Chiral separation effect catalyzed by heavy impurities

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(Received 20 January 2021; accepted 23 February 2021; published 29 March 2021)

We investigate the influence of the Kondo effect, namely, the nonperturbative effect induced by heavy impurities, on the chiral separation effect (CSE) in quark matter. We employ a simple effective model incorporating the Kondo condensate made of a light quark and a heavy quark, and compute the response function of the axial current to the magnetic field in the static and dynamical limits. As a result, we find that the Kondo effect catalyzes the CSE in both of the limits, and in particular the CSE in the dynamical limit can be enhanced by a factor of approximately 3. Our findings clearly show that the presence of heavy impurities in quark matter can play an important role in the transport phenomena of light quarks induced by a magnetic field.

DOI: [10.1103/PhysRevD.103.054041](https://doi.org/10.1103/PhysRevD.103.054041)

I. INTRODUCTION

The investigation of currents induced by an external magnetic field is one of the most important subjects in association with heavy-ion collision (HIC) and neutron star (NS) physics. For example, it was proposed that a magnetic field applied in chirality-imbalanced matter gives rise to a vector current, which is called the chiral magnetic effect (CME) [1–4] (see also Refs. [5,6] for earlier works). Another example is the chiral separation effect (CSE) [7–9] which refers to an axial current induced by a magnetic field in ordinary baryonic or quark matter. The CME has been mostly studied with respect to high-energy HICs in which a chirality-imbalanced environment may be created by metastable states related to the θ vacua [10–12]. The CSE is expected to play an important role in low-energy HICs in which a finite baryon density can be created, as well as in NSs.

In quark matter, in addition to light quarks (u , d) with the Fermi surface, heavy quarks (c) may exist as impurities.¹

In such a situation, light quarks near the Fermi surface can be correlated with the heavy quarks in a nonperturbative way due to the non-Abelian interaction with $SU(N_c)$ symmetry (where N_c is the number of colors) of quantum chromodynamics (QCD). As a result, a condensate made of a light quark and a heavy quark is created, which is the so-called *Kondo condensate*. The emergence of the Kondo condensate and phenomena induced by the condensate are called the *Kondo effect*.² While the Kondo effect was originally found in metal including impurities in the context of condensed matter physics [13–17], it has been theoretically shown to emerge in quark matter [18,19] (see Refs. [20–36] for related discussions).³

The Kondo effect is known to characteristically change the transport phenomena. For example, this effect tends to suppress the electric conductivity of charged particles as we lower the temperature. Similarly, we expect that the Kondo effect also influences the transport phenomena induced by a magnetic field. Based on this expectation, in this paper we discuss a modification of the CSE due to the Kondo effect in quark matter by focusing on the response function of the axial current to a magnetic field. In ordinary quark matter where the Kondo condensate is absent, the magnitude of the CSE was found to be slightly modified by perturbative

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¹In low-energy HICs, heavy quarks can be created by hard processes of gluons, and they do not participate in the equilibration of light quarks.

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²The Kondo effect driven in QCD is sometimes called the *QCD Kondo effect*.

³The Kondo effect in high-energy physics emerges not only in quark matter with heavy quarks, but also in nuclear matter with heavy hadrons [18,37–39]. While the former is caused by the QCD interaction, the latter is caused by spin and/or isospin interactions.

corrections in quantum electrodynamics (QED) [40]. In this paper we investigate a modification of the CSE from nonperturbative corrections of QCD driven by the Kondo effect. Hence, we expect a significant modification of the CSE, and in fact this is the case, as we will see later.

To focus on the impact of the Kondo effect on the CSE, we employ a simple effective model that contains light quarks with a finite chemical potential and heavy quarks defined within the heavy quark effective theory (HQET) [41–44], in the presence of the Kondo condensate. To evaluate the response function of the CSE, we make use of the linear response theory [45]. This method yields the CSE in two types of limits, which are the so-called *static limit* and *dynamical limit*. The former (latter) describes an induced current under an external magnetic field whose time dependence is slower (faster) than the equilibration of the spatial part of the system, and in general these two limits give different values of the induced current [46–50]. Thus, we investigate the response function of the CSE in both the static limit and dynamical limit.

This paper is organized as follows. In Sec. II we present the model employed in our analysis and derive the Green's function of fermions modified by the Kondo condensate. In Sec. III we briefly explain the linear response theory and show our strategy to evaluate the CSE. In Sec. IV we present numerical results for the response function of the CSE in both the static limit and dynamical limit. In Sec. V we discuss the gauge invariance of our computation, and in Sec. VI we conclude the present work.

II. MODEL

Here we introduce an effective Lagrangian that we use to investigate the CSE in the presence of the Kondo condensate. In the present work, we are particularly interested in the response of the axial current to an external magnetic field, and thus we do not derive the Kondo condensate in a concrete model, such as a Nambu–Jona-Lasinio–type (NJL-type) model which is a powerful tool for the study of phase structure [21,27]. Alternatively, we simply assume the presence of the Kondo condensate in a reasonable form, for clarity of discussion.

The kinetic terms of massless light quarks at finite density and heavy quarks acting as impurities are given by

$$\mathcal{L}_{\text{kin}} = \bar{\psi}(i\not{D} + \mu\gamma_0)\psi + \Psi_v^\dagger iD_0\Psi_v. \quad (1)$$

In this Lagrangian, while the light-quark field ψ is given as an ordinary Dirac fermion, the heavy-quark field Ψ_v is given within the HQET [41–44], i.e., Ψ_v is related to the ordinary Dirac field Ψ as $\Psi_v = \frac{1+\gamma_0}{2}e^{iM_Q t}\Psi$ (where M_Q is the mass of Ψ and t describes the time of the system) in the rest frame of the heavy quark, which allows us to take into account only the particle component of the heavy quark together with the energy and momentum of order $\Lambda_{\text{QCD}} \cdot \mu$ is a chemical potential related to the density of light quarks.

In Eq. (1), we have introduced covariant derivatives for the quarks to incorporate a magnetic field as

$$\begin{aligned} D_\mu\psi &= (\partial_\mu + ie_q A_\mu)\psi, \\ D_0\Psi_v &= (\partial_0 + ie_Q A_0)\Psi_v \end{aligned} \quad (2)$$

[where $A^\mu = (A_0, \mathbf{A})$]. In Eq. (2), e_q and e_Q are the electric charges of light and heavy quarks, respectively. We note that the magnetic coupling between the heavy quark and the gauge field is absent in the Lagrangian (1) because of the heavy-quark spin symmetry of the HQET [41–44]. This is explicitly shown by the lack of a spatial derivative of Ψ_v in Eq. (1) in the rest frame. In other words, the heavy quarks do not couple with the magnetic field directly.

Field theoretically, the Kondo effect can be described by a condensate formed by a light quark and a heavy quark, referred to as the Kondo condensate, which is analogous to the diquark condensate for color superconductivity [51–54] or the chiral condensate for spontaneous chiral-symmetry breaking [55]. One of the most concise effective terms incorporating the Kondo condensate can be given as

$$\mathcal{L}_\Delta = \Delta_S(\bar{\psi}\Psi_v) + \Delta_V \cdot (\bar{\psi}\boldsymbol{\gamma}\Psi_v) + (\text{c.c.}). \quad (3)$$

Here we have included the vector condensate Δ_V in addition to the scalar one Δ_S , as naturally suggested by the chiral partner structure of HQET [56,57]. Namely, when the chiral symmetry is significantly restored at finite density, the masses of scalar and vector heavy-light modes tend to degenerate due to the heavy-quark spin symmetry and chiral symmetry [58–60], which suggests that these modes play comparable roles in quark matter. We note that pseudoscalar and axial-vector condensates could also be present in Eq. (3), but we have not included them due to the assumption of parity invariance of the ground state. According to Refs. [21,27] based on an NJL-type analysis, the condensates in momentum space may take the ansatz

$$\begin{aligned} \Delta_S &= \Delta, \\ \Delta_V &= \Delta\hat{p} \end{aligned} \quad (4)$$

($\hat{p} = \mathbf{p}/|\mathbf{p}|$), where Δ is a constant providing an order parameter in the ground state. In particular, the second line in Eq. (4) is called the *hedgehog ansatz*.

In what follows, we assume that the light and heavy quarks carry identical electric charge,

$$e \equiv e_q = e_Q, \quad (5)$$

which allows us to avoid complexity caused by the spontaneous breakdown of the electromagnetic $U(1)_{\text{EM}}$ symmetry and the appearance of the so-called Nambu–Goldstone (NG) mode. For example, if we regard the heavy quark as the charm quark, then the light quark is identified as the up quark, leading to $e \equiv e_q = e_Q = +\frac{2}{3}\hat{e}$, where $\hat{e}$ is

the elementary charge. For the bottom quark and down quark, $e \equiv e_q = e_Q = -\frac{1}{3}\hat{e}$.

Next, we derive a Green's function of the fermions incorporating Δ in the absence of the external gauge field $\mathbf{A}$. The effective Lagrangian employed in this study is given by the sum of Eqs. (1) and (3):

$$\tilde{G}_0(p_0, \mathbf{p}) = \frac{i}{(p_0 - E_p^a)(p_0 - E_p^+)(p_0 - E_p^-)} \times \begin{pmatrix} p_0(p_0 + \mu) - |\Delta|^2 & -(|\mathbf{p}|p_0 + |\Delta|^2)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & -\Delta^*(p_0 + |\mathbf{p}| + \mu) \\ (|\mathbf{p}|p_0 + |\Delta|^2)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & -p_0(p_0 + \mu) + |\Delta|^2 & -\Delta^*(p_0 + |\mathbf{p}| + \mu)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \\ -\Delta(p_0 + |\mathbf{p}| + \mu) & \Delta(p_0 + |\mathbf{p}| + \mu)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & (p_0 - |\mathbf{p}| + \mu)(p_0 + |\mathbf{p}| + \mu) \end{pmatrix} \quad (7)$$

in the Dirac representation, with the dispersion relations for the fermions

$$\begin{aligned} E_p^+ &= \frac{1}{2}(|\mathbf{p}| - \mu + \sqrt{(|\mathbf{p}| - \mu)^2 + 8|\Delta|^2}), \\ E_p^- &= \frac{1}{2}(|\mathbf{p}| - \mu - \sqrt{(|\mathbf{p}| - \mu)^2 + 8|\Delta|^2}), \\ E_p^a &= -|\mathbf{p}| - \mu. \end{aligned} \quad (8)$$

We note that the inverse of the Green's function is a 6×6 matrix because the light-quark field ψ is a four-component Dirac spinor while the heavy-quark field Ψ_v is essentially a two-component one carrying only its particle component. Throughout this paper, the modes carrying the dispersion of E_p^+ , E_p^- , and E_p^a are denoted by “q.p.” (Kondo quasiparticle), “q.h.” (Kondo quasihole), and “a.p.” (light antiparticle), respectively. For later use, we plot the schematic behavior of these dispersions in Fig. 1. In this figure, the solid red, dashed blue, and dotted purple curves correspond to $p_0 = E_p^+$ (q.p.), $p_0 = E_p^-$ (q.h.), and $p_0 = E_p^a$ (a.p.), respectively. The gray horizontal line indicates $p_0 = 0$,

FIG. 1. Schematic behavior of the dispersions in Eq. (8) ($p = |\mathbf{p}|$). The solid red, dashed blue, and dotted purple curves correspond to $p_0 = E_p^+$ (q.p.), $p_0 = E_p^-$ (q.h.), and $p_0 = E_p^a$ (a.p.), respectively. The gray horizontal line indicates $p_0 = 0$, corresponding to the Fermi level.

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\Delta}. \quad (6)$$

Thus, by reading quadratic terms of the light and heavy quarks from the Lagrangian (6) without $\mathbf{A}$ in momentum space, we can easily get the Green's function of the fermions as

corresponding to the Fermi level. We find that E_p^+ is always positive, while E_p^- and E_p^a are always negative as long as Δ is finite.

Equation (7) is the Green's function found in previous works [21,27]. In the present paper, we rewrite Eq. (7) in a more convenient form that clearly shows its physical meaning. Namely, we transform Eq. (7) into the following form:

$$\tilde{G}_0(p_0, \mathbf{p}) = \begin{pmatrix} \tilde{G}_0^{\bar{\psi}\psi}(p_0, \mathbf{p}) & \tilde{G}_0^{\bar{\psi}\Psi_v}(p_0, \mathbf{p}) \\ \tilde{G}_0^{\Psi_v^\dagger\psi}(p_0, \mathbf{p}) & \tilde{G}_0^{\Psi_v^\dagger\Psi_v}(p_0, \mathbf{p}) \end{pmatrix}, \quad (9)$$

where the elements are given as

$$\begin{aligned} \tilde{G}_0^{\bar{\psi}\psi}(p_0, \mathbf{p}) &= i \left[\frac{U_+(\mathbf{p})}{p_0 - E_p^+} + \frac{U_-(\mathbf{p})}{p_0 - E_p^-} \right] \Lambda_p + i \frac{U_a(\mathbf{p})}{p_0 - E_p^a} \Lambda_a, \\ \tilde{G}_0^{\bar{\psi}\Psi_v}(p_0, \mathbf{p}) &= i \left[\frac{V_+(\mathbf{p})}{p_0 - E_p^+} + \frac{V_-(\mathbf{p})}{p_0 - E_p^-} \right] \Lambda_{\text{pH}}, \\ \tilde{G}_0^{\Psi_v^\dagger\psi}(p_0, \mathbf{p}) &= i \left[\frac{V_+(\mathbf{p})}{p_0 - E_p^+} + \frac{V_-(\mathbf{p})}{p_0 - E_p^-} \right] \Lambda_{\text{Hp}}, \\ \tilde{G}_0^{\Psi_v^\dagger\Psi_v}(p_0, \mathbf{p}) &= i \left[\frac{W_+(\mathbf{p})}{p_0 - E_p^+} + \frac{W_-(\mathbf{p})}{p_0 - E_p^-} \right] \mathbf{1}. \end{aligned} \quad (10)$$

In these expressions, $\tilde{G}_0^{\bar{\psi}\psi}$, $\tilde{G}_0^{\bar{\psi}\Psi_v}$, $\tilde{G}_0^{\Psi_v^\dagger\psi}$, and $\tilde{G}_0^{\Psi_v^\dagger\Psi_v}$ are 4×4 , 4×2 , 2×4 , and 2×2 matrices, respectively. Also,

$$\Lambda_p \equiv \frac{1 + \hat{\mathbf{p}} \cdot \boldsymbol{\alpha}}{2} \gamma_0, \quad \Lambda_a \equiv \frac{1 - \hat{\mathbf{p}} \cdot \boldsymbol{\alpha}}{2} \gamma_0 \quad (11)$$

($\alpha = \gamma_0 \boldsymbol{\gamma}$) are the projection operators for positive- and negative-energy states of the light quark, respectively. Similarly,

$$\Lambda_{\text{pH}} \equiv \begin{pmatrix} 1 \\ \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}, \quad \Lambda_{\text{Hp}} \equiv (1 \quad -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \quad (12)$$

are the operators mixing positive-energy states of the light quark and heavy quark.⁴ 1 is a 2×2 unit matrix. The weight factors $U(\mathbf{p})$, $V(\mathbf{p})$, and $W(\mathbf{p})$ in Eq. (10) are

$$\begin{aligned} U_+(\mathbf{p}) &= \frac{2(|\Delta|^2 + |\mathbf{p}|E_p^+)}{(E_p^- - E_p^+)(E_p^a - E_p^+)}, \\ U_-(\mathbf{p}) &= \frac{2(|\Delta|^2 + |\mathbf{p}|E_p^-)}{(E_p^+ - E_p^-)(E_p^a - E_p^-)}, \\ U_a(\mathbf{p}) &= 1, \\ V_+(\mathbf{p}) &= \frac{\Delta}{E_p^- - E_p^+}, \quad V_-(\mathbf{p}) = \frac{\Delta}{E_p^+ - E_p^-}, \\ W_+(\mathbf{p}) &= \frac{E_p^+ - |\mathbf{p}| + \mu}{E_p^+ - E_p^-}, \quad W_-(\mathbf{p}) = \frac{E_p^- - |\mathbf{p}| + \mu}{E_p^- - E_p^+}. \end{aligned} \quad (14)$$

Equations (10) and (14) clearly show that only positive-energy components of the light quark correlate with the heavy quark. Namely, $U_a(\mathbf{p})$ in Eq. (14) is not changed from unity. In what follows we assume Δ to be real.

III. ANALYSIS

A. Linear response theory

In Sec. II we introduced an effective Lagrangian with an appropriate ansatz describing the Kondo condensate, and derived the Green's function of fermions. Here, we show our strategy to investigate the CSE in the presence of the Kondo condensate based on these results. In the following calculation, we set $A_0 = 0$ since the magnetic field is not generated by A_0 .

The axial current is defined by $j_5^i = \bar{\psi}\gamma^i\gamma_5\psi$ ($i = 1, 2, 3$) including only the light-quark degrees of freedom. Its expectation value can be given by

$$\langle j_5^i(t, \mathbf{x}) \rangle_\beta = \frac{1}{\Omega} \lim_{q_0, \mathbf{q} \rightarrow 0} \langle \tilde{j}_5^i(q_0, \mathbf{q}) \rangle_\beta, \quad (15)$$

where $\langle \cdots \rangle_\beta$ stands for the thermodynamic expectation value, $\Omega = \beta V$ (where $\beta = 1/T$ is the inverse of temperature and V is the three-dimensional volume), and $\tilde{j}_5^i(q_0, \mathbf{q})$ is the Fourier transformation of $j_5^i(t, \mathbf{x})$. Thus, first we need to evaluate the expectation value of the axial current in momentum space. In the present study, we are particularly interested in quark matter in a magnetic field with magnitude smaller than that of the Fermi energy: $\sqrt{eB} \ll \mu$.

⁴One can understand these operators easily by

$$\Lambda_p \Lambda_H = \begin{pmatrix} 1 & 0 \\ \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \Lambda_H \Lambda_p = \begin{pmatrix} 1 & -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \\ 0 & 0 \end{pmatrix}, \quad (13)$$

with $\Lambda_H = (1 + \gamma_0)/2$.

Therefore, we can evaluate the CSE within the framework of the linear response theory [45].⁵ The linear response theory tells us that the response function to an external field takes the form of a *retarded* function. The retarded function can be given by an analytic continuation of the one computed within the imaginary-time formalism. Thus, $\langle \tilde{j}_5^i(q_0, \mathbf{q}) \rangle_\beta$ on the rhs of Eq. (15) is provided by

$$\langle \tilde{j}_5^i(q_0, \mathbf{q}) \rangle_\beta = \langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta|_{i\bar{\omega}_n \rightarrow q_0 + i\eta}, \quad (16)$$

where η is an infinitesimal positive number, and $\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta$ is given by

$$\begin{aligned} \langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta &\approx e N_c T \sum_m \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\Gamma^i \Gamma_5 \tilde{\mathcal{G}}_0(i\omega_m + i\bar{\omega}_n, \mathbf{p} + \mathbf{q}) \\ &\quad \times \tilde{A}^j(i\bar{\omega}_n, \mathbf{q}) \Gamma^j \tilde{\mathcal{G}}_0(i\omega_m, \mathbf{p})]. \end{aligned} \quad (17)$$

Here, the 6×6 gamma matrices Γ^i and Γ_5 are defined by

$$\begin{aligned} \Gamma^i &= \begin{pmatrix} \gamma^i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i & 0 \\ -\sigma^i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \Gamma_5 &= \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (18)$$

respectively, which act on only light-quark components. Note that $\bar{\omega}_n = 2\pi n/\beta$ is the Matsubara frequency for bosons which shows the periodic boundary condition for the imaginary-time direction, while $\omega_n = (2n+1)\pi/\beta$ is for fermions which shows the antiperiodic boundary condition. N_c is the number of colors, and we set $N_c = 3$ throughout this paper. $\tilde{\mathcal{G}}_0(i\omega_m, \mathbf{p})$ in Eq. (17) can be obtained by simply replacing p_0 with $p_0 \rightarrow i\omega_m$ in Eq. (9).

Before moving on to the detailed calculation of Eq. (17), we mention the zero-momentum limit in Eq. (15). In medium, due to the lack of Lorentz invariance, it is possible to take the following two types of zero-momentum limits:

- (1) *Static limit*: $\langle j_5^i \rangle_\beta^{\text{sta}} \equiv \frac{1}{\Omega} \lim_{\mathbf{q} \rightarrow 0} \langle \tilde{j}_5^i(0, \mathbf{q}) \rangle_\beta$.
- (2) *Dynamical limit*: $\langle j_5^i \rangle_\beta^{\text{dyn}} \equiv \frac{1}{\Omega} \lim_{q_0 \rightarrow 0} \langle \tilde{j}_5^i(q_0, \mathbf{0}) \rangle_\beta$.

The static limit (dynamical limit) describes the response to a magnetic field whose time dependence is slower (faster) than the equilibration of the spatial part of the system. These two limits yield different results in general [46–50]. Therefore, in the following analysis we will investigate the CSE in both limits.

⁵The axial current can be calculated directly, but the computation is rather complicated in general. See Appendix A for another derivation of the static limit of the axial current in the case of $e = e_q = e_Q$.

B. Calculation of $\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta$

In Sec. III A we showed our strategy to evaluate the axial current $\langle j_5^i(t, \mathbf{x}) \rangle_\beta$ within the framework of the linear response theory and presented a rather general expression for the axial current $\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta$ in momentum space in Eq. (17) in the imaginary-time formalism. In this subsection, we continue the analytical evaluation of $\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta$.

By inserting Eq. (9) into Eq. (17), the trace with respect to the Dirac indices in Eq. (18) is reduced to that for the light quark as

$$\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta = eN_c T \sum_m \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\gamma^i \gamma_5 \tilde{G}_0^{\bar{\psi}\psi}(i\omega'_m, \mathbf{p}') \times \tilde{A}^j(i\bar{\omega}_n, \mathbf{q}) \gamma^j \tilde{G}_0^{\psi\bar{\psi}}(i\omega_m, \mathbf{p})] \quad (19)$$

($\mathbf{p}' = \mathbf{p} + \mathbf{q}$ and $i\omega'_m = i\omega_m + i\bar{\omega}_n$). This reduction can be done because the axial current is defined with respect to only the light-quark degrees of freedom, and the magnetic field only couples with them now. Recalling the first line of Eq. (10), the trace calculation is straightforwardly carried out, which yields

$$\begin{aligned} \langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta &= -ieN_c \epsilon^{ijk} \tilde{A}^k(i\bar{\omega}_n, \mathbf{q}) T \sum_m \int \frac{d^3 p}{(2\pi)^3} \\ &\times \sum_{\zeta, \zeta' = +, -, a} (\epsilon_{\zeta'} \hat{p}'^j - \epsilon_\zeta \hat{p}^j) \frac{U_{\zeta'}(\mathbf{p}') U_\zeta(\mathbf{p})}{(i\omega'_m - E_{\mathbf{p}'}^{\zeta'})(i\omega_m - E_{\mathbf{p}}^\zeta)}. \end{aligned} \quad (20)$$

In Eq. (20), the “sign function” ϵ_ζ is defined as

$$\epsilon_+ = \epsilon_- = +1, \quad (21)$$

while

$$\epsilon_a = -1, \quad (22)$$

with the subscripts “+,” “−,” and “a” corresponding to those in Eq. (8). In deriving Eq. (20), we have used the identity

$$\text{tr}[\gamma^i \gamma_5 \Lambda_{\zeta'}^j \gamma^j \Lambda_\zeta^k] = -i\epsilon^{ijk} (\epsilon_{\zeta'} \hat{p}'^k - \epsilon_\zeta \hat{p}^k), \quad (23)$$

with the help of the formula $\text{tr}[\gamma_5 \gamma^0 \gamma^i \gamma^j \gamma^k] = -4i\epsilon^{ijk}$, where $\xi, \xi' = p, a$. In Eq. (23), the “sign function” is

$$\epsilon_p = +1, \quad \epsilon_a = -1, \quad (24)$$

with the subscripts “p” and “a” corresponding to those in Eq. (11).

Equation (20) includes an infinite sum with respect to the Matsubara frequency. This summation can be done by transforming the infinite series into a contour integral in the complex plane. Namely, we make use of the Abel-Plana formula [45]

$$\begin{aligned} T \sum_m F(p_0 = i\omega_m) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dp_0 \frac{1}{2} [F(p_0) + F(-p_0)] \\ &- \frac{1}{2\pi i} \int_{-\infty+i\eta}^{\infty+i\eta} dp_0 [F(p_0) + F(-p_0)] f_F(p_0), \end{aligned} \quad (25)$$

where $F(p_0)$ is an arbitrary function that is analytic on the imaginary axis, and $f_F(p_0) = \frac{1}{e^{p_0} + 1}$ is the Fermi-Dirac distribution function. The calculation can be carried out without difficulty by means of the Cauchy residue theorem, yielding

$$\langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta = ie\epsilon^{ijk} \tilde{A}^k(i\bar{\omega}_n, \mathbf{q}) \tilde{\Pi}_5^j(i\bar{\omega}_n, \mathbf{q}), \quad (26)$$

in which we have defined

$$\tilde{\Pi}_5^j(i\bar{\omega}_n, \mathbf{q}) \equiv -N_c \int \frac{d^3 p}{(2\pi)^3} \sum_{\zeta, \zeta' = +, -, a} (\epsilon_{\zeta'} \hat{p}'^j - \epsilon_\zeta \hat{p}^j) \frac{U_{\zeta'}(\mathbf{p}') U_\zeta(\mathbf{p})}{i\bar{\omega}_n - E_{\mathbf{p}'}^{\zeta'} + E_{\mathbf{p}}^\zeta} [f_F(E_{\mathbf{p}}^\zeta) - f_F(E_{\mathbf{p}'}^{\zeta'})]. \quad (27)$$

In Eq. (26), $\tilde{\Pi}_5^j(i\bar{\omega}_n, \mathbf{q})$ must be proportional to q^j due to its Lorentz structure. Thus, by defining $\tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q})$ in terms of

$$\tilde{\Pi}_5^j(i\bar{\omega}_n, \mathbf{q}) \equiv \tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q}) q^j, \quad (28)$$

Eq. (26) now reads

$$\begin{aligned} \langle \tilde{j}_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta &= ie\epsilon^{ijk} q^j \tilde{A}^k(i\bar{\omega}_n, \mathbf{q}) \tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q}) \\ &= e\tilde{B}^i(i\bar{\omega}_n, \mathbf{q}) \tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q}). \end{aligned} \quad (29)$$

In the second line of Eq. (29), we have made use of $\tilde{B}^i(i\bar{\omega}_n, \mathbf{q}) = ie\epsilon^{ijk} q^j \tilde{A}^k(i\bar{\omega}_n, \mathbf{q})$ [$\tilde{B}^i(q_0, \mathbf{q})$ is defined by the Fourier transformation of $B^i(t, \mathbf{x})$]. We note that $\tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q})$ can be “solved” as

$$\tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q}) = \frac{1}{|\mathbf{q}|^2} \tilde{\Pi}_5^j(i\bar{\omega}_n, \mathbf{q}) q^j \quad (30)$$

from Eq. (28).

Equation (29) shows that $\tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q})$ is the response function of the axial current to a magnetic field.

Concretely, from Eqs. (27) and (30), the response function is expressed as

$$\tilde{\Pi}_5(i\bar{\omega}_n, \mathbf{q}) = \sum_{\zeta, \zeta' = +, -, a} \tilde{\Pi}_5^{\zeta\zeta'}(i\bar{\omega}_n, \mathbf{q}), \quad (31)$$

with

$$\tilde{\Pi}_5^{\zeta\zeta'}(i\bar{\omega}_n, \mathbf{q}) \equiv -\frac{N_c}{|\mathbf{q}|^2} \int \frac{d^3 p}{(2\pi)^3} [\epsilon_{\zeta'}(\hat{\mathbf{p}}' \cdot \mathbf{q}) - \epsilon_{\zeta}(\hat{\mathbf{p}} \cdot \mathbf{q})] \frac{U_{\zeta'}(\mathbf{p}') U_{\zeta}(\mathbf{p})}{i\bar{\omega}_n - E_{\mathbf{p}'}^{\zeta'} + E_{\mathbf{p}}^{\zeta}} (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}'}^{\zeta'})). \quad (32)$$

After performing the analytic continuation to the real-time formalism and properly taking the zero limit of the external momentum ($q_0, \mathbf{q}$), the static and dynamical limits for $\langle j_5^i \rangle_{\beta}$ can be evaluated from Eq. (29) as

$$\begin{aligned} \langle j_5^i(t, \mathbf{x}) \rangle_{\beta}^{\text{sta}} &= eB^i \tilde{\Pi}_5^{\text{sta}}, \\ \langle j_5^i(t, \mathbf{x}) \rangle_{\beta}^{\text{dyn}} &= eB^i \tilde{\Pi}_5^{\text{dyn}}, \end{aligned} \quad (33)$$

respectively, from Eqs. (15) and (16), with the definitions

$$\begin{aligned} \tilde{\Pi}_5^{\text{sta}} &\equiv \lim_{q \rightarrow 0} \tilde{\Pi}_5(0, \mathbf{q}), \\ \tilde{\Pi}_5^{\text{dyn}} &\equiv \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5(q_0, \mathbf{0}). \end{aligned} \quad (34)$$

We also used $\tilde{B}(q_0 = 0, \mathbf{q} = \mathbf{0}) = \Omega B$. By carrying out the angular integral in Eq. (31) in the limits, the response function can be analytically obtained apart from the $|\mathbf{p}|$

integral. We note that a rather detailed discussion of the response function in the static and dynamical limits is given in Appendix B.

At the end of this subsection, we explain the details of the nine contributions in Eq. (32) obtained by setting $\zeta, \zeta' = +, -, a$. Diagrammatically, these contributions can be understood as in Fig. 2. In this figure, the arrows are placed in such a way that the corresponding modes carry their positive energies. We note that the momentum integrals of the loops in Fig. 2 do not include any ultraviolet divergences. According to Eqs. (B11) and (B12) in Appendix B, each contribution in Eq. (32) in the static limit and the dynamical limit after the analytic continuation is found to satisfy

$$\begin{aligned} \lim_{q \rightarrow 0} \tilde{\Pi}_5^{\zeta\zeta'}(0, \mathbf{q}) &= -N_c \int \frac{d^3 p}{(2\pi)^3} \mathcal{I}_{\zeta}(\mathbf{p}) \frac{\partial f_F(E_{\mathbf{p}}^{\zeta})}{\partial |\mathbf{p}|}, \\ \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta\zeta'}(q_0, \mathbf{0}) &= 0, \end{aligned} \quad (35)$$

FIG. 2. Diagrammatic interpretation of each term in Eq. (32). The arrows are placed such that the corresponding modes carry their positive energies.

FIG. 3. μ dependence of the response function in the static limit ($\tilde{\Pi}_5^{\text{sta}}$) with $\Delta = 0.02$ GeV (red dash-dotted), $\Delta = 0.05$ GeV (purple dashed), and $\Delta = 0.1$ GeV (blue solid). The black dashed curve indicates the response function in normal matter given in Eq. (37).

$[\mathcal{I}_\zeta(\mathbf{p})$ is a function of $\mathbf{p}^6$] for $\tilde{\Pi}_5^{++}$, $\tilde{\Pi}_5^{--}$, and $\tilde{\Pi}_5^{\text{aa}}$, while

$$\lim_{\mathbf{q} \rightarrow \mathbf{0}} \tilde{\Pi}_5^{\zeta'\zeta}(0, \mathbf{q}) = \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{0}) \quad (36)$$

for the remaining ones ($\zeta \neq \zeta'$). In other words, only diagrams (i), (ii), and (iii) in Fig. 2—which correspond to the scattering processes of identical particle by the magnetic field, $q.p. \rightarrow q.p.$, $q.h. \rightarrow q.h.$, and $a.p. \rightarrow a.p.$ (or *intra-band scattering processes* in the terminology of condensed matter physics)—can generate a difference between the results in the static limit and the dynamical limit. Such a difference between the static and dynamical limits was also found in Ref. [35]. There we investigated the spin-orbital crossed susceptibility, namely, the response of spin polarization to a magnetic field, in which a relativistic (linear) band and a nonrelativistic band coexist.

IV. NUMERICAL RESULTS

In Sec. III we showed the detailed procedure to compute the axial current induced by a magnetic field in the presence of the Kondo condensate in the static and dynamical limits. In this section we present numerical results for Eq. (34) and examine the impact of the Kondo effect on CSE. For the sake of clarity, here we do not solve the gap equation to determine the magnitude and density dependence of Δ , but simply regard it as a constant parameter.

A. Density dependence of the CSE

First, we show the density dependence of the response function of the CSE obtained in the present work.

In Figs. 3 and 4 we plot the μ dependences of the response function $\tilde{\Pi}_5^{\text{sta}}$ in the static limit and $\tilde{\Pi}_5^{\text{dyn}}$ in the dynamical limit, respectively, with $\Delta = 0.02$ GeV (red), $\Delta = 0.05$ GeV (purple), and $\Delta = 0.1$ GeV (blue). The black dashed lines indicate the response function in normal matter ($\Delta = 0$), i.e.,

⁶The expression for $\mathcal{I}_\zeta(\mathbf{p})$ may be intricate, so we do not show it explicitly here.

$$\tilde{\Pi}_5^{\text{sta}}|_{\Delta=0} = \frac{N_c}{2\pi^2} \mu, \quad (37)$$

$$\tilde{\Pi}_5^{\text{dyn}}|_{\Delta=0} = \frac{N_c}{6\pi^2} \mu, \quad (38)$$

at any temperature. We have plotted this line to more clearly compare the results with and without the Kondo effect. We show the results in the range $0.3 \text{ GeV} < \mu < 0.6 \text{ GeV}$ in which the Kondo condensate is expected to be significant within reasonable model parameters, as suggested by previous works [21,27].

Figures 3 and 4 show that the Kondo effect enhances the CSE in both the static and dynamical limits. In particular, the response function in the dynamical limit can be enhanced by a factor of approximately 3. Besides, the figures show that the results in the static and dynamical limits at $T = 0$ coincide with each other: $\tilde{\Pi}_5^{\text{sta}} = \tilde{\Pi}_5^{\text{dyn}}$. Note that the magnitude of Δ was found to be $\Delta \approx 0.085 \text{ GeV}$ at $\mu = 0.5 \text{ GeV}$ in the previous work based on the NJL-type analysis [21,27]. Therefore, we can conclude that the influence of the Kondo effect on the CSE with a reasonable value of Δ is significantly large, showing that the heavy impurities in quark matter are expected to play an important role in the transport phenomena induced by a magnetic field. The large enhancement of the response function in the dynamical limit is consistent with the finding in Ref. [35].⁷

B. Δ dependence of the CSE

Next, we study the Δ dependence of the CSE to examine the influence of the Kondo condensate Δ on the CSE in more detail.

In Fig. 5 we display the Δ dependence of the response function in the static limit (upper panel) and the dynamical limit (lower panel) at $\mu = 0.5 \text{ GeV}$ and at several temperatures. The black dashed line indicates the response

⁷In Ref. [35] we saw that the response function to a magnetic field in the dynamical limit gets enhanced for general dispersions with a condensate. The large enhancement found in the present study can be understood as the extreme case in the heavy-quark limit, i.e., the heavy-quark dispersion is flat in momentum space.

FIG. 4. μ dependence of the response function in the dynamical limit ($\tilde{\Pi}_5^{\text{dyn}}$) with $\Delta = 0.02$ GeV (red dash-dotted), $\Delta = 0.05$ GeV (purple dashed), and $\Delta = 0.1$ GeV (blue solid). The black dashed curve indicates the response function in normal matter given in Eq. (38).

function in normal matter given in Eq. (37). The lower panel tells us that $\tilde{\Pi}_5^{\text{dyn}}$ for small Δ with $\Delta \neq 0$ is greatly enhanced due to the effect generated by the Kondo condensate, although it converges to the result for normal matter in the

FIG. 5. Gap dependence of the response function in the static limit ($\tilde{\Pi}_5^{\text{sta}}$) (upper panel) and dynamical limit ($\tilde{\Pi}_5^{\text{dyn}}$) (lower panel) at $\mu = 0.5$ GeV and at several temperatures. The black dashed line indicates the response function in normal matter given in Eq. (37). The blue filled and open circles in the dynamical limit show the discontinuous behavior at $T = 0$ (see the main text).

dynamical limit provided in Eq. (38) at $\Delta = 0$. Such a sudden and large enhancement around $\Delta = 0$ stems from the fact that the effect of Δ is nonperturbative around the Fermi level, which can be understood as follows. The CSE is mostly generated by the response of fermions in the vicinity of the Fermi level ($|\mathbf{p}| \approx \mu$) in the momentum integrals of the loops. However, the factor of $\sqrt{(|\mathbf{p}| - \mu)^2 + 8\Delta^2}$ appearing in $E_p^\pm$ prohibits us from expanding $E_p^\pm$ with respect to Δ around $|\mathbf{p}| = \mu$, because the magnitudes of $(|\mathbf{p}| - \mu)^2$ and $8\Delta^2$ are comparable in this case. As a result, the Taylor expansion of the response function with respect to Δ should fail, and hence the modification of $\tilde{\Pi}_5^{\text{dyn}}$ at small Δ is significant.

The signature of the nonperturbative effect from Δ can be seen prominently at $T = 0$ in the lower panel of Fig. 5. Namely, at $T = 0$, $\tilde{\Pi}_5^{\text{dyn}}$ at $\Delta \neq 0$ varies continuously, but at $\Delta = 0$ it discontinuously jumps to the result for normal matter in the dynamical limit. Here we provide some mathematical explanations for the origin of the discontinuity. As shown in Eqs. (35) and (36), the difference between the results in the static and dynamical limits originates from the scattering process of identical particles, i.e., from diagrams (i), (ii), and (iii) in Fig. 2. Below we focus on diagram (i). At zero temperature, from Eq. (35) the response functions from diagram (i) in the static and dynamical limits are of the forms

$$\begin{aligned} \lim_{q \rightarrow 0} \tilde{\Pi}_5^{++}(0, \mathbf{q}) &\stackrel{T=0}{=} -N_c \int \frac{d^3 p}{(2\pi)^3} \mathcal{I}_+(p) \frac{E_p^+}{E_p^+ - E_p^-} \delta(E_p^+), \\ \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{++}(q_0, \mathbf{0}) &\stackrel{T=0}{=} 0, \end{aligned} \quad (39)$$

respectively. These results coincide with each other, even when Δ is infinitesimally small, since E_p^+ and $E_p^+ - E_p^-$ are always positive in this case, resulting in

$$\lim_{q \rightarrow 0} \tilde{\Pi}_5^{++}(0, \mathbf{q}) \stackrel{T=0}{=} 0. \quad (40)$$

Hence, $\tilde{\Pi}_5^{\text{sta}} = \tilde{\Pi}_5^{\text{dyn}}$ holds for $\Delta \neq 0$. On the other hand, when Δ is exactly zero, the upper equation of Eq. (39) is no

longer zero because $E_p^+|_{\Delta=0} = \theta(|p| - \mu)(|p| - \mu)$ and $E_p^-|_{\Delta=0} = \theta(\mu - |p|)(|p| - \mu)$ can reach zero, yielding $\tilde{\Pi}_5^{\text{sta}} \neq \tilde{\Pi}_5^{\text{dyn}}$. As a result, the discontinuous behavior of the blue curve in the lower panel in Fig. 5 at $\Delta \rightarrow 0$ emerges. A similar explanation can also be applied to diagrams (ii) and (iii). The discontinuity at $\Delta = 0$, however, should not be taken too seriously, because in realistic measurements this discontinuity would be smeared by the finite relaxation time.

In association with the sudden and large enhancement at $T \neq 0$ (and the discontinuity at $T = 0$) in the dynamical limit, in Ref. [61] it was found that in normal matter the damping rate in the quark propagator enhances the CSE in the dynamical limit to coincide with that in the static limit. Thus, by focusing on the fact that a nonzero Δ causes a sudden and large enhancement of the response function in the dynamical limit, resulting in behavior similar to the static limit (as shown in Fig. 5), we speculate that there is an analogy between the inclusion of the damping rate in normal matter and the presence of the Kondo condensate Δ . A detailed comparison between the present analysis and that in Ref. [61] is left for a future study.

C. Separate contributions to the CSE

Finally, we give a physical interpretation of how each diagram in Fig. 2 contributes to the response function.

Again, for the sake of clarity, we restrict ourselves to vanishing temperature. At $T = 0$, the Pauli principle forces the intermediate states of the loops to be excited above the Fermi level, i.e., at least one of the modes inside the loop must be q.p. because only this mode lies above the Fermi level, as shown in Fig. 1. In fact, we can confirm that only diagrams (iv) and (v) for q.p. + q.h. and (viii) and (ix) for q.p. + a.p. in Fig. 2 contribute to the CSE through the pair-creation and pair-annihilation processes. We note that the contribution from diagram (i) vanishes at zero temperature, as explained in detail around Eqs. (39) and (40).

In Fig. 6 we show the μ dependence of each process contributing to the response function at $T = 0$ and $\Delta = 0.1$ GeV (upper panel), and their Δ dependence at $T = 0$ and $\mu = 0.5$ GeV (lower panel). We note that the point for $\Delta = 0$ is excluded in the lower panel to avoid the discontinuity explained in Sec. IV B. In Fig. 6, the blue curve indicates the total value of the response function, the magenta curve corresponds to the sum of q.p. + q.h. pair creation and pair annihilation, and the brown curve corresponds to the sum of q.p. + a.p. pair creation and pair annihilation. Only these processes contribute to the response function due to the reason stated above. Note that at zero temperature the response functions in the static and dynamical limits coincide, $\tilde{\Pi}_5^{\text{sta}} = \tilde{\Pi}_5^{\text{dyn}}$, as found in Sec. IV A. Figure 6 shows that the ratio of the contributions from the q.p. + q.h. process and q.p. + a.p. process is approximately 2:1 for any values of μ or Δ . This ratio is easily

FIG. 6. μ dependence at $T = 0$ and $\Delta = 0.1$ GeV of each process contributing to the response function (upper panel), and their Δ dependence at $T = 0$ and $\mu = 0.5$ GeV (lower panel). We compare the total value of the response function (blue solid), the sum of pair-creation and pair-annihilation processes of q.p. + q.h. (magenta dashed), and the sum of pair-creation and pair-annihilation processes of q.p. + a.p. (brown dash-dotted).

understood because the former process includes two modes (both q.p. and q.h.) lying in the vicinity of the Fermi level, while the latter includes only one mode (only q.p.) there.

At finite temperature, the above picture does not change significantly as long as the temperature is smaller than the chemical potential μ . In fact, this is the case in the present work so that the contributions from diagrams (i), (ii), (iii), (vi), and (vii) in Fig. 2 remain small.

V. GAUGE INVARIANCE

Here we discuss the gauge invariance in our calculations. If we assume a *charged* Kondo condensate with $e_q \neq e_Q$, then the condensate carries an electric charge and leads to the spontaneous breakdown of the electromagnetic $U(1)_{\text{EM}}$ invariance. In this case, we have to correct the vertex for the coupling between the quarks and the magnetic field in order to recover the gauge invariance, leading to the appearance

of the NG mode. This treatment is an analogue of the derivation of the NG mode appearing in the theory of superconductivity [62], or that of the pion in chiral symmetry breaking [63]. On the other hand, even when we assume a *charge-neutral* Kondo condensate as in the present study [recall Eq. (5)], the nontrivial structure of the hedgehog ansatz (4) still requires correcting the vertex for the gauge invariance. In actuality, such a vertex correction is found not to considerably affect our results, as will be explained later. Rigorous arguments on the gauge invariance are, however, important and useful in the study of the response to the external gauge field.

The underlying theory of the effective model (6), i.e., QCD and QED, must possess the electromagnetic $U(1)_{\text{EM}}$ gauge invariance. Thus, the Ward-Takahashi identity together with the Dyson-Schwinger equation with respect to the vector-axial vector current correlator tells us that the vertex Γ^μ between the gauge field and the quarks must obey the following identity [64]:

$$q_\mu \Gamma^\mu = -i\mathcal{G}^{-1}(p'_0, \mathbf{p}')Q + Qi\mathcal{G}^{-1}(p_0, \mathbf{p}) \quad (41)$$

($p' = p + q$), where Γ^μ is the 6×6 vertex matrix

$$\Gamma^\mu \equiv \begin{pmatrix} \Gamma_{A\bar{\psi}\psi}^\mu & \Gamma_{A\bar{\psi}\Psi_v}^\mu \\ \Gamma_{A\Psi_v^\dagger\psi}^\mu & \Gamma_{A\Psi_v^\dagger\Psi_v}^\mu \end{pmatrix}. \quad (42)$$

Here, for example, the subscript of $\Gamma_{A\bar{\psi}\psi}^\mu$ represents the type of gauge interaction, such as $A^\mu\bar{\psi}\psi$. Q is a charge matrix,

$$Q = \begin{pmatrix} e_q & 0 \\ 0 & e_Q \end{pmatrix}, \quad (43)$$

where we have assumed that the electric charges of light and heavy quarks are not identical for our general discussion. $i\mathcal{G}^{-1}(p_0, \mathbf{p})$ is the inverse of the full Green's function in general. In the present work, it can be read from the Lagrangian (6) [where we do not necessarily assume Eq. (5)]:

$$i\mathcal{G}^{-1}(p_0, \mathbf{p}) = \begin{pmatrix} i\mathcal{G}_{\bar{\psi}\psi}^{-1}(p_0, \mathbf{p}) & i\mathcal{G}_{\bar{\psi}\Psi_v}^{-1}(p_0, \mathbf{p}) \\ i\mathcal{G}_{\Psi_v^\dagger\psi}^{-1}(p_0, \mathbf{p}) & i\mathcal{G}_{\Psi_v^\dagger\Psi_v}^{-1}(p_0, \mathbf{p}) \end{pmatrix}, \quad (44)$$

with

$$\begin{aligned} i\mathcal{G}_{\bar{\psi}\psi}^{-1}(p_0, \mathbf{p}) &= \not{p} + m\gamma_0, \\ i\mathcal{G}_{\bar{\psi}\Psi_v}^{-1}(p_0, \mathbf{p}) &= \Delta \begin{pmatrix} 1 \\ -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}, \\ i\mathcal{G}_{\Psi_v^\dagger\psi}^{-1}(p_0, \mathbf{p}) &= \Delta(1 \quad \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}), \\ i\mathcal{G}_{\Psi_v^\dagger\Psi_v}^{-1}(p_0, \mathbf{p}) &= p_0 \mathbf{1}. \end{aligned} \quad (45)$$

Thus, each component of the Ward-Takahashi identity in Eq. (41) reads

$$\begin{aligned} q_\mu \Gamma_{A\bar{\psi}\psi}^\mu &= -e_q \not{q}, \\ q_\mu \Gamma_{A\bar{\psi}\Psi_v}^\mu &= \Delta \begin{pmatrix} e_q - e_Q \\ e_Q \hat{\mathbf{p}}' \cdot \boldsymbol{\sigma} - e_q \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}, \\ q_\mu \Gamma_{A\Psi_v^\dagger\psi}^\mu &= \Delta(e_Q - e_q \quad -e_q \hat{\mathbf{p}}' \cdot \boldsymbol{\sigma} + e_Q \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}), \\ q_\mu \Gamma_{A\Psi_v^\dagger\Psi_v}^\mu &= -e_Q q_0 \mathbf{1}. \end{aligned} \quad (46)$$

For the diagonal components, $A^\mu\bar{\psi}\psi$ and $A^\mu\Psi_v^\dagger\Psi_v$, we can safely use their bare vertices:

$$\Gamma_{A\bar{\psi}\psi}^\mu = -e_q \gamma^\mu, \quad \Gamma_{A\Psi_v^\dagger\Psi_v}^\mu = -e_Q \delta^{\mu 0} \mathbf{1}. \quad (47)$$

On the other hand, for the off-diagonal components $A^\mu\bar{\psi}\Psi_v$ and $A^\mu\Psi_v^\dagger\psi$, we cannot use their bare vertices, and they need to be corrected in such a way that they satisfy the identities in Eq. (46).

When $e_q \neq e_Q$, the second and third identities in Eq. (46) suggest the existence of a collective mode (NG mode) coupling with the vertex, because the rhs in the limit of $q_0, \mathbf{q} \rightarrow 0$ does not vanish in this case [64]. The appearance of the NG mode can be also understood by the fact that $\Delta \sim \langle \Psi_v^\dagger \psi \rangle$ with $e_q \neq e_Q$ carries a nonzero electric charge, leading to the spontaneous breakdown of $U(1)_{\text{EM}}$ symmetry. This collective mode (NG mode) is analogous to that in the theory of superconductivity [62].

On the other hand, when $e \equiv e_q = e_Q$ as we have assumed in the present study, the NG mode related to the spontaneous breakdown of $U(1)_{\text{EM}}$ symmetry does not appear. However, modifications of the vertices are still required, as can be seen on the rhs of the second and third identities in Eq. (46):

$$\begin{aligned} q_\mu \Gamma_{A\bar{\psi}\Psi_v}^\mu &= e\Delta \begin{pmatrix} 0 \\ (\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \cdot \boldsymbol{\sigma} \end{pmatrix}, \\ q_\mu \Gamma_{A\Psi_v^\dagger\psi}^\mu &= e\Delta(0 \quad -(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \cdot \boldsymbol{\sigma}). \end{aligned} \quad (48)$$

Obviously, the modifications originate from the nontrivial momentum alignment caused by the hedgehog ansatz (4). Equation (48) can be solved with respect to $\Gamma_{A\bar{\psi}\Psi_v}^\mu$ and $\Gamma_{A\Psi_v^\dagger\psi}^\mu$ by imposing some assumptions. For example, if we restrict ourselves to the vicinity of $q^\mu = 0$, then the Ward-Takahashi identities for $\Gamma_{A\bar{\psi}\Psi_v}^\mu$ and $\Gamma_{A\Psi_v^\dagger\psi}^\mu$ are solved, and we obtain

$$\begin{aligned} \Gamma_{A\bar{\psi}\Psi_v}^\mu &\approx -e\Delta\delta^{i\mu} \begin{pmatrix} 0 \\ \frac{\sigma^i - (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\hat{p}^i}{|\mathbf{p}|} \end{pmatrix}, \\ \Gamma_{A\Psi_v^\dagger\psi}^\mu &\approx e\Delta\delta^{i\mu} \begin{pmatrix} 0 & \frac{\sigma^i - (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\hat{p}^i}{|\mathbf{p}|} \end{pmatrix}. \end{aligned} \quad (49)$$

The vertex corrections in Eq. (49) can give rise to additional contributions to the axial current as

$$\begin{aligned}
& \langle \delta j_5^i(i\bar{\omega}_n, \mathbf{q}) \rangle_\beta \\
& \approx -N_c T \sum_m \int \frac{d^3 p}{(2\pi)^3} \text{tr} \left[\Gamma^i \Gamma_5 \tilde{\mathcal{G}}_0(i\omega_m + i\bar{\omega}_n, \mathbf{p} + \mathbf{q}) \right. \\
& \quad \times \tilde{A}^j(i\bar{\omega}_n, \mathbf{q}) \begin{pmatrix} 0 & \Gamma_{A\bar{\psi}\Psi_v}^j \\ \Gamma_{A\Psi_v^\dagger\psi}^j & 0 \end{pmatrix} \tilde{\mathcal{G}}_0(i\omega_m, \mathbf{p}) \left. \right] \quad (50)
\end{aligned}$$

[compare to Eq. (17)]. Here, we note that the rhs of Eq. (50) is proportional to Δ^2 , because Eq. (49) is proportional to Δ and the computation of the trace in Eq. (50) picks up another Δ in $V_+(\mathbf{p})$ or $V_-(\mathbf{p})$ in Eq. (10). Namely, the additional contributions from the vertex corrections may be small in a realistic situation where Δ is always smaller than μ .

The above discussion suggests that, while the method of deriving the response functions in Sec. IV seems to violate the $U(1)_{\text{EM}}$ gauge invariance due to a lack of vertex corrections, such a violation is expected to be relatively small and has only a subleading effect on our findings as long as we assume $e \equiv e_q = e_Q$. On the other hand, when we take into account the difference between the electric charges of light and heavy quarks, the effects from the vertex correction would become rather important because of the appearance of the NG mode [64]. Therefore, the study of the CSE in the presence of the Kondo condensate with other choices of electric charges would be of interest, and we leave such an examination for a future publication.

VI. CONCLUSIONS

In this paper we studied the axial current induced by a magnetic field, i.e., the chiral separation effect with the Kondo effect. To demonstrate the impact of the Kondo effect on the CSE in a transparent way, we employed a simple effective Lagrangian incorporating a condensate between a light quark and a heavy quark, which is the so-called Kondo condensate.

When we evaluate the CSE by means of the linear response theory, we can define the axial current in two types of limits: the static limit and the dynamical limit. The former (latter) describes an induced current under an external magnetic field whose time dependence is slower (faster) than the equilibration of the spatial part of the system, and in general they yield different results. Thus, we studied the response function of the axial current to the magnetic field in these two limits. We found that the Kondo effect enhances the CSE in both the static and dynamical limits, and in particular the CSE in the dynamical limit can be increased by a factor of approximately 3 in the range of reasonable values of the Kondo condensate. These results clearly show that the Kondo effect arising from heavy impurities in quark matter can play an important role in the transport phenomena of light quarks induced by a magnetic field.

The results obtained in this paper are expected to contribute to a better understanding of transport phenomena induced by a magnetic field in quark matter with heavy impurities. In particular, the CSE for quarks interacting with gluons can be investigated using lattice gauge simulations, as in Refs. [65,66]. It would be interesting to examine the enhancement of the CSE using simulations that include dynamical heavy quarks.

In condensed matter physics, our findings are related to the electron spin polarization (spin magnetization) in the response to a magnetic field in Dirac/Weyl semimetals with a band structure, including both relativistic and nonrelativistic degrees of freedom, since the spin polarization is related to the axial current by spin-momentum locking [35]. In particular, the response function of the CSE to a magnetic field is equivalent to the so-called spin-orbital crossed susceptibility [67–72] which constitutes a part of the magnetic susceptibility, as pointed out in Ref. [35].

We would now like to discuss a few points related to the present study. While we investigated the CSE with the Kondo effect at vanishing magnetic field frequency in this paper, our analysis can be applied to that at finite frequency, as in Refs. [46,49,50,61,73] for the CME and in Ref. [49] for the CSE. Such a study would be important for a more realistic situation, such as (low-energy and peripheral) heavy-ion collisions where the strength of the magnetic field produced after a nucleus-nucleus collision evolves with time.

Moreover, our analysis can be easily applied to the study of the CME [1–4] and the chiral vortical effect (CVE) [74–81] with the Kondo effect. The CME describes the vector current induced by a magnetic field in chirality-imbalanced matter [1–4], whereas the CVE describes the vector or axial current induced by a magnetic field with rotation in ordinary and/or chirality-imbalanced matter [74–81], respectively. Without the Kondo effect, while the CSE and CVE were found to be affected by quark mass [8,40,82–84] or quantum corrections [40], the CME may be protected by the axial anomaly. Thus, the study of the CVE and CME in addition to the CSE, particularly from the viewpoint of the axial anomaly, would enhance our understanding of chiral transport phenomena with the Kondo effect.

ACKNOWLEDGMENTS

D. S. wishes to thank Keio University and Japan Atomic Energy Agency (JAEA) for their hospitalities during his stay there. Y. A. is supported by the Leading Initiative for Excellent Young Researchers (LEADER). K. S. is supported by Japan Society for the Promotion of Science (JSPS) KAKENHI (Grants No. JP17K14277 and No. JP20K14476). S. Y. is supported by JSPS KAKENHI (Grant No. JP17K05435), by the Ministry of Education, Culture, Sports, Science (MEXT)-Supported Program for the Strategic Research Foundation at Private Universities Topological Science (Grant No. S1511006) and by the

Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS) at RIKEN.

APPENDIX A: ANOTHER DERIVATION OF THE CSE IN THE STATIC LIMIT

Here we show another derivation of the CSE for $e = e_q = e_Q$ in the presence of the Kondo condensate in the static limit. It should be noted that the CSE calculated in this Appendix does not include any vertex corrections.

1. Green's function in a magnetic field

To show the CSE in the static limit in an alternative way, we go back to the original Lagrangian (6),

$$\mathcal{L} = \bar{\psi}(i\not{D} + \mu\gamma_0)\psi + \Psi_v^\dagger iD_0 \Psi_v + \Delta[\bar{\psi}(1 - i\boldsymbol{\gamma} \cdot \hat{\mathbf{V}})\Psi_v] + (\text{c.c.}), \quad (\text{A1})$$

and derive the Green's function of a quasiparticle in a magnetic field using a different treatment from that done in the main text. In Eq. (A1), $D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$ is a covariant derivative with respect to the magnetic field and the differential operator $-i\hat{\mathbf{V}} \equiv (-i\nabla)/|-i\nabla|$ corresponds to the hedgehog ansatz defined in the second line of Eq. (4). The presence of a magnetic field breaks the translational invariance perpendicular to the direction of the magnetic field, which suggests that it is not straightforward to obtain the Green's function in momentum space. For this reason, we first consider the Green's function in coordinate space. In what follows, we choose the z direction as the direction of the magnetic field.

The effective Lagrangian (A1) can be written as

$$\mathcal{L} = (\bar{\psi}, \Psi_v^\dagger) i\mathcal{G}^{-1} \begin{pmatrix} \psi \\ \Psi_v \end{pmatrix}, \quad (\text{A2})$$

where

$$i\mathcal{G}^{-1} = i\mathcal{G}_0^{-1} + \mathcal{V} \quad (\text{A3})$$

is the inverse of the Green's function of the fermions (incorporating the gauge field) in coordinate space, with the free quasiparticle part

$$i\mathcal{G}_0^{-1} = \begin{pmatrix} i\partial_0 + \mu & i\nabla \cdot \boldsymbol{\sigma} & \Delta^* \\ -i\nabla \cdot \boldsymbol{\sigma} & -i\partial_0 - \mu & \Delta^* i\hat{\mathbf{V}} \cdot \boldsymbol{\sigma} \\ \Delta & -\Delta i\hat{\mathbf{V}} \cdot \boldsymbol{\sigma} & i\partial_0 \end{pmatrix} \quad (\text{A4})$$

and the contributions from the magnetic field

$$\mathcal{V} = \begin{pmatrix} 0 & e\mathbf{A} \cdot \boldsymbol{\sigma} & 0 \\ -e\mathbf{A} \cdot \boldsymbol{\sigma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A5})$$

Now we take the Landau gauge $\mathbf{A} = (-By, 0, 0)$. In this paper we assume that the magnetic field is so weak that the inverse of the Green's function (A3) can be approximated up to the first order of eB as

$$\mathcal{G} = i(i\mathcal{G}_0^{-1} + \mathcal{V})^{-1} \approx \mathcal{G}_0 + \mathcal{G}_1, \quad (\text{A6})$$

with $\mathcal{G}_1 \equiv \mathcal{G}_0 i\mathcal{V}\mathcal{G}_0$. In coordinate space, the free quasiparticle part $\mathcal{G}_0$ is straightforwardly expressed by means of the Fourier transformation thanks to the translational invariance as

$$\mathcal{G}_0(r, r') = \int \frac{d^4p}{(2\pi)^4} \tilde{\mathcal{G}}_0(p_0, \mathbf{p}) e^{-ip \cdot (r - r')}, \quad (\text{A7})$$

which in momentum space has the form

$$\tilde{\mathcal{G}}_0(p_0, \mathbf{p}) = \frac{i}{(p_0 - E_p^a)(p_0 - E_p^+)(p_0 - E_p^-)} \times \begin{pmatrix} p_0(p_0 + \mu) - |\Delta|^2 & -[|\mathbf{p}|p_0 + |\Delta|^2]\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & -\Delta^*(p_0 + |\mathbf{p}| + \mu) \\ [|\mathbf{p}|p_0 + |\Delta|^2]\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & -p_0(p_0 + \mu) + |\Delta|^2 & -\Delta^*(p_0 + |\mathbf{p}| + \mu)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \\ -\Delta(p_0 + |\mathbf{p}| + \mu) & \Delta(p_0 + |\mathbf{p}| + \mu)\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & (p_0 - |\mathbf{p}| + \mu)(p_0 + |\mathbf{p}| + \mu) \end{pmatrix}, \quad (\text{A8})$$

with the dispersion relations of the quasiparticles being

$$\begin{aligned} E_p^+ &= \frac{1}{2} \left(|\mathbf{p}| - \mu + \sqrt{(|\mathbf{p}| - \mu)^2 + 8|\Delta|^2} \right), \\ E_p^- &= \frac{1}{2} \left(|\mathbf{p}| - \mu - \sqrt{(|\mathbf{p}| - \mu)^2 + 8|\Delta|^2} \right), \\ E_p^a &= -|\mathbf{p}| - \mu. \end{aligned} \quad (\text{A9})$$

These equations coincide with Eqs. (7) and (8). In what follows we assume Δ to be real.

Unlike $\mathcal{G}_0$, the $\mathcal{G}_1$ part in Eq. (A6) is rather complicated due to the violation of the translational invariance. In fact, by defining the 6×6 matrix

$$\mathcal{V} = \begin{pmatrix} 0 & -\sigma^1 & 0 \\ \sigma^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{A10})$$

for convenience, the $\mathcal{G}_1$ part in Eq. (A6) can be evaluated as

$$\begin{aligned}
\mathcal{G}_1(r, r') &= ieB \int d^4 r'' \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \mathcal{G}_0(p) e^{-ip \cdot (r-r'')} \mathcal{V} \mathcal{G}_0(p') y'' e^{-ip' \cdot (r''-r')} \\
&= -eB \int d^4 r'' \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \mathcal{G}_0(p) \mathcal{V} \mathcal{G}_0(p') [\partial_{p_y - p'_y} e^{i(p-p') \cdot r''}] e^{-ip \cdot r + ip' \cdot r'} \\
&= eB \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \partial_{p_y - p'_y} \left[\frac{1}{\mathcal{G}_0(p)} \mathcal{V} \frac{1}{\mathcal{G}_0(p')} e^{-ip \cdot r + ip' \cdot r'} \right] (2\pi)^4 \delta^4(p - p') \\
&= \frac{eB}{2} \int \frac{d^4 p}{(2\pi)^4} \{ [\partial_{p_y} \mathcal{G}_0(p)] \mathcal{V} \mathcal{G}_0(p) - \mathcal{G}_0(p) \mathcal{V} [\partial_{p_y} \mathcal{G}_0(p)] + i(y + y') \mathcal{G}_0(p) \mathcal{V} \mathcal{G}_0(p) \} e^{-ip \cdot (r-r')} \quad (A11)
\end{aligned}$$

with the help of Eq. (A7). To obtain the third line of Eq. (A11), we have made use of integration by parts. The last term in the fourth line in Eq. (A11), which is proportional to $y + y'$, obviously breaks the translational invariance, which stems from the so-called Schwinger phase [85]. Here we ignore such a term [40,86], and Eq. (A11) takes the form

$$\mathcal{G}_1(r, r') = \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{G}}_1(p_0, \mathbf{p}) e^{-ip \cdot (r-r')}, \quad (A12)$$

where the counterpart in momentum space is given by

$$\tilde{\mathcal{G}}_1(p_0, \mathbf{p}) = -\frac{eB}{2} \frac{\mathcal{I}}{|\mathbf{p}|^3 (p_0 - E_p^a)^2 (p_0 - E_p^+)^2 (p_0 - E_p^-)^2} \quad (A13)$$

with

$$\mathcal{I} \equiv \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 & 0 & \mathcal{I}_4 & \mathcal{I}_5 \\ -\mathcal{I}_2 & -\mathcal{I}_1 & 0 & \mathcal{I}_3 & -\mathcal{I}_5 & \mathcal{I}_4^* \\ -\mathcal{I}_3 & 0 & -\mathcal{I}_1 & -\mathcal{I}_2 & -i\mathcal{I}_6 & \mathcal{I}_7 \\ 0 & -\mathcal{I}_3 & \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{I}_7^* & -i\mathcal{I}_6^* \\ -\mathcal{I}_4^* & \mathcal{I}_5 & i\mathcal{I}_6^* & \mathcal{I}_7 & \mathcal{I}_8 & \mathcal{I}_9 \\ -\mathcal{I}_5 & -\mathcal{I}_4 & \mathcal{I}_7^* & i\mathcal{I}_6 & -\mathcal{I}_9 & -\mathcal{I}_8 \end{pmatrix}, \quad (A14)$$

and the matrix elements are

$$\mathcal{I}_1 = -2i[\Delta^2 - p_0(p_0 + \mu)][\Delta^2(p_x^2 + p_z^2) + |\mathbf{p}|^3 p_0], \quad (A15)$$

$$\mathcal{I}_2 = -2p_y p_z \Delta^2 [\Delta^2 - p_0(p_0 + \mu)], \quad (A16)$$

$$\mathcal{I}_3 = -2ip_z |\mathbf{p}| (\Delta^2 + |\mathbf{p}| p_0)^2, \quad (A17)$$

$$\begin{aligned}
\mathcal{I}_4 &= -\Delta \{ |\mathbf{p}| p_y (2\Delta^2 + p_0 |\mathbf{p}|) (p_x - ip_y) \\
&\quad + ip_0 [|\mathbf{p}|^2 (p_0 + |\mathbf{p}| + \mu)^2 + ip_y (p_0 + \mu)^2 (p_x + ip_y)] \\
&\quad + 2p_x p_y \Delta^2 (p_0 + \mu) \}, \quad (A18)
\end{aligned}$$

$$\mathcal{I}_5 = -p_y p_z \Delta \{ 2\Delta^2 |\mathbf{p}| + p_0 [(p_0 + \mu)^2 + |\mathbf{p}|^2] \}, \quad (A19)$$

$$\begin{aligned}
\mathcal{I}_6 &= 2\Delta p_z \{ |\mathbf{p}| (p_0 + |\mathbf{p}| + \mu) (\Delta^2 + |\mathbf{p}| p_0) \\
&\quad - ip_x p_y [\Delta^2 - p_0(p_0 + \mu)] \}, \quad (A20)
\end{aligned}$$

$$\mathcal{I}_7 = 2\Delta p_y [p_z^2 + ip_y (p_x - ip_y)] [\Delta^2 - p_0(p_0 + \mu)], \quad (A21)$$

$$\mathcal{I}_8 = 2i\Delta^2 (p_x^2 + p_z^2) (p_0 + |\mathbf{p}| + \mu)^2, \quad (A22)$$

$$\mathcal{I}_9 = 2\Delta^2 p_y p_z (p_0 + |\mathbf{p}| + \mu)^2. \quad (A23)$$

Summarizing the above derivation, the Green's function of the quasiparticle in a weak magnetic field in momentum space $\tilde{\mathcal{G}}(p_0, \mathbf{p})$ can be written as

$$\tilde{\mathcal{G}}(p_0, \mathbf{p}) \approx \tilde{\mathcal{G}}_0(p_0, \mathbf{p}) + \tilde{\mathcal{G}}_1(p_0, \mathbf{p}), \quad (A24)$$

with $\tilde{\mathcal{G}}_0(p_0, \mathbf{p})$ and $\tilde{\mathcal{G}}_1(p_0, \mathbf{p})$ being given by Eqs. (A8) and (A13), respectively, apart from the Schwinger phase.

Note that when the Kondo condensate is absent, i.e., when we take $\Delta = 0$ in Eq. (A24), the Green's function is reduced to

$$\tilde{\mathcal{G}}(p_0, \mathbf{p}) \rightarrow \begin{pmatrix} \tilde{\mathcal{G}}_\psi(p_0, \mathbf{p}) & 0 \\ 0 & \tilde{\mathcal{G}}_{\Psi_r}(p_0) \end{pmatrix}, \quad (A25)$$

with

$$\tilde{\mathcal{G}}_\psi(p_0, \mathbf{p}) = \frac{i(\not{p} + \mu\gamma^0)}{(p_0 + \mu)^2 - |\mathbf{p}|^2} + eB\gamma^1 \gamma^2 \frac{(p_0 + \mu)\gamma^0 - p_z \gamma^3}{[(p_0 + \mu)^2 - |\mathbf{p}|^2]^2}, \quad (A26)$$

$$\tilde{\mathcal{G}}_{\Psi_r}(p_0) = \frac{i}{p_0}. \quad (A27)$$

Our result (A26) is consistent with that found in Appendix A of Ref. [86].

2. Calculation of the axial current

In this subsection, we calculate the axial current by making use of the Green's function obtained in Appendix A 1.

From Eq. (A24), by employing the imaginary-time formalism the axial current is now given by

$$\langle j_5^i(t, \mathbf{x}) \rangle = -iN_c T \sum_m \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\Gamma^i \Gamma_5 \tilde{\mathcal{G}}(i\omega_m, \mathbf{p})] \quad (\text{A28})$$

$[\omega_m = (2m + 1)\pi T]$, with

$$\Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i & 0 \\ -\sigma^i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A29})$$

Here the nonzero component is given only for $i = 3$, which is of the form

$$\langle j_5^{i=3} \rangle = 4N_c e B T \sum_m \int \frac{d^3 p}{(2\pi)^3} \times \frac{[\Delta^2 - i\omega_m(i\omega_m + \mu)] \left[\frac{2}{3} \Delta^2 |\mathbf{p}|^2 + i\omega_m |\mathbf{p}|^3 \right]}{|\mathbf{p}|^3 (i\omega_m - E_p^a)^2 (i\omega_m - E_p^+)^2 (i\omega_m - E_p^-)^2}. \quad (\text{A30})$$

Therefore, the Abel-Plana formula (25) enables us to carry out the Matsubara summation straightforwardly. The resulting expression is lengthy and not illuminating, so we do not show it explicitly here, but it was found to be the same as that obtained in the static limit within the procedure employed in the main text.

APPENDIX B: GENERAL DISCUSSIONS ON EACH CONTRIBUTION OF THE RESPONSE FUNCTION IN EQ. (32)

In this Appendix we give a general discussion of each contribution of the response function in Eq. (32) in more detail, and explain the difference between the static and dynamical limits. For convenience, we again show Eq. (32) here:

$$\tilde{\Pi}_5^{\zeta'\zeta}(i\bar{\omega}_n, \mathbf{q}) = -\frac{N_c}{|\mathbf{q}|^2} \int \frac{d^3 p}{(2\pi)^3} [\epsilon_{\zeta'}(\hat{\mathbf{p}}' \cdot \mathbf{q}) - \epsilon_{\zeta}(\hat{\mathbf{p}} \cdot \mathbf{q})] \times \frac{U_{\zeta'}(\mathbf{p}') U_{\zeta}(\mathbf{p})}{i\bar{\omega}_n - E_{\mathbf{p}'}^{\zeta'} + E_{\mathbf{p}}^{\zeta}} (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}'}^{\zeta'})), \quad (\text{B1})$$

with $\zeta, \zeta' = +, -, a$.

With respect to Eq. (B1), in order to properly evaluate the response function we need to take the $q_0, \mathbf{q} \rightarrow 0$ limit after the analytic continuation, as seen from Eq. (34) together with Eq. (31). Here, such a limit suggests that the nine contributions in Eq. (B1) are mathematically classified into the following three cases:

- (I) The $\zeta = \zeta'$ case ($\tilde{\Pi}_5^{++}, \tilde{\Pi}_5^{--}, \tilde{\Pi}_5^{aa}$).
- (II) The $\zeta \neq \zeta'$ and $\epsilon_{\zeta} = \epsilon_{\zeta'}$ case ($\tilde{\Pi}_5^{+-}, \tilde{\Pi}_5^{-+}$).
- (III) The $\zeta \neq \zeta'$ and $\epsilon_{\zeta} \neq \epsilon_{\zeta'}$ case ($\tilde{\Pi}_5^{+a}, \tilde{\Pi}_5^{a+}, \tilde{\Pi}_5^{-a}, \tilde{\Pi}_5^{a-}$).

Recall that the physical meaning of these processes is given in Fig. 2.

First, in case (I) we find

$$\epsilon_{\zeta}(\hat{\mathbf{p}}' \cdot \mathbf{q}) - \epsilon_{\zeta}(\hat{\mathbf{p}} \cdot \mathbf{q}) = C_{\zeta}^{(2)} |\mathbf{q}|^2 + \mathcal{O}(\mathbf{q}^3),$$

$$q_0 - E_{\mathbf{p}'}^{\zeta} + E_{\mathbf{p}}^{\zeta} = q_0 + D_{\zeta}^{(1)} |\mathbf{q}| + \mathcal{O}(\mathbf{q}^2), \quad (\text{B2})$$

where the coefficients $C_{\zeta}^{(2)}$ and $D_{\zeta}^{(1)}$ are independent of $\mathbf{q}$ but dependent on $\mathbf{p}$ (the number in the parentheses corresponds to the order of $\mathbf{q}$). In this Appendix we do not show the explicit expressions of the coefficients because they are not important, and it is enough to consider only the order of expansion with respect to $\mathbf{q}$.

From Eq. (B2), the real part of $\tilde{\Pi}_5^{\zeta'\zeta}$ at small $(q_0, \mathbf{q})$ takes the form

$$\begin{aligned} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{q}) &\approx -\frac{N_c}{|\mathbf{q}|^2} \int \frac{d^3 p}{(2\pi)^3} C_{\zeta}^{(2)} |\mathbf{q}|^2 \frac{U_{\zeta}(\mathbf{p}) U_{\zeta}(\mathbf{p})}{q_0 + D_{\zeta}^{(1)} |\mathbf{q}|} \\ &\times \left[-(\hat{\mathbf{p}} \cdot \mathbf{q}) \frac{\partial f_F(E_{\mathbf{p}}^{\zeta})}{\partial |\mathbf{p}|} \right] \\ &= -N_c \int \frac{d^3 p}{(2\pi)^3} \frac{N_{\zeta}^{(1)} |\mathbf{q}|}{q_0 + D_{\zeta}^{(1)} |\mathbf{q}|} \frac{\partial f_F(E_{\mathbf{p}}^{\zeta})}{\partial |\mathbf{p}|} \quad (\text{B3}) \end{aligned}$$

using the Taylor expansion. In Eq. (B3) we have defined $N_{\zeta}^{(1)} \equiv -(\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) C_{\zeta}^{(2)} U_{\zeta}(\mathbf{p}) U_{\zeta}(\mathbf{p})$ ($\hat{\mathbf{q}} \equiv \mathbf{q}/|\mathbf{q}|$). Therefore, Eq. (B3) yields

$$\lim_{\mathbf{q} \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(0, \mathbf{q}) = -N_c \int \frac{d^3 p}{(2\pi)^3} \frac{N_{\zeta}^{(1)}}{D_{\zeta}^{(1)}} \frac{\partial f_F(E_{\mathbf{p}}^{\zeta})}{\partial |\mathbf{p}|},$$

$$\lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{0}) = 0, \quad (\text{B4})$$

which clearly shows that the results in the static and dynamical limits differ due to a singularity caused by the factor $1/(q_0 + D_{\zeta}^{(1)} |\mathbf{q}|)$ in the $q_0, \mathbf{q} \rightarrow 0$ limit.

Next, in case (II) we find

$$\epsilon_{\zeta'}(\hat{\mathbf{p}}' \cdot \mathbf{q}) - \epsilon_{\zeta}(\hat{\mathbf{p}} \cdot \mathbf{q}) = C_{\zeta'\zeta}^{(2)} |\mathbf{q}|^2 + \mathcal{O}(\mathbf{q}^3),$$

$$q_0 - E_{\mathbf{p}'}^{\zeta'} + E_{\mathbf{p}}^{\zeta} = q_0 + D_{\zeta'\zeta}^{(0)} + \mathcal{O}(\mathbf{q}^1), \quad (\text{B5})$$

so that the same expansion as in Eq. (B3) reads

$$\begin{aligned}
\tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{q}) &\approx -\frac{N_c}{|\mathbf{q}|^2} \int \frac{d^3 p}{(2\pi)^3} C_{\zeta'\zeta}^{(2)} |\mathbf{q}|^2 \frac{U_{\zeta'}(\mathbf{p}) U_{\zeta}(\mathbf{p})}{q_0 + D_{\zeta'\zeta}^{(0)}} \\
&\quad \times (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'})) \\
&\equiv -N_c \int \frac{d^3 p}{(2\pi)^3} \frac{N_{\zeta'\zeta}^{(0)}}{q_0 + D_{\zeta'\zeta}^{(0)}} \\
&\quad \times (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'})) \quad (\text{B6})
\end{aligned}$$

$[N_{\zeta'\zeta}^{(0)} \equiv C_{\zeta'\zeta}^{(2)} U_{\zeta'}(\mathbf{p}) U_{\zeta}(\mathbf{p})]$. Namely, in this case $D_{\zeta'\zeta}^{(0)}$ in the denominator acts as a regulator preventing us from getting the singularity, which means that there is no difference between the static and dynamical limits:

$$\begin{aligned}
\lim_{q \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(0, \mathbf{q}) &= \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{0}) \\
&= -N_c \int \frac{d^3 p}{(2\pi)^3} \frac{C_{\zeta'\zeta}^{(0)}}{D_{\zeta'\zeta}^{(0)}} (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'})). \quad (\text{B7})
\end{aligned}$$

Finally, in case (III) we get

$$\begin{aligned}
\epsilon_{\zeta'}(\hat{\mathbf{p}}' \cdot \mathbf{q}) - \epsilon_{\zeta}(\hat{\mathbf{p}} \cdot \mathbf{q}) &= C_{\zeta'\zeta}^{(1)} |\mathbf{q}| + C_{\zeta'\zeta}^{(2)} |\mathbf{q}|^2 + \mathcal{O}(\mathbf{q}^3), \\
q_0 - E_{\mathbf{p}'}^{\zeta'} + E_{\mathbf{p}}^{\zeta} &= q_0 + D_{\zeta'\zeta}^{(0)} + \mathcal{O}(\mathbf{q}^1), \quad (\text{B8})
\end{aligned}$$

so that the same expansion as in Eq. (B3) leads to a slightly complicated form:

$$\begin{aligned}
\tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{q}) &\approx -\frac{N_c}{|\mathbf{q}|^2} \int \frac{d^3 p}{(2\pi)^3} (C_{\zeta'\zeta}^{(1)} |\mathbf{q}| + C_{\zeta'\zeta}^{(2)} |\mathbf{q}|^2) \\
&\quad \times \frac{[U_{\zeta'}(\mathbf{p}) + (\hat{\mathbf{p}} \cdot \mathbf{q}) \frac{\partial U_{\zeta'}(\mathbf{p})}{\partial |\mathbf{p}|}] U_{\zeta}(\mathbf{p})}{q_0 + D_{\zeta'\zeta}^{(0)}} \\
&\quad \times \left[f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'}) - (\hat{\mathbf{p}} \cdot \mathbf{q}) \frac{\partial f_F(E_{\mathbf{p}}^{\zeta'})}{\partial |\mathbf{p}|} \right] \\
&\approx -N_c \int \frac{d^3 p}{(2\pi)^3} \frac{\frac{1}{|\mathbf{q}|} N_{\zeta'\zeta}^{(-1)} + N_{\zeta'\zeta}^{(0)}}{q_0 + D_{\zeta'\zeta}^{(0)}} \\
&\quad \times \left[f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'}) - (\hat{\mathbf{p}} \cdot \mathbf{q}) \frac{\partial f_F(E_{\mathbf{p}}^{\zeta'})}{\partial |\mathbf{p}|} \right]. \quad (\text{B9})
\end{aligned}$$

Here we have defined $N_{\zeta'\zeta}^{(-1)} \equiv C_{\zeta'\zeta}^{(1)} U_{\zeta'}(\mathbf{p}) U_{\zeta}(\mathbf{p})$ and $N_{\zeta'\zeta}^{(0)} \equiv (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) C_{\zeta'\zeta}^{(1)} \frac{\partial U_{\zeta'}(\mathbf{p})}{\partial |\mathbf{p}|} U_{\zeta}(\mathbf{p}) + C_{\zeta'\zeta}^{(2)} U_{\zeta'}(\mathbf{p}) U_{\zeta}(\mathbf{p})$. Equation (B9) seemingly causes a divergence in the $\mathbf{q} \rightarrow 0$ limit due to the $1/|\mathbf{q}|$ in the numerator. However, such a problematic contribution will be removed after the angular integral. Namely, in both the static and dynamical limits, Eq. (B9) converges, leading to the same result as

$$\begin{aligned}
\lim_{q \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(0, \mathbf{q}) &= \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{0}) \\
&= -N_c \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{N_{\zeta'\zeta}^{(0)}}{D_{\zeta'\zeta}^{(0)}} (f_F(E_{\mathbf{p}}^{\zeta}) - f_F(E_{\mathbf{p}}^{\zeta'})) \right. \\
&\quad \left. - (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \frac{N_{\zeta'\zeta}^{(-1)}}{D_{\zeta'\zeta}^{(0)}} \frac{\partial f_F(E_{\mathbf{p}}^{\zeta'})}{\partial |\mathbf{p}|} \right\}. \quad (\text{B10})
\end{aligned}$$

Summarizing the above discussions, we have confirmed that

$$\begin{aligned}
\lim_{q \rightarrow 0} \tilde{\Pi}_5^{\zeta\zeta}(0, \mathbf{q}) &= -N_c \int \frac{d^3 p}{(2\pi)^3} \mathcal{I}_{\zeta}(\mathbf{p}) \frac{\partial f_F(E_{\mathbf{p}}^{\zeta})}{\partial |\mathbf{p}|}, \\
\lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta\zeta}(q_0, \mathbf{0}) &= 0 \quad (\text{B11})
\end{aligned}$$

for case (I) from Eq. (B4), while

$$\lim_{q \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(0, \mathbf{q}) = \lim_{q_0 \rightarrow 0} \tilde{\Pi}_5^{\zeta'\zeta}(q_0, \mathbf{0}) \quad (\text{B12})$$

for cases (II) and (III) from Eqs. (B7) and (B10), respectively. In Eq. (B11), we have defined $\mathcal{I}_{\zeta}(\mathbf{p}) \equiv N_{\zeta}^{(1)}/D_{\zeta}^{(1)}$ to simplify the expression. Equations (B11) and (B12) show that the difference between the response functions in the static and dynamical limits can be generated in only case (I), which corresponds to the scattering process of q.p. $\rightarrow$ q.p., q.h. $\rightarrow$ q.h., or a.p. $\rightarrow$ a.p.

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