On-shell action and the Bekenstein-Hawking entropy of supersymmetric black holes in AdS_6 spacetimes

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Recently, entropy of static supersymmetric black holes in AdS_6 was microscopically counted by the topologically twisted index of five-dimensional superconformal field theories. However, the AdS/CFT dictionary states that the partition function of the boundary field theory corresponds to the on-shell action of the bulk quantum gravity. In this paper, we aim to explain the microscopic counting of black hole entropy by the topologically twisted index. We calculate the renormalized on-shell action and show that the on-shell action equals the Bekenstein-Hawking entropy of the supersymmetric black holes in AdS_6 .

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I. INTRODUCTION AND CONCLUSIONS

Recently, via the AdS/CFT correspondence, [1–3], there has been great success in microscopic counting of the Bekenstein-Hawking entropy of supersymmetric black holes in AdS_4 , [4–6], by a topologically twisted index of three-dimensional quantum field theories, [7,8]. Further progress has been made for supersymmetric black holes in AdS_6 [9–11], by the topologically twisted index of five-dimensional superconformal field theories [12,13].

However, the AdS/CFT dictionary [1–3] states that the partition function of the boundary field theory corresponds to the on-shell action of the bulk quantum gravity,

$$-\log Z_{\text{CFT}} = S_{\text{gravity}}. \quad (1.1)$$

In this paper, we show that the on-shell action equals the Bekenstein-Hawking entropy of supersymmetric black holes by holographic renormalization [14,15]. We closely follow the parallel calculation performed for supersymmetric black holes in AdS_4 in [16]. See [17,18] for the related works in AdS_4 .

In addition, we compute the on-shell actions of the supersymmetric black strings and branes which are interpolating solutions between the boundary of AdS_6 and the horizon of $\text{AdS}_3 \times H^3$ and $\text{AdS}_4 \times \Sigma_g$, respectively, [19]. The on-shell actions of the black strings and branes reproduce the central charge and the three-sphere free energy of 2d and 3d field theories, respectively [20].

We make some comments on our results. First, we do not need the explicit form of the black hole solutions and the boundary asymptotics will be enough. Second, the supersymmetry of the black hole solutions is essential. The calculation uses the supersymmetry equations in the course of holographic renormalization to obtain the on-shell action. Third, we find that the counterterms precisely match the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22]. They were already employed for calculations of supersymmetric Renyi entropy in [23,24]. It gives a nontrivial verification of counterterms we employ. Fourth, we calculate the on-shell actions of supersymmetric AdS_6 black holes, strings and branes. It reveals a uniform structure of on-shell actions of black objects obtained from topological twist in $F(4)$ gauged supergravity [25].

In Sec. II, we present $F(4)$ gauged supergravity in the Euclidean spacetime. In Sec. III, for the supersymmetric AdS_6 black hole solution from $F(4)$ gauged supergravity [9], we calculate the renormalized on-shell action and prove that it equals the Bekenstein-Hawking entropy of the black holes. In Secs. IV and V, we calculate the on-shell actions of supersymmetric black string and brane solutions [19] and show that they reproduce the central charge and the three-sphere free energy of 2d and 3d dual field theories, respectively. In Appendix A, we review the counterterms for pure AdS gravity theories in diverse dimensions. In Appendix B, we review the counterterms for $F(4)$ gauged supergravity in [21,22].

II. EUCLIDEAN $F(4)$ GAUGED SUPERGRAVITY

We present $F(4)$ gauged supergravity [25] in Euclidean spacetime. The Euclidean continuation is performed by a Wick rotation,

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$$t \rightarrow i\tau. \quad (2.1)$$

As the signature of $F(4)$ gauged supergravity in [25] is $(+-----)$, we obtain the Euclidean theory in the signature of $(-----)$. It accompanies the Wick rotation of the time component of gamma matrices,

$$\gamma_t \rightarrow -i\gamma_\tau, \quad (2.2)$$

which renders the anticommutation relations to be

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \quad (2.3)$$

According to the Euclidean continuation, we employ the chirality operator to be

$$\gamma_7 = -\gamma_i \gamma_{\hat{x}_1} \gamma_{\hat{x}_2} \gamma_{\hat{x}_3} \gamma_{\hat{x}_4} \gamma_{\hat{x}_5} \rightarrow i\gamma_i \gamma_{\hat{x}_1} \gamma_{\hat{x}_2} \gamma_{\hat{x}_3} \gamma_{\hat{x}_4} \gamma_{\hat{x}_5}, \quad (2.4)$$

where the hatted indices are of the flat spacetime. Also the time components of the fields are Wick rotated, e.g.,

$$A_t \rightarrow -iA_\tau, \quad B_{t\mu} \rightarrow -iB_{\tau\mu}. \quad (2.5)$$

The bosonic field content consists of the metric, $g_{\mu\nu}$, a real scalar, ϕ , an $SU(2)$ gauge field, A_μ^I , $I = 1, 2, 3$, a $U(1)$ gauge field, $\mathcal{A}_\mu$, and a two-form field, $B_{\mu\nu}$. The fermionic field content is gravitinos, $\psi_{\mu i}$, and dilatino, χ_i , $i = 1, 2$. The field strengths are defined by

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \\ F_{\mu\nu}^I &= \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + \tilde{g}\epsilon^{IJK} A_\mu^J A_\nu^K, \\ G_{\mu\nu\rho} &= 3\partial_{[\mu} B_{\nu\rho]}, \\ \mathcal{H}_{\mu\nu} &= \mathcal{F}_{\mu\nu} + mB_{\mu\nu}. \end{aligned} \quad (2.6)$$

Implicitly implying (2.1) and (2.2), the apparent expressions of the action, equations of motion, and supersymmetry variations in Euclidean spacetime are identical to the Lorentzian ones up to some subtleties we describe. The bosonic Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= -\frac{1}{4}R + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{8}(\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g}me^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}) - \frac{1}{4}e^{-\sqrt{2}\phi}(\mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu} + F_{\mu\nu}^I F^{I\mu\nu}) + \frac{1}{12}e^{2\sqrt{2}\phi}G_{\mu\nu\rho}G^{\mu\nu\rho} \\ &\quad - \frac{i}{8}\epsilon^{\mu\nu\rho\sigma\tau\kappa}B_{\mu\nu}\left(\mathcal{F}_{\rho\sigma}\mathcal{F}_{\tau\kappa} + mB_{\rho\sigma}\mathcal{F}_{\tau\kappa} + \frac{1}{3}m^2 B_{\rho\sigma}B_{\tau\kappa} + F_{\rho\sigma}^I F_{\tau\kappa}^I\right), \end{aligned} \quad (2.7)$$

where $\tilde{g}$ is the $SU(2)$ gauge coupling and m is the mass parameter of the two-form field. The expressions are identical to the Lorentzian supergravity, however, note the factor of i in the Chern-Simons terms [21,22]. It is required for gauge invariance in the path integral with Euclidean measure. It is also implied by supersymmetry.

The scalar potential,

$$V = -\frac{1}{8}(\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g}me^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}), \quad (2.8)$$

can be obtained from a superpotential,

$$W = \frac{1}{2\sqrt{2}}(\tilde{g}e^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}}), \quad (2.9)$$

from

$$V = \frac{1}{2}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{5}{4}W^2. \quad (2.10)$$

We present the equations of motion,

$$\begin{aligned} R_{\mu\nu} &= 2\partial_\mu \phi \partial_\nu \phi + \frac{1}{8}g_{\mu\nu}(\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g}me^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}) - 2e^{-\sqrt{2}\phi}\left(\mathcal{H}_\mu{}^\rho \mathcal{H}_{\nu\rho} - \frac{1}{8}g_{\mu\nu}\mathcal{H}_{\rho\sigma}\mathcal{H}^{\rho\sigma}\right) \\ &\quad - 2e^{-\sqrt{2}\phi}\left(F_\mu{}^\rho F_{\nu\rho}^I - \frac{1}{8}g_{\mu\nu}F_{\rho\sigma}^I F^{I\rho\sigma}\right) + e^{2\sqrt{2}\phi}\left(G_\mu{}^{\rho\sigma}G_{\nu\rho\sigma} - \frac{1}{6}g_{\mu\nu}G_{\rho\sigma\tau}G^{\rho\sigma\tau}\right), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu \phi) &= \frac{1}{4\sqrt{2}}\left(\tilde{g}^2 e^{\sqrt{2}\phi} - 4\tilde{g}me^{-\sqrt{2}\phi} + 3m^2 e^{-3\sqrt{2}\phi}\right) \\ &\quad + \frac{1}{2\sqrt{2}}e^{-\sqrt{2}\phi}(\mathcal{H}_{\mu\nu}\mathcal{H}^{\mu\nu} + F_{\mu\nu}^I F^{I\mu\nu}) + \frac{1}{3\sqrt{2}}e^{2\sqrt{2}\phi}G_{\mu\nu\rho}G^{\mu\nu\rho}, \end{aligned} \quad (2.12)$$

$$\mathcal{D}_\nu(e^{-\sqrt{2}\phi}\mathcal{H}^{\nu\mu}) = \frac{i}{6}\epsilon^{\mu\nu\rho\sigma\tau\kappa}\mathcal{H}_{\nu\rho}G_{\sigma\tau\kappa}, \quad (2.13)$$

$$\mathcal{D}_\nu(e^{-\sqrt{2}\phi}F^{\nu\mu}) = \frac{i}{6}e\epsilon^{\mu\nu\rho\sigma\tau\kappa}F_{\nu\rho}^IF_{\sigma\tau\kappa}, \quad (2.14)$$

$$\begin{aligned} \mathcal{D}_\rho(e^{2\sqrt{2}\phi}G^{\rho\mu\nu}) &= -\frac{i}{4}e\epsilon^{\mu\nu\rho\sigma\tau\kappa}(\mathcal{H}_{\rho\sigma}\mathcal{H}_{\tau\kappa} + F_{\rho\sigma}^IF_{\tau\kappa}^I) \\ &\quad - me^{-\sqrt{2}\phi}\mathcal{H}^{\mu\nu}. \end{aligned} \quad (2.15)$$

Note the factors of i in the last three equations which originate from the factor of i in the Chern-Simons terms in the action.

The supersymmetry transformations of the fermionic fields are

$$\begin{aligned} \delta\psi_{\mu i} &= \nabla_\mu\epsilon_i + \tilde{g}A_\mu^I(T^I)_i\epsilon_j - \frac{1}{8\sqrt{2}}\left(\tilde{g}e^{-\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}}\right)\gamma_\mu\gamma_7\epsilon_i \\ &\quad - \frac{1}{8\sqrt{2}}e^{-\frac{\phi}{\sqrt{2}}}(\mathcal{F}_{\nu\lambda} + mB_{\nu\lambda})(\gamma_\mu^{\nu\lambda} - 6\delta_\mu^{\nu\lambda})\epsilon_i \\ &\quad - \frac{1}{4\sqrt{2}}e^{-\frac{\phi}{\sqrt{2}}}F_{\nu\lambda}^I(\gamma_\mu^{\nu\lambda} - 6\delta_\mu^{\nu\lambda})\gamma_7(T^I)_i{}^j\epsilon_j \\ &\quad - \frac{1}{24}e^{\sqrt{2}\phi}G_{\nu\lambda\rho}\gamma_7\gamma^{\nu\lambda\rho}\gamma_\mu\epsilon_i, \end{aligned} \quad (2.16)$$

$$\begin{aligned} \delta\chi_i &= \frac{1}{\sqrt{2}}\gamma^\mu\partial_\mu\phi\epsilon_i + \frac{1}{4\sqrt{2}}\left(\tilde{g}e^{-\frac{\phi}{\sqrt{2}}} - 3me^{-\frac{3\phi}{\sqrt{2}}}\right)\gamma_7\epsilon_i \\ &\quad + \frac{1}{4\sqrt{2}}e^{-\frac{\phi}{\sqrt{2}}}(\mathcal{F}_{\mu\nu} + mB_{\mu\nu})\gamma^{\mu\nu}\epsilon_i \\ &\quad + \frac{1}{2\sqrt{2}}e^{-\frac{\phi}{\sqrt{2}}}F_{\mu\nu}^I\gamma^{\mu\nu}\gamma_7(T^I)_i{}^j\epsilon_j \\ &\quad - \frac{1}{12}e^{\sqrt{2}\phi}G_{\mu\nu\lambda}\gamma_7\gamma^{\mu\nu\lambda}\epsilon_i, \end{aligned} \quad (2.17)$$

where T^I , $I = 1, 2, 3$, are the $SU(2)$ left-invariant one-forms,

$$T^I = -\frac{i}{2}\sigma^I. \quad (2.18)$$

The expressions are identical to the Lorentzian ones with (2.1) and (2.2) implied implicitly.

III. ON-SHELL ACTION OF AdS_6 BLACK HOLES

We compute the renormalized on-shell action of the supersymmetric AdS_6 black holes from $F(4)$ gauged supergravity [9] and show that it equals the Bekenstein-Hawking entropy of the black holes.

A. Euclidean black holes

We present the supersymmetric black holes with a horizon which is a product of two Riemann surfaces [9] in Euclidean spacetime. We consider the metric,

$$\begin{aligned} ds^2 &= -e^{2f(r)}(dt^2 + dr^2) - e^{2g_1(r)}(d\theta_1^2 + \sinh^2\theta_1 d\phi_1^2) \\ &\quad - e^{2g_2(r)}(d\theta_2^2 + \sinh^2\theta_2 d\phi_2^2), \end{aligned} \quad (3.1)$$

for the $\Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ background with $\mathfrak{g}_1 > 0$ and $\mathfrak{g}_2 > 0$. The only nonvanishing component of the non-Abelian $SU(2)$ gauge field, A_μ^I , $I = 1, 2, 3$, is given by

$$A^3 = a_1 \cosh\theta_1 d\phi_1 + a_2 \cosh\theta_2 d\phi_2, \quad (3.2)$$

where the magnetic charges, a_1 and a_2 , are constant. We turn off the $U(1)$ gauge field, $\mathcal{A}_\mu$.

The final expressions of the supersymmetry equations are identical to the Lorentzian ones. Therefore, we briefly summarize the differences in the derivations. The two-form field is determined by solving the equations of motion,

$$B_{tr} = -\frac{2i}{m^2}a_1a_2e^{\sqrt{2}\phi+2f-2g_1-2g_2}, \quad (3.3)$$

where there is a factor of i in contrast to the Lorentzian solution. As discussed before, time components of the fields are Wick rotated. In the derivation of supersymmetry equations, this factor of i is canceled by the i in the time component of gamma matrices. We present the supersymmetry equations,

$$\begin{aligned} f'e^{-f} &= -\frac{1}{4\sqrt{2}}\left(\tilde{g}e^{\frac{\phi}{\sqrt{2}}} + me^{\frac{3\phi}{\sqrt{2}}}\right) + \frac{k}{2\sqrt{2}\tilde{g}}e^{-\frac{\phi}{\sqrt{2}}}(e^{-2g_1} + e^{-2g_2}) - \frac{3}{\sqrt{2}\tilde{g}^2m}e^{\frac{\phi}{\sqrt{2}}-2g_1-2g_2}, \\ g'_1e^{-f} &= -\frac{1}{4\sqrt{2}}\left(\tilde{g}e^{\frac{\phi}{\sqrt{2}}} + me^{\frac{3\phi}{\sqrt{2}}}\right) - \frac{k}{2\sqrt{2}\tilde{g}}e^{-\frac{\phi}{\sqrt{2}}}(3e^{-2g_1} - e^{-2g_2}) + \frac{1}{\sqrt{2}\tilde{g}^2m}e^{\frac{\phi}{\sqrt{2}}-2g_1-2g_2}, \\ g'_2e^{-f} &= -\frac{1}{4\sqrt{2}}\left(\tilde{g}e^{\frac{\phi}{\sqrt{2}}} + me^{\frac{3\phi}{\sqrt{2}}}\right) - \frac{k}{2\sqrt{2}\tilde{g}}e^{-\frac{\phi}{\sqrt{2}}}(3e^{-2g_2} - e^{-2g_1}) + \frac{1}{\sqrt{2}\tilde{g}^2m}e^{\frac{\phi}{\sqrt{2}}-2g_1-2g_2}, \\ \frac{1}{\sqrt{2}}\phi'e^{-f} &= \frac{1}{4\sqrt{2}}\left(\tilde{g}e^{\frac{\phi}{\sqrt{2}}} - 3me^{\frac{3\phi}{\sqrt{2}}}\right) - \frac{k}{2\sqrt{2}\tilde{g}}e^{-\frac{\phi}{\sqrt{2}}}(e^{-2g_1} + e^{-2g_2}) - \frac{1}{\sqrt{2}\tilde{g}^2m}e^{\frac{\phi}{\sqrt{2}}-2g_1-2g_2}, \end{aligned} \quad (3.4)$$

with

$$a_1 = -\frac{k}{\lambda\tilde{g}}, \quad a_2 = -\frac{k}{\lambda\tilde{g}}, \quad (3.5)$$

where $\lambda = \pm 1$ and $k = -1$ for the $\Sigma_{g_1} \times \Sigma_{g_2}$ background with $g_1 > 0$ and $g_2 > 0$. When checking that the equations of motion are satisfied by the supersymmetry equations, factors of -1 from products of two-form fields are canceled by the change of time component of the metric, $g_{tt} \rightarrow -g_{tt}$.

B. On-shell action

1. The bulk and boundary terms

In this section, we calculate the on-shell action. The calculation is quite parallel to the calculation of on-shell action of AdS₄ black holes in [16]. The Einstein equations for the ansatz in the previous section can be presented by

$$\begin{aligned} 0 &= e^{-2f} R_{tt} + V + F + 3B, \\ 0 &= e^{-2f} R_{rr} + 4\Phi - V - F - 3B, \\ 0 &= \frac{1}{2} (e^{-2g_1} R_{\theta_1\theta_1} + e^{-2g_2} R_{\theta_2\theta_2}) - V + F + B, \end{aligned} \quad (3.6)$$

where we define

$$\begin{aligned} V &= -\frac{1}{8} (\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g} m e^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}), \\ F &= -\frac{1}{2} e^{-\sqrt{2}\phi} (a_1^2 e^{-4g_1} + a_2^2 e^{-4g_2}), \\ B &= -\frac{2}{m^2} a_1^2 a_2^2 e^{\sqrt{2}\phi - 4g_1 - 4g_2}, \\ \Phi &= -\frac{1}{2} e^{-2f} \phi' \phi', \end{aligned} \quad (3.7)$$

and V is the scalar potential. With the Einstein equations, the bosonic Lagrangian can be expressed in terms of the metric functions,

$$\begin{aligned} e^{-1} \mathcal{L} &= -\frac{1}{4} R + \Phi - V + F + B \\ &= -\frac{1}{4} (R + e^{-2f} (R_{tt} + R_{rr}) \\ &\quad + 2(e^{-2g_1} R_{\theta_1\theta_1} + e^{-2g_2} R_{\theta_2\theta_2})) \\ &= -\frac{1}{2} e^{-2f} (f'' + 2f'(g'_1 + g'_2)) \\ &= -\frac{1}{4} e^{-2f-2g_1-2g_2} (e^{-2f+2g_1+2g_2} (e^{2f})')'. \end{aligned} \quad (3.8)$$

The bulk action and the Gibbons-Hawking term [26,27] are, respectively,

$$\begin{aligned} S_{\text{bulk}} &= \frac{1}{4\pi G_N} \int d^6 x \mathcal{L} \\ &= \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{8\pi G_N} \int_{r_h}^{r_0} dr \frac{1}{2} (e^{-2f+2g_1+2g_2} (e^{2f})')', \end{aligned} \quad (3.9)$$

$$\begin{aligned} S_{\text{GH}} &= -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \mathcal{K} \\ &= -\frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{8\pi G_N} e^{2g_1+2g_2} (f' + 2g'_1 + 2g'_2) \Big|_{r=r_0}, \end{aligned} \quad (3.10)$$

where h_{ij} is the induced metric. The extrinsic curvature, normal vector, and trace of the extrinsic curvature are, respectively,

$$\mathcal{K}_{ij} = \frac{1}{2} n^k \partial_k g_{ij}, \quad n^j = \frac{1}{\sqrt{g_{rr}}} \left(\frac{\partial}{\partial r} \right)^j, \quad \mathcal{K} = h^{ij} \mathcal{K}_{ij}. \quad (3.11)$$

We denote the six-dimensional Newton's gravitational constant by G_N . The radius of the horizon and the location of the boundary are denoted by r_h and r_0 , respectively. We obtain the inverse temperature by

$$\beta = \frac{1}{T} = \frac{4\pi}{e^{2f} (-e^{-2f})'}. \quad (3.12)$$

Then the sum of the bulk action and the Gibbons-Hawking term is

$$S_{\text{bulk}} + S_{\text{GH}} = \frac{A}{4G_N} - \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{8\pi G_N} (e^{2g_1+2g_2})' \Big|_{r=r_0}, \quad (3.13)$$

where the area of the horizon is

$$A = e^{2g_1+2g_2} \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2}) \Big|_{r=r_h}. \quad (3.14)$$

Employing the supersymmetry equations in (3.4), we can rewrite

$$\begin{aligned} &(e^{2g_1+2g_2})' \\ &= 2e^{2g_1+2g_2} (g'_1 + g'_2) \\ &= 4e^{f+2g_1+2g_2} \left[-\frac{1}{2} W + \frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} (a_1 e^{-2g_1} + a_2 e^{-2g_2}) \right. \\ &\quad \left. + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2} \right], \end{aligned} \quad (3.15)$$

where W is the superpotential in (2.9). Using the expression in (3.15), the sum of the bulk action and the Gibbons-Hawking term is

$$S_{\text{bulk}} + S_{\text{GH}} = \frac{A}{4G_N} + \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{4\pi G_N} e^{f+2g_1+2g_2} W \Big|_{r=r_0} - \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{2\pi G_N} e^{f+2g_1+2g_2} \left[\frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} (a_1 e^{-2g_1} + a_2 e^{-2g_2}) + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}}-2g_1-2g_2} \right] \Big|_{r=r_0}. \quad (3.16)$$

The full black hole solution is an interpolating geometry between the asymptotic AdS_6 boundary at $r_0 = 0$ and the $\text{AdS}_2 \times H^2 \times H^2$ horizon at $r_h = \infty$. At the boundary, $r_0 = 0$, the metric and the scalar field asymptote to AdS_6 spacetime,

$$e^f = e^{g_1} = e^{g_2} \cong \frac{R_{\text{AdS}_6}}{r}, \quad e^{\frac{\phi}{\sqrt{2}}} \cong 1. \quad (3.17)$$

Hence, we observe that the superpotential term, the term with a_1 and a_2 , the term with $a_1 a_2$ in (3.16) are quintic,

cubic, and linear divergences, respectively, at the boundary, $r = r_0$. Therefore, in the next subsection, we will introduce counterterms to eliminate the divergent terms in the on-shell action.

2. The counterterms

In order to eliminate the divergent terms in the on-shell action, we introduce the counterterms. Accordingly, we introduce the superpotential counterterm and counterterms from the curvature of the boundary, respectively,

$$S_{\text{SUSY}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} [2W] = -\frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{4\pi G_N} e^{f+2g_1+2g_2} W \Big|_{r=r_0}, \quad (3.18)$$

$$S_{\mathcal{R}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{1}{\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}} \mathcal{R} \right] = -\frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{2\pi G_N} e^{f+2g_1+2g_2} \left[\frac{k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}} (e^{-2g_1} + e^{-2g_2}) \right] \Big|_{r=r_0}, \quad (3.19)$$

$$S_{\mathcal{R}^2} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{2}{\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{5}{16} \mathcal{R}^2 \right) \right] = \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{4\pi G_N} e^{f+2g_1+2g_2} \left[\frac{1}{\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}-4g} \right] \Big|_{r=r_0}, \quad (3.20)$$

where the Ricci scalar of the induced metric is

$$\mathcal{R} = 2k(e^{-2g_1} + e^{-2g_2}). \quad (3.21)$$

In (3.20) we set

$$g \equiv g_1 = g_2 \quad (3.22)$$

which is indeed the case for supersymmetric solutions. Combining the counterterms to the bulk action and the Gibbons-Hawking term, we have

$$S_{\text{on-shell}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} = \frac{A}{4G_N} - \frac{\beta \text{vol}(\Sigma_{g_1}) \text{vol}(\Sigma_{g_2})}{4\pi G_N} e^{f+4g} \left[\frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}}-4g} \right] \Big|_{r=r_0}. \quad (3.23)$$

Therefore, the counterterms we introduced have eliminated the quintic and cubic divergences and the half of the linear divergence in the on-shell action. We still have the second term in (3.23) which is linearly divergent at the boundary.

If we set the scalar field to vanish, $\phi = 0$, and use $\tilde{g} = 3m$, we obtain

$$\begin{aligned} S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} &= \frac{1}{8\pi G_N} \int_{\partial} d^5x \sqrt{h} \left[\frac{4}{3\sqrt{2}} \tilde{g} + \frac{1}{\sqrt{2}\tilde{g}} \mathcal{R} + \frac{6}{\sqrt{2}\tilde{g}^3} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{5}{16} \mathcal{R}^2 \right) \right], \\ &= \frac{1}{8\pi G_N} \int_{\partial} d^5x \sqrt{h} \left[\frac{4}{l} + \frac{l}{6} \mathcal{R} + \frac{l^3}{18} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{5}{16} \mathcal{R}^2 \right) \right], \end{aligned} \quad (3.24)$$

where, in the second line, we reparametrized $\tilde{g} \rightarrow 3\sqrt{2}/l$. This precisely coincides with the counterterms introduced for pure gravity in AdS_6 in (A1). This provides a nontrivial check of our counterterms against the counterterms of pure gravity in AdS .

Returning to the on-shell action, in order to eliminate the second term in (3.23), we introduce additional counterterms,

$$S_{F^2} = \frac{1}{8\pi G_N} \int_{\partial} d^5x \sqrt{h} \frac{1}{\sqrt{2}m} F_{ij}^I F^{Iij}, \quad (3.25)$$

$$S_{\phi} = \frac{1}{8\pi G_N} \int_{\partial} d^5x \sqrt{h} \left(-\frac{1}{2} \right) \frac{4\tilde{g}}{3\sqrt{2}} (1 - e^{\frac{\phi}{\sqrt{2}}})^2, \quad (3.26)$$

which we obtain from the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22]. See Appendix B for details. For the counterterm, S_{ϕ} , we have an additional factor of $-1/2$ compared to the one in [21,22]. The factor of this term is freely determined to cancel the divergences. See, e.g., explanation around (4.32) of [28]. Asymptotically solving the supersymmetry equations, (3.4), around the boundary, $r \rightarrow 0$, we obtain the solutions,

$$\begin{aligned} f &= -\log \frac{r}{l} - \frac{1}{18} r^2 + \dots, \\ g_1 &= -\log \frac{r}{l} + \frac{1}{9} r^2 + \dots, \\ g_2 &= -\log \frac{r}{l} + \frac{1}{9} r^2 + \dots, \\ \phi &= -\frac{1}{3\sqrt{2}} r^2 + \dots, \end{aligned} \quad (3.27)$$

where $l = 3\sqrt{2}/\tilde{g}$ is the radius of AdS_6 . By plugging the asymptotic solutions into the counterterms, (3.25) and (3.26), we observe that they indeed eliminate the divergences in the second term of (3.23).

We finally obtain that the renormalized on-shell action,

$$\begin{aligned} S_{\text{on-shell}} &= S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} + S_{F^2} + S_{\phi} \\ &= \frac{A}{4G_N}, \end{aligned} \quad (3.28)$$

which equals the Bekenstein-Hawking entropy of the supersymmetric black holes in AdS_6 .¹ The counterterms precisely match the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22], which we present in Appendix B.²

IV. ON-SHELL ACTION OF AdS_6 BLACK STRINGS

We compute the renormalized on-shell action of the supersymmetric AdS_6 black strings from $F(4)$ gauged supergravity [19] and show that it reproduces the central charge of 2d dual conformal field theories.

A. Euclidean black strings

We present the supersymmetric black strings [19] in Euclidean spacetime. It is obtained from D4-branes wrapped on a compact hyperbolic space, H^3 , which is an interpolating solution between the boundary of AdS_6 and the horizon of $\text{AdS}_3 \times H^3$. We consider the metric,

$$\begin{aligned} ds^2 &= -e^{2f(r)}(dt^2 + du^2 + dr^2) \\ &\quad - e^{2g(r)}(d\varphi^2 + \sinh^2 \varphi d\theta^2 + \sinh^2 \varphi \sin^2 \theta d\psi^2). \end{aligned} \quad (4.1)$$

The nonvanishing components of the non-Abelian $SU(2)$ gauge field, A_{μ}^I , $I = 1, 2, 3$, are given by

$$\begin{aligned} A^1 &= a \cosh \varphi d\theta, & A^2 &= b \cos \theta d\psi, \\ A^3 &= c \sin \theta \cosh \varphi d\psi, \end{aligned} \quad (4.2)$$

where the magnetic charges, a , b and c , are constant. We turn off the $U(1)$ gauge field, A_{μ} , and the two-form field, $B_{\mu\nu}$.

¹As we are in the Euclidean spacetime of $(- - - - -)$ signature, the on-shell action gives the Bekenstein-Hawking entropy, not the minus of it.

²As our two-form field (3.3) has a component on the radial direction, r , its pull-back on the induced metric trivially vanishes. Therefore, all terms involving the two-form field in (B1) identically vanishes.

We present the supersymmetry equations,

$$\begin{aligned} f' e^{-f} &= -\frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{3k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \\ g' e^{-f} &= -\frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{5k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \\ \frac{1}{\sqrt{2}} \phi' e^{-f} &= \frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} - 3m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{3k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \end{aligned} \quad (4.3)$$

with the twist condition,

$$a = -\frac{k}{\tilde{g}}, \quad b = -\frac{k}{\tilde{g}}, \quad c = \frac{k}{\tilde{g}}, \quad (4.4)$$

where $k = -1$ for the H^3 background.

B. On-shell action

1. The bulk and boundary terms

In this section, we calculate the on-shell action. The calculation is quite parallel to the calculation of the on-shell action of AdS₄ black holes in [16]. The Einstein equations for the ansatz in the previous section can be presented by

$$\begin{aligned} 0 &= e^{-2f} R_{tt} - V - F, \\ 0 &= e^{-2f} R_{rr} + 4\Phi - V - F, \\ 0 &= e^{-2g} R_{\phi\phi} - V + \frac{5}{3}F, \end{aligned} \quad (4.5)$$

where we define

$$\begin{aligned} V &= -\frac{1}{8} (\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g} m e^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}), \\ F &= -\frac{3}{2g^2} e^{-\sqrt{2}\phi-4g}, \\ \Phi &= -\frac{1}{2} e^{-2f} \phi' \phi', \end{aligned} \quad (4.6)$$

and V is the scalar potential. With the Einstein equations, the bosonic Lagrangian can be expressed in terms of the metric functions,

$$\begin{aligned} e^{-1} \mathcal{L} &= -\frac{1}{4} R + \Phi - V + F \\ &= -\frac{1}{4} R - \frac{1}{4} (e^{-2f} R_{rr} + 3e^{-2g} R_{\phi\phi}) \\ &= -\frac{1}{2} e^{-2f} (f'' + f' f' + 3f' g') \\ &= -\frac{1}{2} e^{-3f-3g} (e^{3g} (e^f)')'. \end{aligned} \quad (4.7)$$

The bulk action and the Gibbons-Hawking term [26,27] are, respectively,

$$\begin{aligned} S_{\text{bulk}} &= \frac{1}{4\pi G_N} \int d^6 x \mathcal{L} \\ &= l_u \frac{\beta \text{vol}(H^3)}{8\pi G_N} \int_{r_h}^{r_0} dr (e^{3g} (e^f)')', \end{aligned} \quad (4.8)$$

$$\begin{aligned} S_{\text{GH}} &= -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \mathcal{K} \\ &= -l_u \frac{\beta \text{vol}(H^3)}{8\pi G_N} e^{f+3g} (2f' + 3g') \Big|_{r=r_0}, \end{aligned} \quad (4.9)$$

where h_{ij} is the induced metric and l_u is from the integration over the u coordinate of the metric, (4.1). The extrinsic curvature, normal vector, and trace of the extrinsic curvature are, respectively,

$$\mathcal{K}_{ij} = \frac{1}{2} n^k \partial_k g_{ij}, \quad n^j = \frac{1}{\sqrt{g_{rr}}} \left(\frac{\partial}{\partial r} \right)^j, \quad \mathcal{K} = h^{ij} \mathcal{K}_{ij}. \quad (4.10)$$

We denote the six-dimensional Newton's gravitational constant by G_N . The radius of the horizon and the location of the boundary are denoted by r_h and r_0 , respectively. We obtain the inverse temperature by

$$\beta = \frac{1}{T} = \frac{4\pi}{e^{2f} (-e^{-2f})'}. \quad (4.11)$$

Then the sum of the bulk action and the Gibbons-Hawking term is

$$\begin{aligned} S_{\text{bulk}} + S_{\text{GH}} &= l_u \frac{e^{f+3g} \text{vol}(H^3)}{4G_N} \Big|_{r=r_h} \\ &\quad - l_u \frac{\beta \text{vol}(H^3)}{8\pi G_N} (e^{f+3g})' \Big|_{r=r_0}. \end{aligned} \quad (4.12)$$

Employing the supersymmetry equations in (4.3), we can rewrite

$$\begin{aligned} (e^{f+3g})' &= e^{f+3g} (f' + 3g') \\ &= 4e^{2f+3g} \left[-\frac{1}{2} W + \frac{3k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right], \end{aligned} \quad (4.13)$$

where W is the superpotential in (2.9). Using the expression in (4.13), the sum of the bulk action and the Gibbons-Hawking term is

$$S_{\text{bulk}} + S_{\text{GH}} = l_u \frac{e^{f+3g} \text{vol}(H^3)}{4G_N} \Big|_{r=r_h} + \left(l_u \frac{\beta \text{vol}(H^3)}{4\pi G_N} e^{2f+3g} W - l_u \frac{\beta \text{vol}(H^3)}{2\pi G_N} e^{2f+3g} \frac{3k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right) \Big|_{r=r_0}. \quad (4.14)$$

The full solution is an interpolating geometry between the asymptotic AdS_6 boundary at $r_0 = 0$ and the $\text{AdS}_3 \times H^3$ horizon at $r_h = \infty$. At the boundary, $r_0 = 0$, the metric and the scalar field asymptote to AdS_6 spacetime,

$$e^f = e^g \cong \frac{R_{\text{AdS}_6}}{r}, \quad e^{\frac{\phi}{\sqrt{2}}} \cong 1. \quad (4.15)$$

Hence, we observe that the superpotential term and the term with $1/\tilde{g}$ in (4.14) are quintic and cubic divergences, respectively, at the boundary, $r = r_0$. Therefore, in the next subsection, we will introduce counterterms to eliminate the divergent terms in the on-shell action.

2. The counterterms

In order to eliminate the divergent terms in the on-shell action, we introduce the counterterms. Accordingly, we introduce the superpotential counterterm and counterterms from the curvature of the boundary, respectively,

$$S_{\text{SUSY}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} [2W] \\ = -l_u \frac{\beta \text{vol}(H^3)}{4\pi G_N} e^{2f+3g} W \Big|_{r=r_0}, \quad (4.16)$$

$$S_{\mathcal{R}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{1}{\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}} \mathcal{R} \right] \\ = -l_u \frac{\beta \text{vol}(H^3)}{2\pi G_N} e^{2f+3g} \left[\frac{3k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right] \Big|_{r=r_0}, \quad (4.17)$$

$$S_{\mathcal{R}^2} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{2}{\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{5}{16} \mathcal{R}^2 \right) \right] \\ = l_u \frac{\beta \text{vol}(H^3)}{4\pi G_N} e^{2f+3g} \left[\frac{3}{2\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}-4g} \right] \Big|_{r=r_0}, \quad (4.18)$$

where the Ricci scalar of the induced metric is

$$\mathcal{R} = 6ke^{-2g}. \quad (4.19)$$

Combining the counterterms to the bulk action and the Gibbons-Hawking term, we have

$$S_{\text{on-shell}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} \\ = l_u \frac{e^{f+3g} \text{vol}(H^3)}{4G_N} \Big|_{r=r_h} + S_{\mathcal{R}^2}. \quad (4.20)$$

Therefore, the counterterms we introduced have eliminated the quintic and cubic divergences. However, they also introduced the $S_{\mathcal{R}^2}$ term which diverges linearly.

In order to eliminate the $S_{\mathcal{R}^2}$ term in (4.20), we introduce additional counterterms,

$$S_{F^2} = \frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \frac{1}{\sqrt{2}m} F_{ij}^l F^{lij}, \quad (4.21)$$

$$S_{\phi} = \frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left(-\frac{1}{2} \right) \frac{4\tilde{g}}{3\sqrt{2}} (1 - e^{\frac{\phi}{\sqrt{2}}})^2, \quad (4.22)$$

which we obtain from the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22]. See Appendix B for details. For the counterterm, S_{ϕ} , we have an additional factor of $-1/2$ compared to the one in [21,22]. The factor of this term is freely determined to cancel the divergences. See, e.g., explanation around (4.32) of [28]. Asymptotically solving the supersymmetry equations, (4.3), around the boundary, $r \rightarrow 0$, we obtain the solutions,

$$f = -\log \frac{r}{l} - \frac{1}{12} r^2 + \dots, \\ g = -\log \frac{r}{l} + \frac{1}{4} r^2 + \dots, \\ \phi = -\frac{1}{2\sqrt{2}} r^2 + \dots, \quad (4.23)$$

where $l = 3\sqrt{2}/\tilde{g}$ is the radius of AdS_6 . By plugging the asymptotic solutions into the counterterms, (4.21) and (4.22), we observe that they indeed eliminate the divergences in the second term of (4.20).

We finally obtain the renormalized on-shell action,

$$S_{\text{on-shell}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} + S_{F^2} + S_{\phi} \\ = l_u \frac{e^{f+3g} \text{vol}(H^3)}{4G_N} \Big|_{r=r_h}. \quad (4.24)$$

The on-shell action per unit volume of u space, (5.1), reproduces the central charge of 2d dual conformal field theories,

$$\frac{S_{\text{on-shell}}}{V} = \frac{e^{f+3g} \text{vol}(H^3)}{4G_N} \Big|_{r=r_h} = \frac{c_{2d}}{6}, \quad (4.25)$$

where $V = l_u$. See, e.g., (4.34), (B.3) and (B.5) in [20]. The counterterms precisely match the counterterms for general

solutions of $F(4)$ gauged supergravity derived in [21,22], which we present in Appendix B.

V. ON-SHELL ACTION OF AdS_6 BLACK BRANES

We compute the renormalized on-shell action of the supersymmetric AdS_6 black branes from $F(4)$ gauged supergravity [19] and show that it reproduces the three-sphere free energy of 3d dual field theories.

A. Euclidean black branes

We present the supersymmetric black branes with a horizon of a Riemann surface [19] in Euclidean spacetime. It is obtained from D4-branes wrapped on a Riemann surface which is an interpolating solution between the boundary of AdS_6 and the horizon of $\text{AdS}_4 \times \Sigma_g$. It was studied from a ten-dimensional supergravity perspective in [29] and from field theory in [13]. We consider the metric,

$$ds^2 = -e^{2f(r)}(dt^2 + dx^2 + dy^2 + dr^2) - e^{2g(r)}(d\theta^2 + \sinh^2 \theta d\varphi^2), \quad (5.1)$$

for the Σ_g background with $g > 0$. The only nonvanishing component of the non-Abelian $SU(2)$ gauge field, A_μ^I , $I = 1, 2, 3$, is given by

$$A^3 = a \cosh \theta d\varphi, \quad (5.2)$$

where the magnetic charge, a , is constant. We turn off the $U(1)$ gauge field, $\mathcal{A}_\mu$, and the two-form field, $B_{\mu\nu}$.

We present the supersymmetry equations,

$$\begin{aligned} f' e^{-f} &= -\frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \\ g' e^{-f} &= -\frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \\ \frac{1}{\sqrt{2}} \phi' e^{-f} &= \frac{1}{4\sqrt{2}} \left(\tilde{g} e^{\frac{\phi}{\sqrt{2}}} - 3m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g}, \end{aligned} \quad (5.3)$$

with

$$a = -\frac{k}{\lambda \tilde{g}}, \quad (5.4)$$

where $\lambda = \pm 1$ and $k = -1$ for the Σ_g background with $g > 0$.

B. On-shell action

1. The bulk and boundary terms

In this section, we calculate the on-shell action. The calculation is quite parallel to the calculation of the on-shell

action of AdS_4 black holes in [16]. The Einstein equations for the ansatz in the previous section can be presented by

$$\begin{aligned} 0 &= e^{-2f} R_{tt} - V - F, \\ 0 &= e^{-2f} R_{rr} + 4\Phi - V - F, \\ 0 &= e^{-2g} R_{\theta\theta} - V + 3F, \end{aligned} \quad (5.5)$$

where we define

$$\begin{aligned} V &= -\frac{1}{8} (\tilde{g}^2 e^{\sqrt{2}\phi} + 4\tilde{g} m e^{-\sqrt{2}\phi} - m^2 e^{-3\sqrt{2}\phi}), \\ F &= -\frac{1}{2} a^2 e^{-\sqrt{2}\phi-4g}, \\ \Phi &= -\frac{1}{2} e^{-2f} \phi' \phi', \end{aligned} \quad (5.6)$$

and V is the scalar potential. With the Einstein equations, the bosonic Lagrangian can be expressed in terms of the metric functions,

$$\begin{aligned} e^{-1} \mathcal{L} &= -\frac{1}{4} R + \Phi - V + F \\ &= -\frac{1}{4} R + \frac{1}{4} e^{-2f} (R_{tt} - R_{rr}) - \frac{1}{2} (e^{-2f} R_{tt} + e^{-2g} R_{\theta\theta}) \\ &= -\frac{1}{2} e^{-2f} (f'' + 2f' f' + 2f' g') \\ &= -\frac{1}{4} e^{-4f-2g} (e^{2g} (e^{2f})')'. \end{aligned} \quad (5.7)$$

The bulk action and the Gibbons-Hawking term [26,27] are, respectively,

$$\begin{aligned} S_{\text{bulk}} &= \frac{1}{4\pi G_N} \int d^6 x \mathcal{L} \\ &= l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{8\pi G_N} \int_{r_h}^{r_0} dr \frac{1}{2} (e^{2g} (e^{2f})')', \\ S_{\text{GH}} &= -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \mathcal{K} \\ &= -l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{8\pi G_N} e^{2f+2g} (3f' + 2g') \Big|_{r=r_0}, \end{aligned} \quad (5.8)$$

where h_{ij} is the induced metric and l_x and l_y are from the integration over x and y coordinates of the metric (5.1). The extrinsic curvature, normal vector, and trace of the extrinsic curvature are, respectively,

$$\mathcal{K}_{ij} = \frac{1}{2} n^k \partial_k g_{ij}, \quad n^j = \frac{1}{\sqrt{g_{rr}}} \left(\frac{\partial}{\partial r} \right)^j, \quad \mathcal{K} = h^{ij} \mathcal{K}_{ij}. \quad (5.10)$$

We denote the six-dimensional Newton's gravitational constant by G_N . The radius of the horizon and the location

of the boundary are denoted by r_h and r_0 , respectively. We obtain the inverse temperature by

$$\beta = \frac{1}{T} = \frac{4\pi}{e^{2f}(-e^{-2f})'}. \quad (5.11)$$

Then the sum of the bulk action and the Gibbons-Hawking term is

$$S_{\text{bulk}} + S_{\text{GH}} = l_x l_y \frac{e^{2f+2g} \text{vol}(\Sigma_g)}{4G_N} \Big|_{r=r_h} - l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{8\pi G_N} (e^{2f+2g})' \Big|_{r=r_0}. \quad (5.12)$$

Employing the supersymmetry equations in (5.3), we can rewrite

$$(e^{2f+2g})' = 2e^{2f+2g}(f' + g') = 4e^{3f+2g} \left[-\frac{1}{2}W + \frac{\lambda}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right], \quad (5.13)$$

where W is the superpotential in (2.9). Using the expression in (5.13), the sum of the bulk action and the Gibbons-Hawking term is

$$S_{\text{bulk}} + S_{\text{GH}} = l_x l_y \frac{e^{2f+2g} \text{vol}(\Sigma_g)}{4G_N} \Big|_{r=r_h} + \left(l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{4\pi G_N} e^{3f+2g} W - l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{2\pi G_N} e^{3f+2g} \frac{\lambda}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right) \Big|_{r=r_0}. \quad (5.14)$$

The full solution is an interpolating geometry between the asymptotic AdS_6 boundary at $r_0 = 0$ and the $\text{AdS}_4 \times \Sigma_g$ horizon at $r_h = \infty$. At the boundary, $r_0 = 0$, the metric and the scalar field asymptote to AdS_6 spacetime,

$$e^f = e^g \cong \frac{R_{\text{AdS}_6}}{r}, \quad e^{\frac{\phi}{\sqrt{2}}} \cong 1. \quad (5.15)$$

Hence, we observe that the superpotential term and the term with $1/\tilde{g}$ in (5.14) are quintic and cubic divergences, respectively, at the boundary, $r = r_0$. Therefore, in the next subsection, we will introduce counterterms to eliminate the divergent terms in the on-shell action.

2. The counterterms

In order to eliminate the divergent terms in the on-shell action, we introduce the counterterms. Accordingly, we introduce the superpotential counterterm and counterterms from the curvature of the boundary, respectively,

$$S_{\text{SUSY}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} [2W] = -l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{4\pi G_N} e^{3f+2g} W \Big|_{r=r_0}, \quad (5.16)$$

$$S_{\mathcal{R}} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{1}{\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}\mathcal{R}} \right] = -l_x l_y \frac{\beta \text{vol}(\Sigma_g)}{2\pi G_N} e^{3f+2g} \left[\frac{k}{2\sqrt{2}\tilde{g}} e^{-\frac{\phi}{\sqrt{2}}-2g} \right] \Big|_{r=r_0}, \quad (5.17)$$

$$S_{\mathcal{R}^2} = -\frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left[\frac{2}{\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{5}{16} \mathcal{R}^2 \right) \right] = l_x l_y \frac{\beta \text{vol}(\Sigma_{g1})}{4\pi G_N} e^{3f+2g} \left[\frac{3}{2\sqrt{2}\tilde{g}^2 m} e^{\frac{\phi}{\sqrt{2}}-4g} \right] \Big|_{r=r_0}, \quad (5.18)$$

where the Ricci scalar of the induced metric is

$$\mathcal{R} = 2ke^{-2g}. \quad (5.19)$$

Combining the counterterms to the bulk action and the Gibbons-Hawking term, we have

$$S_{\text{on-shell}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} = l_x l_y \frac{e^{2f+2g} \text{vol}(\Sigma_g)}{4G_N} \Big|_{r=r_h} + S_{\mathcal{R}^2}. \quad (5.20)$$

Therefore, the counterterms we introduced have eliminated the quintic and cubic divergences. However, they also introduced the $S_{\mathcal{R}^2}$ term which diverges linearly.

In order to eliminate the $S_{\mathcal{R}^2}$ term in (5.20), we introduce additional counterterms,

$$S_{F^2} = \frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \frac{1}{\sqrt{2}m} F_{ij}^I F^{Iij}, \quad (5.21)$$

$$S_{\phi} = \frac{1}{8\pi G_N} \int_{\partial} d^5 x \sqrt{h} \left(-\frac{1}{2} \right) \frac{4\tilde{g}}{3\sqrt{2}} (1 - e^{\frac{\phi}{\sqrt{2}}})^2, \quad (5.22)$$

which we obtain from the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22]. See Appendix B for details. For the counterterm, S_{ϕ} , we have an additional factor of $-1/2$ compared to the one in [21,22]. The factor of this term is freely determined to cancel the divergences. See, e.g., explanation around (4.32) of [28]. Asymptotically solving the supersymmetry equations, (5.3), around the boundary, $r \rightarrow 0$, we obtain the solutions,

$$\begin{aligned}
f &= -\log \frac{r}{l} - \frac{1}{36} r^2 + \dots, \\
g &= -\log \frac{r}{l} + \frac{5}{36} r^2 + \dots, \\
\phi &= -\frac{1}{6\sqrt{2}} r^2 + \dots,
\end{aligned} \tag{5.23}$$

where $l = 3\sqrt{2}/\tilde{g}$ is the radius of AdS_6 . By plugging the asymptotic solutions into the counterterms, (5.21) and (5.22), we observe that they indeed eliminate the divergences in the second term of (5.20).

We finally obtain the renormalized on-shell action,

$$\begin{aligned}
S_{\text{on-shell}} &= S_{\text{bulk}} + S_{\text{GH}} + S_{\text{SUSY}} + S_{\mathcal{R}} + S_{\mathcal{R}^2} + S_{F^2} + S_{\phi} \\
&= l_x l_y \frac{e^{2f+2g} \text{vol}(\Sigma_g)}{4G_N} \Big|_{r=r_h}.
\end{aligned} \tag{5.24}$$

The on-shell action per unit volume of x and y space, (5.1), reproduces the three-sphere free energy of 3d dual field theories,

$$\frac{S_{\text{on-shell}}}{V} = \frac{e^{2f+2g} \text{vol}(\Sigma_g)}{4G_N} \Big|_{r=r_h} = \frac{F_{S^3}}{2\pi}, \tag{5.25}$$

where $V = l_x l_y$. See, e.g., (4.31), (B.3) and (B.8) in [20]. The counterterms precisely match the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22], which we present in Appendix B.

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APPENDIX A: COUNTERTERMS FOR PURE GRAVITY IN AdS_{n+1}

For pure gravity theories in AdS_{n+1} spacetimes with $n < 7$, the counterterms to eliminate the divergent terms arising from the on-shell action at the boundary were systematically obtained long ago in [30]. The counterterms for generic pure gravity theories in AdS_{n+1} spacetimes with $n < 7$ are [30]

$$\begin{aligned}
S_{ct} &= \frac{1}{8\pi G_N} \int_{\partial} d^n x \sqrt{h} \left[\frac{n-1}{l} + \frac{l}{2(n-2)} \mathcal{R} \right. \\
&\quad \left. + \frac{l^3}{2(n-4)(n-2)^2} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{n}{4(n-1)} \mathcal{R}^2 \right) + \dots \right],
\end{aligned} \tag{A1}$$

where h , $\mathcal{R}$, $\mathcal{R}_{ij}$ are the determinant of the metric, the Ricci scalar, and the Ricci tensor of the boundary metric, respectively, and l is the radius of AdS_{n+1} . According to [30], the first term first appeared in [31], the second term in [32], and the third term in [30]. In odd dimensions, some denominators vanish, and they produce the logarithmic corrections, e.g., [33,34]. In AdS_4 , as the third term is not divergent and vanishes at the boundary, there are counterterms only up to $\mathcal{R}$, e.g., [16,35–37]. In AdS_6 we encounter the $\mathcal{R}^2$ counterterms, as they appeared in [21,22,28].

APPENDIX B: COUNTERTERMS FOR $F(4)$ GAUGED SUPERGRAVITY

We present the counterterms for general solutions of $F(4)$ gauged supergravity derived in [21,22],

$$\begin{aligned}
S_{ct} &= \frac{1}{8\pi G_N} \int_{\partial} \left[\left(\frac{4\sqrt{2}}{3} + \frac{1}{2\sqrt{2}} \mathcal{R}[h] - \frac{1}{6\sqrt{2}} \|\tilde{B}\|_h^2 + \frac{3}{4\sqrt{2}} \mathcal{R}[h]_{mn} \mathcal{R}[h]^{mn} - \frac{15}{64\sqrt{2}} \mathcal{R}[h]^2 - \frac{3}{4\sqrt{2}} \|\tilde{F}^I\|_h^2 + \frac{1}{12\sqrt{2}} \text{Tr}_h \tilde{B}^4 \right. \right. \\
&\quad \left. - \frac{13}{192\sqrt{2}} \|\tilde{B}\|_h^4 - \frac{1}{\sqrt{2}} \|d\tilde{B}\|_h^2 + \frac{5}{8\sqrt{2}} \|d *_h \tilde{B} + \frac{i\sqrt{2}}{3} \tilde{B} \wedge \tilde{B}\|_h^2 - \frac{1}{4\sqrt{2}} \langle \tilde{B}, d\delta_h \tilde{B} + \frac{i\sqrt{2}}{3} d *_h \tilde{B} \wedge \tilde{B} \rangle_h \right. \\
&\quad \left. + \frac{4\sqrt{2}}{3} (1-X)^2 - \frac{1}{\sqrt{2}} \langle \text{Ric}[h] \circ \tilde{B}, \tilde{B} \rangle_h + \frac{9}{32\sqrt{2}} \mathcal{R}[h] \|\tilde{B}\|_h^2 \right) d^5 x \sqrt{h} \\
&\quad \left. - \frac{1}{4\sqrt{2}} \tilde{B} \wedge \left(d *_h d\tilde{B} + \frac{i\sqrt{2}}{3} \tilde{B} \wedge \delta_h \tilde{B} - \frac{2}{9} \tilde{B} \wedge *_h (\tilde{B} \wedge \tilde{B}) \right) \right],
\end{aligned} \tag{B1}$$

where it is defined to be

$$\|F^I\|_h^2 = \frac{1}{2!} F_{mn}^I F^{Imn}. \quad (\text{B2})$$

In the first line of (B1), the first term is quintic and the second and third terms are cubic divergences. The rest of the counterterms are linear divergences.

Beside the definition of (B2), the conventions of [38] are employed in [21,22]. They are related to our conventions of [25] by

$$g = 2\tilde{g}, \quad X = e^{-\frac{\tilde{\phi}}{2\sqrt{2}}} = e^{\frac{\phi}{\sqrt{2}}}, \quad A_\mu^I = \frac{1}{2}\tilde{A}_\mu^I, \quad A_\mu = \frac{1}{2}\tilde{A}_\mu, \quad B_{\mu\nu} = \frac{1}{2}\tilde{B}_{\mu\nu}, \quad (\text{B3})$$

where the tilded ones are from [38]. The signature is chosen to be (++++++) compared to our (-----). The gauge coupling and the radius of AdS_6 are fixed to be $g = 2$ and $l = 3\sqrt{2}/2$, respectively.

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