_1 > 0$ increases, as we can see from Fig. 7(c), two new potential barriers appear away from the origin, and the potential well around the origin increases. As a result, the zero mode wave function splits into two peaks as shown in Fig. 7(d).

For $n_1 = 3$ and $k < 0$ as we can see from Fig. 8(a), when k_1 is decreasing, the potential well splits in two around the origin, with two new potential barriers, and the zero mode wave function divides into three, with a peak at the origin and the other two further away from the origin as seen in Fig. 8(b). When $k_1 > 0$ increases, as we can see from Fig. 8(c), two new potential barriers appear away from the origin, and the potential well around the origin increases.

FIG. 6. Massive modes for $f_2(T, B)$ and $p = \lambda = 1$. (a) $n_1 = 2$. (b) $n_1 = 3$.

FIG. 7. Plots of the effective potential and the zero mode for $n_1 = 2$ and $p = \lambda = 1$. (a,b) $k_1 < 0$. (c,d) $k_1 > 0$.

As a result, the zero mode wave function splits into two peaks as shown in Fig. 8(d).

For $f_2(T, B)$ the superpotential (71) has the form

$$H_2 = -\frac{(n_2 - 1)}{\xi} + \frac{z\lambda^2}{2\xi} [2 + 2n_2(p - 1) - 5p]. \quad (79)$$

As we can see from Eq. (79), there is no dependency on the k_2 parameter. Figure 9 shows us how the potential and the zero mode recognize the division of the brane by varying the parameters that control the torsion term of $f_2(T, B)$. When $n_2 = 1$ we have the same solution as $f_1(T, B)$ with $n_1 = 1$. As we can see in Fig. 9(a), the potential goes from a well to a delta-type barrier when we increase the parameter n_2 . As a result, zero modes become nonlocalized for values of $n_2 = 2, 4, \dots$ (even numbers) as shown in Fig. 9(b).

IV. FINAL REMARKS

We studied the torsion and boundary term effects on a braneworld in the context of the $f(T, B)$ modified teleparallel gravity. For this, we propose two particular cases for $f(T, B)$, namely, $f_1(T, B) = T + k_1 B^{n_1}$ and $f_2(T, B) = B + k_2 T^{n_2}$. We note that the linear term for B in f_2 may be

omitted as it does not contribute to the equations of motion; in this case f_2 is equivalent to a $f(T)$ case where $f(T) = kT^n$. In both cases the torsion and boundary term produce an inner brane structure tending to split the brane. Furthermore, $f(T, B)$ modified the exterior region, making the solutions depend on the parameters that control the torsion and boundary term $n_{1,2}$ and $k_{1,2}$. Even with the cosmological constant being zero, for $f_1(T, B)$ it is possible to obtain solutions.

The vacuum expectation value and the profile of the scalar field inside the core are controlled by the parameters that control torsion and the boundary term. The profile of the scalar field suggests a topological stability for $f_1(T, B)$. However, for $f_2(T, B)$ the stability is only observed for the values of the parameter $n_2 = 1, 3, 5, \dots$ odd. The thick brane undergoes a phase transition evinced by the energy density components. Similar behavior was found for $f(T)$ in Refs [33,36]. As the parameters $n_{1,2}$ and $k_{1,2}$ increase, the source violates the dominant energy condition, which reflects on the negative density responsible for the brane splitting.

For $f_1(T, B)$ with the first massive values, the massive mode is dependent on the parameters that control the boundary term, which is well evidenced for $n_1 = 3$ where decreasing the value of k_1 increases the amplitude of the

FIG. 8. Plots of the effective potential and the zero mode for $n_1 = 3$ and $p = \lambda = 1$. (a,b) $k_1 < 0$. (c,d) $k_1 > 0$.

ripples, making them more intense and presenting ripples within the brane. For $f_2(T, B)$ with the first massive values, the massive mode is dependent on the parameter n_2 , which controls the torsion term and presents no amplitude within the brane. Therefore, the brane splitting process leads to modifications of the massive gravitons inside the thick brane. The interaction of the massive modes with the torsion and boundary term is more intense inside the brane core where the amplitude and the rate of growth depend on the parameters $n_{1,2}$ and $k_{1,2}$.

The analysis of the Schrödinger-like potential reveals the effects of the torsion and boundary term on the KK modes. For $f_1(T, B)$ with $n_1 = 2$, increasing k_1 , two new potential barriers appear away from the origin, and the potential well around the origin increases. As a result, the zero mode wave function splits into two peaks. For $n_1 = 3$, when k_1 decreases, the potential well splits in two around the origin, with two new potential barriers, and the zero mode wave function divides into three, with a peak at the origin and the other two further away from the origin.

FIG. 9. Plots of the effective potential (a) and zero mode (b), for $p = \lambda = 1$.

When k_1 increases, two new potential barriers appear away from the origin, and the potential well around the origin increases. As a result, the zero mode wave function splits into two peaks. We find an interesting configuration for $f_2(T, B)$ where the potential goes from a well to a delta-type barrier when we increase the parameter n_2 . As a result, zero modes become nonlocalized for even values of $n_2 = 2, 4, \dots$

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