Query complexity and cutoff dependence of the CFT_2 ground state

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We present the vacuum of a two-dimensional conformal field theory as a network of Wilson lines in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory, which is conventionally used to study gravity in three-dimensional anti-de Sitter space. The position and shape of the network encode the cutoff scale at which the ground state density operator is defined. In the framework of query complexity, a general argument relates the “density of complexity” of the network to the extrinsic curvature of the cutoff surface in AdS_3 , which accords with the holographic complexity = volume proposal.

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I. INTRODUCTION

Complexity has attracted much interest in holographic duality, with several proposals for how the concept is realized on the anti-de Sitter (AdS) [1–3] and conformal field theory (CFT) [4,5] sides of the AdS/CFT correspondence. In some cases, the two sides agree, but the match is not compelling because the CFT proposals involve many free parameters. We propose a new approach to complexity in holographic CFT s based on the notion of query complexity. Our proposal has no free parameters and is more directly related to the bulk than are current, circuit complexity-based proposals.

The motivation for studying complexity comes primarily from time-dependent processes such as the growth of black hole interiors [1,6]. However, if the core idea that complexity manifests itself holographically as spatial size is correct, it must also apply to static configurations. (In such contexts, we speak of “complexity of formation”; see e.g., [7].) We set out to study one such example: the pure three-dimensional AdS_3 spacetime (AdS_3) that is holographically dual to the ground state of a two-dimensional conformal field theory (CFT_2). Although pure AdS_3 is static and not a black hole, it is a perfect arena for testing ideas about complexity because (i) it is the best understood bulk geometry that arises in holography, (ii) its properties are fixed by symmetry so the discussion does not depend on details of the CFT or the field content of the bulk theory, and (iii) many prior discussions of the complexity of formation studied AdS_3 as an example [5,7,8] so it is a standard test case.

One consequence of conformal symmetry is that the CFT_2 vacuum “looks the same” over an infinite range of

scales, so its complexity can only be infinite. For a finite measure of complexity, the ground state must be subject to an ultraviolet (UV) cutoff and the subcutoff data appropriately truncated. In holography, the choice of cutoff manifests itself as a large scale (infrared, IR) cutoff in the bulk; any gravity-side proposal for complexity must have a built-in dependence on the cutoff. But the meaning and implementation of the bulk cutoff have been extensively debated [9] and it is at present unclear what notion of a bulk cutoff is most conducive for studying complexity.

II. OVERVIEW

We start with some heuristics of what *cutoff* and *state* should mean in the context of holographic *state complexity*. We emphasize that a state is a functional over the space of CFT operators and as such can be characterized by query complexity [10]: how many times a subroutine must be called (how many times an oracle is queried) before the functional can be computed on an arbitrary input. For evaluating the ground state functional—for computing CFT vacuum correlators—the oracle will supply applications of the fusion algebra.

On a technical level, we present an object in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory whose properties match the stated heuristics about the CFT_2 ground state defined at a cutoff. Symmetry singles out a unique candidate for quantifying the query complexity of this object, which in bulk language ends up matching the spatial volume encircled by the cutoff surface, in agreement with [1]. Our emphasis is not on the specifics of the CFT_2 ground state but on conceptualizing field theory complexity as query complexity instead of circuit complexity.

III. THE CUTOFF

Consider a semiclassical, static, asymptotically AdS_3 geometry dual to some state of a holographic CFT_2 .

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We stipulate the following about how an IR cutoff in the former relates to a UV cutoff in the latter:

- (1) In longitudinally homogeneous geometries, radial curves manifest constant CFT cutoffs.

Examples include $u = \text{const}$ lines in the Poincaré patch,

$$ds^2 = (dx^+ dx^- + du^2)/u^2, \quad (1)$$

and $\chi = \text{const}$ curves in the BTZ/AdS₃-Rindler metric,

$$ds^2 = -\sinh^2 \chi dt^2 + d\chi^2 + \cosh^2 \chi dx^2. \quad (2)$$

- (2) Coarsening the CFT UV cutoff pushes the bulk cutoff away from the boundary (deeper in the interior).
- (3) Maximally coarsening the CFT cutoff (so it encompasses the entire Cauchy slice) reduces the theory to s-waves (constant modes). This places the cutoff surface on the horizon if one exists.

This follows as a limit of point 2 and from subregion duality [11]. If s-waves defined a bulk cutoff shy of the horizon, how would we interpret the region between the horizon and the cutoff? Yet the cutoff cannot reach beyond the horizon lest it violate subregion duality.

- (4) In locally AdS₃ geometries, if the bulk cutoff follows a geodesic γ , then the corresponding CFT cutoff scheme truncates the modes localized in the interval subtended by γ to its s-wave sector.

The last point follows from remark 3 after recognizing the geodesic γ as the horizon of an AdS₃-Rindler geometry. The appropriate way of smearing local operators to produce s-waves on CFT intervals was discussed in [12].

IV. THE STATE

A density operator is a functional, which sends operators to real numbers,

$$\rho: \mathcal{O} \rightarrow \text{tr}(\rho \mathcal{O}). \quad (3)$$

In field theory, the state's dependence on the cutoff restricts the domain of functional (3). Specializing to the holographic context, suppose the cutoff is presented as a smooth and convex curve in the bulk, parametrized by proper length λ . We do not assume that the cutoff is homogeneous, so the profile of the cutoff curve generically depends on space. We require the map ρ —the CFT₂ ground state at the given cutoff scale—to satisfy the following:

- (i) ρ eats up multilocal operators $\mathcal{O}_1(\lambda_1) \dots \mathcal{O}_n(\lambda_n)$ and returns $\langle 0 | \mathcal{O}_1(\lambda_1) \dots \mathcal{O}_n(\lambda_n) | 0 \rangle$.
- (ii) The arguments $\mathcal{O}_i(\lambda_i)$ of ρ are labeled by their locations λ_i on the bulk cutoff curve.
- (iii) If the curve follows a geodesic over some range of λ 's, ρ takes at most one input $\mathcal{O}(\lambda)$ from that range.

- (iv) Shifts in the bulk curve and the attendant transformations of ρ enact local changes of scale.

Requirement (iii) follows from point 4 in the cutoff discussion: more than one independent input from a geodesic segment would be outside the s-wave sector of the AdS₃-Rindler space lying beyond the geodesic.

We now show that three-dimensional $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory contains an object, which automatically satisfies demands (i)–(iv). The Chern-Simons language is useful because it accommodates locally varying $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ frames and CFT operators are sections of an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ bundle [13]—a technicality we explain below. Emphatically, we will make no reference to gravitational dynamics, for which the Chern-Simons formalism is conventionally deployed.

V. A REVIEW OF $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ GAUGE FIELDS

We are interested in Lorentzian CFT₂s on a space with metric $ds^2 = dx^+ dx^-$. Introduce an orthogonal semi-infinite axis u and place the CFT at $u = 0$. Let A and $\bar{A}$ be two $\mathfrak{sl}(2, \mathbb{R})$ -valued connection one-forms in (x^+, x^-, u) -space, i.e., gauge fields whose components are linear combinations of L_{-1}, L_0, L_1 . The latter generate the left- and right-moving global conformal symmetries and satisfy

$$[L_n, L_m] = (n - m)L_{n+m}. \quad (4)$$

We choose a *flat* configuration of A and $\bar{A}$, which satisfies boundary conditions

$$\lim_{u \rightarrow 0} u A_+ = L_{-1} \quad \text{and} \quad \lim_{u \rightarrow 0} A_- = 0, \quad (5)$$

$$\lim_{u \rightarrow 0} u \bar{A}_- = L_1 \quad \text{and} \quad \lim_{u \rightarrow 0} \bar{A}_+ = 0. \quad (6)$$

With this choice, Wilson lines (and their networks) compute CFT₂ ground state correlation functions [14, 15].

VI. THE CFT MEANING OF WILSON LINES

In gauge theories, gauge-invariant quantities are supplied by path-ordered exponentials along paths $(x^+(\sigma), x^-(\sigma), u(\sigma))$,

$$P \exp \int_{\sigma_b}^{\sigma_e} d\sigma (\partial_\sigma x^+ A_+ + \partial_\sigma x^- A_- + \partial_\sigma u A_u) \times (A \leftrightarrow \bar{A}).$$

Ordinarily, the path must be closed and this is called a Wilson loop. In the present case, the flatness of A and $\bar{A}$ makes all Wilson loops trivial. When the path is open, there is a residual gauge dependence at the end points, so arbitrary Wilson *lines* are not gauge invariant. The exception is when the end points are taken to the boundary where

the gauge dependence is frozen by boundary conditions. Consequently, we focus below on Wilson lines with end points at the asymptotic boundary $u \rightarrow 0$.

Consider a path from (x_b^+, x_b^-, u_0) to (x_e^+, x_e^-, u_0) . Because A and $\bar{A}$ are flat, the exact course of the path is immaterial so we can pull it onto the $u = u_0$ plane. The Wilson line of A evaluates to $\exp(L_{-1}(x_e^+ - x_b^+)/u_0)$, a finite translation by $(x_e^+ - x_b^+)/u_0$. The $\bar{A}$ -Wilson line performs a similar translation, only in $1/x^-$. Indeed, any Wilson line is a finite $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ transformation because it is a path-ordered exponential of an $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ one-form. Due to the boundary conditions (5) and (6), Wilson lines that begin and end on the boundary are always finite translations in x^+ and in $1/x^-$.

We would like to compute expectation values of such Wilson lines in various $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representations. Representations of $SL(2, \mathbb{R})$ are classified by conformal dimensions h . Their multiplets are built up by acting with L_{-1} s on the highest weight state $|h\rangle \equiv \mathcal{O}_h(0)|0\rangle$. One example of an expectation value of our Wilson line is therefore

$$\langle h | e^{L_{-1}(x_e^+ - x_b^+)/u_0} | h \rangle = \langle 0 | \mathcal{O}_h(0) \mathcal{O}_h((x_e^+ - x_b^+)/u_0) | 0 \rangle. \quad (7)$$

We recognize that all CFT two-point functions are expectation values of translations, taken over $SL(2, \mathbb{R})$ multiplets in various representations. Once again, Eq. (5) ensures that boundary-pegged A -Wilson lines evaluate to the correct translations up to a rescaling by u_0 , which we comment on below. For the $\bar{A}$ -components of our Wilson lines, the conclusion is the same except the $SL(2, \mathbb{R})$ multiplets must be built in the $1/x^-$ conformal frame—by acting with L_{+1} s from the right on $\langle h | \equiv |h\rangle^\dagger$.

VII. THE CFT MENAING OF NETWORKS

For higher-point correlators, we combine Wilson lines to form networks—collections of lines joined by junctions. Such junctions were previously described in [15]. In the discussion above, the flatness of $(A, \bar{A})$ allowed us to pull arbitrary Wilson lines onto $u = \text{const}$ slices. We need to make sure that networks with junctions are similarly deformable.

A junction merges two incoming Wilson lines into one outgoing line. Recall that an individual Wilson line carries an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representation—a vector space populated by $|h, \bar{h}\rangle \equiv \mathcal{O}_{h, \bar{h}}|0\rangle$ and descendants. Two incoming Wilson lines carry two independent such multiplets, which a junction then sends to one multiplet,

$$\begin{aligned} & \mathcal{O}_{h_1, \bar{h}_1}(v, \bar{v})|0\rangle_v \otimes \mathcal{O}_{h_2, \bar{h}_2}(w, \bar{w})|0\rangle_w \\ & \xrightarrow{\text{junction}} \int dz d\bar{z} \sum_{h, \bar{h}} c_{h_1, \bar{h}_1; h_2, \bar{h}_2}^{h, \bar{h}}(z, \bar{z}, v, \bar{v}, w, \bar{w}) \mathcal{O}_{h, \bar{h}}(z, \bar{z})|0\rangle_z. \end{aligned} \quad (8)$$

The notation $|0\rangle_v$ distinguishes the “ground state” on v -space from the ground states on w - and z -spaces—local copies of the CFT on which the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ multiplets live. Defining a junction boils down to choosing appropriate kernels $c_{h_1, \bar{h}_1; h_2, \bar{h}_2}^{h, \bar{h}}(z, \bar{z}, v, \bar{v}, w, \bar{w})$ in (8).

With an arbitrarily chosen kernel, networks of Wilson lines will not be deformable. A condition that guarantees deformability is this: if an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ transformation is simultaneously applied to the v - and w -space inputs in (8), its z -space output must be identically transformed [15]. The same condition governs the operator product expansion (OPE) of local operators in the CFT. Therefore, in a deformable junction, the dependence of $c_{h_1, \bar{h}_1; h_2, \bar{h}_2}^{h, \bar{h}}(z, \bar{z}, v, \bar{v}, w, \bar{w})$ on coordinate arguments is fixed by conformal kinematics up to overall constants $c_{h_1, \bar{h}_1; h_2, \bar{h}_2}^{h, \bar{h}}$. Evidently, defining a deformable set of junctions leaves out the same flexibility as defining a CFT does: the undetermined data are OPE coefficients.

Naturally, we choose Wilson line junctions to match the fusion algebra of the CFT. Thus, a network shown in the top left panel of the Fig. 1 represents a repeated application of the OPE expansion. Once again, this network can be pulled

FIG. 1. All pictures are equal time snapshots of (x^+, x^-, u) -space. Top left: a network of Wilson lines, which computes CFT six-point functions. Top right: a network which computes multi-point functions. After the red *tentacles* are amputated, it becomes the ground state at the scale defined by the black line. The density of inputs on the network is $dC \propto K d\lambda$ so the network accepts no inputs on geodesic segments. Bottom: two amputated networks in global AdS_3 , which differ only by crossing symmetry, are shown in black. Crossing symmetry is implemented by shifting the feature $-00-$, which implements the projector $|0\rangle\langle 0|$ in computing correlation functions. The blue features—Wilson lines and projections—convert the black network into one at a coarser, uniform cutoff. Projections are necessary to accord with $dC \propto K d\lambda$ because coarse-graining eliminates degrees of freedom. The coarse-graining scheme is highly nonunique; it varies with gauge choice for $(A, \bar{A})$ and with crossing symmetry.

onto the boundary, in which case it is the OPE expansion of the CFT. When the network is contracted at all end points with member states of various $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ multiplets, it will by construction return the relevant multipoint correlation function.

VIII. THE CHERN-SIMONS FIELD ENCODES THE AdS_3 GEOMETRY

The $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ connection we exploit is most famous for a reformulation of pure gravity in asymptotically AdS_3 spacetimes [16]. When a flat $(A, \bar{A})$ which satisfies boundary conditions (5) and (6) is substituted into

$$g_{\mu\nu} = (\frac{1}{2})\text{tr}_{\text{fundamental}}(A - \bar{A})_\mu(A - \bar{A})_\nu, \quad (9)$$

it returns the metric of the Poincaré patch of AdS_3 . The nonvanishing traces entering (9) are $\text{tr}L_0^2 = 1/2$ and $\text{tr}L_{-1}L_1 = -1$. Other locally AdS_3 geometries, including global AdS_3 , arise from (9) with modified boundary conditions [14,17]. (The assertion that networks of Wilson lines compute CFT correlators still applies, with expectation values now taken in the vacuum on a cylinder or in Virasoro descendants.) As an example, the choice

$$A = (L_{-1}/u)dx^+ + L_0d(\log u), \quad (10)$$

$$\bar{A} = (L_1/u)dx^- - L_0d(\log u) \quad (11)$$

produces metric (1) while other, gauge-equivalent choices produce the same geometry in different coordinates. The flatness of $(A, \bar{A})$, which expresses vacuum Einstein's equations with $\Lambda = -1$, follows from the classical equations of motion of Chern-Simons theory.

Our discussion of CFT correlation functions made no reference to the quantum Chern-Simons theory and $(A, \bar{A})$ were only used as auxiliary technical tools. But because the AdS_3 spacetime dual to the ground state of any holographic CFT₂ is described by metric (9), we may explain the meaning of bulk cutoff curves with reference to classical solutions such as (10) and (11). In the following, we write down a combination of Chern-Simons Wilson lines, which computes CFT correlation functions at a given scale. This construction could be done for arbitrary CFTs, in which case the requisite Chern-Simons configurations would remain purely auxiliary. In holographic theories, however, we can use classical solutions of the Chern-Simons equations of motion to identify our network with a curve living in the bulk.

IX. THE CFT VACUUM AT A HOLOGRAPHICALLY SPECIFIED SCALE

Consider the network of Wilson lines pictured in the top right panel of the Fig. 1. Assume the upper, black part of the network follows the bulk cutoff curve. Now sever all the red

tentacles of this network to produce the *amputated network*. (Amputation creates stumps that transform in dual representations—objects that eat up incoming representations and return numbers.) We claim that this amputated network is the ground state at the scale specified by the cutoff surface. More examples, which represent the vacuum on a cylinder, are shown in the bottom panels of the Fig. 1. (Our amputated network shares certain features with the tensor network of [18].)

The amputated network manifestly satisfies conditions (i)–(iii). It eats up $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representations (delivered by incoming Wilson lines) to return appropriate multipoint correlators. Incoming lines are sprinkled over the cutoff curve except—as marked in the Fig. 1—on geodesic segments. To verify (iv), inspect one of the amputated tentacles, say in the u -direction in gauge (10),

$$\exp \int_{u_0}^{u_*} (du/u) L_0 = (u_*/u_0)^{L_0}. \quad (12)$$

We recognize that the job of the tentacles is to bring $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representations from some reference scale u_0 to the scale of the cutoff curve, u_* . If we shifted the cutoff curve from u_* to a new scale u'_* , individual stumps would get rescaled by $(u_*/u'_*)^{L_0}$ to absorb the changed scales of their inputs; this is just what requirement (iv) stipulates. (Under coarse-graining, some stumps are also projected out; see the bottom right panel of the Fig. 1 and text below. Such projections are routine in tensor networks that model the RG behavior of CFT states [19].) Equation (12) explains why the end points of the Wilson line in (7) were put at some arbitrary u_0 . Sending them to the true asymptotic boundary ($u_0 \rightarrow 0$) would have dilated the computation by an overall infinite factor familiar from AdS/CFT .

The amputated network is not gauge-invariant. Its gauge dependence, dual to the gauge dependence of tentacles like (12), is an inalienable trait of renormalization. There is no preordained way of renormalizing CFT fields; doing so with $(u_*/u_0)^{L_0}$ is an artifact of gauge (10) and (11) and, by extension, of metric (1). This ambiguity of holographic RG manifests the technicality we mentioned—that CFT operators are sections of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ bundles—and reflects the rule of thumb that boundary global symmetries are bulk gauge symmetries.

Finally, we comment on the loose ends of the amputated network. They carry trivial representations, which institute $\langle 0|$ and $|0\rangle$ in the computation of $\langle 0|\mathcal{O}_1 \dots \mathcal{O}_n|0\rangle$. Had we joined the ends, $|0\rangle\langle 0|$ would be replaced with a sum over representations and the network would compute torus correlators instead. Shifting the loose ends implements crossing symmetry by reordering applications of the fusion algebra. We illustrate this with the two black networks in the bottom panels of the Fig. 1.

X. WHAT IS THE COMPLEXITY OF THE AMPUTATED NETWORK?

The amputated network is an *algorithm* for computing CFT correlation functions—an algorithm for evaluating ρ in Eq. (3). The traditional framework of circuit complexity, which counts applications of some preferred unitary gates to a (presumably unentangled) reference state, is difficult to apply to an algorithm. It is contrived to go about constructing map $\mathcal{O} \rightarrow \langle 0|\mathcal{O}|0\rangle$ from a more exotic map $\mathcal{O} \rightarrow \text{tr}(\rho\mathcal{O})$ which, if ρ is unentangled, is marred by singularities not present in $|0\rangle\langle 0|$. Moreover, the amputated network is not even composed of unitaries.

Query complexity [10] applies to the amputated network in a natural way: it counts how many times an algorithm calls a particular subroutine (queries an oracle). The network computes correlation functions by repeatedly applying OPE fusion; thus, our oracle will be tasked with enacting the fusion algebra. Note that counting queries of a “fusion algebra oracle” amounts to counting Wilson lines the network absorbs; because no algorithm runs faster than linearly in the inputs, we need not prove that our algorithm is optimal.

Does any principle dictate how many Wilson lines the network can absorb? Thus far, our considerations have strongly relied on the global symmetry of the CFT_2 . Because AdS is dual to the ground states of *all* holographic CFTs, a geometric representation of their complexities—if it exists—must be fixed by conformal symmetry. If so, the measure of complexity $d\mathcal{C}$ on our amputated network must be an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ invariant of the cutoff curve. Curves confined to a static slice of AdS_3 have only two local invariants: the proper length element $d\lambda$ and $Kd\lambda$, where K is the extrinsic curvature. (Higher powers of K are local invariants on smooth curves, but become ill-defined if we discretize the curve into a sequence of geodesic segments.) The density of complexity on the cutoff curve must therefore be a linear combination of the two: $d\mathcal{C}/d\lambda = \#_0 + \#_1 K$. As our network takes no inputs on geodesic segments where $K = 0$, $\#_0$ must vanish so $d\mathcal{C} \propto Kd\lambda$. For a sanity check, note that all equal time $u = \text{const}$ lines in metric (1) have $K = 1$ and $d\lambda = dx/u$, so as expected $d\mathcal{C}/dx \propto 1/u$ in that case.

XI. COMPARISON WITH OTHER PROPOSALS

Specializing again to cutoff curves confined to the static slice, the Gauss-Bonnet theorem relates the complexity of the amputated network to the volume enclosed by the cutoff surface

$$\mathcal{C} \propto \int Kd\lambda = \int (-R)dV + 2\pi = \int dV + 2\pi. \quad (13)$$

In the convention of Eq. (9), the Ricci scalar on the static slice is $R = -1$. We will not attempt to explain the additive Euler term, which does not scale with the cutoff.

Our argument seems to support the complexity = volume proposal [1], albeit in a novel way: the amputated Wilson line network does so not by filling out the volume, but with an appropriate density of nodes along the boundary. However, this agreement is not compelling evidence for [1]. When we take the cutoff curve off the static slice, symmetry allows a new contribution $d\mathcal{C}/d\lambda \propto \#_1 K + \#_2 \tau$, where τ is the Lorentzian “torsion” of the curve. Whatever combination of extrinsic curvature and torsion might quantify complexity, it will not match the maximal volume inside the curve, which is a more erratic quantity. In this way, nonstatic cutoffs provide a transparent distinction between our CFT query complexity and the bulk proposal of [1].

We wish to emphasize the advantages of conceptualizing field theory complexity as query complexity. It dispenses with many arbitrary choices that mire applications of circuit complexity to holography, such as the choice of gates, their costs, and a reference state. Query complexity is also more faithful to the original motivation for the complexity conjectures, which involved counting nodes in hypothetical space-filling tensor networks [1]. In an algorithmic language, nodes perform some subroutine and query complexity counts how many times it is called.

XII. OTHER STATES

We have presented the CFT_2 ground state as an algorithm, which queries a “fusion algebra oracle.” A similar algorithm applicable to other states should transport and fuse modular frequency modes of cutoff-sized intervals instead of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representations. (Transport of modular modes was sketched in [20].) Calling an oracle that fuses modular modes consistently generalizes the present analysis because local operators at interval end points, as well as their fusions, are automatically vacuum modular modes of intervals [12,21]. We will explore these ideas in future work.

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