DDK system in finite volumeJin-Yi Pang¹, Jia-Jun Wu^{2,*} and Li-Sheng Geng^{3,4,5,6,†}¹*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China*²*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*³*School of Physics, Beihang University, Beijing 100191, China*⁴*Beijing Key Laboratory of Advanced Nuclear Materials and Physics, Beihang University, Beijing 100191, China*⁵*Beijing Advanced Innovation Center for Big Data-Based Precision Medicine, School of Medicine and Engineering, Beihang University, Beijing, 100191, China*⁶*School of Physics and Microelectronics, Zhengzhou University, Zhengzhou, Henan 450001, China*

(Received 2 September 2020; accepted 23 November 2020; published 23 December 2020)

The DDK three-body system is supposed to be bound due to the strongly attractive interaction between the D meson and the K meson in the isospin zero channel. The minimum quark content of this three-body bound state is $cc\bar{q}\bar{s}$ with $q = u, d$. It will be an explicitly exotic tetraquark state once discovered. In order to confirm the phenomenological study of the DDK system, we can refer to lattice QCD as a powerful theoretical tool parallel to the experiment measurement. In this paper, a three-body quantization condition scheme is derived via the nonrelativistic effective theory and the particle-dimer picture in a finite volume. The lattice spectrum of this three-body system is calculated within the existing model inputs. The spectrum shows various interesting properties of the DDK system, and it may reveal the nature of the $D^*(2317)$. This predicated spectrum is expected to be tested in future lattice simulations.

DOI: [10.1103/PhysRevD.102.114515](https://doi.org/10.1103/PhysRevD.102.114515)**I. INTRODUCTION**

The discovery of the $D_{s0}^*(2317)$ [1–3] and subsequent phenomenological studies [4–17] imply that there is a strongly attractive interaction between D and K . The prevailing idea that the $D_{s0}^*(2317)$ can be understood as a molecular state has been confirmed by many Weinberg-Tomozawa potential calculations [18–34] and lattice QCD simulations [35–43]. It is also found that there is a long-range attractive potential between D and $D_{s0}^*(2317)$. This potential can be explained by the exchange of a nearly on shell K meson [44]. Subsequently, it is very natural to ask what happens in the DDK three-body system. Till now, many works [44–47] have been devoted to studying this problem. The Ref. [46] found that a DDK bound state exists with a binding energy of about 70 MeV due to the attractive DK interaction. In this scenario, the K meson

behaves like a glue attracting both D mesons. In this three-body system, there is also the DD_{s0}^* interaction which can, at the same time, make the whole system bind more tightly. Whereas, in principle, the original physical reason of the DD_{s0}^* interaction is through a K exchange, it still originates from the DK interaction. In Ref. [46], the quantum numbers of the DDK bound state are $J^P = 0^-, I = 1/2, S = 1$, and $C = 2$. It turns out that the minimum quark content of this state is at least $cc\bar{q}\bar{s}$ with $q = u, d$. Therefore, once this state is discovered, it should be an explicitly exotic mesonic state, which is beyond the traditional $q\bar{q}$ picture.

Lattice QCD provide an alternative tool to understand considerable hadronic processes based on the QCD first principle. Recently, lattice simulations have started to probe three-body systems [48–51]. A lot of progress has been made in the analysis of these three-body lattice data [52–89] (for a recent review, see Ref. [90]). The finite volume spectrum produced by lattice simulations can help determine the low energy constants (LECs) in effective field theory and further test the phenomenological works. The Refs. [68,69,71,80] supply a three-body quantization condition scheme via the nonrelativistic effective theory and the particle-dimer picture in finite volume. This method can be applied in many hadronic processes transparently

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and connect lattice QCD simulations with established phenomenological models.

In the current paper, we focus on the DDK system and apply the phenomenological model [46] in the finite volume formalism of [69] to calculate the corresponding spectrum. This work starts from DK and DD two-body scattering, then builds a cutoff independent three-body formalism of a DDK system, aiming to obtaining the energy levels in finite volume in the end. This spectrum is automatically divided into three parts, above the DDK threshold, below the $DD_s^*(2317)$ threshold, and between these two thresholds. Such a spectrum provides various interesting physical information of the DDK system. Once the spectrum is confirmed by lattice simulations, the hadronic molecular picture will be strongly supported in the three-body sector. In addition, we expect to resolve more detailed dynamics from lattice QCD studies beyond experiments, such as the three-body amplitude coefficient $H(\Lambda)$ introduced in our formalism and lots of properties in the lattice spectrum, which only emerge in the finite volume system.

Especially, we emphasize that the formation of the DDK bound state is dominantly due to the strongly attractive interaction between the D meson and K meson. The three-body amplitude coefficient $H(\Lambda)$ encodes all the short range information of both two body and three body. In addition, it helps us to reproduce the same pole position as that obtained in the phenomenological studies. Although we have used the same two-body parameters as [46,91], the inputs of the two-body sector are actually the scattering amplitude, which has been resolved by lattice studies [25,37,40–43]. In this viewpoint, the coefficient $H(\Lambda)$ is the only parameter that we are concerned with. $H(\Lambda)$ runs with the high energy scale Λ , and for certain scales, it is very small and therefore, can be dealt with perturbatively in infinite volume.

The paper is organized as follows. In Sec. II, we give the basic formalism to describe the three-body system of DDK and derive the quantization condition. In Sec. III, the matching of the two-body scattering amplitude including DK and DD channels is given based on the setup of Refs. [46,91]. We build a three-body amplitude coefficient to determine the pole position of the DDK bound state predicted by [46] and use quantization condition to produce its lattice spectrum. The calculation up to $O(p^0)$, $O(p^2)$, and $O(p^4)$ are compared to guarantee the stability of the nonrelativistic framework. Finally, the summary and outlook are given in Sec. IV.

II. FORMALISM

A. Particle-dimer formalism

We derive the three-body quantization condition for the DDK system in the particle-dimer formalism used in [68,69,71]. In the particle-dimer picture, the DDK system

is described by a DK dimer with the other D meson as a spectator. In principle, the dimer DK can give all the two-body dynamics in the DK subsystem, not necessarily constrained as a two-body bound state or resonance, e.g., the $D_{s0}^*(2317)$. To complete a full physical picture, we also need to consider the DD dimer with the K meson as the spectator. Therefore, the effective Lagrangian reads

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \quad (1)$$

where

$$\begin{aligned} \mathcal{L}_1 = & D^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m_D} \right) D + K^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m_K} \right) K \\ & + T_{DK}^\dagger \sigma_{DK} T_{DK} + T_{DD}^\dagger \sigma_{DD} T_{DD}; \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{L}_2 = & T_{DK}^\dagger [D\mathcal{F}_{DK}K] + T_{DD}^\dagger [D\mathcal{F}_{DD}D] + \text{H.c.}, \\ & \text{where the operator reads } \mathcal{F} = f_0 + f_2 \overset{\leftrightarrow}{\nabla}^2; \end{aligned} \quad (3)$$

$$\mathcal{L}_3 = h_0 [T_{DK}^\dagger D^\dagger] [T_{DK} D]. \quad (4)$$

In the Lagrangian, we have the kinematic part $\mathcal{L}_1$, including both single particle fields and dimer fields, saying D , K and T_{DD} , T_{DK} . The dynamics of a two-body sector are given in $\mathcal{L}_2$, where the operators encode the interactions between the dimer fields and their constituents, e.g., DK dimer and D meson, K meson. In the formalism, the two-body LECs are denoted by f_0 and f_2 . The remaining interaction is a three-body contact term induced by the dimer and a spectating single particle, e.g., a DK dimer and another D meson just as we have shown in $\mathcal{L}_3$. Here, for all the possible three-body contact interaction, we choose only one operator for the channel $(DK) + D \rightarrow (DK) + D$ and a corresponding LEC, i.e., h_0 . In principle, the contact term can also appear in the interaction between the DD dimer and the spectating K meson, as well as the crossed channel $(DK) + D \rightarrow (DD) + K$ or the inverse case. It seems to be more complete that we parametrize a three-body force in the form of

$$\begin{aligned} & h_0 (T_{DK}^\dagger D^\dagger) (T_{DK} D) + h'_0 (T_{DD}^\dagger K^\dagger) (T_{DD} K) \\ & + h''_0 ((T_{DD}^\dagger K^\dagger) (T_{DK} D) + \text{H.c.}). \end{aligned} \quad (5)$$

As a matter of fact, considering the equivalent effective field theory without dimers,¹ we have only one operator to build up the three-body interaction for the DDK system, that is $h(D^\dagger D^\dagger K^\dagger)(DDK)$. Consequently, the relationship between the general LEC, h , and the LECs, h_0 , h'_0 , h''_0 in particle-dimer formalism is

¹The equivalence can be seen by integrating the dimer fields out in the Lagrangian (1).

$$h = \frac{f_{0,DK}^2}{\sigma_{DK}^2} h_0 + \frac{f_{0,DD}^2}{\sigma_{DD}^2} h'_0 + \frac{2f_{0,DD}f_{0,DK}}{\sigma_{DK}\sigma_{DD}} h''_0. \quad (6)$$

Since dimer fields are, nevertheless, auxiliary fields, physical observables can only determine h but have nothing to do with the distribution of h_0 , h'_0 , and h''_0 ; we are allowed to choose $h'_0 = h''_0 = 0$ and use the contact term in $(DK) + D \rightarrow (DK) + D$ to parametrize the three-body force.

There are many other ways to parametrize the particle-dimer formalism, for example, using the N/D method with the analytic properties derived from three-body unitarity [72]. Nevertheless, all these parametrizations give the same three-body quantization condition in finite volume.

B. Particle-dimer scattering equation

The scattering amplitude of $DDK \rightarrow DDK$ encodes all the information of the three-body system. In infinite volume, we can introduce the particle-dimer scattering equation to resolve the analytic properties of the amplitude (see Fig. 1). The equation reads

$$\mathcal{M}(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + 4\pi \int^\Lambda \frac{d^3k}{(2\pi)^3} Z(\mathbf{p}, \mathbf{k}; E) \times \tau(\mathbf{k}; E) \mathcal{M}(\mathbf{k}, \mathbf{q}; E). \quad (7)$$

$\mathcal{M}$ denotes the particle-dimer scattering amplitude, which resolves the three-body information equivalently. Since there are two kinds of dimers, $\mathcal{M}$ takes the form of 2×2 matrix as

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_2 \end{pmatrix}. \quad (8)$$

The subscript 1 represents the channel of scattering between dimer DK and spectating D meson, while 2 means the channel of DD and K . The off diagonal term is thus the amplitude of the corresponding cross channel. We introduce a momentum cutoff Λ for the spectating particle to treat UV divergence.

In the two-body sector, τ stands for the dressed dimer propagators in two channels,

$$\tau = \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}. \quad (9)$$

We can write down the explicit forms for these propagators within nonrelativistic kinematics and S -wave projection,

$$\tau_1(\mathbf{k}; E) = \frac{1}{k_{*,1} \cot \delta_{DK} - ik_{*,1}}, \quad k_{*,1}^2 = 2\mu E - \beta^2 k^2; \quad (10)$$

$$\tau_2(\mathbf{k}; E) = \frac{2Z_{DD}}{k_{*,2} \cot \delta_{DD} - ik_{*,2}}, \quad k_{*,2}^2 = 2\mu E / Z_{DD} - \alpha^2 k^2, \quad (11)$$

where μ is the DK two-body reduced mass; i.e., $\mu = \frac{m_K m_D}{m_K + m_D}$, $\gamma = \frac{\mu}{m_K}$, $\beta^2 = 1 - \gamma^2$, $Z_{DD} = 2(1 - \gamma)$, and $\alpha^2 = Z_{DD}^{-1} - \frac{1}{4}$. Here, the momentum $k_{*,1(2)}$ denotes the relative momentum of the corresponding pair in the rest frame.

The kernel of the scattering equation is denoted by Z in three channels,

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{pmatrix}. \quad (12)$$

In principle, the potential Z should include both particle exchange terms and contact terms. As we have discussed before, the proper contact term is introduced in the form of nonderivative coupling, i.e., h_0 in the $(DK) + D \rightarrow (DK) + D$ channel. This is just one choice to parametrize the three-body force of the DDK system. More explicitly, in the potential, we introduce the three-body amplitude coefficient to reparametrize the three-body force,

$$\frac{H_0}{\Lambda^2} = -\frac{h_0}{2\mu f_{0,DK}^2}. \quad (13)$$

FIG. 1. Schematic three-body scattering equation.

In the practical matching, this term will absorb the two-body off shell amplitude on the attaching vertex of the corresponding exchange process [69]. Therefore, it encodes both the two-body and three-body information of the detailed dynamics. Additionally, the component Z_1 includes also the potential of $(DK) + D \rightarrow (DK) + D$ by exchanging the K meson; Z_{12} includes the potential $(DK) + D \rightarrow (DD) + K$ by exchanging the D meson and Z_{21} inversely. In a closed form, Z_2 is designed for the potential of $(DD) + K \rightarrow (DD) + K$. However, there is no one-particle exchange potential in this reaction, and therefore, we are allowed to set $Z_2 = 0$ at the beginning. Finally, we arrive at the elements of Z , which can be written as

$$Z_1(\mathbf{p}, \mathbf{q}; E) = \frac{1}{p^2 + q^2 + 2\gamma\mathbf{p}\mathbf{q} - 2\mu E} + \frac{H_0}{\Lambda^2}; \quad (14)$$

$$Z_{12}(\mathbf{p}, \mathbf{q}; E) = \frac{1}{2(1-\gamma)p^2 + q^2 + 2(1-\gamma)\mathbf{p}\mathbf{q} - 2\mu E}; \quad (15)$$

$$Z_{21}(\mathbf{p}, \mathbf{q}; E) = Z_{12}(\mathbf{q}, \mathbf{p}; E). \quad (16)$$

$$Z_2(\mathbf{p}, \mathbf{q}; E) = 0. \quad (17)$$

Corresponding to Eq. (7), the finite volume amplitude $\mathcal{M}_L$ satisfies the following scattering equation:

$$\mathcal{M}_L(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + \frac{4\pi}{L^3} \sum_{\mathbf{k}} Z(\mathbf{p}, \mathbf{k}; E) \tau_L(\mathbf{k}; E) \times \mathcal{M}_L(\mathbf{k}, \mathbf{q}; E). \quad (18)$$

L is the spatial size of the cubic box; E is, thus, the finite-volume energy level. Here, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}$ are the discretized relative dimer-spectator three momenta, i.e., $\mathbf{p}, \mathbf{q}, \mathbf{k} \in \{2\pi\mathbf{n}/L | \mathbf{n} \in \mathbb{Z}^3\}$. The two-body finite-volume correction is encoded in τ_L as

$$\tau_{L,1}(\mathbf{k}; E) = \frac{1}{k_{*,1} \cot \delta_{DK} - S_L^{(DK)}(\mathbf{k}; E)}; \quad (19)$$

$$\tau_{L,2}(\mathbf{k}; E) = \frac{2Z_{DD}}{k_{*,2} \cot \delta_{DD} - S_L^{(DD)}(\mathbf{k}; E)}, \quad (20)$$

where the finite volume corrections read

$$S_L^{(DK)}(\mathbf{k}; E) = \left(\frac{1}{L^3} \sum_{\mathbf{l}} -\text{PV} \int \frac{d^3l}{(2\pi)^3} \right) \frac{1}{(\mathbf{l} + \gamma\mathbf{k})^2 - k_{1*}^2}; \quad (21)$$

$$S_L^{(DD)}(\mathbf{k}; E) = \left(\frac{1}{L^3} \sum_{\mathbf{l}} -\text{PV} \int \frac{d^3l}{(2\pi)^3} \right) \frac{1}{(\mathbf{l} + \mathbf{k}/2)^2 - k_{2*}^2}. \quad (22)$$

The three-body amplitude coefficient does not need to be modified in Eq. (18) because it describes short-range interaction in the system, which is not sensitive to the finite volume effect.

C. Quantization condition

The energy levels on lattice are determined by the pole position of finite volume amplitude $\mathcal{M}_L$. Since these energy levels are always discrete, the scattering equation can be reduced to a homogeneous equation. Near the pole of $\mathcal{M}_L$ at E_* , the amplitude can be written in the form of

$$\mathcal{M}_L(\mathbf{p}, \mathbf{q}; E) = \frac{\phi(\mathbf{p})\phi^*(\mathbf{q})}{E - E_*} + \text{reg.} \quad (23)$$

So the homogeneous equation is

$$\phi(\mathbf{p}) = \frac{4\pi}{L^3} \sum_{\mathbf{k}} Z(\mathbf{p}, \mathbf{k}) \tau_L(\mathbf{k}) \phi(\mathbf{k}). \quad (24)$$

To be consistent with Z and τ_L , the factorized component ϕ also consists of two channels, i.e., $\phi = (\phi_1, \phi_2)^T$. Using the cubic symmetry projection of [71], we have the following equation in the A_1^+ irreducible representation (A_1^+ -irreps.):

$$\phi_r = \frac{4\pi}{L^3} \sum_s^{s_\Lambda} \vartheta_s Z_{rs}^{(A_1^+)} \tau_s \phi_s. \quad (25)$$

The indices r, s denote the shells to which the discrete momenta belong. Correspondingly, s_Λ is the cutoff shell. We call ϑ_s as the number of momenta in shell s . The projected potential and dimer propagator are

$$Z_{rs}^{(A_1^+)} = \frac{1}{48} \sum_{g \in G} Z(g\mathbf{p}_0^{(r)}, \mathbf{k}_0^{(s)}), \quad \tau_s = \tau_L(\mathbf{k}_0^{(s)}), \quad (26)$$

where $\mathbf{p}_0^{(r)}, \mathbf{k}_0^{(s)}$ are reference momenta of r, s shells, respectively. g is octahedral transformation of the group G . At last, we obtain the three-body quantization condition of the DDK system in the A_1^+ -irreps.,

$$\det \begin{bmatrix} \delta_{rs} \begin{pmatrix} \tau_{L,1}^{-1}(s) & \\ & \tau_{L,1}^{-1}(s) \end{pmatrix} \\ -\frac{4\pi}{L^3} \vartheta_s \begin{pmatrix} Z_{1,rs}^{(A_1^+)} & Z_{12,rs}^{(A_1^+)} \\ Z_{21,rs}^{(A_1^+)} & 0 \end{pmatrix} \end{bmatrix} = 0. \quad (27)$$

FIG. 2. DK scattering amplitudes. FIG. 3. DD scattering amplitudes.

III. RESULTS AND DISCUSSIONS

A. Two-body part

In order to resolve the three-body dynamics, it is necessary to calculate the two-body scattering amplitude firstly. Therefore, we solve the Lippmann-Schwinger (LS) equation based on the DK and DD potential. The equation is written down as

$$T_\ell(p, q) = V_\ell(p, q) + \frac{1}{2\pi^2} \int_0^\infty k^2 dk V_\ell(p, k) G(k) T_\ell(k, q). \quad (28)$$

Here, T_ℓ is the partial-wave scattering amplitude. The corresponding potential is projected as

$$V_\ell(p, k) = \frac{1}{2} \int_{-1}^1 d\cos\theta P_\ell(\cos\theta) V(|\mathbf{p} - \mathbf{k}|), \quad \text{with} \\ \mathbf{p} \cdot \mathbf{k} = pk \cos\theta. \quad (29)$$

We can calculate the potential in momentum space $V(|\mathbf{p} - \mathbf{k}|)$ by Fourier transformation in the form of

$$V(|\mathbf{p} - \mathbf{k}|) = \int d^3x \tilde{V}(\mathbf{x}) e^{-i(\mathbf{p}-\mathbf{k})\mathbf{x}}. \quad (30)$$

To complete the LS equation, the two-body propagator takes the form of² $G(k) = 1/(E - \frac{k^2}{2\mu} + i\epsilon)$.

The two-body interaction inside the DDK system is described in terms of effective potentials in [46]. We introduce the DK interacting subsystem by considering a contact-range effective field theory, where the leading order contribution is

$$\tilde{V}_{DK}(r; R_c) = C'_L e^{-(r/R_c)^2}. \quad (31)$$

Here, the interaction coupling constant takes the value of $C'_L = -320.1$ MeV, and the typical interaction length is $R_c = 1$ fm. At the same time, the DD interaction is described by the one boson exchange (OBE) potential involving σ , ρ , and ω mesons [91],

$$V_{DD}(\mathbf{q}) = \sum_{V=\sigma, \rho, \omega} C_{\text{iso.}}(V) \frac{g_V^2}{\mathbf{q}^2 + m_V^2} \left(\frac{\tilde{\Lambda}^2 - m_V^2}{\mathbf{q}^2 + \tilde{\Lambda}^2 - q_0^2} \right)^2. \quad (32)$$

Here, the isospin factors are $C_{\text{iso.}}(\sigma) = -1$, $C_{\text{iso.}}(\rho) = 1$, and $C_{\text{iso.}}(\omega) = +1$. m_V is the mass of the vector meson.

²For the DD system, we also need to consider the additional symmetry factor, $1/2$.

TABLE I. Parameters of the scattering amplitude up to NLO. The scattering length of DK is consistent with the result of the [39] result in a bracket and also other lattice simulation results in [25,37,40–43]. Note that the sign convention of the scattering length in lattice simulations is opposite to that of our phenomenological studies. On the other hand, the scattering length of DD is consistent the result of [91] in a bracket as well.

	a (fm)	r (fm)	κ (MeV)	C	R (MeV $^{-1}$)	$R^{(1)}$ (MeV $^{-1}$)	$R^{(2)}$ (MeV $^{-1}$)
DK	1.683 (1.58 $^{+0.22}_{-0.17}$)	0.791	187.795	3.881	0.0121	3.73×10^{-5}	1.11×10^{-7}
DD	-0.392 (-0.4 $^{+0.1}_{-0.2}$)	3.236	-195.166	0.243	7.43×10^{-4}	2.43×10^{-6}	7.52×10^{-9}

The couplings are chosen as $g_\sigma = 3.4$, $g_\rho = g_\omega = 2.6$ with the corresponding cutoff parameter $\tilde{\Lambda} = 1$ GeV.

By solving the S -wave LS equation with the DK and DD interaction potential, we can obtain the two-body scattering

amplitudes (see Figs. 2 and 3). Here, the amplitude below the threshold is the analytic continuation since it is required in the three-body calculation. We need to reexpress the amplitude in terms of momentum expansion as follows:

$$T_{DK}(E) = -\frac{2\pi}{\mu} \begin{cases} \left(-a_{DK}^{-1} + \frac{1}{2} r_{DK} p^2 - ip \right)^{-1}, & p = \sqrt{+2\mu E}, \quad E > 0; \\ C_{DK}(\kappa - \kappa_{DK})^{-1} + R_{DK} + R_{DK}^{(1)}\kappa + R_{DK}^{(2)}\kappa^2, & \kappa = \sqrt{-2\mu E}, \quad E < 0, \end{cases} \quad (33)$$

and

$$T_{DD}(E) = -\frac{8\pi}{m_D} \begin{cases} \left(-a_{DD}^{-1} + \frac{1}{2} r_{DD} p^2 - ip \right)^{-1}, & p = \sqrt{+m_D E}, \quad E > 0; \\ C_{DD}(\kappa - \kappa_{DD})^{-1} + R_{DD} + R_{DD}^{(1)}\kappa + R_{DD}^{(2)}\kappa^2, & \kappa = \sqrt{-m_D E}, \quad E < 0. \end{cases} \quad (34)$$

In the above definitions, μ is the reduced mass of DK . The corresponding expansion parameters are listed in Table I, which are all fitted from the amplitude obtained from the LS equation. One should note that above the threshold, we use the normal effective range expansion, while the expression below the threshold is different because the quantity $p \cot \delta - ip$ induces additional spurious poles far below threshold, where p is purely imaginary. These spurious poles emerge at the hard scale of the theory, i.e., outside the range, where the effective field theory is applicable. In order to protect the unitarity in our formalism, we give an alternative expression, and we can explicitly see that in the proper expression below threshold, there is only one physical pole for the DK system (no pole for the DD system),

$$m_D + m_K - \frac{\kappa_{DK}^2}{2\mu} \simeq 2318. \quad (35)$$

This pole is interpreted as the DK bound state, $D_{s0}^*(2317)$ which has been supported by both lattice QCD [35–43] and phenomenological studies [18–34]. Based on the DD interaction potential, Eq. (32) referring to [46,91], we also find a virtual state in the DD system, $\kappa_{DD} \simeq -195$ MeV. We emphasize that the DD amplitude on the physical Riemann sheet is not sensitive to this pole. Here, we just use this pole to adjust the scattering amplitude below the threshold. In principle, the energy region below the threshold will not be influenced by the existence of this pole. Therefore, we are

allowed to extend the form for higher n of κ^n ; such terms can be absorbed into the parameters $R_{DD}^{(i)}$. The final conclusion, however, will not be influenced much.

B. Three-body part

1. Three-body system in infinite volume

We introduced a three-body amplitude coefficient to describe the short range interaction in the DDK system. This LEC encodes dynamics information of both the two-body and three-body part.

It turns out that the three-body amplitude coefficient H_0 should be running periodically as shown in Fig. 4. Consequently, the solution of the particle-dimer scattering equation (7) shows that the pole position of the three-body bound state pole in the DDK system is precisely consistent with the prediction of [46] (see Table II). Actually, the behavior of the three-body amplitude coefficient as shown in Fig. 4 is determined from the binding energy of the three-body bound state, which can be obtained in either the phenomenological study—that we refer to—or an experimental measurement. This relationship between H_0 and Λ can keep our predictions totally cutoff independent.³

³Sometimes, one can choose a proper value for the cutoff, Λ , so that H_0 is typically small, then this three-body amplitude coefficient will be effectively suppressed in the corresponding model.

FIG. 4. *DDK* three-body force as a function of the cutoff. Left: without the *DD* interaction; Right: with the meson exchange interaction between the *D* mesons.

2. Three-body system in finite volume

Inputting the three-body amplitude coefficient $H_0(\Lambda)$ into the quantization condition (27), we can obtain the lattice spectrum of the *DDK* three-body system, (see Fig. 5). Two thresholds are presented. The threshold at $\Delta E = 0$ denotes the three-body threshold, which in fact means the energy at $m_K + 2m_D$. There is an additional two-body threshold at $\Delta E = -45$ MeV, which denotes the $DD_{s0}^*(2317)$ threshold. This is the consequence of involving the *DK* two-body bound state in our formalism. The existence of two thresholds is consistent with our previous toy model [68,69,71,80]. Therefore, we also find the avoided level crossing between spectra tending to the three-body threshold and the two-body threshold. Tracing back these spectra to the small box region, we find that the *DDK* spectrum is always located between two 3-body free lines, which is obtained by turning off all the interaction among hadrons. In addition, the energy levels between the two 2-body free lines can be recognized as the discrete spectra of DD_{s0}^* scattering states, where the free lines are calculated from the total kinematic energy of the particles *D* and D_{s0}^* with the lattice momenta. The two kinds of free lines have explicit crossing. Naturally, this crossing behavior is inherited by the *DDK* spectrum and DD_{s0}^* spectrum once the full interactions are turned on. The level crossing appearing in the free case apparently implies that there is degeneracy between the *DDK* three-body picture and the DD_{s0}^* two-body picture at the crossing point, consistent with the fact that $R(4140)$ can be obtained in either the explicit three-body calculation [45,46] or the quasi-two-body study [44]. Although the level crossing is avoided in the lattice spectra, we can still conclude that the lattice energy eigenstates around the crossing points are the

explicit superpositions of two components correspondingly. This is similar to the scenario where two-body systems present several coupled level channels. In that case, lattice spectra also show avoided level crossing. Besides the three-body threshold and the DD_{s0}^* two-body threshold, we do not observe additional thresholds; this implies that the *DD* system is unbound. Otherwise, there should be a third threshold which is related to the *DD* bound state and the spectating *K* meson. In summary, the three-body scattering states of *DDK* live above the three-body threshold, while the two-body scattering states of $DD_{s0}^*(2317)$ live above the two-body threshold. Finally, below the two-body $DD_{s0}^*(2317)$ threshold, the three-body bound state in a finite volume is found as well by checking the lowest energy level of Fig. 5. A closer look at Fig. 6 reveals that the finite volume energy level exhibits an exponential behavior tending to the bound state (-65.8 MeV without *DD* interaction and -71.2 MeV with both *DK* and *DD* interaction) in an infinite volume as predicted by [46].

C. Comparing the $O(p^2)$ results with the $O(p^0)$ and $O(p^4)$ results

The two-body interaction involving *DK* and *DD* are resolved at $O(p^2)$. We also perform the calculation by truncating the two-body part at $O(p^0)$ and expanding it up to $O(p^4)$. At $O(p^4)$, the parameters are listed in Table III. We find that the lattice spectrum of the *DDK* three-body system turns to be stable in a general profile (see Fig. 7).

The *DK* two-body amplitude fitted up to different orders are shown in Fig. 8. The higher order fitting, i.e., $O(p^4)$ expansion, gives better precision near the threshold; however, the part far away presents a large discrepancy. The three-body spectrum turns out to depend on the two-body phase shifts both near and far away from threshold. We do not mean that the final spectrum is strongly related to the high-momentum dynamics, but it is fair to argue that up to very high order, the large discrepancy far away from threshold is able to disturb the result competing with the convergence near threshold. Since up to $O(p^2)$, the result has stabilized, it is reasonable to stick to the result at $O(p^2)$.

TABLE II. The three-body *DDK* bound state predicted by [46].

C'_S (MeV)	C'_L (MeV)	E_2 (MeV)	E_3 (only <i>DK</i>)	E_3 (both <i>DK</i> and <i>DD</i>)
$R_L = 1$ fm, $R_S = 0.5$ fm				
0	-320.1	-45.0	-65.8	-71.2

FIG. 5. DDK states in finite volume. Left: only the DK interaction is considered. Right: both DK and DD interactions are taken into account. The upper blue regions indicate the case of three free particles in finite volume. The red and yellow regions indicate the case of free $D^*_{s0}(2317)$ and D . The lower blue regions indicate the DDK bound state below the $DD^*_{s0}(2317)$ threshold.

For the different expansion orders of the two-body interaction, the running of the three-body amplitude coefficient also shows different behavior. Since here the

three-body amplitude coefficient is in fact a counterterm, the strength of this term will be sensitive to the precision of the two-body interaction.

FIG. 6. DDK bound states in finite volume. Left: only the DK interaction is considered. Right: both DK and DD interactions are taken into account. The deeper blue regions indicate the DDK bound state below the $DD^*_{s0}(2317)$ threshold.

TABLE III. Parameters of the scattering amplitude at $N^2\text{LO}$. The effective range expansion at $N^2\text{LO}$ reads $p \cot \delta = -a^{-1} + \frac{1}{2}rp^2 + P_4p^4$.

	P_4 (MeV^{-3})	$R^{(3)}$ (MeV^{-4})	$R^{(4)}$ (MeV^{-5})
DK	-3.228×10^{-10}	3.17×10^{-10}	8.49×10^{-13}
DD	2.49×10^{-9}	2.33×10^{-11}	7.27×10^{-14}

FIG. 7. DDK states in finite volume. The orange curves are calculated at $O(p^0)$, the blue curves at $O(p^2)$, and the purple curves at $O(p^4)$.

FIG. 8. Two-body amplitude and the fitting up to $O(p^0)$, $O(p^2)$, and $O(p^4)$. The orange line denotes the numerical solution of the LS equation. The black line is at $O(p^0)$, the blue line is at $O(p^2)$, and the blue dashed line is at $O(p^4)$.

IV. SUMMARY

We have successfully produced the three-body bound state of DDK on lattice (see Fig. 5). Our calculation is based on effective field theory referring to the phenomenological setup in [46,91]. The phenomenological studies [46] predict that there is a DDK three-body bound state with about 70 MeV binding energy and quantum numbers $J^P = 0^-, I = 1/2, S = 1$, and $C = 2$. The formation of this bound state is dominantly due to the strongly attractive interaction between the D meson and K meson. By inputting the two-body scattering amplitude, we present this typical three-body hadronic molecular picture in the finite volume system. The three-body amplitude coefficient H_0 , which encodes short range information, has been introduced in order to fix the pole position of the three-body bound state to be the same as that of the phenomenological studies. In the infinite volume limit, our result tends to the realistic energy predicted by [46]. The energy level of the spectrum for the bound state of three bodies shows the exponential behavior, which is consistent with many previous calculations of Efimov states on a lattice [53,68,71]. The prediction for the lattice spectrum of a bound state can be tested by future lattice simulations. We can find out whether lattice QCD will generate this stable three-body bound state as a hadronic molecule, and furthermore, we will be able to use lattice spectrum to constrain the three-body amplitude coefficient H_0 , which is a very important LEC in the effective field theory. This cross-check presents a connection between phenomenological methods and lattice calculations on the three-body dynamics, thus supplies an *ab initio* way to study the DDK system.

In the lattice spectrum, there exist energy levels, which tend to the DD^*_{s0} threshold in the infinite volume limit. This is the consequence of the two-body bound state $D^*_{s0}(2317)$. In our formalism, we assume that the $D^*_{s0}(2317)$ is a $D - K$

molecule. This will influence deeply the three-body dynamics. That is to say, we can find the avoided level crossing in the spectrum. In other words, three-body lattice simulations are able to confirm the existences of $D_{s0}^*(2317)$ and its behavior in the finite volume of three-body system. Since the DD system is unbound, we do not see any energy level that belongs to the pattern of $(DD) - K$. We believe that future lattice simulations of DDK will open a new window to reveal these dynamics in the charm sector.

Our prediction for the DDK three-body lattice spectrum is derived from the nonrelativistic effective field theory. It is reasonable because the calculation is carried out near the threshold. In order to justify this, we have checked the stability of the spectrum by including the two-body dynamics at $O(p^0)$, $O(p^2)$, and $O(p^4)$. It shows that the three-body spectrum is not sensitive to the physics in the high momentum region of the relevant particles. Higher order calculations in principle can be done but are unnecessary because when the order is taken to too high, the effective range expansion, which we have used in the two-body

matching, will break the precision of the three-body result. So the result up to $O(p^2)$ is already good enough.

The lattice spectrum below the DD_{s0}^* threshold can give more detail to study the physical picture of the DDK bound state as we have done in the toy model [71]. The analytic expression for the three-body ground state in [80] can also be derived for the DDK system and compared with the corresponding numerical result. Additionally, the work can be improved by introducing a three-body amplitude coefficient beyond nonderivative coupling. This is also a relevant topic for the cross-check with future lattice simulations.

ACKNOWLEDGMENTS

The authors thank Meng-Lin Du, Li Ma, and Akaki Rusetsky for useful discussions. J.-J.W. acknowledges the support from the Fundamental Research Funds for the Central Universities. L.-S.G. is partly supported by the National Natural Science Foundation of China under Grants No. 11735003, No. 11975041, and No. 11961141004.

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