Second Gegenbauer moment of a ρ -meson distribution amplitude

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Using the soft pion theorem, crossing, and the dispersion relations for the two-pion distribution amplitude (DA), we argue that the second Gegenbauer moment, the ρ -meson DA ($a_2^{(\rho)}$), is most probably negative. This result is at variance with the majority of the model calculations for $a_2^{(\rho)}$. Using the instanton theory of the QCD vacuum, we computed $a_2^{(\rho)}$ at a low normalization point and obtain for the ratio $a_2^{(\rho)}/M_3^{(\pi)}$ a *definitely negative value* in the range of $a_2^{(\rho)}/M_3^{(\pi)} \in [-2, -1]$. The range of values corresponds to a generous variation of the parameters of the instanton vacuum. The value of the second Gegenbauer moment of pion DA is positive in the whole range and is compatible with its the most advanced lattice measurement. It seems that the topologically nontrivial field configurations in the QCD vacuum (instantons) lead to qualitatively different shapes of the pion and the ρ -meson DAs.

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I. INTRODUCTION

The meson distribution amplitudes (DAs) describe the lowest Fock component of the meson light-cone wave function. They provide us with invaluable information about the quark-gluon structure of mesons. The meson DAs enter the QCD description of the amplitudes of various hard exclusive processes; see, e.g., the review [1].

The chiral-even leading-twist ρ -meson DA ϕ_ρ is defined by the following matrix element¹:

$$\begin{aligned} \langle \rho^0(P) | \bar{\psi}(x) \not{n} \frac{\tau^3}{2} \psi(0) | 0 \rangle \Big|_{x^+ = x_\perp = 0} \\ = (n \cdot \epsilon^\lambda) \frac{f_\rho m_\rho}{\sqrt{2}} \int_0^1 dz e^{-iz(P \cdot x)} \phi_\rho(z). \end{aligned} \quad (1)$$

Here, ϵ^λ is the polarization vector of the ρ meson and n is a lightlike vector. The dimensional constant f_ρ is defined in a way that the DA follows the normalization condition:

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¹Note that this definition is not, rigorously speaking, correct: the ρ meson, being a resonance, cannot be an asymptotic out state. For a definition of resonance DA see Ref. [2].

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$$\int_0^1 dz \phi_\rho(z) = 1. \quad (2)$$

The ρ -meson DA can be expanded in terms of Gegenbauer polynomials

$$\phi_\rho(z) = 6z(1-z) \left(1 + \sum_{n=2, \text{even}}^\infty a_n^{(\rho)} C_n^{3/2}(2z-1) \right). \quad (3)$$

In the above expression, the first term in the parentheses $1 (= a_0^{(\rho)})$ corresponds to the normalization (2). The expansion is related to the deviation from the perturbative DA $\phi_\rho(z) = 6z(1-z)$ and the leading coefficient $a_2^{(\rho)}$ provides information about the width of the DA.

Through the light-cone QCD sum rules, the ρ -meson DA ϕ_ρ plays an important role to describe the processes such as exclusive semileptonic decays of the B mesons ($B \rightarrow \rho l \bar{\nu}$) [3–6]. The resulting $B \rightarrow \rho$ transition form factors can be used to extract the CKM matrix element $|V_{ub}|$, providing supplementary information to the process $B \rightarrow \pi l \bar{\nu}$ [7,8] about $|V_{ub}|$. In the extraction of $|V_{ub}|$ from $B \rightarrow \rho$ transition form factors, the coefficient $a_2^{(\rho)}$ provides a nontrivial and prominent contribution; for detailed discussion, see [6].

Various nonperturbative methods have been applied to the study of the ρ -meson DA: the QCD sum rule [6,9–14], lattice simulation of the QCD [15–17], and other model approaches [2,18–24]. In Table I we collected various model predictions for $a_2^{(\rho)}$. One sees that the majority of the predictions corresponds to the positive value of the second Gegenbauer coefficient.

TABLE I. Values of $a_2^{(\rho)}$ obtained in various approaches.

References	Approach	$a_2^{(\rho)}$	Scale
Chernyak <i>et al.</i> [9]	SR	0.17	$\mu^2 = 1.5 \text{ GeV}^2$
Ball and Braun [10]	SR	0.18 ± 0.10	$\mu = 1 \text{ GeV}$
Ball <i>et al.</i> [11]	SR	0.18 ± 0.10	$\mu = 1 \text{ GeV}$
Bakulev and Mikhailov [12]	SR	0.079(20)	$\mu = 1 \text{ GeV}$
Pimikov <i>et al.</i> [13]	SR	0.047(61)	$\mu = 1 \text{ GeV}$
Stefanis and Pimikov [25]	SR	0.017(24)	$\mu = 2 \text{ GeV}$
Fu <i>et al.</i> [14]	SR	0.119(82)	$\mu = 1 \text{ GeV}$
Boyle <i>et al.</i> [15]	lattice	0.108(105)(35)	$\mu = 2 \text{ GeV}$
Arthur <i>et al.</i> [16]	lattice	0.204(29)(58)	$\mu = 2 \text{ GeV}$
Braun <i>et al.</i> [17]	lattice	0.132(27)	$\mu = 2 \text{ GeV}$
Ji <i>et al.</i> [18]	LCQM	-0.03	$\mu = 1 \text{ GeV}$
Choi and Ji [19]	LFQM	(+0.02, -0.02)	$\mu = 1 \text{ GeV}$
Qian <i>et al.</i> [26]	light front quantization	0.058	$\mu = 1 \text{ GeV}$
Forshaw and Sandapen [20]	exp. fit with LCWF	(0.079, 0.085, -0.053)	$\mu = 1 \text{ GeV}$
Forshaw and Sandapen [21]	exp. fit with AdS/QCD	0.082	$\mu = 1 \text{ GeV}$
Choi and Ji [22]	LFQM	-0.02	$\mu = 1 \text{ GeV}$
Gao <i>et al.</i> [23]	Dyson-Schwinger	0.088	$\mu = 1 \text{ GeV}$
Polyakov [2]	instantons	-0.14	$\mu \simeq 0.6 \text{ GeV}$
Shuryak [24]	instanton induced interaction	$\simeq -0.5$	$\mu = ?$
Clerbaux and Polyakov [27]	exp. fit	-0.1 ± 0.2	$\mu = 4.5 \text{ GeV}$

In Ref. [2] it was shown that with the help of the dispersion relations the Gegenbauer moments of the ρ -meson DA can be expressed in terms of the two-pion distribution amplitudes (2π DAs). In particular, $a_2^{(\rho)}$ is related to the single pion observables thanks to the crossing symmetry and the low-energy theorem. In Ref. [2], using the instanton approach, the ratio of the ρ and pion DA Gegenbauer moments $a_2^{(\rho)}/a_2^{(\pi)} \simeq -2.3$ was obtained, whereas the majority of the other works predicts the opposite sign for the ratio. We argue here that the negative sign of the ratio $a_2^{(\rho)}/a_2^{(\pi)}$ is deeply rooted in chiral dynamics and general properties of quantum field theory such as unitarity, crossing, and dispersion relations.

In this work, we first explain briefly how the $a_2^{(\rho)}$ is expressed in terms of the single pion observables owing to low-energy theorems, crossing symmetry, and the dispersion relations. After that, we adopt the pion observables from a recent lattice calculation and global data analysis to obtain the model independent relations for the ratio $a_2^{(\rho)}/a_2^{(\pi)}$. Finally, we present the results from the instanton model of the QCD vacuum.

II. RELATION BETWEEN $a_2^{(\rho)}$ MOMENT AND SINGLE PION OBSERVABLES

The twist-2 chiral-even 2π DA is defined as follows [28]:

$$\Phi^{ab}(z, \zeta, W^2) = \frac{1}{4\pi} \int dx^- \exp(-izP^+x^-/2) \langle \pi^a(p_1) \pi^b(p_2) | \bar{\psi}(x) \not{T} \psi(0) | 0 \rangle \Big|_{x^+ = x_1 = 0}. \quad (4)$$

Here we introduce the light-cone coordinate for a vector $v^\pm = n \cdot v = v^0 \pm v^3$, with the lightlike vector $n^2 = 0$, represented as $n \rightarrow (1, 0, 0, 1)$. T is the isospin matrix and in our case of interest, the isovector, $T = \tau^3/2$. The 2π DA has three independent variables: z , the quark momentum fraction with respect to $P = p_1 + p_2$, $\zeta = p_1^+/P^+$, the longitudinal momentum distribution of two pions, and $W^2 = P^2 = (p_1 + p_2)^2$, the invariant mass. The isovector ($I = 1$) part of the 2π DA can be projected out and expanded in terms of the Gegenbauer polynomials $C_n^\alpha(x)$ in the following form:

$$\Phi^{I=1}(z, \zeta, W^2) = 6z(1-z) \sum_{n=0}^{\infty} \sum_{l=0}^{n+1} B_{nl}(W^2) C_n^{3/2}(2z-1) C_l^{1/2}(2\zeta-1). \quad (5)$$

In Ref. [2] the dispersion relations for the generalized Gegenbauer moments $B_{nl}(W^2)$ were derived. Moreover, the solution of these dispersion relations for $W^2 \leq 16m_\pi^2$ was found:

$$B_{nl}(W^2) = B_{nl}(0) \exp \left[\sum_{k=1}^{N-1} a_k^{(nl)} W^{2k} + \frac{W^{2N}}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\delta_l^{I=1}(s)}{s^N (s - W^2 - i0)} \right]. \quad (6)$$

Here $\delta_l^{I=1}(s)$ is the isospin one $\pi\pi$ scattering phase shift with the orbital momentum l , N is the number of the subtractions in the dispersion relations, and $a_k^{(nl)}$ are the corresponding low-energy subtraction constants. For the ρ -meson channel ($I = 1$ and $l = 1$) it is known [29] that the dispersion relations with two subtraction gives an excellent description of the pion form factor for invariant mass until $W^2 \simeq 2.5 \text{ GeV}^2$. Therefore, we restrict ourselves to $N = 2$ in Eq. (6) for the studies of generalized Gegenbauer moments at W around the ρ -meson mass.

The ρ meson is a resonance in the $\pi\pi$ scattering amplitude in the channel with $l = 1$, $I = 1$, the scattering phase shift $\delta_1^1(s)$ in Eq. (6) crosses rapidly the value of $\pi/2$ near $s = m_\rho^2$. The scattering phase in the vicinity of the ρ meson mass can be parametrized in the following form:

$$\delta_1^1(s \sim m_\rho^2) = \arctg\left(\frac{m_\rho \Gamma_\rho}{m_\rho^2 - s}\right) + \text{nonresonant contributions.} \quad (7)$$

As it was shown in Ref. [2] such behavior of the scattering phase leads to the appearance of the pole in $B_{n1}(W^2)$ at $W = m_\rho - i\Gamma_\rho/2$ and the residue in this pole corresponds to the Gegenbauer moment of the resonance DA; see a detailed derivation in Ref. [2]. In this derivation it was important that the pole in $B_{n1}(W^2)$ coefficients for all n appear solely from the dispersion integral in Eq. (6); therefore, the phase of the residue in the resonance pole is the same for all Gegenbauer moments; this common phase can be factored out in the Gegenbauer series (3) and the values of the Gegenbauer moments of the resonance DA are fixed by the values of $B_{n1}(0)$ at zero and by the subtraction constants $a_k^{(nl)}$ in Eq. (6). For example, the second Gegenbauer moment of the ρ -meson DA can be obtained as [2]

$$a_2^{(\rho)} = B_{21}(0) \exp(c_1^{(21)} m_\rho^2), \quad (8)$$

where $c_1^{(21)} = a_1^{(21)} - a_1^{(01)}$ is the subtraction constant. Its value is not known *a priori*; however, it can be estimated in the low-energy models or determined from the shape of the $\pi\pi$ mass spectrum in hard exclusive processes. See a detailed discussion and fits to experimental data in Ref. [27].

Expression (8) can be further reduced to the single pion observables by the soft pion theorem and the crossing symmetry [2]. The soft pion theorem for 2π DAs [2] relates various Gegenbauer coefficients $B_{nl}(0)$ at zero invariant mass to the Gegenbauer moments of the pion DA²:

²This equation has a correction of order $\sim m_\pi^2$; there are no corrections enhanced by chiral logarithms [30]. Numerically such corrections are of order $\sim 2\%$. In what follows we shall consider consistently the chiral limit.

$$\sum_{l=1}^{n+1} B_{nl}(0) = a_n^{(\pi)}. \quad (9)$$

Another important ingredient in our analysis is the crossing relations between $B_{nn+1}(0)$ and the $(n+1)$ th Mellin moments of parton distributions in the pion [2]:

$$B_{nn+1}(0) = \frac{2(2n+3)}{3(n+2)} M_{n+1}^{(\pi)}, \quad (10)$$

with $M_{n+1}^{(\pi)} = \int_0^1 dx x^n (q_\pi(x) - \bar{q}_\pi(x))$. The above results allow us to relate the pion observables to the ρ -meson ones:

$$a_2^{(\rho)} = B_{21}(0) \exp(c_1^{(21)} m_\rho^2) = \left(a_2^{(\pi)} - \frac{7}{6} M_3^{(\pi)}\right) \exp(c_1^{(21)} m_\rho^2), \quad (11)$$

where $a_2^{(\pi)}$ is the second Gegenbauer moment of the π -meson light-cone DA and $M_3^{(\pi)}$ is the third Mellin moment of the parton distribution functions in the pion,

$$M_3^{(\pi)} = \int_0^1 dx x^2 (q_\pi(x) - \bar{q}_\pi(x)). \quad (12)$$

We note that for higher Gegenbauer moments $n \geq 4$ there are no such simple relations to single pion observables as for the second moment in Eq. (11). The reason is that for $n \geq 4$, the soft pion theorem (9) involves the Gegenbauer coefficients of resonances with the spins $l = 3, 5$, etc., i.e., new unknown quantities.

In order to calculate the second Gegenbauer moment of the ρ -meson DA with help of Eq. (11) we use

- (i) the results for $a_2^{(\pi)}$ from the lattice calculation,
- (ii) $M_3^{(\pi)}$ from global phenomenological analysis,
- (iii) the value of the subtraction constant $c_1^{(21)}$ from low-energy effective theory derived from the instanton model of QCD vacuum. We note that the precise value of the subtraction constant $c_1^{(21)}$ in (11) does not influence the sign of $a_2^{(\rho)}$.

The value of $a_2^{(\pi)}$ we take from the lattice simulation of Bali *et al.* (RQCD) [31] at the next-to-leading order (NLO)³:

$$a_2^{(\pi)}(\mu = 2 \text{ GeV}) = 0.078 \pm 0.028 \quad (\text{RQCD at NLO}). \quad (13)$$

The central value is smaller by a factor of $\sim 1/2$ compared to the central values of older lattice results [15,16,32,33]

³The main result of Bali *et al.* [31] is presented at the next-to-next-to-leading order as $a_2^{(\pi)}(\mu = 2 \text{ GeV}) = 0.101 \pm 0.024$. In the present work, we take their NLO result as the pion PDF we adopt [31] is extracted at the NLO, see below.

TABLE II. Phenomenologically determined ratios $B_{21}(0)/M_3^{(\pi)}$ and $a_2^{(\rho)}/a_2^{(\pi)}$ at QCD normalization point $\mu = 2$ GeV. The ratios are stable under scale evolution at the one-loop order. For calculation of the ratio $a_2^{(\rho)}/a_2^{(\pi)}$ the subtraction constant is taken in the range $c_1^{(21)} \in [0.7, 0.9]$ GeV⁻²; this range is calculated within the instanton model in the present work. The uncertainty related to the variation of $c_1^{(21)}$ is shown in the last bracket in the bottom row (note that this bracket is always positive).

	RQCD/xFitter
$B_{21}(0)/M_3^{(\pi)}$	-0.48 ± 0.27
$a_2^{(\rho)}/a_2^{(\pi)}$	$(-1.15 \pm 0.86)(1.0 \pm 0.1)$

evaluated at the same renormalization point or higher. We note that the lattice simulations [31] are the most advanced in terms of the extrapolation to the chiral and continuum limits. For the $M_3^{(\pi)}$, we consider the results of the recent phenomenological analysis by I. Novikov *et al.* (xFitter) [34], which supersedes the old analysis by M. Gluck *et al.* [35]. At the renormalization scale $\mu = 2$ GeV they obtained at the NLO:

$$M_3^{(\pi)}(\mu = 2 \text{ GeV}) = 0.114 \pm 0.020 \text{ (xFitter)}. \quad (14)$$

The error bar in the above equation is obtained by expressing $M_3^{(\pi)}$ in terms of the model parameters of Ref. [34] and then computing the uncertainty of $M_3^{(\pi)}$ from error bars for the parameters given in Ref. [34].

In Table II, the ratios $B_{21}(0)/M_3^{(\pi)}$ and $a_2^{(\rho)}/a_2^{(\pi)}$ obtained with the help of Eq. (11) are shown. We give the results for ratios, instead of individual observables, as the former stay stable under the scale evolution at the one-loop order. Note that the positive factor $\exp(c_1^{(21)} m_\rho^2)$ should be multiplied to $B_{21}(0)$ to obtain the $a_2^{(\rho)}$, as seen in Eq. (11). As the factor is not yet determined phenomenologically,⁴ we use the model value calculated in this work, $c_1^{(21)} \in [0.7, 0.9]$ GeV⁻². The detailed discussion of this constant will be given in the following section. Note that the smaller value of $a_2^{(\pi)}$ results in a cancellation for $B_{21}(0)$. Even though it is difficult at this point to conclude on the sign of $a_2^{(\rho)}$ decisively due to relatively large statistical errors in the studies, the results still strongly suggest that the $a_2^{(\rho)}$ is negative.⁵

⁴The subtraction constant $c_1^{(21)}$ can be measured in the hard exclusive two-pion production; see the discussion in Ref. [27].

⁵In what follows we shall implicitly assume that $a_2^{(\pi)} \geq 0$, for a negative $a_2^{(\pi)}$ the second Gegenbauer moment of ρ -meson DA is obviously negative; see Eq. (11).

III. $a_2^{(\rho)}$ FROM INSTANTONS

The low-energy effective action of quarks interacting with (pseudo)Goldstone bosons derived from the theory of the instanton vacuum [36,37] has the following form in the Euclidean space:

$$S_{\text{eff}} = \int d^4x \bar{\Psi}(x) \left[i \not{\partial} + i \sqrt{M(i\partial)} U^{\gamma_5}(x) \sqrt{M(i\partial)} \right] \Psi(x). \quad (15)$$

U^{γ_5} is the nonlinear chiral $SU(2)_f$ field defined by

$$U^{\gamma_5}(x) = U(x) \frac{1 + \gamma^5}{2} + U^\dagger(x) \frac{1 - \gamma^5}{2}, \quad (16)$$

with

$$U(x) = \exp \left[\frac{i}{F_\pi} \pi^a(x) \tau^a \right]. \quad (17)$$

The momentum dependence of the dynamical quark mass arises from the Fourier transform of the quark zero mode and has the following representation in the momentum space [37]:

$$M(k) = M_0 F^2(k), \quad (18)$$

$$F(k) = 2t \left[I_0(t) K_1(t) - I_1(t) K_0(t) - \frac{1}{t} I_1(t) K_1(t) \right] \Big|_{t=\frac{k\rho}{2}}. \quad (19)$$

In the above expression K and I are modified Bessel functions and ρ is the average instanton size. The zero-momentum quark mass $M_0 = M(k=0)$ generated by the spontaneous breakdown of the chiral symmetry is calculated by using the gap equation:

$$\frac{N}{V} = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)}, \quad (20)$$

where N is the average instanton number and V is the 4-Euclidean volume. In the above expression, the average instanton density N/V can be reexpressed as $N/V = 1/R^4 = (\rho^4/R^4)/\rho^4$, where R is the average instanton inter-distance. Hence, M_0 is obtained with a given average instanton packing fraction and average instanton size. Typically, the set of values $\rho/R \approx 1/3$ and $R \approx 1$ fm was obtained from the Feynman variational principle for QCD partition function [36,38]. In principle, the values of ρ and R are related to the Λ_{QCD} and can be matched to perturbative QCD to provide the model renormalization point; see Ref. [38].

The effective action (15) can be used to compute many low-energy observables for the pions. For example, the

pion decay constant, which is well established experimentally $F_\pi \simeq 93$ MeV ($F_\pi \simeq 88$ MeV in the chiral limit [39]), can be computed as the loop integral in 4D-Euclidean momentum space which has the following parametrical dependence:

$$F_\pi^2 = \frac{C}{\rho^2} \left(\frac{\rho}{R} \right)^4 \ln \left(\frac{R^2}{\rho^2} \right), \quad (21)$$

with the constant $C \sim 1$. We see that in the theory of the instanton vacuum the pion decay constant has a natural suppression by the small instanton packing fraction; hence, the instanton mechanism of the spontaneous chiral symmetry breaking explains the “accidental” smallness of the pion decay constant. With the typical choice of $\rho/R \approx 1/3$ with $R = 1$ fm, we obtain $F_\pi \sim 100$ MeV. The best values of the instanton vacuum parameters were determined with the help of the Feynman variational principle in Refs. [36,37]; the values are $\rho/R \simeq 1/3$ and $\rho \simeq 0.33$ fm. In this paper we vary these parameters in the range which corresponds to the pion decay constant (in the chiral limit) $F_\pi = 88 \pm 15$ MeV.

The calculations of the pion and two-pion DAs were pioneered in Ref. [40] and Ref. [41], respectively. The main finding in these papers was that the momentum dependence of the quark mass is very important to determine the shape of (two)pion DAs. Using the simplifying assumption about the momentum dependence of the quark mass, the first estimates of the DAs were performed. The technique was refined in Refs. [42–45], and we refer to these papers for details of calculations of pion DAs. In all these previous work, a pole-type quark-mass momentum dependence was used to simplify the calculation and to obtain some analytic results. In this work, we will use (19) directly to explore the instanton parameter dependences of the observables as it follows from the theory of the instanton vacuum.

In the limit $\rho \rightarrow 0$ with fixed $\rho/R \ll 1$ one obtains that

$$a_2^{(\pi)} = \frac{7}{18}, \quad M_3^{(\pi)} = \frac{1}{3}, \quad [\text{limit } \rho \rightarrow 0]. \quad (22)$$

Therefore, owing to the soft pion theorem and crossing relations, we have, in this limit,

$$B_{21}(0) = 0 \quad [\text{limit } \rho \rightarrow 0]. \quad (23)$$

For small W^2 , one can also easily obtain in the limit of small ρ [2]

$$B_{21}(W^2) = -\frac{7N_c}{1440\pi^2 F_\pi^2} W^2 + O(W^4) \quad [\text{limit } \rho \rightarrow 0]. \quad (24)$$

The result indicates that $B_{21}(W^2)$ is getting negative with the increasing of the two-pion invariant mass.

FIG. 1. Comparison of the ratio $B_{21}(0)/M_3^{(\pi)}$, calculated for various values of the parameters ρ/R and ρ to the phenomenological values from Table II (horizontal band). The vertical band shows the range of ρ/R resulting to $F_\pi = 88 \pm 15$ MeV for $\rho = 0.33$ fm.

The above limiting results correspond to flat pion DA and to a flat quark distribution function in the pion. Also, these limiting results imply that $a_2^{(\rho)} \rightarrow 0$ for $\rho \rightarrow 0$; see Eq. (19). Furthermore, we made an important observation that beyond the limit of zero instanton size $B_{21}(0)$ is always negative. Therefore, independent of the details of the dynamics [e.g., the shape of the form factor $F(k)$] the second Gegenbauer moment for the ρ -meson DA is always negative.

In Fig. 1 we show the results of our calculations of the ratio $B_{21}(0)/M_3^{(\pi)}$ for various values of instanton parameters ρ and ρ/R . We also show on this figure by the vertical shaded band the range of instanton parameters which leads to reasonable values of $F_\pi = 88 \pm 15$ MeV, which is about 20% around the phenomenological value of the decay constant in the chiral limit for $\rho = 0.33$ fm. From Fig. 1 we clearly see that the ratio $B_{21}(0)/M_3^{(\pi)}$ in the instanton model is definitely negative and is compatible (within 2σ) with the phenomenological analysis from the previous section. The latter is shown by the horizontal shaded 1σ band.

In order to obtain the second Gegenbauer moment of the ρ -meson DA from Eq. (11), we need to calculate the subtraction constant $c_1^{(21)}$. In Fig. 2 the result of the calculation of the dimensionless combination $c_1^{(21)} F_\pi^2$ is shown for various values of instanton vacuum parameters ρ and ρ/R . We see that the subtraction constant has a rather strong dependence on ρ/R . This prevents us from gaining a precise determination of this constant. Therefore, for the phenomenological analysis presented in the previous section, we assumed generously that $c_1^{(21)} \in [0.7, 0.9]$ GeV⁻², which follows from results presented in Fig. 2. We note that the sharp dependence of the subtraction constant on ρ/R can be used for the phenomenological determination of this instanton parameter from a measurement of $c_1^{(21)}$.

Eventually, combining the calculations of $B_{21}(0)$ and the subtraction constant, we present in Fig. 3 the ratio

FIG. 2. Dimensionless combination of the subtraction constant $c_1^{(21)}$ and F_π calculated for various ρ and ρ/R . The range for the latter corresponds to $F_\pi = 88 \pm 15$ MeV.

FIG. 3. The ratio $a_2^{(\rho)}/M_3^{(\pi)}$ as a function of ρ/R is plotted for various ρ values. The green dotted horizontal line indicates the result $a_2^{(\rho)}/M_3^{(\pi)} = -0.63$ from Ref. [2]. The presented range of ρ/R corresponds to $F_\pi = 88 \pm 15$ MeV for $\rho = 0.33$ fm.

$a_2^{(\rho)}/M_3^{(\pi)}$ at various values of the instanton parameters. The value of the ratio $a_2^{(\rho)}/M_3^{(\pi)}$ has a rather wide range, roughly from -2 to -1 , depending on the given values of the packing fraction ρ/R and the average instanton size ρ . An important observation is that the ratio $a_2^{(\rho)}/M_3^{(\pi)}$, and hence $a_2^{(\rho)}$, is *always negative* in the instanton model. This is at variance with the majority of the results for this quantity in various models; see Table I. It is also very interesting that the recent calculation of the ρ -meson DA in an approach which emphasises the role of topologically (instanton) induced quark interactions [24] also predicts the negative $a_2^{(\rho)}$.

IV. SUMMARY

The main findings of the present work are

- (i) Using the soft pion theorem, crossing, and the dispersion relations for 2π DA, we performed a

phenomenological analysis for the ratio of the second Gegenbauer moments of the pion and the ρ -meson DAs with the result at $\mu = 2$ GeV⁶:

$$a_2^{(\rho)}/a_2^{(\pi)} = (-1.15 \pm 0.86)(1.0 \pm 0.1). \quad (25)$$

As the input for our analysis, we used the value of $a_2^{(\pi)}$ from the most advanced to date lattice simulation in Ref. [31] and the most recent phenomenological analysis of the pion PDFs [34]. Eventually the yet experimentally unknown subtraction constant for the dispersion relations is calculated here using the effective low-energy theory derived from the instanton theory of the QCD vacuum. The range for the subtraction constant obtained within the model leads to the uncertainty of the numerical estimate (25) and is reflected in the last brackets of Eq. (25). Note that this bracket is always positive—uncertainty in the subtraction constant does not influence the sign in Eq. (25)

- (ii) We computed the ratio $a_2^{(\rho)}/M_3^{(\pi)}$ in the instanton model of the QCD vacuum:

$$a_2^{(\rho)}/M_3^{(\pi)} \in [-1, -2]. \quad (26)$$

Here the range of the ratio reflects a rather generous range of variation for the parameters of the instanton vacuum. The model calculations demonstrate that $a_2^{(\rho)}$ is negatively defined; this is in qualitative disagreement, especially with what concerns the sign, with the majority of calculations of $a_2^{(\rho)}$ in the literature (see the summary of results in Table I). It is important to find the reasons for this qualitative discrepancy.

From these results we may conjecture that the topologically nontrivial field configurations in the QCD vacuum (instantons) lead to qualitatively different shapes of the pion and the ρ -meson distribution amplitudes. A similar picture was obtained, although from a different perspective, by E. Shuryak in Ref. [24].

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⁶Note that the ratio $a_2^{(\rho)}/a_2^{(\pi)}$ is renormalization scale independent at one-loop order.

- [1] V. L. Chernyak and A. R. Zhitnitsky, *Phys. Rep.* **112**, 173 (1984).
- [2] M. V. Polyakov, *Nucl. Phys.* **B555**, 231 (1999).
- [3] P. Ball and V. M. Braun, *Phys. Rev. D* **55**, 5561 (1997).
- [4] A. Khodjamirian and R. Ruckl, *Adv. Ser. Dir. High Energy Phys.* **15**, 345 (1998).
- [5] P. Ball and V. M. Braun, *Phys. Rev. D* **58**, 094016 (1998).
- [6] P. Ball and R. Zwicky, *Phys. Rev. D* **71**, 014029 (2005).
- [7] P. del Amo Sanchez *et al.* (BABAR Collaboration), *Phys. Rev. D* **83**, 032007 (2011).
- [8] A. Sibidanov *et al.* (Belle Collaboration), *Phys. Rev. D* **88**, 032005 (2013).
- [9] V. L. Chernyak, A. R. Zhitnitsky, and I. R. Zhitnitsky, *Nucl. Phys.* **B204**, 477 (1982); **B214**, 547(E) (1983).
- [10] P. Ball and V. M. Braun, *Phys. Rev. D* **54**, 2182 (1996).
- [11] P. Ball, V. M. Braun, Y. Koike, and K. Tanaka, *Nucl. Phys.* **B529**, 323 (1998).
- [12] A. P. Bakulev and S. V. Mikhailov, *Phys. Lett. B* **436**, 351 (1998).
- [13] A. V. Pimikov, S. V. Mikhailov, and N. G. Stefanis, *Few Body Syst.* **55**, 401 (2014).
- [14] H. B. Fu, X. G. Wu, W. Cheng, and T. Zhong, *Phys. Rev. D* **94**, 074004 (2016).
- [15] P. A. Boyle *et al.* (RBC and UKQCD Collaborations), *Proc. Sci.*, LATTICE2008 (2008) 165 [[arXiv:0810.1669](#)].
- [16] R. Arthur, P. A. Boyle, D. Brommel, M. A. Donnellan, J. M. Flynn, A. Juttner, T. D. Rae, and C. T. C. Sachrajda, *Phys. Rev. D* **83**, 074505 (2011).
- [17] V. M. Braun *et al.*, *J. High Energy Phys.* **04** (2017) 082.
- [18] C. R. Ji, P. L. Chung, and S. R. Cotanch, *Phys. Rev. D* **45**, 4214 (1992).
- [19] H. M. Choi and C. R. Ji, *Phys. Rev. D* **75**, 034019 (2007).
- [20] J. R. Forshaw and R. Sandapen, *J. High Energy Phys.* **11** (2010) 037.
- [21] J. R. Forshaw and R. Sandapen, *Phys. Rev. Lett.* **109**, 081601 (2012).
- [22] H. M. Choi and C. R. Ji, *Phys. Rev. D* **89**, 033011 (2014).
- [23] F. Gao, L. Chang, Y. X. Liu, C. D. Roberts, and S. M. Schmidt, *Phys. Rev. D* **90**, 014011 (2014).
- [24] E. Shuryak, *Phys. Rev. D* **100**, 114018 (2019).
- [25] N. G. Stefanis and A. V. Pimikov, *Nucl. Phys.* **A945**, 248 (2016).
- [26] W. Qian, S. Jia, Y. Li, and J. P. Vary, [arXiv:2005.13806](#).
- [27] B. Clerbaux and M. V. Polyakov, *Nucl. Phys.* **A679**, 185 (2000).
- [28] M. Diehl, T. Gousset, B. Pire, and O. Teryaev, *Phys. Rev. Lett.* **81**, 1782 (1998).
- [29] F. Guerrero and A. Pich, *Phys. Lett. B* **412**, 382 (1997).
- [30] N. Kivel and M. V. Polyakov, *Nucl. Phys.* **A711**, 211 (2002).
- [31] G. S. Bali, V. M. Braun, S. Bürger, M. Göckeler, M. Gruber, F. Hutzler, P. Korcyl, A. Schäfer, A. Sternbeck, and P. Wein, *J. High Energy Phys.* **08** (2019) 065.
- [32] L. Del Debbio, M. Di Pierro, and A. Dougall, *Nucl. Phys. B, Proc. Suppl.* **119**, 416 (2003).
- [33] V. M. Braun *et al.*, *Phys. Rev. D* **74**, 074501 (2006).
- [34] I. Novikov *et al.*, *Phys. Rev. D* **102**, 014040 (2020).
- [35] M. Gluck, E. Reya, and I. Schienbein, *Eur. Phys. J. C* **10**, 313 (1999).
- [36] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* **B245**, 259 (1984).
- [37] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* **B272**, 457 (1986).
- [38] D. Diakonov, M. V. Polyakov, and C. Weiss, *Nucl. Phys.* **B461**, 539 (1996).
- [39] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984).
- [40] V. Y. Petrov, M. V. Polyakov, R. Ruskov, C. Weiss, and K. Goeke, *Phys. Rev. D* **59**, 114018 (1999).
- [41] M. V. Polyakov and C. Weiss, *Phys. Rev. D* **59**, 091502 (1999).
- [42] M. Praszalowicz and A. Rostworowski, *Phys. Rev. D* **64**, 074003 (2001).
- [43] M. Praszalowicz and A. Rostworowski, *Phys. Rev. D* **66**, 054002 (2002).
- [44] E. R. Arriola and W. Broniowski, *Phys. Rev. D* **66**, 094016 (2002).
- [45] M. Praszalowicz and A. Rostworowski, *Acta Phys. Pol. B* **34**, 2699 (2003).