Loop corrections to celestial amplitudes

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We study the effect of loop corrections to conformal correlators on the celestial sphere at null infinity. We first analyze finite one-loop celestial amplitudes in pure Yang-Mills theory and Einstein gravity. We then turn to our main focus: infrared divergent loop amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory. We compute the celestial one-loop amplitude in dimensional regularization and show that it can be recast as an operator acting on the celestial tree-level amplitude. This extends to any loop order, and the resummation of all planar loops enables us to write down an expression for the all-loop celestial amplitude. Finally, we show that the exponentiated all-loop expression given by the Bern-Dixon-Smirnov (BDS) formula gets promoted on the celestial sphere to an operator acting on the tree-level conformal correlation function, thus yielding, the *celestial* BDS formula.

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I. INTRODUCTION

Celestial amplitudes reveal conformal properties of four-dimensional scattering amplitudes of massless particles as the standard plane wave basis is replaced by a basis of boost eigenstates. This is achieved by a Mellin transform applied to each external state in the scattering amplitude, which maps plane waves labeled by the null momenta of the particles, or equivalently, their energy and a point on the 2-sphere as well as their helicity to so-called conformal primary wave functions. The latter are labeled by the conformal dimension Δ and the spin J under the two-dimensional global conformal group which arises from the action of the four-dimensional $SL(2, \mathbb{C})$ Lorentz group on the celestial sphere at null infinity.

Conformal wave functions have already been considered by Dirac [1], but recent years have seen a surge in interest in part due to their role for a potentially holographic description of asymptotically flat spacetimes [2–5]. This has in part been driven by the realization that the asymptotic symmetry group of Einstein gravity at null infinity should include an extension of the Bondi-Metzner-Sachs (BMS) group that enhances the global conformal group on the celestial sphere to the full Virasoro symmetry group.¹

¹The Virasoro symmetry may be further enhanced to $\text{Diff}(S^2)$ —see Ref. [6] for a discussion of this point. See also Ref. [7–10] for further results on symmetries of celestial amplitudes.

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To understand the properties of a putative holographically dual *celestial* conformal field theory (CFT) at null infinity of asymptotically Minkowski spacetimes, a bottom-up approach has been pursued, starting with the identification of a conformal basis of wave functions in Refs. [7,11–14] and the computation of celestial amplitudes for various spins [15–19], which all take the form of two-dimensional correlation functions on the celestial sphere. Furthermore, the celestial analog of various soft theorems in quantum field theory have been obtained in Refs. [20–25], while collinear limits of scattering amplitudes have been shown in Refs. [26–30] to extract from celestial amplitudes the operator product expansion of conformal primaries in the putative celestial CFT. The formalism for a relativistic partial wave expansion for celestial four-point amplitudes was developed in Ref. [31] (see also Ref. [20]). A procedure for a celestial double copy relating celestial gravity and gauge theory amplitudes has been discussed in Ref. [32].

Most work so far has focused on celestial tree-level amplitudes, while quantum effects have largely been neglected. One may thus wonder how the conformal structure uncovered at tree level is affected once loop corrections are taken into account. The Mellin transform involves a sum over all energies that mixes the infrared and ultraviolet regimes. The main difficulty in computing celestial amplitudes at loop level is thus due to the integration over internal momentum loops which render the Mellin integrals divergent.²

²These difficulties are circumvented when considering the celestial analog of finite loop amplitudes whose tree-level counterparts vanish. Such amplitudes were considered in Ref. [33] for gluons and gravitons, and we will discuss the case of one-loop four-point amplitudes in pure Yang-Mills theory and Einstein gravity below.

Nevertheless, there have been first attempts at understanding the conformal structure of flat space scattering amplitudes beyond tree level in Ref. [34], which studies celestial loop effects in massless scalar field theory with ϕ^4 interaction.

Here, we would like to initiate the study of divergent loop corrections to celestial tree-level amplitudes involving gluons as external massless states. Due to gauge invariance, the external states are guaranteed to reach null infinity at any order in perturbation theory. We will focus on the case of maximal helicity violation (MHV) four-point amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory. This has the benefit that the four-gluon amplitude is known to all orders in the loop expansion. To control infrared divergences, we employ the dimensional regularization method known as the four-dimensional helicity (FDH) scheme [35]. The momenta of the internal particles are taken to be in $D = 4 - 2\epsilon$ dimensions, while the momenta of the external particles remain in $D = 4$, as do the polarization vectors for both external and internal particles. The four-gluon MHV amplitude in planar $\mathcal{N} = 4$ super-Yang-Mills can be conveniently written in factorized form in terms of the tree-level amplitude containing the helicity structure and an infrared divergent piece containing the information about the loop order.

Our main result is that this infrared divergent factor gets promoted, after the Mellin transform from the momentum basis to the conformal basis, to a differential operator. Thus, the celestial one-loop amplitude in planar $\mathcal{N} = 4$ super-Yang-Mills can be recast as an operator acting on the celestial tree-level amplitude.³ This structure is shown to persist at any loop order. The effect of this *celestial loop operator* is to simultaneously shift the conformal dimensions of all the external particles. The operator is dressed with a loop-order dependent function of the conformal cross-ratio and diverges as the dimensional regulator is taken to zero.

Given that the Mellin transform effectively replaces the notion of energy with that of conformal scaling dimension, it is not too surprising that loop effects should manifest themselves as a change in the latter. Interestingly enough, though, celestial loop effects can be recast as an operator statement. This also resonates with the recent presentation in Ref. [32] of tree-level graviton and gluon amplitudes as operators acting on scalar amplitudes when the asymptotic states are taken to be in the conformal basis.

In momentum space, the four-gluon amplitude is known to all orders in the loop expansion. Moreover, the resummation of all loop contributions exponentiates yielding the BDS formula found by Bern *et al.* [37] based on an iterative relation uncovered by Anastasiou *et al.* [38].⁴ Here, we

provide its celestial analog. At any loop order, there exists an operator acting on the celestial tree-level amplitude which results in a loop-order dependent shift in the conformal dimensions. The resummation of all celestial loop operators exponentiates, thus yielding the celestial BDS formula.

This paper is organized as follows. We begin with a review in Sec. II, where we set up the necessary notation. A convenient basis for discussing celestial amplitudes at tree and loop levels is introduced in Sec. II A. In Sec. II B, we review some of the salient features of celestial amplitudes, and we give a tree-level example in Sec. II C. We discuss loop amplitudes in Sec. III. First, we compute in Sec. III A celestial amplitudes for one-loop processes in Yang-Mills theory for which the tree-level amplitudes vanish. We then move on to the main focus of this paper in Sec. III B and analyze how divergent loop corrections in planar $\mathcal{N} = 4$ super-Yang-Mills theory correct the corresponding tree-level celestial correlators. We show in Sec. III B 1 that the one-loop four-gluon celestial amplitudes can be recast as an operator acting on the celestial tree-level amplitude and generalize this result to ℓ loops in Sec. III B 2. The exponentiated resummation of the all-loop result then gets promoted to an operator statement yielding the celestial analog of the BDS formula. In the Appendix A, we make use of the general expression for celestial amplitudes introduced in Sec. II to analyze tree-level amplitudes in massless QED in Appendix A 1 and finite one-loop amplitudes in Einstein gravity in Appendix A 2.

II. SETUP

In this section, we lay out a convenient basis for expressing celestial amplitudes at tree and loop levels. We consider scattering amplitudes in four dimensions⁵ with external massless particles. Each external particle is labeled by a momentum p_i^μ , a helicity ℓ_i , and a sign distinguishing incoming from outgoing states. We parametrize p_i^μ in terms of points $(z_i, \bar{z}_i)$ on the two-dimensional celestial sphere through the map

$$p_i^\mu = \frac{1}{2} \omega_i (1 + |z_i|^2, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - |z_i|^2), \quad (2.1)$$

with $\omega_i \geq 0$.

A. Convenient basis for four-point amplitudes

Our focus will be on four-point amplitudes

$$\mathcal{A}(\{\omega_i, \ell_i, z_i, \bar{z}_i\}) = A(1^{\ell_1}, 2^{\ell_2}, 3^{\ell_3}, 4^{\ell_4}) \times \delta(p_1 + p_2 - p_3 - p_4). \quad (2.2)$$

³Similar conclusions have been recently hinted at for QED and gravity [36].

⁴This was later confirmed in the strong coupling regime by Alday and Maldacena [39].

⁵We work in the $(+ - - -)$ signature.

Without loss of generality, we assume that the stripped amplitude can be split as

$$A(1^{\ell_1}, 2^{\ell_2}, 3^{\ell_3}, 4^{\ell_4}) = \mathcal{B}(s, t) \mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)}, \quad (2.3)$$

where $\mathcal{B}(s, t)$ is a (not necessarily analytic) function of two of the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_4)^2$, and $u = (p_1 - p_3)^2$, and the rational function $\mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)}$ carries the helicity structure of the external particles. The latter is a function of the spinor products $\langle ij \rangle$ and $[ij]$ defined by⁶

$$\begin{aligned} \langle ij \rangle &= \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta = \sqrt{\omega_i \omega_j} z_{ij}, \\ [ij] &= -\epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}} = -\sqrt{\omega_i \omega_j} \bar{z}_{ij}, \end{aligned} \quad (2.4)$$

where $z_{ij} = z_i - z_j$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$, as well as the conformally invariant cross-ratio

$$r = \frac{z_{12} z_{34}}{z_{23} z_{41}}, \quad (2.5)$$

which is related to the four-dimensional scattering angle in the center of mass frame θ through

$$r = -\frac{s}{t} = \csc^2\left(\frac{\theta}{2}\right). \quad (2.6)$$

It follows that $r > 1$.

Since all the information about the helicities of the external states is encoded in the function $\mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)}$, it must satisfy

$$\hat{\ell}_i \mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)} = \ell_i \mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)}, \quad (2.7)$$

for the helicity operator for the i th particle

$$\hat{\ell}_i = \frac{1}{2} \left(-\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right). \quad (2.8)$$

We may express a solution to (2.7) in terms of powers of Lorentz invariant functions R_i , with the defining property that $\hat{\ell}_i R_j = \delta_{ij} R_j$, as

$$\mathcal{R}_{(\ell_1, \ell_2, \ell_3, \ell_4)} = r^{\alpha_1} (r-1)^{\alpha_2} \prod_{i=1}^4 R_i^{\ell_i}, \quad (2.9)$$

⁶The spinors λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$ can be read off from

$$p^{\dot{\alpha}\alpha} = \bar{\sigma}_\mu^{\dot{\alpha}\alpha} p^\mu = \omega \begin{bmatrix} 1 & z \\ \bar{z} & |z|^2 \end{bmatrix} = \underbrace{\sqrt{\omega} \begin{bmatrix} 1 \\ \bar{z} \end{bmatrix}}_{\tilde{\lambda}^{\dot{\alpha}}} \underbrace{\sqrt{\omega} \begin{bmatrix} 1 & z \end{bmatrix}}_{\lambda^\alpha}.$$

with α_1 and α_2 real numbers. A basis of functions R_i that depend on the energies ω_i of the external particles was given in Ref. [41]. For our purposes, it is more convenient to express $\mathcal{R}$ in terms of functions that depend exclusively on the differences of points $(z_{ij}, \bar{z}_{ij})$ on the celestial sphere. Such a set of R_i is given by

$$\begin{aligned} R_1 &= \left(\frac{[12][13]\langle 23 \rangle}{\langle 12 \rangle \langle 13 \rangle [32]} \right)^{\frac{1}{2}}, & R_2 &= \left(\frac{[12][23]\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle [31]} \right)^{\frac{1}{2}}, \\ R_3 &= \left(\frac{[13][23]\langle 12 \rangle}{\langle 13 \rangle \langle 23 \rangle [21]} \right)^{\frac{1}{2}}, & R_4 &= \frac{[24]}{\langle 24 \rangle} \left(\frac{[31]\langle 12 \rangle \langle 23 \rangle}{\langle 13 \rangle [12][23]} \right)^{\frac{1}{2}}. \end{aligned} \quad (2.10)$$

The four-point amplitude (2.3) can then be expressed as

$$\begin{aligned} A(1^{\ell_1}, 2^{\ell_2}, 3^{\ell_3}, 4^{\ell_4}) &= r^{\alpha_1} (r-1)^{\alpha_2} \mathcal{B}(s, t) \left(\frac{z_{12}}{\bar{z}_{12}} \right)^{-\frac{1}{2}(\ell_1 + \ell_2 - \ell_3 - \ell_4)} \\ &\times \left(\frac{z_{13}}{\bar{z}_{13}} \right)^{-\frac{1}{2}(\ell_1 - \ell_2 + \ell_3 + \ell_4)} \left(\frac{z_{23}}{\bar{z}_{23}} \right)^{\frac{1}{2}(\ell_1 - \ell_2 - \ell_3 + \ell_4)} \\ &\times \left(\frac{z_{24}}{\bar{z}_{24}} \right)^{-\ell_4}. \end{aligned} \quad (2.11)$$

Notice that no mention to perturbation theory has been made, and as long as the amplitude is of the form (2.11), the statements made so far apply to the exact four-point $\mathcal{S}$ -matrix element.

B. Celestial amplitudes

Celestial amplitudes are obtained by performing a Mellin transform on each of the n external particles in a scattering process,

$$\begin{aligned} \tilde{\mathcal{A}}(\{\Delta_i, J_i, z_i, \bar{z}_i\}) &= \prod_{k=1}^n \left(\int_0^\infty d\omega_k \omega_k^{\Delta_k-1} \right) \mathcal{A}(\{\omega_i, \ell_i, z_i, \bar{z}_i\}). \end{aligned} \quad (2.12)$$

Under $SL(2, \mathbb{C})$ Lorentz transformations, they have been shown to transform as [11, 16, 19]

$$\begin{aligned} \tilde{\mathcal{A}}\left(\left\{\Delta_j, J_j; \frac{az_j + b}{cz_j + d}, \frac{\bar{a}\bar{z}_j + \bar{b}}{\bar{c}\bar{z}_j + \bar{d}}\right\}\right) &= \prod_{j=1}^n ((cz_j + d)^{\Delta_j + J_j} (\bar{c}\bar{z}_j + \bar{d})^{\Delta_j - J_j}) \tilde{\mathcal{A}}(\{\Delta_i, J_i; z_i, \bar{z}_i\}), \end{aligned} \quad (2.13)$$

where Δ_j are the conformal dimensions and $J_j \equiv \ell_j$ are the spins of operators inserted at the points $(z_j, \bar{z}_j) \in S^2$. Celestial amplitudes thus share conformal properties with

correlation functions on the celestial sphere. Indeed, evaluating (2.12) for the four-point amplitude (2.2) using

$$\begin{aligned} & \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \\ &= \frac{4}{\omega_4 |z_{14}|^2 |\bar{z}_{23}|^2} \delta(r - \bar{r}) \delta\left(\omega_1 - \frac{z_{24} \bar{z}_{34}}{z_{12} \bar{z}_{13}} \omega_4\right) \\ & \quad \times \delta\left(\omega_2 + \frac{z_{14} \bar{z}_{34}}{z_{12} \bar{z}_{23}} \omega_4\right) \delta\left(\omega_3 + \frac{z_{24} \bar{z}_{14}}{z_{23} \bar{z}_{13}} \omega_4\right) \end{aligned} \quad (2.14)$$

yields the celestial four-point amplitude (up to a numerical factor)

$$\tilde{\mathcal{A}}(\{\Delta_i, J_i, z_i, \bar{z}_i\}) = f(r, \bar{r}) \prod_{i < j}^4 z_{ij}^{\frac{h_i - h_j}{2} - h_j} \bar{z}_{ij}^{\frac{\bar{h}_i - \bar{h}_j}{2} - \bar{h}_j}. \quad (2.15)$$

The conformally invariant expression $f(r, \bar{r})$ is given by

$$\begin{aligned} f(r, \bar{r}) &= 2\delta(r - \bar{r}) \Theta(r - 1) r^{\alpha_1 + \frac{\Delta}{6}} (r - 1)^{\alpha_2 + \frac{\Delta}{6}} \\ & \quad \times \int_0^\infty dw w^{\frac{\Delta-6}{2}} \mathcal{B}(rw, -w), \end{aligned} \quad (2.16)$$

with $\Delta \equiv \sum_{i=1}^4 \Delta_i$. The conformal weights of the external wave functions introduced in (2.15) are given by

$$h_k = \frac{1}{2}(\Delta_k + J_k), \quad \bar{h}_k = \frac{1}{2}(\Delta_k - J_k), \quad (2.17)$$

where the two-dimensional spins J_k are identified with the helicities ℓ_k of the four-dimensional particles crossing null infinity and the conformal dimensions Δ_k of finite energy wave functions are restricted to lie on the principal continuous series of the $SL(2, \mathbb{C})$ Lorentz group[11], namely

$$\Delta_k = 1 + i\lambda_k \quad \text{with} \quad \lambda_k \in \mathbb{R}, \quad J_k \equiv \ell_k. \quad (2.18)$$

C. Example: Celestial gluons at tree level

We illustrate the above prescription for celestial four-gluon amplitudes at tree level in pure Yang-Mills theory which were first discussed in Refs. [16,19]. The MHV four-gluon amplitude is given by (2.2) with the stripped amplitude

$$A_{\text{tree}}(1^-, 2^-, 3^+, 4^+) = g^2 \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = g^2 r \frac{z_{12} \bar{z}_{34}}{\bar{z}_{12} z_{34}}, \quad (2.19)$$

where in the last step we used the constraint (2.14). Comparing with (2.11) for $(\ell_1, \ell_2, \ell_3, \ell_4) = (-1, -1, 1, 1)$, we read off $\alpha_1 = 1$, $\alpha_2 = 0$, and the function $\mathcal{B}(s, t) = g^2$

just corresponds to the coupling.⁷ The corresponding celestial amplitude is given by the two-dimensional four-point correlation function (2.15) of gluons with conformal weights $(h_k, \bar{h}_k) = (\frac{i}{2}\lambda_k, 1 + \frac{i}{2}\lambda_k)$ for negative helicity and $(h_k, \bar{h}_k) = (1 + \frac{i}{2}\lambda_k, \frac{i}{2}\lambda_k)$ for positive helicity. The conformally invariant expression $f(r, \bar{r})$ is given by

$$f_{\text{tree}}(r, \bar{r}) = 2g^2 \delta(r - \bar{r}) \Theta(r - 1) r^{1 + \frac{\Delta}{6}} (r - 1)^{\frac{\Delta}{6}} \mathcal{I}(\lambda), \quad (2.20)$$

where

$$\mathcal{I}(\lambda) \equiv \int_0^\infty \frac{dw}{w} w^{i\frac{\lambda}{2}} = 4\pi \delta(\lambda), \quad (2.21)$$

with $\lambda = \sum_{k=1}^4 \lambda_k$. Enforcing (2.21) sets $\Delta = 4$ and thus yields

$$f_{\text{tree}}(r, \bar{r}) = 8\pi g^2 \delta(r - \bar{r}) \Theta(r - 1) r^{\frac{5}{3}} (r - 1)^{\frac{2}{3}} \delta(\lambda). \quad (2.22)$$

Because the integral in (2.16) is marginally convergent for celestial tree-level Yang-Mills amplitudes, it can be interpreted as a distribution given by (2.21). This is no longer true for celestial amplitudes at loop level, which we will discuss in the next section. Before doing so, let us comment that, while we have focused here on tree-level amplitudes in pure Yang-Mills theory in order to set the stage for the discussion of celestial gluon amplitudes at loop level, the prescription discussed in Secs. II A–II B is more broadly applicable. In Appendix A, we analyze tree-level amplitudes in massless QED theory and finite one-loop Einstein gravity processes.

III. CELESTIAL LOOP AMPLITUDES

In this section, we discuss how loop corrections modify celestial amplitudes. We first consider in Sec. III A processes in pure Yang-Mills theory that vanish at tree level and yield finite one-loop amplitudes. In Sec. III B, we move on to the more interesting case of MHV amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory. We compute the one-loop celestial amplitude in Sec. III B 1 and extend this result to all loops in Sec. III B 2.

A. Finite one-loop amplitudes in Yang-Mills

We consider loop corrected amplitudes for theories involving only external gluons. The simplest processes are the ones that vanish at tree level.⁸ This is the case for gluon amplitudes whose helicities are $++++$ and $-+++$ as well as the opposite helicity cases. For the

⁷Here and henceforth, we omit a factor $i(2\pi)^4$.

⁸Recently, we learned about the work of Albayrak *et al.* [33], which has some overlap with this subsection and with Appendix A 2.

all helicity plus and all helicity minus amplitudes, we have [42–44]

$$A(1^\pm, 2^\pm, 3^\pm, 4^\pm) = g^4 \left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} \right)^{\pm 1} = g^4 \left(\frac{\bar{z}_{12} \bar{z}_{34}}{z_{12} z_{34}} \right)^{\pm 1}, \quad (3.1)$$

while the mixed helicity amplitudes are given by

$$A(1^-, 2^+, 3^+, 4^+) = g^4 \frac{\langle 13 \rangle [31] [24]^2}{[12] \langle 23 \rangle \langle 34 \rangle [41]} = g^4 \frac{z_{13} \bar{z}_{31} \bar{z}_{24}^2}{\bar{z}_{12} z_{23} \bar{z}_{34} \bar{z}_{41}}, \quad (3.2)$$

with the opposite helicity amplitude obtained by exchanging $z_{ij} \leftrightarrow \bar{z}_{ij}$. In pure Yang-Mills theory, these amplitudes correspond to the single trace and color ordered contribution to four-gluon processes. They also represent the leading contribution to photon-photon or gluon-gluon scattering from the box diagram of massless QED or QCD with a fermion running inside the loop [45–47].

Because the amplitudes (3.1) and (3.2) are again of the form (2.3), the conformal weights are directly obtained from (2.17). The new information from these loop processes in the bulk lies in the computation of the conformally invariant factor (2.16), which yields

$$f_{1\text{-loop}}(r, \bar{r}) = 8\pi g^4 \delta(\lambda) \delta(r - \bar{r}) \Theta(r - 1) (r - 1)^{\frac{2}{3}} \times \begin{cases} r^{\frac{2}{3}}, & (\pm, \pm, \pm, \pm) \\ r^{\frac{5}{3}}, & (\mp, \pm, \pm, \pm) \end{cases}. \quad (3.3)$$

Notice that here we were again able to make use of (2.21), reflecting the fact that the above amplitudes are finite even at one loop. For infrared divergent amplitudes which we will discuss in the following, the distribution involving the sum of the conformal dimensions will be promoted to an operator statement.

B. Infrared divergent amplitudes in planar $\mathcal{N}=4$ super-Yang-Mills

We now turn to the more interesting case of MHV amplitudes and how loop corrections modify the celestial tree-level correlators involving gluons as external states. Because these amplitudes do not vanish at tree level, their loop corrections (in momentum space) typically suffer from both UV and IR divergences. While the former may be taken care of by renormalization, the latter are more subtle.⁹ Here, to control infrared divergences, we employ the dimensional regularization method known as the FDH scheme [35]. The momenta of the internal particles are in $D = 4 - 2\epsilon$ dimensions, whereas all polarization vectors (internal and external) and the momenta of

the external states remain in $D = 4$. We focus on planar amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. Besides being UV finite, this has the benefit that the four-gluon amplitude is known to all orders in the loop expansion (in the 't Hooft coupling). The MHV four-gluon amplitude can be conveniently written as¹⁰

$$A_{\text{all loops}} = M_\epsilon A_{\text{tree}}, \quad (3.4)$$

where the scalar factor

$$M_\epsilon = 1 + \sum_{\ell=1}^{\infty} a^\ell M_\epsilon^{(\ell)} \quad (3.5)$$

depends on the Mandelstam invariants s and t as well as the dimensional regulator. Here, $a \equiv \frac{g^2 N}{8\pi^2} (4\pi e^{-\gamma_E})^\epsilon$ is the 't Hooft coupling, and γ_E is the Euler-Mascheroni constant. The tree-level amplitude A_{tree} is the same as in pure Yang-Mills, which for the choice of helicities $(1^-, 2^-, 3^+, 4^+)$ is given in (2.19). As shown in Ref. [37], the amplitude (3.5) exponentiates according to the BDS formula

$$M_\epsilon = \exp \left(\sum_{\ell=1}^{\infty} a^\ell (f_\epsilon^{(\ell)} M_{\ell\epsilon}^{(1)} + C^{(\ell)} + E_\epsilon^{(\ell)}) \right), \quad (3.6)$$

where $f_\epsilon^{(\ell)}$ are regular functions of ϵ and are directly related to the cusp and collinear anomalous dimensions, $C^{(\ell)}$ are numerical constants, and $E_\epsilon^{(\ell)}$ is of order $\mathcal{O}(\epsilon)$. In the following, we will discuss the celestial analog of the above statements.

1. Celestial gluons at one loop

The one-loop contribution [51]

$$A_{1\text{-loop}} = a M_\epsilon^{(1)} A_{\text{tree}} \quad (3.7)$$

is obtained by solving the scalar box integral

$$M_\epsilon^{(1)} = -\frac{1}{2} (\mu^2 e^{\gamma_E})^\epsilon \int \frac{d^D p}{i\pi^{D/2}} \times \frac{st}{p^2(p+p_1)^2(p+p_1+p_2)^2(p+p_4)^2}, \quad (3.8)$$

where μ is the dimensional regularization scale. This integral can be explicitly evaluated in terms of hypergeometric functions (see, for instance, Appendix E of Ref. [52])

⁹For a recent discussion of IR divergences, see Refs. [48,49].

¹⁰See, for instance, Sec. IV 3 in Ref. [50].

$$M_\epsilon^{(1)} = -\frac{1}{\epsilon^2} e^{\epsilon\gamma_E} \gamma_\Gamma \frac{t}{\mu^2} \left[\left(\frac{\mu^2}{-t} \right)^{1+\epsilon} {}_2F_1 \left(-\epsilon, 1; 1-\epsilon; 1 + \frac{t}{s} \right) - \left(\frac{\mu^2}{-s} \right)^{1+\epsilon} {}_2F_1 \left(1, 1; 1-\epsilon; 1 + \frac{t}{s} \right) \right], \quad (3.9)$$

where $\gamma_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$. Note that we can express (3.9) as

$$M_\epsilon^{(1)} = \left(\frac{\mu^2}{-t} \right)^\epsilon \mathcal{F}_1(r, \epsilon), \quad (3.10)$$

where $\mathcal{F}_1(r, \epsilon)$ depends on the Mandelstam invariants s and t only through the conformally invariant cross-ratio $r = -s/t$ introduced in (2.6) and has double poles in ϵ inherited from the dimensionally regularized one-loop integral whose explicit form can be read off from (3.9).

In computing the celestial four-gluon amplitude (2.15) at one loop in planar $\mathcal{N} = 4$ super-Yang-Mills, we thus see that the main difference to the tree-level result lies in the (now divergent) Mellin integral in (2.16) where $w = -t$, namely

$$\int_0^\infty \frac{dw}{w} w^{i\frac{1}{2}} \mathcal{B}(rw, -w) = a\mu^{2\epsilon} \mathcal{F}_1(r, \epsilon) \int_0^\infty \frac{dw}{w} w^{i\frac{1}{2}-\epsilon}. \quad (3.11)$$

The one-loop contribution to the conformally invariant factor is given by

$$f_{1\text{-loop}}(r, \bar{r}, \epsilon) = 2ag^2\mu^{2\epsilon} \mathcal{F}_1(r, \epsilon) \delta(r - \bar{r}) \Theta(r - 1) \times r^{1+\frac{\Delta}{6}}(r-1)^{\frac{\Delta}{6}} \mathcal{I}(\lambda + 2i\epsilon). \quad (3.12)$$

While this result shares some similarities with f_{tree} given in (2.20), it notably differs from the tree-level result of the Mellin integral (2.21) by a shift in the argument. Interestingly, this shift can be reexpressed through the action of the differential operator

$$\mathcal{I}(\lambda + 2i\epsilon) = e^{2i\epsilon\partial_\lambda} \mathcal{I}(\lambda). \quad (3.13)$$

This suggests that celestial amplitudes at loop level may be obtainable from tree-level ones through the action of appropriate differential operators. Indeed, we find that the celestial one-loop amplitude can be written as

$$\tilde{\mathcal{A}}_{1\text{-loop}} = a\hat{\mathcal{M}}_\epsilon^{(1)} \tilde{\mathcal{A}}_{\text{tree}}, \quad (3.14)$$

for the *celestial one-loop operator*

$$\hat{\mathcal{M}}_\epsilon^{(1)} = \mathcal{F}_1(r, \epsilon) \hat{\mathcal{P}}^\epsilon, \quad (3.15)$$

where we defined

$$\hat{\mathcal{P}} = \mu^2 r^{\frac{1}{3}} (r-1)^{\frac{1}{3}} \prod_{i<j} (z_{ij} \bar{z}_{ij})^{-\frac{1}{6}} \exp \left(\frac{i}{2} \sum_{k=1}^4 \frac{\partial}{\partial \lambda_k} \right). \quad (3.16)$$

Notice that $\hat{\mathcal{P}}$ is related to the operator $P_{+,k} = e^{\frac{1}{2}(\partial_{h_k} + \partial_{\bar{h}_k})}$ introduced in Ref. [9] whose effect is to shift the conformal dimension $\Delta_k \rightarrow \Delta_k + 1$ of the individual gluons. There, the *sum* over all $P_{+,k}$ was shown to annihilate celestial amplitudes as expected by translation invariance. Here, $\hat{\mathcal{P}}$ involves instead the *product* over all $P_{+,k}$. Moreover, the prefactor in (3.16) can be recast as a correlation function of four scalar primaries with conformal weights $h_i = \bar{h}_i = \frac{1}{4}$. Its appearance is not surprising, as the action of the exponential operator shifts all the conformal weights by the same amount $h_i \rightarrow h_i - \frac{\epsilon}{4}$ and hence the role of the prefactor in (3.16) is precisely to cancel these extra factors.

Thus, we see that the celestial one-loop amplitude in planar $\mathcal{N} = 4$ super-Yang Mills can be recast as an infrared divergent operator acting on the celestial tree-level amplitude. The one-loop factor $M_\epsilon^{(1)}$ multiplying the tree-level amplitude in (3.7) gets promoted to the celestial one-loop operator $\hat{\mathcal{M}}_\epsilon^{(1)}$ acting on the celestial tree-level amplitude (3.14). In the following, we generalize this statement to all loops.

2. Celestial gluons at all loops

The ℓ -loop contribution to the four-gluon planar amplitude is

$$A_{\ell\text{-loop}} = a^\ell M_\epsilon^{(\ell)} A_{\text{tree}}, \quad (3.17)$$

where $M_\epsilon^{(\ell)}$ can be written as [53,54]

$$M_\epsilon^{(\ell)} = \left(\frac{\mu^4}{st} \right)^{\frac{\ell\epsilon}{2}} \mathcal{G}_\ell(r, \epsilon). \quad (3.18)$$

For our purposes, using $s = -t/r$, it is more convenient to write this instead as

$$M_\epsilon^{(\ell)} = \left(\frac{\mu^2}{-t} \right)^{\ell\epsilon} \mathcal{F}_\ell(r, \epsilon), \quad (3.19)$$

where $\mathcal{F}_\ell = (-r)^{-\frac{\ell\epsilon}{2}} \mathcal{G}_\ell$ has poles starting at $\epsilon^{-2\ell}$. Explicit expression for $\mathcal{F}_\ell$ to all orders in ϵ can be obtained in terms of Mellin-Barnes integral representations for $\ell = 2$ in Ref. [55], for $\ell = 3$ in Ref. [37], and for $\ell = 4$ in Ref. [56]. Notice that, as in the one-loop case, the dependence of the ℓ -loop amplitude on the Mandelstam variables is a simple power of t factored out in (3.19), while $\mathcal{F}_\ell(r, \epsilon)$ depends only on the conformal cross-ratio, thus making the Mellin integral in (2.16) straightforward to compute. The celestial one-loop expression (3.14) generalizes to ℓ loops as

$$\tilde{\mathcal{A}}_{\ell\text{-loop}} = a^\ell \hat{\mathcal{M}}_\epsilon^{(\ell)} \tilde{\mathcal{A}}_{\text{tree}}, \quad (3.20)$$

with the celestial ℓ -loop operator given by

$$\hat{\mathcal{M}}_\epsilon^{(\ell)} = \mathcal{F}_\ell(r, \epsilon) \hat{\mathcal{P}}^{\ell\epsilon}. \quad (3.21)$$

We can immediately resum the perturbative expansion of the four-gluon celestial amplitude to all loop orders, yielding

$$\tilde{\mathcal{A}}_{\text{all loops}} = \hat{\mathcal{M}}_\epsilon \tilde{\mathcal{A}}_{\text{tree}}, \quad (3.22)$$

with

$$\hat{\mathcal{M}}_\epsilon = 1 + \sum_{\ell=1}^{\infty} a^\ell \hat{\mathcal{M}}_\epsilon^{(\ell)}. \quad (3.23)$$

Inspired by the BDS exponentiation [37], we now show that the infinite sum (3.23) can be recast as an exponential operator acting on the celestial tree-level amplitude. To do so, we make use of the identity

$$1 + \sum_{\ell=1}^{\infty} a^\ell \hat{\mathcal{M}}_\epsilon^{(\ell)} = \exp \left(\sum_{L=1}^{\infty} a^L (\hat{\mathcal{M}}_\epsilon^{(L)} - \hat{X}^{(L)}[\hat{\mathcal{M}}_\epsilon^{(\ell)}]) \right), \quad (3.24)$$

where we introduced

$$\hat{X}^{(L)}[\hat{\mathcal{M}}_\epsilon^{(\ell)}] = \hat{\mathcal{M}}_\epsilon^{(L)} - \log \left(1 + \sum_{\ell=1}^{\infty} a^\ell \hat{\mathcal{M}}_\epsilon^{(\ell)} \right) \Big|_{a^L\text{-term}}, \quad (3.25)$$

which only depends on the lower-loop operator $\hat{\mathcal{M}}_\epsilon^{(\ell)}$ with $\ell < L$. We obtain

$$\hat{\mathcal{M}}_\epsilon = \exp \left(\sum_{L=1}^{\infty} a^L (\mathcal{F}_L(r, \epsilon) - X^{(L)}[\mathcal{F}_\ell(r, \epsilon)]) \hat{\mathcal{P}}^{L\epsilon} \right). \quad (3.26)$$

Note that $X^{(L)}$ in the above expression is no longer an operator, as opposed to $\hat{X}^{(L)}$ in (3.24). At this point, we can make use of the explicit form of $\mathcal{F}_\ell$ known from the BDS formula [37]

$$\mathcal{F}_L(r, \epsilon) = X^{(L)}[\mathcal{F}_\ell(r, \epsilon)] + f_\epsilon^{(L)} \mathcal{F}_1(r, L\epsilon) + C^{(L)} + \mathcal{E}_L(r, \epsilon), \quad (3.27)$$

where $\mathcal{E}_L$ are noniterating $\mathcal{O}(\epsilon)$ contributions. With this, we find the celestial BDS formula

$$\tilde{\mathcal{A}}_{\text{all loops}} = \exp \left(\sum_{L=1}^{\infty} a^L (f_\epsilon^{(L)} \mathcal{F}_1(r, L\epsilon) + C^{(L)} + \mathcal{E}_L(r, \epsilon)) \hat{\mathcal{P}}^{L\epsilon} \right) \tilde{\mathcal{A}}_{\text{tree}}. \quad (3.28)$$

This demonstrates, for the case of MHV amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills that celestial amplitudes at loop level can be obtained through the action of an exponential operator on the celestial tree-level amplitudes.

IV. CONCLUSIONS

We have analyzed the role played by loop corrections on celestial correlators. We first showed that for finite loop amplitudes, celestial amplitudes can be defined unambiguously. Then, we have considered infrared divergent amplitudes in $\mathcal{N} = 4$ superYang-Mills, where we have shown that planar celestial four-point gluon amplitudes can be defined as a differential operator (containing the infrared divergences) acting on the tree level correlator. At l -loop this expression is given by (3.14) and at all loop orders we have obtained expression (3.28).

While we have focused here on the four-gluon amplitude, one could analyze higher-point amplitudes at loop level in $\mathcal{N} = 4$ super-Yang-Mills as well as other theories with less symmetry. A preliminary study shows that a similar operator structure as the one we uncovered here seems to arise for pure Yang-Mills theory [40]. It would be interesting to investigate whether this pattern persists in other theories. In Ref. [19], the authors computed celestial tree-level amplitudes of four massless states in the open sector of the type I string. A natural next step is to compute the celestial one-loop string amplitude and see if it can be recast as an operator acting on the celestial tree-level string amplitude. If that is the case, and since conformal invariance on the worldsheet intimately ties infrared divergences to ultraviolet ones through the open/closed string duality, it would be appealing to explore how this duality manifests itself on the celestial sphere. We leave these interesting questions for the future.

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APPENDIX A: MORE CELESTIAL AMPLITUDES

In this Appendix, we make use of (2.11) to analyze tree-level amplitudes in massless QED and finite one-loop Einstein gravity processes.

1. Celestial tree-level massless QED

The asymptotic states of massless QED are massless electrons, positrons, and photons. We first analyze processes involving only fermions in the external states. The nonvanishing amplitudes correspond to two fermions of positive helicities and two fermions of negative helicities. These are given by (see, e.g., Ref. [57])

$$\begin{aligned} A(1^{\frac{1}{2}}, 2^{-\frac{1}{2}}, 3^{-\frac{1}{2}}, 4^{\frac{1}{2}}) &= -2e^2 \frac{[14]\langle 23 \rangle}{u} = \frac{2e^2}{r-1} \left(\frac{z_{23}\bar{z}_{14}}{\bar{z}_{23}z_{14}} \right)^{\frac{1}{2}}, \\ A(1^{\frac{1}{2}}, 2^{-\frac{1}{2}}, 3^{\frac{1}{2}}, 4^{-\frac{1}{2}}) &= -2e^2 \frac{[13]\langle 24 \rangle}{t} = \frac{2e^2}{r-1} \left(\frac{z_{24}\bar{z}_{13}}{\bar{z}_{24}z_{13}} \right)^{\frac{1}{2}}, \\ A(1^{\frac{1}{2}}, 2^{\frac{1}{2}}, 3^{-\frac{1}{2}}, 4^{-\frac{1}{2}}) &= -2e^2 [12]\langle 34 \rangle \frac{s}{tu} = \frac{2e^2 r^2}{r-1} \left(\frac{z_{34}\bar{z}_{12}}{\bar{z}_{34}z_{12}} \right)^{\frac{1}{2}}, \end{aligned} \quad (\text{A1})$$

with the remaining cases given by the complex conjugation. Comparing these expressions with (2.11) yields $\mathcal{B}(s, t) = 2e^2$ for all massless QED amplitudes with external fermions. The conformally invariant expressions for the amplitudes in (A1) are given by

$$\begin{aligned} f_{\frac{1}{2}}(r, \bar{r}) &= 8\pi e^2 \delta(r - \bar{r}) \Theta(r - 1) \delta(\lambda) \\ &\times \begin{cases} r^{\frac{2}{3}}(r-1)^{\frac{2}{3}}, & (\pm \frac{1}{2}, \mp \frac{1}{2}, \mp \frac{1}{2}, \pm \frac{1}{2}) \\ r^{\frac{2}{3}}(r-1)^{-\frac{1}{3}}, & (\pm \frac{1}{2}, \mp \frac{1}{2}, \pm \frac{1}{2}, \mp \frac{1}{2}) \\ r^{\frac{8}{3}}(r-1)^{\frac{2}{3}}, & (\pm \frac{1}{2}, \pm \frac{1}{2}, \mp \frac{1}{2}, \mp \frac{1}{2}) \end{cases} \end{aligned} \quad (\text{A2})$$

Next, we consider amplitudes combining two external photons and two external fermions,

$$\begin{aligned} A(1^{\frac{1}{2}}, 2^{-\frac{1}{2}}, 3^+, 4^-) &= 2e^2 \frac{\langle 24 \rangle^2}{\langle 13 \rangle \langle 23 \rangle} \\ &= 2e^2 (r-1)^{\frac{1}{2}} \frac{z_{24}}{\bar{z}_{24}} \left(\frac{\bar{z}_{23}\bar{z}_{13}}{z_{23}z_{13}} \right)^{\frac{1}{2}}, \\ A(1^{\frac{1}{2}}, 2^{-\frac{1}{2}}, 3^-, 4^+) &= 2e^2 \frac{\langle 23 \rangle^2}{\langle 14 \rangle \langle 24 \rangle} \\ &= 2e^2 (r-1)^{-\frac{1}{2}} \frac{z_{23}}{\bar{z}_{23}} \left(\frac{\bar{z}_{24}\bar{z}_{14}}{z_{24}z_{14}} \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{A3})$$

Again, we have $\mathcal{B}(s, t) = 2e^2$, and the conformally invariant expression are

$$\begin{aligned} f_{(1, \frac{1}{2})}(r, \bar{r}) &= 8\pi e^2 \delta(r - \bar{r}) \Theta(r - 1) \delta(\lambda) r^{\frac{2}{3}} \\ &\times \begin{cases} (r-1)^{\frac{5}{3}}, & (\frac{1}{2}, -\frac{1}{2}, +, -) \\ (r-1)^{-\frac{1}{3}}, & (\frac{1}{2}, -\frac{1}{2}, -, +) \end{cases}. \end{aligned} \quad (\text{A4})$$

Hence, celestial amplitudes in massless QED are correlators of two-dimensional conformal primaries with spin 1 and spin $\frac{1}{2}$.¹¹

2. Celestial pure gravity at one loop

Finite amplitudes in pure gravity at one loop are [59,60] given by

$$\begin{aligned} A(1^{++}, 2^{++}, 3^{++}, 4^{++}) &= \kappa^4 \frac{s^2 - st + t^2}{960} A(1^+, 2^+, 3^+, 4^+)^2, \\ A(1^{--}, 2^{++}, 3^{++}, 4^{++}) &= \kappa^4 \frac{s^2 - st + t^2}{2880} \left(\frac{st}{u^2} \right)^2 A(1^-, 2^+, 3^+, 4^+)^2, \end{aligned} \quad (\text{A5})$$

where $\kappa = \sqrt{32\pi G_N}$ and we have used the double copy when writing these amplitudes in terms of the four-gluon stripped amplitudes $A(1^\pm, 2^+, 3^+, 4^+)$ given in (3.1) and (3.2). For the corresponding celestial amplitudes, we find

$$\begin{aligned} f_{1\text{-loop}}(r, \bar{r}) &= \frac{\kappa^4}{240} \mathcal{I}(\lambda - 2i) \delta(r - \bar{r}) \Theta(r - 1) \\ &\times \begin{cases} r^2 + r + 1, & (++, ++, ++, ++) \\ \frac{r^2 + r + 1}{3(r-1)^2}, & (--, ++, ++, ++) \end{cases}, \end{aligned} \quad (\text{A6})$$

with the Mellin integral $\mathcal{I}(\lambda)$ defined in (2.21). Note that this is the same type of divergence observed in celestial tree-level graviton amplitudes [19].¹² In string theory, these are rendered finite due to the exponentially damped behavior of the amplitudes at high energies [19]. Since the soft UV behavior holds perturbatively to all orders in the string loop expansion for both closed [61,62] and open strings [63], one would expect convergence of the Mellin integrals of the corresponding celestial string amplitudes. In the $r \rightarrow \infty$ limit, one should then recover the above celestial one-loop gravity amplitude.

¹¹Spin $\frac{1}{2}$ conformal wave packets were recently discussed in Ref. [58] to compute celestial amplitudes in $\mathcal{N} = 1$ super-Yang-Mills theory.

¹²A regularization of classical celestial graviton amplitudes is given by an imaginary shift of λ corresponding to the insertion of gravitons of integer conformal dimension such as the stress tensor or its shadow [23].

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