$\mathcal{N}=4$ super-Schwarzian derivative via nonlinear realizationsAnton Galajinsky^{1,*} and Sergey Krivonos^{2,†}¹*Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia*²*Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

(Received 12 September 2020; accepted 14 October 2020; published 13 November 2020)

Current studies of supersymmetric extensions of the Sachdev–Ye–Kitaev model stimulate a renewed interest in super-Schwarzian derivatives. In this work, we apply the method of nonlinear realizations to the finite-dimensional superconformal group $SU(1, 1|2)$ and link its invariants to the $\mathcal{N} = 4$ super-Schwarzian.

DOI: [10.1103/PhysRevD.102.106015](https://doi.org/10.1103/PhysRevD.102.106015)**I. INTRODUCTION**

Current studies of supersymmetric extensions of the Sachdev–Ye–Kitaev model (see [1–3] and references therein) stimulate a renewed interest in super-Schwarzian derivatives [4–7]. An $\mathcal{N}$ -extended super-Schwarzian acts upon a fermionic superfield which specifies superconformal diffeomorphisms of the odd sector of $\mathcal{S}^{1|\mathcal{N}}$ superspace [8]. It enjoys a remarkable composition law which implies invariance of the $\mathcal{N} = 1, 2, 3, 4$ super-Schwarzian under finite transformations forming $OSp(1|2)$, $SU(1, 1|1)$, $OSp(3|2)$, and $SU(1, 1|2)$ superconformal group, respectively.

A conventional way of introducing a super-Schwarzian derivative is to compute a (finite) superconformal transformation of the super stress-energy tensor underlying a $2d$ $\mathcal{N}$ -extended conformal field theory, in which it shows up as the anomalous term [4–7]. Alternatively, one can study the cocycles describing central extensions of infinite dimensional Lie superalgebras (see, e.g., [9]). Because for $\mathcal{N} \geq 5$ the construction of the central term operator is problematic [10], the $\mathcal{N} = 1, 2, 3, 4$ instances mentioned above seem to exhaust all available options.

In a recent work [11], a third alternative was studied, which consists in applying the method of nonlinear realizations [12] to finite-dimensional superconformal groups. In this setting, a super-Schwarzian derivative is linked to the supergroup invariants.

Within the method of nonlinear realizations, one usually starts with a coset space element $\tilde{g}$, on which a (super)group representative g acts by the left multiplication $\tilde{g}' = g \cdot \tilde{g}$. Then one constructs the Maurer–Cartan one-forms $\tilde{g}^{-1}d\tilde{g}$,

where d is the (super)differential, which automatically hold invariant under the transformation. These invariants can be used to impose constraints allowing one to express some of (super)fields parametrizing the coset element $\tilde{g}$ in terms of the other [13]. If the algebra at hand is such that all but one (super)fields can be linked to a single unconstrained (super) field, then the last remaining Maurer–Cartan invariant describes a derivative of the latter, which holds invariant under the action of the (super)group one started with. In particular, in [11] the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Schwarzian derivatives were obtained by applying the procedure to $OSp(1|2)$ and $SU(1, 1|1)$ superconformal groups, respectively. A nonsupersymmetric case was previously studied in [14].

The goal of this work is to provide a similar derivation of the $\mathcal{N} = 4$ super-Schwarzian associated with $SU(1, 1|2)$ superconformal group.

The paper is organized as follows. In the next section, superconformal diffeomorphisms of $\mathcal{S}^{1|4}$ superspace are considered and conditions which follow from the requirement that the covariant derivatives transform homogeneously are analysed. For the fermionic superfield, which describes superconformal diffeomorphisms of the odd sector of $\mathcal{S}^{1|4}$, one reveals the chirality condition and an extra quadratic constraint [6]. The restrictions are solved explicitly and it is demonstrated that the latter is actually equivalent to a simpler linear equation (see Eq. (19) below).

In Sec. III, the method of nonlinear realizations is applied to the finite-dimensional superconformal group $SU(1, 1|2)$ with the aim to link its invariants to an $\mathcal{N} = 4$ super-Schwarzian. First, each generator in $su(1, 1|2)$ superalgebra is accompanied by a Goldstone superfield of the same Grassmann parity, which all together give rise to a group-theoretic element $\tilde{g}$. Then the Maurer–Cartan invariants $\tilde{g}^{-1}\mathcal{D}^\alpha\tilde{g}$, where $\mathcal{D}^\alpha$ is the covariant derivative, are computed. After that, constraints are imposed, which enable one to link all the Goldstone superfields entering $\tilde{g}$ to a single fermionic superfield (a companion of the supersymmetry generator). Substituting the resulting relations back into the

*a.galajinsky@tusur.ru

†krivonos@theor.jinr.ru

Maurer–Cartan invariants, one unambiguously reproduces the $\mathcal{N} = 4$ super-Schwarzian derivative introduced in [7]. Finally, properties of the super-Schwarzian, including the finite $SU(1, 1|2)$ transformations which leave it invariant, are discussed. In the concluding Sec. IV we summarize our results and discuss possible further developments. Our spinor conventions are gathered in the Appendix.

Throughout the text summation over repeated indices is understood.

II. SUPERCONFORMAL DIFFEOMORPHISMS OF $\mathcal{S}^{1|4}$

Consider $\mathcal{S}^{1|4}$ superspace parametrized by a real bosonic coordinate t and a pair of Hermitian conjugate anticommuting $SU(2)$ -spinors $(\theta_\alpha, \bar{\theta}^\alpha)$, $(\theta_\alpha)^\dagger = \bar{\theta}^\alpha$, $\alpha = 1, 2$ (see Appendix for our spinor conventions). The $d=1$, $\mathcal{N} = 4$ supersymmetry algebra

$$\{q_\alpha, \bar{q}^\beta\} = 2h\delta_\alpha^\beta \quad (1)$$

allows one to represent $\mathcal{S}^{1|4}$ as the supergroup manifold

$$\tilde{g} = e^{i\theta} e^{\theta^\alpha q_\alpha + \bar{\theta}_\alpha \bar{q}^\alpha}. \quad (2)$$

The left action of the supergroup on the superspace, $\tilde{g}' = e^{i\theta} e^{\epsilon^\alpha q_\alpha + \bar{\epsilon}_\alpha \bar{q}^\alpha} \cdot \tilde{g}$, where a and $(\epsilon^\alpha, \bar{\epsilon}_\alpha)$ are even and odd supernumbers, respectively, generates the $d=1$, $\mathcal{N} = 4$ supersymmetry transformations

$$\begin{aligned} t' &= t + a; & \theta'_\alpha &= \theta_\alpha + \epsilon_\alpha, & \bar{\theta}'^\alpha &= \bar{\theta}^\alpha + \bar{\epsilon}^\alpha, \\ t' &= t - i(\epsilon_\alpha \bar{\theta}^\alpha + \bar{\epsilon}^\alpha \theta_\alpha). \end{aligned} \quad (3)$$

Covariant derivatives, which anticommute with the supersymmetry generators, read

$$\mathcal{D}^\alpha = \partial^\alpha + i\bar{\theta}^\alpha \partial_t, \quad \bar{\mathcal{D}}_\alpha = \bar{\partial}_\alpha + i\theta_\alpha \partial_t, \quad (4)$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial^\alpha = \frac{\partial}{\partial \theta_\alpha}$, $\bar{\partial}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha}$. They satisfy the relations

$$\begin{aligned} \{\mathcal{D}^\alpha, \bar{\mathcal{D}}_\beta\} &= 2i\delta_\beta^\alpha \partial_t, & \mathcal{D}^\alpha \mathcal{D}^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta} \mathcal{D}^2, \\ \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}_\beta &= -\frac{1}{2}\epsilon_{\alpha\beta} \bar{\mathcal{D}}^2, & [\mathcal{D}^2, \bar{\mathcal{D}}_\alpha] &= -4i\mathcal{D}_\alpha \partial_t, \\ [\bar{\mathcal{D}}^2, \mathcal{D}^\alpha] &= -4i\bar{\mathcal{D}}^\alpha \partial_t, \\ [\bar{\mathcal{D}}^2, \mathcal{D}^2] &= -4i(\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha - \bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha) \partial_t, \end{aligned} \quad (5)$$

with $\mathcal{D}^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha$, $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}^\alpha$.

In what follows, we will need a component decomposition of a chiral fermionic superfield ψ_β , which obeys the equation

$$\bar{\mathcal{D}}_\beta \psi_\gamma = 0. \quad (6)$$

Taking into account the identity $\bar{\mathcal{D}}_\beta = e^{-i\bar{\theta}\theta\partial_t} \bar{\partial}_\beta e^{i\bar{\theta}\theta\partial_t}$, one gets

$$\begin{aligned} \psi_\gamma(t, \theta, \bar{\theta}) &= \alpha_\gamma(t) + \theta_\beta b_\gamma^\beta(t) - i\bar{\theta}\theta \dot{\alpha}_\gamma(t) + \theta^2 \beta_\gamma(t) \\ &\quad + \frac{i}{2} \bar{\theta}_\beta \theta^2 \dot{b}_\gamma^\beta(t) - \frac{1}{4} \theta^2 \bar{\theta}^2 \ddot{\alpha}_\gamma(t), \end{aligned} \quad (7)$$

where $\alpha_\gamma(t)$, $\beta_\gamma(t)$ are complex fermionic components and $b_\gamma^\beta(t)$ is a complex bosonic matrix-valued function of t . The Hermitian conjugation rules

$$\begin{aligned} (\mathcal{D}^\alpha \rho)^\dagger &= -\bar{\mathcal{D}}_\alpha \rho, & (\mathcal{D}^\alpha \psi_\beta)^\dagger &= \bar{\mathcal{D}}_\alpha \bar{\psi}^\beta, \\ (\mathcal{D}^\alpha \bar{\psi}^\beta)^\dagger &= -\bar{\mathcal{D}}_\alpha \bar{\psi}^\beta, \end{aligned} \quad (8)$$

which involve a real bosonic superfield ρ , a complex fermionic superfield ψ_α , and its Hermitian conjugate $\bar{\psi}^\alpha = (\psi_\alpha)^\dagger$, will be heavily used below.

Similarly to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases [4,5] (see also [15]), superconformal diffeomorphisms of $\mathcal{S}^{1|4}$ are introduced as the transformations

$$t' = \rho(t, \theta, \bar{\theta}), \quad \theta'_\alpha = \psi_\alpha(t, \theta, \bar{\theta}), \quad \bar{\theta}'^\alpha = \bar{\psi}^\alpha(t, \theta, \bar{\theta}), \quad (9)$$

where ρ is a real bosonic superfield and ψ_α is a complex fermionic superfield, under which the covariant derivatives transform homogeneously

$$\mathcal{D}^\alpha = (\mathcal{D}^\alpha \psi_\beta) \mathcal{D}'^\beta, \quad \bar{\mathcal{D}}_\alpha = (\bar{\mathcal{D}}_\alpha \bar{\psi}^\beta) \bar{\mathcal{D}}'_\beta. \quad (10)$$

Equation (10) yields the constraints

$$\begin{aligned} \bar{\mathcal{D}}_\alpha \psi_\beta &= 0, & \mathcal{D}^\alpha \rho - i(\mathcal{D}^\alpha \psi_\beta) \bar{\psi}^\beta &= 0, \\ \mathcal{D}^\alpha \bar{\psi}^\beta &= 0, & \bar{\mathcal{D}}_\alpha \rho - i(\bar{\mathcal{D}}_\alpha \bar{\psi}^\beta) \psi_\beta &= 0, \end{aligned} \quad (11)$$

which also imply

$$\partial_t \rho = -i\bar{\psi}^\alpha \partial_t \psi_\alpha + i\partial_t \bar{\psi}^\alpha \psi_\alpha + \frac{1}{2} (\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha \rho), \quad (12)$$

where we abbreviated $\mathcal{D}^\alpha \psi_\beta \bar{\mathcal{D}}_\alpha \bar{\psi}^\beta = (\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha \rho)$. Thus, ρ is fixed provided ψ_α is known.

The compatibility of (10) with the properties of the covariant derivatives (5) imposes further restrictions on ψ_α . From $\{\mathcal{D}^\alpha, \bar{\mathcal{D}}_\beta\} = 2i\delta_\beta^\alpha \partial_t$ and the identity

$$\partial_t = \partial_t \psi_\alpha \mathcal{D}'^\alpha + \partial_t \bar{\psi}^\alpha \bar{\mathcal{D}}'_\alpha + \frac{1}{2} (\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha \rho) \partial_t, \quad (13)$$

one gets the quadratic constraint [6]

$$\mathcal{D}^\alpha \psi_\lambda \bar{\mathcal{D}}_\beta \bar{\psi}^\lambda = \frac{1}{2} \delta_\beta^\alpha (\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha \rho). \quad (14)$$

Thus, up to a factor, $\mathcal{D}^\alpha \psi_\lambda$ is a unitary matrix, which also implies

$$\begin{aligned} \mathcal{D}^\lambda \psi_\alpha \bar{\mathcal{D}}_\lambda \bar{\psi}^\beta &= \frac{1}{2} \delta_\alpha^\beta (\mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi}) \Rightarrow \\ \mathcal{D}^\alpha \psi_\beta &= -\frac{1}{2} \bar{\mathcal{D}}^\alpha \bar{\psi}_\beta \frac{(\mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi})}{\det(\bar{\mathcal{D}} \bar{\psi})}, \end{aligned} \quad (15)$$

with $\det(\bar{\mathcal{D}} \bar{\psi}) = -\frac{1}{2} \epsilon^{\alpha\beta} \epsilon_{\gamma\delta} \bar{\mathcal{D}}_\alpha \bar{\psi}^\gamma \bar{\mathcal{D}}_\beta \bar{\psi}^\delta$. Computing the covariant derivatives of (14), one gets a chain of relations, two of which

$$\begin{aligned} \mathcal{D}^\alpha (\mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi}) &= 4i \partial_t \bar{\psi}^\beta \mathcal{D}^\alpha \psi_\beta, \\ \bar{\mathcal{D}}_\alpha (\mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi}) &= 4i \partial_t \psi_\beta \bar{\mathcal{D}}_\alpha \bar{\psi}^\beta \end{aligned} \quad (16)$$

will be important for what follows. Because the last two relations in (5) result from $\{\mathcal{D}^\alpha, \bar{\mathcal{D}}_\beta\} = 2i\delta_\beta^\alpha \partial_t$, they do not produce further restrictions on ψ_α .

Using the covariant projection method, in which components of a superfield are linked to its covariant derivatives evaluated at $\theta_\alpha = 0$, $\bar{\theta}^\alpha = 0$, one can verify that Eq. (14) connects the fermionic components of the chiral superfield (7) to each other and reduces the matrix-valued bosonic function $b_\alpha^\beta(t)$ to a single unknown scalar

$$\begin{aligned} \beta_\gamma(t) &= i\dot{\alpha}_\gamma(t) e^{2iv}, \\ b_\alpha^\beta(t) &= u(t) e^{iv} \left(\exp \left[\frac{i}{2} \xi_c \sigma_c \right] \right)_\alpha^\beta. \end{aligned} \quad (17)$$

Here $u(t)$ is an arbitrary real function of t , v and ξ_a , $a = 1, 2, 3$, are real bosonic constants, and σ_a are the Pauli matrices (see Appendix).

Substituting (17) into the covariant derivatives of ψ_α and $\bar{\psi}^\alpha$, one reveals the identity

$$e^{-iv} \mathcal{D}^\alpha \psi_\beta + e^{iv} \bar{\mathcal{D}}^\alpha \bar{\psi}_\beta = 0. \quad (18)$$

Because one can always redefine the fermionic superfield $e^{-iv} \psi_\alpha \rightarrow \psi_\alpha$, in what follows we set the parameter v in (17) to vanish, thus reducing the quadratic constraint (14) to the linear restriction

$$\mathcal{D}^\alpha \psi_\beta + \bar{\mathcal{D}}^\alpha \bar{\psi}_\beta = 0. \quad (19)$$

The latter also implies $\mathcal{D}^2 \psi_\alpha = -4i \partial_t \bar{\psi}_\alpha$. The fact that the quadratic equation (14) is equivalent to (19) seems to have escaped attention thus far.

The Taylor series expansion of $u(t)$ and $\alpha_\gamma(t)$ involves an infinite number of constant parameters, which all together represent an infinite-dimensional extension of $SU(1,1|2)$.

III. $\mathcal{N} = 4$ SUPER-SCHWARZIAN VIA NONLINEAR REALIZATIONS

As was mentioned in the Introduction, the goal of this work is to link the $\mathcal{N} = 4$ super-Schwarzian derivative to invariants of $SU(1,1|2)$ superconformal group. To this end, let us first consider the structure relations of the superconformal algebra $su(1,1|2)$

$$\begin{aligned} [P, D] &= iP, & [P, K] &= 2iD, \\ [D, K] &= iK, & [\mathcal{J}_a, \mathcal{J}_b] &= i\epsilon_{abc} \mathcal{J}_c, \\ \{Q_\alpha, \bar{Q}^\beta\} &= 2P\delta_\alpha^\beta, & \{Q_\alpha, \bar{S}^\beta\} &= 2i(\sigma_a)_\alpha^\beta \mathcal{J}_a - 2D\delta_\alpha^\beta, \\ \{S_\alpha, \bar{S}^\beta\} &= 2K\delta_\alpha^\beta, & \{\bar{Q}^\alpha, S_\beta\} &= -2i(\sigma_a)_\beta^\alpha \mathcal{J}_a - 2D\delta_\beta^\alpha, \\ [D, Q_\alpha] &= -\frac{i}{2} Q_\alpha, & [D, S_\alpha] &= \frac{i}{2} S_\alpha, \\ [K, Q_\alpha] &= iS_\alpha, & [P, S_\alpha] &= -iQ_\alpha, \\ [\mathcal{J}_a, Q_\alpha] &= -\frac{1}{2} (\sigma_a)_\alpha^\beta Q_\beta, & [\mathcal{J}_a, S_\alpha] &= -\frac{1}{2} (\sigma_a)_\alpha^\beta S_\beta, \\ [D, \bar{Q}^\alpha] &= -\frac{i}{2} \bar{Q}^\alpha, & [D, \bar{S}^\alpha] &= \frac{i}{2} \bar{S}^\alpha, \\ [K, \bar{Q}^\alpha] &= i\bar{S}^\alpha, & [P, \bar{S}^\alpha] &= -i\bar{Q}^\alpha, \\ [\mathcal{J}_a, \bar{Q}^\alpha] &= \frac{1}{2} \bar{Q}^\beta (\sigma_a)_\beta^\alpha, & [\mathcal{J}_a, \bar{S}^\alpha] &= \frac{1}{2} \bar{S}^\beta (\sigma_a)_\beta^\alpha. \end{aligned} \quad (20)$$

Here $(P, D, K, \mathcal{J}_a)$, $a = 1, 2, 3$, are (Hermitian) bosonic generators of translations, dilatations, special conformal transformations, and $su(2)$ rotations, respectively. Q_α and S_α are fermionic generators of supersymmetry transformations and superconformal boosts, $\bar{Q}^\alpha$, $\bar{S}^\alpha$ being their Hermitian conjugates. $(\sigma_a)_\beta^\alpha$ are the Pauli matrices (see Appendix).

Following the recipe in [12], each generator in the superalgebra is then accompanied by a Goldstone superfield of the same Grassmann parity and the group-theoretic element is introduced

$$\begin{aligned} \tilde{g} &= e^{i\theta h} e^{\theta^\alpha q_\alpha + \bar{\theta}_\alpha \bar{q}^\alpha} e^{i\rho P} e^{\psi^\alpha Q_\alpha + \bar{\psi}_\alpha \bar{Q}^\alpha} e^{\phi^\alpha S_\alpha + \bar{\phi}_\alpha \bar{S}^\alpha} e^{i\mu K} \\ &\times e^{ivD} e^{i\lambda_a \mathcal{J}_a}, \end{aligned} \quad (21)$$

in which $(\rho, \mu, \nu, \lambda_a)$ are real bosonic superfields and $(\psi_\alpha, \phi_\alpha, \bar{\psi}^\alpha, \bar{\phi}^\alpha)$ are complex fermionic superfields. Here ρ and ψ_α are identified with those in the preceding section and the constraints (11), (19) are assumed to hold. The choice of $\tilde{g}$ is prompted by the study of the $d = 1$, $\mathcal{N} = 4$ superconformal mechanics in [16].

The left multiplication by a group element $\tilde{g} \rightarrow g \cdot \tilde{g}$

$$g = e^{iaP} e^{\epsilon^\alpha Q_\alpha + \bar{\epsilon}_\alpha \bar{Q}^\alpha} e^{\kappa^\alpha S_\alpha + \bar{\kappa}_\alpha \bar{S}^\alpha} e^{icK} e^{ibD} e^{i\xi_a \mathcal{J}_a}, \quad (22)$$

in which (a, b, c, ξ_a) and $(\epsilon_\alpha, \kappa_\alpha)$ are bosonic and fermionic parameters, respectively, generates a finite $SU(1,1|2)$

transformation. In practical applications, it proves sufficient to focus on the infinitesimal transformations

$$\begin{aligned}
\rho' &= \rho + a, & \psi'_\alpha &= \psi_\alpha, & \bar{\psi}'^\alpha &= \bar{\psi}^\alpha; \\
\rho' &= \rho + b\rho, & \psi'_\alpha &= \psi_\alpha + \frac{1}{2}b\psi_\alpha, & \bar{\psi}'^\alpha &= \bar{\psi}^\alpha + \frac{1}{2}b\bar{\psi}^\alpha; \\
\rho' &= \rho + c\rho^2 - \frac{1}{2}c\psi^2\bar{\psi}^2, & \psi'_\alpha &= \psi_\alpha + c\rho\psi_\alpha + \frac{i}{2}c\psi^2\bar{\psi}_\alpha, \\
\bar{\psi}'^\alpha &= \bar{\psi}^\alpha + c\rho\bar{\psi}^\alpha + \frac{i}{2}c\bar{\psi}^2\psi^\alpha; \\
\rho' &= \rho, & \psi'_\alpha &= \psi_\alpha + \frac{i}{2}\xi_a(\sigma_a)_\alpha{}^\beta\psi_\beta, \\
\bar{\psi}'^\alpha &= \bar{\psi}^\alpha - \frac{i}{2}\xi_a\bar{\psi}^\beta(\sigma_a)_\beta{}^\alpha; \\
\rho' &= \rho + i(\bar{\psi}\epsilon - \bar{\epsilon}\psi), & \psi'_\alpha &= \psi_\alpha + \epsilon_\alpha, & \bar{\psi}'^\alpha &= \bar{\psi}^\alpha + \bar{\epsilon}^\alpha; \\
\rho' &= \rho - i\rho(\bar{\psi}\kappa - \bar{\kappa}\psi) + \bar{\psi}\psi(\bar{\psi}\kappa + \bar{\kappa}\psi), \\
\psi'_\alpha &= \psi_\alpha - \rho\kappa_\alpha + i\bar{\psi}\psi\kappa_\alpha - i\psi^2\bar{\kappa}_\alpha, \\
\bar{\psi}'^\alpha &= \bar{\psi}^\alpha - \rho\bar{\kappa}^\alpha - i\bar{\psi}\psi\bar{\kappa}^\alpha - i\bar{\psi}^2\kappa^\alpha,
\end{aligned} \tag{23}$$

which are obtained with the aid of the Baker–Campbell–Hausdorff formula

$$e^{iA}Te^{-iA} = T + \sum_{n=1}^{\infty} \frac{i^n}{n!} \underbrace{[A, [A, \dots [A, T] \dots]]}_{n \text{ times}}. \tag{24}$$

Note that both the original and transformed superfields depend on the same arguments $(t, \theta, \bar{\theta})$ such that the transformations affect the form of the superfields only, e.g., $\delta\rho = \rho'(t, \theta, \bar{\theta}) - \rho(t, \theta, \bar{\theta})$. Computing the algebra of the infinitesimal transformations (23), one can verify that it does reproduce the structure relations (20).

As the next step, one builds the odd analogues of the Maurer–Cartan one-forms

$$\begin{aligned}
\tilde{g}^{-1}\mathcal{D}^\alpha\tilde{g} &= i\omega_D^\alpha D + i\omega_K^\alpha K + \omega_Q^\alpha Q_\beta + \omega_S^\alpha S_\beta \\
&+ \omega_{\bar{S}\beta}{}^\alpha \bar{S}^\beta + i\omega_{\mathcal{J}}^\alpha \mathcal{J}_a - q^\alpha,
\end{aligned} \tag{25}$$

where $\mathcal{D}^\alpha$ is the covariant derivative (4), which give rise to the $SU(1,1|2)$ invariants

$$\begin{aligned}
\omega_D^\alpha &= \mathcal{D}^\alpha\nu + 2i\bar{\phi}^\beta\mathcal{D}^\alpha\psi_\beta, \\
\omega_K^\alpha &= e^\nu(\mathcal{D}^\alpha\mu - 2i\mu\bar{\phi}^\beta\mathcal{D}^\alpha\psi_\beta - i\bar{\phi}^\beta\mathcal{D}^\alpha\phi_\beta \\
&\quad - \bar{\phi}^2\phi^\beta\mathcal{D}^\alpha\psi_\beta - i\phi_\beta\mathcal{D}^\alpha\bar{\phi}^\beta), \\
\omega_Q^\alpha &= e^{-\frac{k}{2}}\mathcal{D}^\alpha\psi^\beta \left(\exp \left[\frac{i}{2}\lambda_c\sigma_c \right] \right)_\beta{}^\alpha, \\
\omega_S^\alpha &= e^{\frac{k}{2}}(\mathcal{D}^\alpha\phi^\beta + \mu\mathcal{D}^\alpha\psi^\beta - i\bar{\phi}\phi\mathcal{D}^\alpha\psi^\beta) \left(\exp \left[\frac{i}{2}\lambda_c\sigma_c \right] \right)_\beta{}^\alpha, \\
\omega_{\bar{S}\gamma}{}^\alpha &= e^{\frac{k}{2}}(\mathcal{D}^\alpha\bar{\phi}_\beta + i\bar{\phi}^2\mathcal{D}^\alpha\psi_\beta) \left(\exp \left[-\frac{i}{2}\lambda_c\sigma_c \right] \right)_\gamma{}^\beta,
\end{aligned} \tag{26}$$

while $i\omega_{\mathcal{J}}^{aa}\mathcal{J}_a$ yields

$$e^{i\lambda_c\mathcal{J}_c}(\mathcal{D}^\alpha e^{-i\lambda_k\mathcal{J}_k}) + 2i(\mathcal{D}^\alpha\psi^\beta(\sigma_a)_\beta{}^\gamma\bar{\phi}_\gamma)\mathcal{J}_a. \tag{27}$$

The invariant $\tilde{g}^{-1}\bar{\mathcal{D}}_\alpha\tilde{g}$ could be considered likewise, which would result in the Hermitian conjugates of (26) and (27).

Because the $\mathcal{N}=4$ super-Schwarzian derivative is expected to involve the fermionic superfield ψ_α only, one is led to use the invariants (26) so as to eliminate $(\nu, \mu, \lambda_a, \phi_\alpha, \bar{\phi}^\alpha)$ from the consideration. Guided by a recent analysis of the $\mathcal{N}=1$ and $\mathcal{N}=2$ super-Schwarzian derivatives in a similar setting [11], let us impose the following constraints

$$\omega_D^\alpha = 0, \quad \omega_Q^{\alpha\gamma} = r^{\alpha\gamma}, \quad \omega_S^{\alpha\gamma} = 0, \tag{28}$$

where $r^{\alpha\gamma}$ is a constant matrix with even supernumber elements (coupling constants). Note that the consistency requires $r^{\alpha\beta}$ and its Hermitian conjugate $(r^{\alpha\beta})^\dagger = \bar{r}_{\beta\alpha}$ to obey the restrictions $\bar{r}_{\alpha\beta}r^{\beta\gamma} = \frac{1}{2}\delta_\alpha{}^\gamma(r\bar{r})$, $r^{\alpha\beta}\bar{r}_{\beta\gamma} = \frac{1}{2}\delta_\gamma{}^\alpha(r\bar{r})$ (see Eq. (30) below).

Making use of the Hermitian conjugates, one can express ν and μ in terms of the fermionic superfields ψ_α and ϕ_α

$$\begin{aligned}
e^\nu &= \frac{(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi})}{r\bar{r}}, \\
\mu(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi}) + (\mathcal{D}\phi\bar{\mathcal{D}}\bar{\psi}) - i\bar{\phi}\phi(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi}) &= 0,
\end{aligned} \tag{29}$$

where $r\bar{r} = r^{\alpha\beta}\bar{r}_{\beta\alpha}$, while $\omega_Q^{\alpha\gamma}$ links λ_a to ψ_α

$$\left(\exp \left[\frac{i}{2}\lambda_a\sigma_a \right] \right)_\alpha{}^\beta = -\frac{2e^{\frac{k}{2}}r^{\beta\gamma}\bar{\mathcal{D}}_\gamma\bar{\psi}_\alpha}{\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi}}. \tag{30}$$

In order to link $(\phi_\alpha, \bar{\phi}^\alpha)$ to $(\psi_\alpha, \bar{\psi}^\alpha)$, it suffices to contract $\omega_D^\alpha = 0$ with $\bar{\mathcal{D}}_\alpha\bar{\psi}^\gamma$ and to make use of the quadratic constraints (15). The result reads

$$\phi_\alpha = -\frac{2\partial_t\psi_\alpha}{(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi})}, \quad \bar{\phi}^\alpha = -\frac{2\partial_t\bar{\psi}^\alpha}{(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi})}, \tag{31}$$

which also simplifies the expression for μ

$$\mu = 2\frac{(\mathcal{D}^\alpha\partial_t\psi_\beta)(\bar{\mathcal{D}}_\alpha\bar{\psi}^\beta)}{(\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi})^2}. \tag{32}$$

Finally, substituting (29), (30), (31), (32) back into the invariants (26), one reveals that they either vanish identically (it proves helpful to evaluate $\omega_{\mathcal{J}}^{aa}\mathcal{J}_a$ at $\mathcal{J}_a = \frac{1}{2}\sigma_a$), or produce the second-rank tensor

$$\bar{\mathcal{D}}_\alpha\bar{\psi}^\gamma\mathcal{D}^\beta\left(\frac{\partial_t\psi_\gamma}{\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi}}\right) - \frac{1}{2}\delta_\alpha{}^\beta\left(\bar{\mathcal{D}}_\mu\bar{\psi}^\nu\mathcal{D}^\mu\left(\frac{\partial_t\psi_\nu}{\mathcal{D}\psi\bar{\mathcal{D}}\bar{\psi}}\right)\right) := \mathcal{I}_\alpha{}^\beta, \tag{33}$$

or, lowering the upper index,

$$\mathcal{I}_{\alpha\beta} = \frac{1}{8i} (\mathcal{D}_\alpha \bar{\mathcal{D}}_\beta + \mathcal{D}_\beta \bar{\mathcal{D}}_\alpha) \ln (\mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}). \quad (34)$$

Thus, after the auxiliary fields $(\mu, \nu, \lambda_a, \phi_a, \bar{\phi}^a)$ have been eliminated from the consideration, the only $SU(1, 1|2)$ invariant combination involving ψ_α is given by (34). Because $\mathcal{I}_{\alpha\beta}$ is traceless, it contains three independent components which can be extracted by contracting with the (traceless) Pauli matrices

$$\mathcal{D}^\alpha (\sigma_a)_\alpha{}^\beta \bar{\mathcal{D}}_\beta \ln (\mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}) := \mathcal{I}_a[\psi(t, \theta, \bar{\theta}); t, \theta, \bar{\theta}]. \quad (35)$$

Equation (35) coincides with the $\mathcal{N} = 4$ super-Schwarzian derivative introduced in [7]. Note that it transforms as a vector under $SU(2)$ transformations acting in $\mathcal{S}^{1|4}$ superspace. It also satisfies the conditions

$$\mathcal{D}^2 \mathcal{I}_a = \bar{\mathcal{D}}^2 \mathcal{I}_a = 0. \quad (36)$$

A few comments are in order. First, focusing on the infinitesimal $SU(1, 1|2)$ transformations (23) and taking into account the constraints (11) and the equalities

$$\begin{aligned} \mathcal{D}\psi' \bar{\mathcal{D}}\bar{\psi}' &= (1 + 2c\rho) \mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}, & \mathcal{D}\psi' \bar{\mathcal{D}}\bar{\psi}' &= (1 + b) \mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}, \\ \mathcal{D}\psi' \bar{\mathcal{D}}\bar{\psi}' &= (1 + 2i(\bar{\kappa}\psi - \bar{\psi}\kappa)) \mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}, \end{aligned} \quad (37)$$

one can verify that (35) does hold invariant.

Secondly, considering the superconformal diffeomorphism described by Eqs. (9), (11), and (14) above, changing the argument $\psi_\alpha(t, \theta, \bar{\theta}) \rightarrow \Omega_\alpha(\rho, \psi, \bar{\psi})$ of the super-Schwarzian (35), and taking into account Eq. (10), which gives rise to the identity

$$(\mathcal{D}\Omega \bar{\mathcal{D}}\bar{\Omega}) = \frac{1}{2} (\mathcal{D}\psi \bar{\mathcal{D}}\bar{\psi}) (\mathcal{D}'\Omega \bar{\mathcal{D}}'\bar{\Omega}), \quad (38)$$

one reveals the composition law [7]

$$\begin{aligned} \mathcal{I}_a[\Omega(t', \theta', \bar{\theta}'); t, \theta, \bar{\theta}] &= \mathcal{I}_a[\psi(t, \theta, \bar{\theta}); t, \theta, \bar{\theta}] \\ &+ \frac{1}{2} M_{ab} \mathcal{I}_b[\Omega(t', \theta', \bar{\theta}'); t', \theta', \bar{\theta}'], \end{aligned} \quad (39)$$

with $M_{ab} = \mathcal{D}^\alpha \psi_\lambda (\sigma_a)_\alpha{}^\beta (\sigma_b)_\beta{}^\lambda \bar{\mathcal{D}}_\beta \bar{\psi}^\nu$. In deriving Eq. (39), the properties of the Pauli matrices exposed in Appendix have been used. In particular, the $\mathcal{N} = 4$ super-Schwarzian holds invariant under the change of the argument $\psi_\alpha(t, \theta, \bar{\theta}) \rightarrow \Omega_\alpha(\rho, \psi, \bar{\psi})$, provided the last term in (39) vanishes.

Third, assuming the constraints (11) and (19) to hold, which result in the restrictions (17) upon the components of the chiral superfield (7) (recall that v in (17) was set to zero), and analyzing the equation $\mathcal{I}_a[\psi(t, \theta, \bar{\theta}); t, \theta, \bar{\theta}] = 0$, or equivalently $\mathcal{I}_{\alpha\beta} = 0$, one can fix $u(t)$ and $\alpha_\gamma(t)$

$$u(t) = \frac{1}{ct + d}, \quad \alpha_\gamma(t) = \epsilon_\gamma + \frac{(\bar{\kappa}\kappa)\kappa_\gamma}{ct + d}, \quad (40)$$

where (c, d) and $(\epsilon_\gamma, \kappa_\gamma)$ are bosonic and fermionic parameters, respectively. The resulting superfield (7) determines a finite $SU(1, 1|2)$ transformation acting in the odd sector of $\mathcal{S}^{1|4}$ superspace. In particular, it correctly reduces to Eqs. (23) in the infinitesimal limit [17]. A finite $SU(1, 1|2)$ transformation acting in the even sector of $\mathcal{S}^{1|4}$ can be found by integrating Eq. (12).

IV. CONCLUSION

To summarize, in this work we applied the method of nonlinear realizations to the superconformal group $SU(1, 1|2)$ and reproduced the $\mathcal{N} = 4$ super-Schwarzian derivative in [7]. As compared to other approaches in the literature, our consideration relied upon invariants of $SU(1, 1|2)$ alone. Neither infinite-dimensional extension of the supergroup, nor conformal field theory techniques, nor the analysis of central charges/cocycles were needed.

Turning to possible further developments, it would be interesting to study a variant of an $\mathcal{N} = 4$ supersymmetric extension of the Sachdev–Ye–Kitaev model which is based upon the effective action involving $\mathcal{I}_a \bar{\mathcal{I}}_a$.

Apart from the $\mathcal{N} = 4$ super-Schwarzian associated with $SU(1, 1|2)$, one can define a similar derivative which exhibits $OSp(4|2)$ superconformal invariance [6]. Both the supergroups are known to be particular instances of the most general superconformal group in one dimension $D(2, 1; \alpha)$. To the best of our knowledge, a $D(2, 1; \alpha)$ super-Schwarzian has not yet been constructed and we hope to report on its peculiarities elsewhere.

Another interesting problem is to obtain an $\mathcal{N} = 3$ super-Schwarzian along similar lines.

ACKNOWLEDGMENTS

This work is supported by the Russian Foundation for Basic Research, Grant No 20-52-12003.

APPENDIX: $SU(2)$ SPINOR CONVENTIONS

Throughout the text we use a lower Greek index to designate an $SU(2)$ -doublet representation. Hermitian conjugation yields an equivalent representation to which one assigns an upper index

$$(\psi_\alpha)^\dagger = \bar{\psi}^\alpha, \quad \alpha = 1, 2.$$

As usual, spinor indices are raised and lowered with the use of the $SU(2)$ -invariant antisymmetric matrices

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}_\alpha = \epsilon_{\alpha\beta} \bar{\psi}^\beta,$$

where $\epsilon_{12} = 1$, $\epsilon^{12} = -1$. For spinor bilinears we stick to the notation

$$\psi^2 = (\psi^\alpha \psi_\alpha), \quad \bar{\psi}^2 = (\bar{\psi}_\alpha \bar{\psi}^\alpha), \quad \bar{\psi}\psi = (\bar{\psi}^\alpha \psi_\alpha),$$

such that

$$\begin{aligned} \psi_\alpha \psi_\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \psi^2, & \bar{\psi}^\alpha \bar{\psi}^\beta &= \frac{1}{2} \epsilon^{\alpha\beta} \bar{\psi}^2, \\ \psi_\alpha \bar{\psi}_\beta - \psi_\beta \bar{\psi}_\alpha &= \epsilon_{\alpha\beta} (\bar{\psi}\psi), & \psi^\alpha \psi^\beta &= -\frac{1}{2} \epsilon^{\alpha\beta} \psi^2, \\ \bar{\psi}_\alpha \bar{\psi}_\beta &= -\frac{1}{2} \epsilon_{\alpha\beta} \bar{\psi}^2, & (\bar{\psi}\psi)^2 &= \frac{1}{2} \psi^2 \bar{\psi}^2. \end{aligned}$$

The Pauli matrices $(\sigma_a)_\alpha{}^\beta$ are taken in the standard form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which obey

$$\begin{aligned} (\sigma_a \sigma_b)_\alpha{}^\beta + (\sigma_b \sigma_a)_\alpha{}^\beta &= 2\delta_{ab} \delta_\alpha{}^\beta, \\ (\sigma_a \sigma_b)_\alpha{}^\beta - (\sigma_b \sigma_a)_\alpha{}^\beta &= 2i\epsilon_{abc} (\sigma_c)_\alpha{}^\beta, \\ (\sigma_a \sigma_b)_\alpha{}^\beta &= \delta_{ab} \delta_\alpha{}^\beta + i\epsilon_{abc} (\sigma_c)_\alpha{}^\beta, \\ (\sigma_a)_\alpha{}^\beta (\sigma_a)_\gamma{}^\rho &= 2\delta_\alpha{}^\rho \delta_\gamma{}^\beta - \delta_\alpha{}^\beta \delta_\gamma{}^\rho, \\ (\sigma_a)_\alpha{}^\beta \epsilon_{\beta\gamma} &= (\sigma_a)_\gamma{}^\beta \epsilon_{\beta\alpha}, \\ \epsilon^{\alpha\beta} (\sigma_a)_\beta{}^\gamma &= \epsilon^{\gamma\beta} (\sigma_a)_\beta{}^\alpha, \end{aligned}$$

where ϵ_{abc} is the totally antisymmetric Levi-Civita tensor, $\epsilon_{123} = 1$. From the last line one finds

$$\begin{aligned} \left(\exp \left[-\frac{i}{2} \xi_a \sigma_a \right] \right)_\alpha{}^\gamma \epsilon_{\gamma\beta} &= - \left(\exp \left[\frac{i}{2} \xi_a \sigma_a \right] \right)_\beta{}^\gamma \epsilon_{\gamma\alpha}, \\ \epsilon^{\alpha\gamma} \left(\exp \left[-\frac{i}{2} \xi_a \sigma_a \right] \right)_\gamma{}^\beta &= -\epsilon^{\beta\gamma} \left(\exp \left[\frac{i}{2} \xi_a \sigma_a \right] \right)_\gamma{}^\alpha, \end{aligned}$$

where ξ_a is a real vector parameter.

Throughout the text we use the abbreviation $\bar{\psi}\sigma_a\psi = \bar{\psi}^\alpha (\sigma_a)_\alpha{}^\beta \psi_\beta$. Our convention for the Hermitian conjugation adopted above imply

$$(\bar{\psi}_\alpha)^\dagger = -\psi^\alpha, \quad (\psi^2)^\dagger = \bar{\psi}^2, \quad (\bar{\psi}\sigma_a\chi)^\dagger = \bar{\chi}\sigma_a\psi.$$

-
- [1] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, *Phys. Rev. D* **95**, 026009 (2017).
[2] T. Mertens, G. Turiaci, and H. Verlinde, *J. High Energy Phys.* **08** (2017) 136.
[3] M. Berkooz, N. Brukner, V. Narovlansky, and A. Raz, [arXiv:2003.04405](https://arxiv.org/abs/2003.04405).
[4] D. Friedan, Notes on string theory and two-dimensional conformal field theory, in *Unified String Theories*, edited by M. Green and D. Gross (World Scientific, Singapore, 1985), pp. 162–213.
[5] J. Cohn, *Nucl. Phys.* **B284**, 349 (1987).
[6] K. Schoutens, *Nucl. Phys.* **B295**, 634 (1988).
[7] S. Matsuda and T. Uematsu, *Mod. Phys. Lett. A* **05**, 841 (1990).
[8] Because conformal transformations in $\mathcal{R}^1$ involve the inversion $t \rightarrow \frac{1}{t}$, the conformal group $SL(2, R)$ does not act globally on $\mathcal{R}^1$, but rather on $\mathcal{S}^1 = \mathcal{RP}^1$.
[9] J. Michel and C. Duval, *Int. Math. Res. Not.* **2008**, 054 (2008).
[10] D. Chang and A. Kumar, *Phys. Lett. B* **193**, 181 (1987).
[11] A. Galajinsky, *J. High Energy Phys.* **06** (2020) 027.
[12] S. Coleman, J. Wess, and B. Zumino, *Phys. Rev.* **177**, 2239 (1969).
[13] E. Ivanov and V. Ogievetsky, *Theor. Math. Phys.* **25**, 1050 (1975).
[14] A. Galajinsky, *Phys. Lett. B* **795**, 277 (2019).
[15] E. Ivanov, S. Krivonos, and V. Leviant, *J. Phys. A* **22**, 4201 (1989).
[16] E. Ivanov, S. Krivonos, and O. Lechtenfeld, *J. High Energy Phys.* **03** (2003) 014.
[17] In order to reproduce the infinitesimal form of the superconformal boosts entering (23), one sets $d = 1$, considers c to be small, such that $\frac{1}{1+ct} \approx 1 - ct$, and identifies $c(\bar{\kappa}\kappa)\kappa_\gamma$ with the infinitesimal κ_γ in (23). The resulting transformation is a superposition of the supersymmetry transformation, special conformal transformation parametrized by c and the superconformal boost associated with $c(\bar{\kappa}\kappa)\kappa_\gamma$.