Confluent conformal blocks and the Teukolsky master equation

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Quasinormal modes of usual, four-dimensional, Kerr black holes are described by certain solutions of a confluent Heun differential equation. In this work, we express these solutions in terms of the connection matrices for a Riemann-Hilbert problem, which was recently solved in terms of the Painlevé V transcendent. We use this formulation to generate small-frequency expansions for the angular spheroidal harmonic eigenvalue and derive conditions on the monodromy properties for the radial modes. Using exponentiation, we relate the accessory parameter to a semiclassical conformal description and discuss the properties of the operators involved. For the radial equation, while the operators at the horizons have Liouville momenta proportional to the entropy intake, we find that spatial infinity is described by a Whittaker operator.

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I. INTRODUCTION

The Kerr black hole is described by the metric, in Boyer-Lindquist coordinates [1]:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dt d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2, \quad (1)$$

where

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad a = \frac{J}{M}, \quad (2)$$

referring to a solution of the four-dimensional, vacuum Einstein equations that is asymptotically flat and has mass M and angular momentum $J = aM$. It has two event horizons at $r_{\pm}$ and the region $r < r_+$ cannot causally affect the region $r > r_+$. The importance of this solution to the development of general relativity and all theories that generalize it can hardly be overestimated, since it is shown to be the most general vacuum metric with mass and angular momenta. Subsequent studies trying to reconcile its

apparent simplicity with the multitude of processes which can in principle surround a black hole led to the concept of black hole entropy. The microscopic description of the latter for generic black holes remains an outstanding problem in theoretical physics.

However, the significance of the Kerr metric goes beyond formal developments due to the various astrophysical applications of phenomena in the Kerr background, especially the experimental detection of the black hole ringdown after a black hole merging event [2], as well as the recent image of a supermassive black hole whose shadow region [3] gives strong evidence of the existence of an event horizon. As a matter of fact, both experiments are interpreted as direct evidence of a Kerr black hole, and from the raw data the black hole parameters, such as M and a , are measured.

All of these phenomena underscore the importance of the study of fluctuations of the Kerr metric. Their evolution is described by the linearized Einstein equations, with the metric fluctuations decomposed into linear (spin 0), vector (spin 1), and tensor (spin 2) parts. The resulting partial differential equations are linear and separable, and the spin s solution can be written as a sum of solutions of two ordinary differential equations,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dS}{d\theta}\right] + \left[a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + s + \lambda\right]S(\theta) = 0, \quad (3)$$

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR(r)}{dr}\right) + \left(\frac{K^2(r) - 2is(r-M)K(r)}{\Delta} + 4is\omega r - s\lambda_{\ell,m} - a^2\omega^2 + 2am\omega\right)R(r) = 0, \quad (4)$$

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where

$$K(r) = (r^2 + a^2)\omega - am, \\ \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \quad (5)$$

which are called the (vacuum) *Teukolsky master equations* [4]. The spin 1 version can be seen to describe the coupling of the electromagnetic field to the black hole, and one can make $s = 1/2$ to describe massless spinorial particles. The derivation and the behavior of the solutions for the Teukolsky master equation are the subject of many articles and monographs, e.g., [5].

Equations (3) and (4) have been studied for decades now, the problem of scattering and the quasinormal modes being the main topics of interest. Quasinormal modes were dealt with extensively in [6,7], and fast numerical techniques exist to compute the spectra of perturbations. From the analytical side, less is known about the behavior of the solutions and, while asymptotic formulas for the angular eigenvalue exist [8,9], not much can be said about the analytic behavior of the spectrum of eigenmodes. A better analytic grasp on these would be invaluable to the study of phenomena such as stability, superradiance, and degeneracy properties of the spectrum.

The purpose of this article is to study eigenmodes of the Teukolsky master equation analytically, using the isomonodromy method and its relation to classical conformal blocks. The application of the isomonodromy method to black holes was developed from early extensions of the Wentzel-Kramers-Brillouin method using monodromy techniques [10,11], developed also in [12,13]. In [14], the isomonodromy symmetry was introduced, and the relation between the latter and $c = 1$ conformal blocks was pointed to give a formal solution to the scattering problem. Parallel developments allowed for eigenmode expansions for scalar perturbations (of generic mass) to the five-dimensional Kerr-AdS black hole [15] and generic massless perturbations of the four-dimensional Kerr-dS black hole [16]. These ordinary differential equations involved in these problems are Fuchsian, and the relation between the connection and monodromy property of their solutions was outlined in [17].

Generically, the relation between the parameters of a Fuchsian equation and the monodromy properties of their solutions is the oldest form of the Riemann-Hilbert problem. This consists to determine a particular complex function from its singular behavior. Since the inception of this problem, the solution of the Riemann-Hilbert problem has been related to solutions of the classical Liouville equation. Surprisingly, the *quantum* version of Liouville provided the window to the procedural construction of these solutions. In 2009, Alday, Gaiotto, and Tachikawa [18] conjectured that the correlation functions of conformal primaries in quantum Liouville theory would be the same as the instanton partition function of some

four-dimensional supersymmetric Yang-Mills theories, which were given by Nekrasov functions [19]. The relation was proven in [20], through combinatorial means solving recursion conditions on the representations of the Virasoro algebra.

Liouville field theory is also related to the theory of flat holomorphic connections, and the monodromy data of the latter is encoded in the isomonodromic τ functions, of which the simplest nontrivial examples were introduced by Jimbo, Miwa, and Ueno in [21–23]. These τ functions have the Painlevé property [24], and the simplest examples are guises of the six Painlevé transcendents, solutions of ordinary differential nonlinear equations of second order with rational coefficients whose essential singularities are determined from the equation itself, and the remaining singularities are single poles. Following the Liouville field theory realization, expansions of the sixth, fifth, and third Painlevé transcendents were given in terms of $c = 1$ conformal blocks in [25]. The relation was further explored in [26], and the Fredholm determinant expression for a generic class of Painlevé transcendents was given in [27]. The Fredholm determinant formulation allows for faster numerical calculations, as well as a more direct contact between the usual applications of the Riemann-Hilbert problem, such as those in matrix models, and conformal field theory methods.

The overall program of phrasing perturbations of gravitational backgrounds in terms of conformal blocks has a holographic flavor which was also explored by a number of authors; see, for instance, [28]. The purpose, however, can be understood to be what conditions in the purported dual theory one gets from the integrable structure of the gravitational perturbations, rather than the other way around. The latter program of framing the dual theory from the particular integrable structure of the perturbations is bound to be valuable for asymptotically flat spaces, where the dual theory is not so clear cut. In the five-dimensional Kerr-AdS case studied in [29], such conditions were observed to arise from a *unitary* conformal field theory. The relation between the $c = 1$ blocks used to construct the relation between monodromy parameters and scattering coefficients and the semiclassical prescription outlined in [30,31] remains mysterious and may yet shed light in a true quantum description of the black hole states.

In this article we will carry on the analysis of the quasinormal modes of the Teukolsky master equation by exploring the interpretation of the differential equations involved with conformal blocks. We will see that the relevant conformal blocks are irregular, as studied by the authors of [32,33], and the relevant Painlevé transcendent the fifth type [34]. We will see that, while the $c = 1$ blocks give an analytic solution to the accessory parameter problem, exponentiation also allows for a description in terms of semiclassical irregular conformal blocks.

The paper is structured as follows. In Sec. II we will introduce the monodromy data associated with the relevant

differential equation, as well as phrase the connection problem in terms of the isomonodromic τ function. In Sec. III we will apply the method to the angular differential equation (3) and obtain expansions for the spin-weighted spheroidal harmonic eigenvalue. In Sec. IV we will revise the conformal block version of the construction and apply it to the radial equation (4), interpreting the semiclassical conformal block as a correlation function of an unitary theory and using the method to obtain some quasinormal modes. We close by remarking on the future prospects in Sec. V.

II. PREAMBLE: THE CONFLUENT HEUN EQUATION

Both Eqs. (3) and (4) can be brought to the confluent Heun canonical form:

$$\frac{d^2 y}{dz^2} + \left[\frac{1 - \theta_0}{z} + \frac{1 - \theta_{t_0}}{z - t_0} \right] \frac{dy}{dz} + \left[-\frac{1}{4} + \frac{\theta_\infty}{2z} - \frac{t_0 c_{t_0}}{z(z - t_0)} \right] y(z) = 0. \quad (6)$$

The differential equation (6) has three singular points: two regular at $z = 0$ and $z = t_0$ and an irregular singular point of Poincaré rank 1 at $z = \infty$. Series expansions for the solutions $y(z)$ at the regular points can be obtained from the Frobenius method. The point at infinity is trickier, because the solutions present the *Stokes phenomenon*: convergence is conditional to sectors of the complex plane, depending on the direction one takes the limit $z \rightarrow \infty$.

Near a regular singular point z_i , the Frobenius method allows us, in general, to construct two solutions, whose local behavior is

$$y_i^\pm(z) = (z - z_i)^{\frac{1}{2}\alpha_i \pm \frac{1}{2}\theta_i} (1 + \mathcal{O}(z - z_i)), \quad (7)$$

which we will call the local Frobenius solutions at $z = z_i$. In general, one given Frobenius solution at $z = z_i$ will be expressed as a linear combination of the Frobenius solutions constructed at a different point $z = z_j$. For a particular set of parameters in the differential equation (6), namely discrete values of the accessory parameter c_{t_0} , there will be a solution which has definite behavior at *both* $z = z_i$ and $z = z_j$, for instance,

$$y(z) = \begin{cases} (z - z_i)^{\frac{1}{2}\alpha_i + \frac{1}{2}\theta_i} (1 + \mathcal{O}(z - z_i)), & z \rightarrow z_i; \\ (z - z_j)^{\frac{1}{2}\alpha_j + \frac{1}{2}\theta_j} (1 + \mathcal{O}(z - z_j)), & z \rightarrow z_j. \end{cases} \quad (8)$$

Finding the (discrete) values of c_{t_0} for which such a $y(z)$ exists will be referred to as the *eigenvalue problem*. The formulation of the eigenvalue problem at the irregular singular point $z = \infty$ is a bit more complicated and will be dealt with later.

Let us briefly describe the solution to the eigenvalue problem proposed in [14]. The second order differential equation (6) can be cast as a first order matrix equation,

$$\begin{aligned} \frac{d\Phi}{dz} \Phi^{-1}(z) &= A(z) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{A_0}{z} + \frac{A_t}{z - t} \\ &= \frac{1}{2} \sigma_3 + \frac{A_0}{z} + \frac{A_t}{z - t}, \end{aligned} \quad (9)$$

where we introduced the *fundamental matrix* of solutions $\Phi(z)$,

$$\Phi(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ w_1(z) & w_2(z) \end{pmatrix}, \quad (10)$$

with $y_{1,2}(w)$ satisfying our original Eq. (6) and $w_{1,2}(z)$ related to $y_{1,2}(z)$ by differentiation and multiplication by a rational function,

$$w_i(z) = \frac{1}{A_{12}(z)} \left(\frac{dy_i}{dz} - A_{11}(z) y_i(z) \right). \quad (11)$$

We note that any two solutions of (9) are related by right multiplication. We also note that one can change the value of α_i at will by multiplication of the solution (10) by a factor $\prod_i (z - z_i)^{\frac{1}{2}\alpha_i}$, with the exception of the singular point at infinity.

The basis of the method is to see the parameter t in (9) as a gauge parameter in the space of flat holomorphic connections $A(z, t)$, and to recover the differential equation (6) as we take t to t_0 . The usefulness of this deformation stems from the fact that we can translate conditions such as the quantization condition (68) in terms of gauge-invariant properties of (9), called *monodromy data*.

A. Monodromy data

Let us first describe the latter. The monodromy data associated with the matrix of solution $\Phi(z)$ of (9) is its behavior under analytical continuation around the singular points,

$$\Phi((z - z_0)e^{2\pi i} + z_0) = \Phi(z) M_{z_0}, \quad (12)$$

which defines the monodromy matrix as the decomposition of the analytic continuation of each of the solutions in terms of themselves. As defined above, the matrices M_i are independent of the homotopy class of the curve we choose for analytic continuation. The matrices M_i are also independent of the sum of the indicial exponents at each singular point, the α_i in (8), due to the fact that these can be changed by multiplication of a scalar function.

For the irregular singular point $z = \infty$ there is a subtlety, due to the Stokes phenomenon. Let us follow [35] (see also [36]) and define sectors $\mathcal{S}_k$ as

$$\mathcal{S}_k = \left\{ z \in \mathbb{C}, -\frac{1}{2}\pi + (k-2)\pi < \arg z < \frac{3}{2}\pi + (k-2)\pi \right\},$$

$$k \in \mathbb{Z}; \quad (13)$$

in each the asymptotic solution for (9) is

$$\Phi_k(z) = (1 + \mathcal{O}(z^{-1}))$$

$$\times \exp \left[\frac{1}{2}\sigma_3 z + \frac{1}{2}((\hat{\theta}_0 + \hat{\theta}_t)\mathbb{1} - \hat{\theta}_\infty \sigma_3) \log z \right],$$

$$z \rightarrow \infty, \quad z \in \mathcal{S}_k, \quad (14)$$

where θ_i are defined as

$$\hat{\theta}_0 = \text{Tr} A_0, \quad \hat{\theta}_t = \text{Tr} A_t, \quad \hat{\theta}_\infty = -\text{Tr}[\sigma_3(A_0 + A_t)].$$

$$(15)$$

The analytic continuation of the solution $\Phi_k(z)$ can now be described as the connection between $\Phi_k(z)$ in different sectors,

$$\Phi_{k+1}(z) = \Phi_k(z) S_k, \quad (16)$$

where S_k are the *Stokes matrices*. By (14),

$$S_{k+2} = e^{i\pi\hat{\theta}_\infty\sigma_3} S_k e^{-i\pi\hat{\theta}_\infty\sigma_3}, \quad (17)$$

so only two of the Stokes matrices are independent. It can be checked from the discussion that they have the structure

$$S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \quad S_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1} & 1 \end{pmatrix}, \quad k \in \mathbb{Z},$$

$$(18)$$

where the parameters s_{2k} and s_{2k+1} are called *Stokes multipliers*. It is customary to define the monodromy matrix at $z = \infty$ in the sector $k = 2$,

$$\Phi_2(ze^{-2\pi i}) = \Phi_2(z) M_\infty(k=2) = \Phi_2(z) M_\infty, \quad (19)$$

with the corresponding matrices for generic k defined through the recursion $M_\infty(k+1) = S_k^{-1} M_\infty(k) S_k$. The monodromy matrix M_∞ can be obtained from the Stokes matrices by

$$M_\infty = S_2 e^{i\pi\hat{\theta}_\infty\sigma_3} S_1 \quad (20)$$

and satisfies the relation

$$M_\infty M_t M_0 = \mathbb{1}. \quad (21)$$

With these definitions, we define the monodromy data ρ associated with the matrix equation (9) as the basis independent data in the matrices M_i :

$$\rho = \{\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_\infty; s_1, s_2\}. \quad (22)$$

It will be convenient to define the trace of M_∞ as an independent parameter

$$2 \cos \pi \hat{\sigma} = \text{Tr} M_\infty = 2 \cos \pi \hat{\theta}_\infty + s_1 s_2 e^{-i\pi \hat{\theta}_\infty}. \quad (23)$$

B. Connection matrix and the quantization condition

We can now phrase the eigenvalue problem (8) in terms of monodromy data. Let us choose the fundamental solution at $z = 0$, $\Phi(z; z_0 = 0)$ with $y_1(z)$ and $y_2(z)$ in (10) constructed using the Frobenius method at $z = 0$,

$$\Phi(z; 0) = (1 + \mathcal{O}(z)) \exp \left[\left(\frac{1}{2} \alpha_0 \mathbb{1} + \frac{1}{2} \hat{\theta}_0 \sigma_3 \right) \log z \right],$$

$$(24)$$

where $\frac{1}{2}(\alpha_0 \pm \hat{\theta}_0)$ are the eigenvalues of A_0 . It is clear that the monodromy matrix around $z_0 = 0$ for this basis is diagonal:

$$\Phi(ze^{2\pi i}; 0) = \Phi(z; 0) e^{i\pi \alpha_0} e^{i\pi \hat{\theta}_0 \sigma_3}. \quad (25)$$

The α_0 , Abelian part of the monodromy can be removed by a “ s -homotopic transformation” like (70) and can be taken to be zero. We therefore have that in this basis of solutions $M_0 = e^{i\pi \hat{\theta}_0 \sigma_3}$. The monodromy around $z = t$ is likewise diagonal with the fundamental solution $\Phi(z; z_0 = t)$, but in terms of $\Phi(z, z_0 = 0)$ above

$$\Phi((z-t)e^{2\pi i} + t; 0) = \Phi(z; 0) C_{t0}^{-1} e^{i\pi \hat{\theta}_t \sigma_3} C_{t0},$$

$$\text{where } C_{t0} = \Phi(z; t)^{-1} \Phi(z; 0), \quad (26)$$

is called the *connection matrix* between the singular points at $z = 0$ and $z = t$.

Now, we can see that if the parameters in the matrix system (9) are such that the conditions (68) are satisfied, then the connection matrix C_{t0} is either lower triangular or upper triangular. Simple algebra shows that, if this is the case, then

$$\text{Tr} M_0 M_t = 2 \cos \pi (\hat{\theta}_0 + \hat{\theta}_t). \quad (27)$$

It can be checked that the converse is also true: if this trace property is satisfied, then C_{t0} is either lower or upper triangular. This is a condition to be satisfied when λ corresponds to the angular eigenvalue. Using the property (21), we have $\text{Tr} M_0 M_t = \text{Tr} M_\infty^{-1}$, and, by the definition of σ above (23), we arrive at

$$\cos \pi (\hat{\theta}_0 + \hat{\theta}_t) = \cos \pi \hat{\sigma}, \Rightarrow \hat{\sigma}(\lambda_\ell) = \hat{\theta}_0 + \hat{\theta}_t + 2j, \quad j \in \mathbb{Z},$$

$$(28)$$

where we underscored the dependence of the $\hat{\sigma}$ parameter on λ , but in fact it depends on all parameters in (6).

The condition (28) does not provide a full solution of the system, however, because it may involve non-normalizable solutions of the differential equation (6). In our applications below, it will be clear from the context which values of θ_i lead to the proper modes.

C. The τ function and Painlevé V system

To calculate σ as a function of the differential equation parameters is a version of the Riemann-Hilbert problem, whose solution we will make use of. The idea goes back to the theory of isomonodromic deformations as introduced by [21–23], and is based on interpreting t as a gauge parameter. If we accompany (9) by its Lax pair

$$\frac{\partial \Phi}{\partial t} [\Phi(z, t)]^{-1} = -\frac{A_t}{z - t}, \quad (29)$$

the existence of the mixed derivative $\partial_z \partial_t \Phi = \partial_t \partial_z \Phi$ requires that A_0 and A_t satisfy the *Schlesinger equations*,

$$\frac{\partial A_0}{\partial t} = \frac{1}{t} [A_t, A_0], \quad \frac{\partial A_t}{\partial t} = -\frac{1}{t} [A_t, A_0] - \frac{1}{2} [A_t, \sigma_3], \quad (30)$$

whose solution gives a one-parameter family of matrix systems with different values of t but the same monodromy data. Since A_0 and A_t are now arbitrary, let us consider the generic differential equation satisfied by the first row of $\Phi(z)$ in (9)

$$\frac{d^2 y}{dz^2} + p(z) \frac{dy}{dz} + q(z)y = 0, \quad (31)$$

$$p(z) = \frac{1 - \hat{\theta}_0}{z} + \frac{1 - \hat{\theta}_t}{z - t} - \frac{1}{z - \lambda},$$

$$q(z) = -\frac{1}{4} + \frac{\hat{\theta}_\infty - 1}{2z} - \frac{tc_t}{z(z - t)} + \frac{\lambda\mu}{z(z - \lambda)}, \quad (32)$$

where λ is the root of $A_{12}(z)$ and $\mu = A_{11}(z = \lambda)$. c_t is related to λ and μ by

$$\mu^2 - \left[\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_t - 1}{\lambda - t} \right] \mu + \frac{\hat{\theta}_\infty - 1}{2\lambda} - \frac{tc_t}{\lambda(\lambda - t)} = \frac{1}{4}. \quad (33)$$

The algebraic condition (33) tells us that the singularity at $z = \lambda$ in (32) is an apparent one: the indicial equation gives integer exponents 0 and 2, and there is no logarithmic behavior due to (33). The monodromy matrix around $z = \lambda$ is then trivial. The Schlesinger equations induce a flow to λ and μ , and the corresponding differential equation for λ is equivalent to the Painlevé V transcendent.

The family of isomonodromic connections will include our original equation (6) if

$$\hat{\theta}_0 = \theta_0, \quad \hat{\theta}_t = \theta_{t_0} - 1, \quad \hat{\theta}_\infty = \theta_\infty + 1,$$

$$\lambda(t_0) = t_0, \quad \mu(t_0) = -\frac{c_{t_0}}{\theta_{t_0} - 1}, \quad (34)$$

and note that, per (23), $\hat{\sigma} = \sigma - 1$. These conditions are more conveniently written in terms of the Jimbo-Miwa-Ueno (JMU) τ function

$$\frac{d}{dt} \log \tau(\rho; t) = \frac{1}{2} \text{Tr} \sigma_3 A_t + \frac{1}{t} \text{Tr} \left(A_0 - \frac{1}{2} \hat{\theta}_0 \mathbb{1} \right) \left(A_t - \frac{1}{2} \hat{\theta}_t \mathbb{1} \right), \quad (35)$$

where we left explicitly the dependence of the JMU τ function on the monodromy data ρ due to its expansions [25,35]. Therefore, Eq. (34) is

$$\frac{d}{dt} \log \tau(\hat{\rho}; t_0) = c_{t_0} + \frac{\hat{\theta}_0 \hat{\theta}_t}{2t_0}, \quad \frac{d}{dt} t \frac{d}{dt} \log \tau(\hat{\rho}; t_0) + \frac{\hat{\theta}_t}{2} = 0. \quad (36)$$

The second condition (36) stems from the second derivative of the τ function, calculated using the Schlesinger equations and imposing (34). The left-hand side can be related through the *Toda equation* [37] to a product of τ functions,

$$\frac{d}{dt} t \frac{d}{dt} \log \tau(\hat{\rho}; t_0) + \frac{\hat{\theta}_t}{2} = K_V \frac{\tau(\hat{\rho}^+; t) \tau(\hat{\rho}^-; t)}{\tau^2(\hat{\rho}; \tau)}, \quad (37)$$

where K_V is independent of t and the $\rho^\pm$ are related to ρ by simple shifts:

$$\hat{\rho}^\pm = \{\hat{\theta}_0, \hat{\theta}_t \pm 1, \sigma \pm 1, \hat{\theta}_\infty \mp 1; s_1, s_2\}. \quad (38)$$

Miwa's theorem [24] tells us that τ defined by (35) is analytic in t except at the critical points $t = 0$ and $t = \infty$. Therefore either factor of the numerator in (37) has to vanish.

The proof of (37) is straightforward; from a fundamental solution $\Phi(z)$ one defines the derived solutions

$$\Phi^\pm(z) = \exp[p^\pm \sigma^\mp] \begin{pmatrix} (z - t)^{\pm 1} & 0 \\ 0 & 1 \end{pmatrix} \exp[q^\pm \sigma^\pm] \Phi(z), \quad (39)$$

where $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent combinations of Pauli matrices. Given $\Phi^\pm(z)$, one can establish the Toda equation (37) by comparing the corresponding expressions for each τ function (35), and choosing $p^\pm$ and $q^\pm$ in order to keep the form of the new connection, defined through (9), and maintain the partial fraction form at $z = t$ and $z = \infty$. It is clear that the monodromy data of $\Phi^\pm(z)$ are related to that of $\Phi(z)$ by (38). Further algebraic manipulation shows that

$$\frac{d}{dt} \log \frac{\tau(\rho^+; t)}{\tau(\rho; t)} = -\frac{1}{2} - \frac{\lambda}{t} \left(\mu - \frac{1}{2} \right) + \frac{\lambda}{t(\lambda - t)} \hat{\theta}_t, \quad (40)$$

$$\begin{aligned} \frac{d}{dt} \log \frac{\tau(\rho^-; t)}{\tau(\rho; t)} &= \frac{1}{2} - \frac{(\lambda - t)(\mu - \frac{1}{2}) - \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_t - \hat{\theta}_\infty)}{\lambda(\mu - \frac{1}{2}) - \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_t - \hat{\theta}_\infty)} \\ &\times \left(\frac{\lambda}{t} \left(\mu - \frac{1}{2} \right) - \frac{\hat{\theta}_0}{t} \right). \end{aligned} \quad (41)$$

Given that the first line has a divergent limit $\lambda \rightarrow t$, we conclude that we can substitute the second condition in (36) by the simpler one

$$\tau(\rho; t_0) = 0, \quad (42)$$

where the monodromy data are that of (6),

$$\rho = \{\theta_0, \theta_{t_0}, \theta_\infty; s_1, s_2\}, \quad (43)$$

whereas, in terms of ρ , the first condition in (36) is given by

$$c_{t_0} = \frac{d}{dt} \log \tau(\rho^-; t_0) - \frac{\theta_0(\theta_{t_0} - 1)}{2t_0}, \quad (44)$$

with the shift in ρ^- analogous to that of $\hat{\rho}$ above.

D. The Nekrasov expansion

In this section we are going to drop the “hatted” notation in order not to overburden the formulas. The Nekrasov expansion of the Painlevé V τ function is given by [25]

$$\begin{aligned} \tau(\rho; t) &= \sum_{n \in \mathbb{Z}} C_V(\vec{\theta}, \sigma + 2n) s_V^n t^{\frac{1}{4}(\sigma + 2n)^2 - \frac{1}{4}(\theta_0^2 + \theta_t^2)} \\ &\times \mathcal{B}_V(\vec{\theta}; \sigma + 2n; t). \end{aligned} \quad (45)$$

Here $\rho = \{\theta_0, \theta_t, \theta_\infty; \sigma, s_V\}$ is the monodromy data. The definition of the parameter σ in terms of the Stokes parameters is given by (23), and we will discuss the parameter s_V below. The function $\mathcal{B}_V$ is analytic near $t = 0$ and closely related to the irregular conformal blocks of the first kind [32,34]. It is based on the Nekrasov expansion, which is a scalar function associated with a pair of Young diagrams λ, μ , a complex parameter b , as well as a complex number α ,

$$\begin{aligned} Z_{\lambda, \mu}(\alpha) &= \prod_{(i, j) \in \lambda} (\alpha + b^{-1} a_\lambda(i, j) - b(l_\mu(i, j) + 1)) \\ &\times \prod_{(i', j') \in \mu} (\alpha - b^{-1}(a_\mu(i', j') + 1) + b l_\lambda(i', j')), \end{aligned} \quad (46)$$

where $a_\lambda(i, j)$ and $l_\lambda(i, j)$ are, respectively, the arm length and the leg length of the box (i, j) in the diagram λ . The parameter b is related to the central charge of the

Virasoro algebra by $c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2$ (79). As it can be checked in [38], the expansion of the irregular conformal block of the first kind is given by

$$\begin{aligned} \mathcal{B}\left(P_\infty; P_\sigma; \frac{P_t}{P_0}; t\right) \\ = t^{\Delta_\sigma - \Delta_0 - \Delta_t} e^{-i(\frac{Q}{2} - iP_t)t} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}(\vec{P}; P_\sigma) t^{|\lambda| + |\mu|}, \end{aligned} \quad (47)$$

where $\Delta_i = \frac{Q^2}{4} + P_i^2$ and $\mathcal{B}_{\lambda, \mu}$ is given by ratios of Nekrasov functions,

$$\begin{aligned} \mathcal{B}_{\lambda, \mu}(\vec{P}; P_\sigma) &= \frac{Z_{\lambda, \emptyset}(\frac{Q}{2} - i(P_\infty - P_\sigma)) Z_{\mu, \emptyset}(\frac{Q}{2} - i(P_\infty + P_\sigma))}{Z_{\lambda, \lambda}(0) Z_{\mu, \mu}(0) Z_{\lambda, \mu}(2iP_\sigma) Z_{\mu, \lambda}(-2iP_\sigma)} \\ &\times \prod_{\epsilon = \pm} Z_{\lambda, \emptyset}\left(\frac{Q}{2} + i(P_t + \epsilon P_0 + P_\sigma)\right) \\ &\times Z_{\emptyset, \mu}\left(\frac{Q}{2} + i(P_t + \epsilon P_0 - P_\sigma)\right). \end{aligned} \quad (48)$$

As stated in [25], the expansion of the τ function for the Painlevé V near $t = 0$ is given in terms of $c = 1$ irregular conformal blocks. These are obtained taking $b = \sqrt{-1}$ —and therefore $Q = 0$ —in the expressions above, as well as setting the parameters P_i to the monodromy parameters:

$$P_0 = \frac{\theta_0}{2}, \quad P_t = \frac{\theta_t}{2}, \quad P_\infty = \frac{\theta_\infty}{2}, \quad P_\sigma = \frac{\sigma}{2}. \quad (49)$$

Coming back to (45), one can recognize in the $\mathcal{B}_V$ expansion the terms of the same functions $\mathcal{B}_{\lambda, \mu}$ appearing in the expansion of the irregular conformal blocks (48),

$$\mathcal{B}_V(\vec{\theta}, \sigma; t) = e^{-\frac{\theta_t}{2}t} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}(\vec{\theta}, \sigma) t^{|\lambda| + |\mu|}, \quad (50)$$

where, again, the sum runs over all pairs of Young diagrams (λ, μ) , with each coefficient in the series given by the appropriate reduction of (48),

$$\begin{aligned} \mathcal{B}_{\lambda, \mu}(\vec{\theta}, \sigma) \\ = \prod_{\lambda \in \mathbb{Y}} \frac{(2(i - j) + \sigma - \theta_\infty)((\sigma + \theta_t + 2(i - j))^2 - \theta_0^2)}{8h_\lambda^2(i, j)(l_\lambda(i, j) + a_\mu(i, j) + 1 + \sigma)^2} \\ \times \prod_{\mu \in \mathbb{Y}} \frac{(2(i - j) - \sigma - \theta_\infty)((\theta_t - \sigma + 2(i - j))^2 - \theta_0^2)}{8h_\mu^2(i, j)(a_\lambda(i, j) + l_\mu(i, j) + 1 - \sigma)^2}, \end{aligned} \quad (51)$$

and the hook length is defined by $h_\lambda(i, j) = a_\lambda(i, j) + l_\lambda(i, j) + 1$. The structure constants C_V in (45) are rational products of Barnes functions

$$C_V(\vec{\theta}, \sigma) = \mathcal{N}(\vec{\theta}, \sigma) \mathcal{N}(\vec{\theta}, -\sigma), \quad (52)$$

where

$$\mathcal{N}(\vec{\theta}, \sigma) = \frac{G(1 + \frac{1}{2}(\sigma - \theta_\infty))G(1 + \frac{1}{2}(\theta_t + \theta_0 + \sigma))G(1 + \frac{1}{2}(\theta_t - \theta_0 + \sigma))}{G(1 + \sigma)}, \quad (53)$$

where the Barnes function $G(z)$ is defined by functional equation $G(1+z) = \Gamma(z)G(z)$ plus some convexity requirements. The functional equation is its only property required to recover the results in this paper.

E. Monodromy matrices

The parameter s_V in (45) has a geometrical interpretation in terms of the monodromy data. Following [35,36], we will introduce an explicit representation for the monodromy matrices. Let

$$M_0 = C_0^{-1} e^{i\pi\theta_0\sigma_3} C_0, \quad M_t = C_t^{-1} e^{i\pi\theta_t\sigma_3} C_t, \quad M_\infty = S_2 e^{i\pi\theta_\infty\sigma_3} S_1. \quad (54)$$

The connection matrices C_0 and C_t allow the following parametrization:

$$D_t C_t D = \left(\frac{\Gamma(1-\sigma)\Gamma(-\theta_t)}{\Gamma(-\frac{1}{2}(\theta_t+\theta_0+\sigma))\Gamma(-\frac{1}{2}(\theta_t-\theta_0+\sigma))} \frac{\Gamma(1+\sigma)\Gamma(-\theta_t)}{\Gamma(-\frac{1}{2}(\theta_t+\theta_0-\sigma))\Gamma(-\frac{1}{2}(\theta_t-\theta_0-\sigma))} \frac{e^{-\pi i\theta_t}\Gamma(1-\sigma)\Gamma(\theta_t)}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\sigma))} \frac{e^{-\pi i\theta_t}\Gamma(1+\sigma)\Gamma(\theta_t)}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\sigma))} \right) \times \kappa^{-\frac{1}{2}\sigma_3} C_\infty, \quad (55)$$

$$D_0 C_0 D = \left(\frac{e^{\frac{1}{2}\pi i(\theta_t-\theta_0+\sigma)}\Gamma(1-\sigma)\Gamma(-\theta_0)}{\Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\sigma))\Gamma(-\frac{1}{2}(\theta_t+\theta_0+\sigma))} \frac{e^{\frac{1}{2}\pi i(\theta_t-\theta_0-\sigma)}\Gamma(1+\sigma)\Gamma(-\theta_0)}{\Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\sigma))\Gamma(-\frac{1}{2}(\theta_t+\theta_0-\sigma))} \frac{e^{\frac{1}{2}\pi i(\theta_t+\theta_0+\sigma)}\Gamma(1-\sigma)\Gamma(\theta_0)}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\sigma))\Gamma(-\frac{1}{2}(\theta_t-\theta_0+\sigma))} \frac{e^{\frac{1}{2}\pi i(\theta_t+\theta_0-\sigma)}\Gamma(1+\sigma)\Gamma(\theta_0)}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\sigma))\Gamma(-\frac{1}{2}(\theta_t-\theta_0-\sigma))} \right) \times \kappa^{-\frac{1}{2}\sigma_3} C_\infty, \quad (56)$$

with D_t , D_0 , and D diagonal matrices and

$$C_\infty = \begin{pmatrix} -e^{-i\frac{\pi}{2}(\sigma+\theta_\infty)} \frac{\Gamma(1-\sigma)}{\Gamma(-\frac{1}{2}(\sigma-\theta_\infty))} & -\frac{\Gamma(1-\sigma)}{\Gamma(1-\frac{1}{2}(\sigma+\theta_\infty))} \\ e^{i\frac{\pi}{2}(\sigma-\theta_\infty)} \frac{\Gamma(1+\sigma)}{\Gamma(\frac{1}{2}(\sigma+\theta_\infty))} & \frac{\Gamma(1+\sigma)}{\Gamma(1+\frac{1}{2}(\sigma-\theta_\infty))} \end{pmatrix}. \quad (57)$$

The κ parameter in the monodromy matrix is related to the s_V parameter in the Nekrasov expansion (45) by a string of gamma functions,

$$\kappa = \frac{\Gamma^2(1-\tilde{\sigma})\Gamma(1+\frac{1}{2}(\tilde{\sigma}-\theta_\infty))\Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\tilde{\sigma}))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\tilde{\sigma}))}{\Gamma^2(1+\tilde{\sigma})\Gamma(1-\frac{1}{2}(\tilde{\sigma}+\theta_\infty))\Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\tilde{\sigma}))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\tilde{\sigma}))} s_V. \quad (58)$$

As a comment, the diagonal matrices D_0 and D_t represent the ambiguity in diagonalizing M_t and M_0 , which is in turn tied to the choice of normalization of the Frobenius basis $y_i(z; z_0)$ at each point. Likewise, C_∞ diagonalizes M_∞ and D represents the ambiguity in the basis normalization at ∞ . The parameter κ (or s_V) then has the interpretation of the relative normalization between the system at ∞ and the system at $0, t$, which is an isomonodromy invariant as can be checked from the asymptotic analysis such as that in [35] or [36]. Alternatively, one can relate the $s_V = e^{i\eta}$ to the relative twist between the “gluing” of the three-point Riemann-Hilbert problem with monodromies $\{\theta_0, \theta_t, \sigma\}$ —which is solved by hypergeometric functions—to the two-point irregular Riemann-Hilbert

problem $\{-\sigma, \theta_\infty, s_1, s_2\}$, solved by confluent hypergeometrics, as was defined in [38].

F. The accessory parameter for the confluent Heun equation

Solving (44) involves finding the root of the JMU τ function and then using the value of this root to find c_{t_0} as the derivative of the logarithm of the shifted function. Given the structure of (45), it is interesting to write

$$\tau(\rho; t) = C_V(\vec{\theta}; \sigma) t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} e^{-\frac{1}{2}\theta_t t} \hat{\tau}(\rho; t), \quad (59)$$

where $\hat{\tau}$ involves only the combinatorial expansion of the irregular conformal blocks (50) and ratios of Barnes

functions, which can be written in terms of Euler's gamma functions. The asymptotics of $\hat{\tau}$ is given by [35]

$$\begin{aligned} \hat{\tau}(\rho; t) = 1 &+ \left(\frac{\theta_t}{2} - \frac{\theta_\infty}{4} + \frac{\theta_\infty(\theta_0^2 - \theta_t^2)}{4\tilde{\sigma}^2} \right) t \\ &+ \frac{(\theta_\infty - \tilde{\sigma})((\tilde{\sigma} + \theta_t)^2 - \theta_0^2)}{8\tilde{\sigma}^2(\tilde{\sigma} - 1)^2} \kappa^{-1} t^{1-\tilde{\sigma}} \\ &+ \frac{(\theta_\infty + \tilde{\sigma})((\tilde{\sigma} - \theta_t)^2 - \theta_0^2)}{8\tilde{\sigma}^2(\tilde{\sigma} + 1)^2} \kappa t^{1+\tilde{\sigma}} + \mathcal{O}(t^2, t^{2\pm 2\Re\tilde{\sigma}}), \end{aligned} \quad (60)$$

where parameter κ is as above. The $\tilde{\sigma}$ appearing in (60) is related to the monodromy parameter by the addition of an even integer $\tilde{\sigma} = \sigma - 2p$, $p \in \mathbb{Z}$. This indeterminacy stems

from the quasiperiodicity of the Nekrasov expansion (45) with respect to σ ,

$$\tau(\vec{\theta}, \sigma, s_V; t) = s_V^{-p} \tau(\vec{\theta}, \sigma - 2p, s_V; t), \quad p \in \mathbb{Z}. \quad (61)$$

This quasiperiodicity will impose a multivaluedness in the monodromy parameters found by solving (42). The non-trivial zeros of τ are those of $\hat{\tau}$, but, to work the asymptotics, we have to make sure that the terms in the expansion (60) are indeed dominant. To that end, it is useful to define the variable $\tilde{\kappa} = \kappa t^\sigma$. Seen as a function of $\tilde{\kappa}$ and t , $\hat{\tau}$ is meromorphic in $\tilde{\kappa}$, and so $\hat{\tau}(\tilde{\kappa}, t_0) = 0$ can be inverted to give $\tilde{\kappa}(\vec{\theta}, \sigma; t_0)$. The quasiperiodicity means that, from one such solution, we can create a series labeled by the integer p ,

$$s_V(\vec{\theta}, \sigma; t_0; p) = Y(\vec{\theta}, \sigma - 2p) t_0^{1-\sigma+2p} X(\vec{\theta}, \sigma - 2p; t_0), \quad p \in \mathbb{Z}, \quad (62)$$

where $Y(\vec{\theta}, \sigma)$ is related to the string of gamma functions in (58),

$$Y(\vec{\theta}, \sigma) = \frac{\Gamma^2(\sigma) \Gamma(\frac{1}{2}(2 - \sigma - \theta_\infty)) \Gamma(\frac{1}{2}(2 - \sigma + \theta_t + \theta_0)) \Gamma(\frac{1}{2}(2 - \sigma + \theta_t - \theta_0))}{\Gamma^2(2 - \sigma) \Gamma(\frac{1}{2}(\sigma - \theta_\infty)) \Gamma(\frac{1}{2}(\sigma + \theta_t + \theta_0)) \Gamma(\frac{1}{2}(\sigma + \theta_t - \theta_0))}, \quad (63)$$

and $X(\vec{\theta}, \sigma; t_0)$ is analytic, obtained by inverting (60). We quote the first three terms, valid if $\Re\sigma > 0$,

$$X(\vec{\theta}, \sigma; t_0) = 1 + \chi_1 t_0 + \chi_2 t_0^2 + \cdots + \chi_n t_0^n + \cdots \quad (64a)$$

with

$$\chi_1 = (\sigma - 1) \frac{\theta_\infty(\theta_0^2 - \theta_t^2)}{\sigma^2(\sigma - 2)^2} \quad (64b)$$

and

$$\begin{aligned} \chi_2 = & \frac{\theta_\infty^2(\theta_0^2 - \theta_t^2)^2}{64} \left(\frac{5}{\sigma^4} - \frac{1}{(\sigma - 2)^4} - \frac{2}{(\sigma - 2)^2} + \frac{2}{\sigma(\sigma - 2)} \right) \\ & - \frac{(\theta_0^2 - \theta_t^2)^2 + 2\theta_\infty^2(\theta_0^2 + \theta_t^2)}{64} \left(\frac{1}{\sigma^2} - \frac{1}{(\sigma - 2)^2} \right) \\ & + \frac{(1 - \theta_\infty^2)((\theta_0 - 1)^2 - \theta_t^2)((\theta_0 + 1)^2 - \theta_t^2)}{128} \left(\frac{1}{(\sigma + 1)^2} - \frac{1}{(\sigma - 3)^2} \right). \end{aligned} \quad (64c)$$

The value of p in (62) will be determined later by the requirement that the quantities have a sensible limit as $t_0 \rightarrow 0$. For the accessory parameter (44), this ambiguity is just the shift on σ by an even integer, which will play no further role. In order to use (44) and find the accessory parameter, we must shift the monodromy parameters by one unit. A simple calculation using (58) yields

$$\tilde{\kappa}(\rho^-; t) = \frac{8\sigma^2(\sigma - 1)^2}{(\sigma - \theta_\infty)((\sigma + \theta_t)^2 - \theta_0^2)t} \tilde{\kappa}(\rho; t). \quad (65)$$

Now, using (44)

$$c_{t_0} = \frac{(\sigma - 1)^2 - (\theta_0 + \theta_t - 1)^2}{4t_0} - \frac{\theta_t - 1}{2} + \frac{d}{dt} \log \hat{\tau}(\rho^-; t_0), \quad (66)$$

and expanding the $\hat{\tau}$ term, we find the asymptotic formula for the accessory parameter

$$t_0 c_{t_0} = k_0 + k_1 t_0 + k_2 t_0^2 + \cdots + k_n t_0^n + \cdots, \quad (67a)$$

with the three first terms in the expansion given by

$$k_0 = \frac{(\sigma - 1)^2 - (\theta_0 + \theta_t - 1)^2}{4}, \quad k_1 = -\frac{\theta_\infty(\sigma(\sigma - 2) - \theta_0^2 + \theta_t^2)}{4\sigma(\sigma - 2)}, \quad (67b)$$

$$k_2 = \frac{1}{32} + \frac{\theta_\infty^2(\theta_0^2 - \theta_t^2)^2}{64} \left(\frac{1}{\sigma^3} - \frac{1}{(\sigma - 2)^3} \right) + \frac{(1 - \theta_\infty^2)(\theta_0^2 - \theta_t^2)^2 + 2\theta_\infty^2(\theta_0^2 + \theta_t^2)}{32\sigma(\sigma - 2)} - \frac{(1 - \theta_\infty^2)((\theta_0 - 1)^2 - \theta_t^2)((\theta_0 + 1)^2 - \theta_t^2)}{32(\sigma + 1)(\sigma - 3)}, \quad (67c)$$

where we assumed $\Re\sigma > 0$. The corresponding expression for $\Re\sigma < 0$ can be obtained by sending $\sigma \rightarrow -\sigma$. Higher order terms can be consistently computed using (45). Although the terms become increasingly complicated, we have the structure where the term k_n is a rational function of the monodromy parameters and is analytic in the single monodromy parameters $\vec{\theta}$. As a function of σ it is meromorphic, with poles at integer values. The structure of the poles at order n is

- (i) poles of order $2n - 1$ and below at $\sigma = 0$ and $\sigma = \pm 2$;
- (ii) single poles at $\sigma = \pm 3, \dots, \pm(n + 1)$ —note that the structure of (67a) for negative σ is illusory, since it is only valid for $\Re\sigma > 0$;
- (iii) analytic at $\sigma = 1$.

This structure mirrors that of the accessory parameter for the (nonconfluent) Heun equation found in [15]. There, the structure was inherited from the corresponding structure of conformal blocks [39]. It seems that irregular conformal blocks display the same traits.

It should be stressed that (42) and (44) are exact relations, even though their usefulness stems from our ability to compute the τ function for Painlevé V efficiently. Miwa's theorem [24] shows that the τ function is analytic in the whole complex plane except at $t = 0$ and $t = \infty$. Thus, the expansion (45) has an infinite radius of convergence, even if it becomes exponentially hard to compute the higher order coefficients in t , due to their combinatorial nature. These limitations should be overcome by the Fredholm determinant formulation of the τ function proposed recently [38], which would be of great help for numerical studies.

At $t = \infty$, the expansion of the Painlevé V τ function is substantially more complicated. No general expansion exists, but formulas for $t \rightarrow \infty$ along specific rays, such as $\arg t = 0, \pi/2, \pi, 3\pi/2$, have been proposed; see [38] for a review as well as the relation between these expansions and the different types of irregular conformal blocks at $c = 1$. In the application of interest in this work, however, the parameter t_0 depends on ω , which will be complex for

the general case, therefore straying from these rays. We hope to study the large frequency asymptotic of the quasinormal modes in the context presented here in future work.

III. SPHEROIDAL HARMONICS

We are interested in solutions of (6) which are regular at both the South and the North poles:

$$y(z) = \begin{cases} z^0(1 + \mathcal{O}(z)), & z \rightarrow 0; \\ (z - t_0)^0(1 + \mathcal{O}(z - 1)), & z \rightarrow t_0; \end{cases} \quad (68)$$

which will place a restriction on the value of λ , allowing only a discrete set as possible values $\lambda_\ell(s, m)$, $\ell \in \mathbb{N}$. Finding these corresponds to the *eigenvalue problem* for the angular equation.

We are going to define the single monodromy parameters

$$\theta_0 = -m - s, \quad \theta_{t_0} = m - s, \quad \theta_\infty = 2s. \quad (69)$$

Upon the change of variables

$$y(z) = (1 + \cos \theta)^{\theta_{t_0}/2} (1 - \cos \theta)^{\theta_0/2} S(\theta), \quad z = -2a\omega(1 - \cos \theta), \quad (70)$$

we bring the differential equation to a canonical confluent Heun form (6), with $\vec{\theta}$ as above and

$$t_0 = -4a\omega, \quad t_0 c_{t_0} = \lambda + 2a\omega s + a^2\omega^2. \quad (71)$$

Given the expansion (67a), it is a matter of direct substitution of the parameters of the spheroidal harmonic equations (69) and (71), using the quantization condition (28):

$$\begin{aligned} \theta_0 &= -m - s, & \theta_{t_0} &= m - s, & \theta_\infty &= 2s, \\ t_0 &= -4a\omega, & \sigma &= -2s + 2j. \end{aligned} \quad (72)$$

The result is

$$\begin{aligned} {}_s\lambda_{\ell,m}(a\omega) &= (\ell - s)(\ell + s + 1) - \frac{2ms^2}{\ell(\ell + 1)}a\omega \\ &+ \left(\frac{2((\ell + 1)^2 - m^2)((\ell + 1)^2 - s^2)^2}{(2\ell + 1)(\ell + 1)^3(2\ell + 3)} - \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell - 1)\ell^3(2\ell + 1)} - 1 \right) a^2\omega^2 + \mathcal{O}(a^3\omega^3), \end{aligned} \quad (73)$$

which can be checked to agree with the literature [40]—see [8] for a thorough review. In order to recover the asymptotics, we chose $j = \ell + s + 1$ in (28). As anticipated in [4], the minimum eigenvalue of ℓ is $|s|$ and the azimuthal momentum is constrained so $|m| \leq \ell$.

IV. CONFORMAL BLOCKS AND THE RADIAL EQUATION

The Nekrasov expansion for some of the Painlevé τ functions has been interpreted in terms of $c = 1$ conformal blocks in [25,41]. The details of the structure stems from the Alday-Gaiotto-Tachikawa conjecture [18,20] and can be checked in the references. For Painlevé VI, the structure of the corresponding τ function is similar to (45), with “instanton sectors” labeled by n , and regular conformal blocks, defined as

$$\begin{aligned} \mathcal{F}\left(\begin{matrix} P_1 & P & P_t \\ P_\infty & P & P_0 \end{matrix}; t\right) &= \langle \Delta_\infty | V_{\Delta_1}(1) \Pi_\Delta V_{\Delta_t}(t) | \Delta_0 \rangle, \\ \Delta_k &= \frac{c-1}{24} + P_k^2, \quad \Delta = \frac{c-1}{24} + P^2, \end{aligned} \quad (74)$$

where $V_{\Delta_i}(z_i)$ are primary vertex operators, acting on the primary state $|\Delta_j\rangle$ and its descendants (the Verma module built on the primary state) with an operator of dimension Δ_i and Π_Δ a projector onto the Verma module generated from $|\Delta\rangle$ (see [42] for details and notation). The conformal blocks are dependent on the Virasoro algebra central charge c , which enters through the Kac-Shapovalov matrix of inner products of descendant states of $|\Delta\rangle$. $\mathcal{F}$ can be seen to have the asymptotic expansion

$$\begin{aligned} \mathcal{F}\left(\begin{matrix} P_1 & P & P_t \\ P_\infty & P & P_0 \end{matrix}; t\right) &= t^{\Delta - \Delta_t - \Delta_0} (1 - t)^{-2(\frac{Q}{2} - iP_t)\frac{Q}{2} - iP_1} \\ &\times \left(1 + \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta - \Delta_4 + \Delta_3)}{2\Delta} t + \mathcal{O}(t^2) \right), \end{aligned} \quad (75)$$

where again $c = 1 + 6Q^2$. The higher order terms in t can be computed either recursively [39] or via the Nekrasov functions [19,20].

The correspondence between $c = 1$ conformal blocks and accessory parameters to Fuchsian ordinary differential

equations have been established in [17]. On the other hand, in [31] it was outlined as a method to compute the same accessory parameters using *semiclassical blocks*. These are obtained in the $c \rightarrow \infty$ limit of the conformal blocks defined above. The analysis at the semiclassical limit is based on the property of exponentiation:

$$\mathcal{F}\left(\begin{matrix} P_1 & P & P_t \\ P_\infty & P & P_0 \end{matrix}; t\right) \simeq \exp\left[\frac{c}{6}\mathcal{F}(\delta_k; \delta; t)\right], \quad (76)$$

where δ_k and δ are obtained from a scaling procedure from Δ_k and Δ . With the parametrization $Q = b + 1/b$, we have

$$\delta_k = \lim_{b \rightarrow 0} b^2 \Delta_k, \quad \delta = \lim_{b \rightarrow 0} b^2 \Delta. \quad (77)$$

It can be checked by applying the Virasoro algebra that the Verma module constructed from the “light” operator $V_{(2,1)}(z)$ with $\Delta_{(2,1)} = -\frac{1}{2} - \frac{3b^2}{4}$ has a null vector at level 2. Requiring that this vector decouples from correlation functions implies the condition

$$\frac{1}{b^2} \frac{\partial^2}{\partial z_0^2} V_{(2,1)}(z_0) + :T(z)V_{(2,1)}(z_0): = 0. \quad (78)$$

When this condition is applied to correlation functions involving primary operators, we find Fuchsian differential equations, essentially due to the fact that the operator product expansion (OPE) between $T(z)$ and primary operators at z_i have no terms diverging faster than $(z - z_i)^{-2}$.

To describe irregular singular points we need to take confluent limits of two primary operators [32], which are associated with Whittaker modules of the Virasoro algebra (see [34] for a review and [38] for the relation between these conformal blocks to the asymptotics of Painlevé V). The confluent limit of the two colliding primary operators generates a nonprimary operator, and in order to derive the corresponding Ward identity related to the null condition we will work with the Feigin-Fuchs representation of Liouville field theory,

$$\begin{aligned} \phi(z_0)\phi(z_1) &= -\frac{1}{2}\log|z_0 - z_1|^2 + :\phi(z_0)\phi(z_1): \\ T(z) &= -:(\partial\phi)^2(z): + Q\partial^2\phi(z), \\ \Delta(:e^{2\alpha\phi(z)}:) &= \alpha(Q - \alpha), \end{aligned} \quad (79)$$

which can be seen to generate a central charge of $c = 1 + 6Q^2$. The confluent primary vertex operator of rank 1, as defined in [33], is given by

$$V_{\alpha,\beta}(z) =: \exp(2\alpha\phi(z) + 2\beta\partial\phi(z)):, \quad (80)$$

and by analogy with primary operators and Verma modules, $V_{\alpha,\beta}(0)$ is associated with a Whittaker module of states. We find for singular terms of the OPE with the stress-energy tensor

$$\begin{aligned} T(z)V_{\alpha,\beta}(z_i) = & -\left(\frac{\alpha}{z-z_i} + \frac{\beta}{(z-z_i)^2}\right)^2 V_{\alpha,\beta}(z_i) + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} V_{\alpha,\beta}(z_i) \\ & + \frac{1}{(z-z_i)^2} \beta \frac{\partial}{\partial \beta} V_{\alpha,\beta}(z_i) + Q\left(\frac{\alpha}{(z-z_i)^2} + \frac{2\beta}{(z-z_i)^3}\right) V_{\alpha,\beta}(z_i) + \text{reg.}, \end{aligned} \quad (81)$$

and then

$$[L_n, V_{\alpha,\beta}(z_i)] = \left[z_i^{n+1} \frac{\partial}{\partial z_i} + (n+1)z_i^n \left(\Delta + \beta \frac{\partial}{\partial \beta} \right) + n(n+1)z_i^{n-1} \beta(Q - \alpha) + \frac{n(n^2-1)}{6} z_i^{n-2} \beta^2 \right] V_{\alpha,\beta}(z_i), \quad (82)$$

where $\Delta = \alpha(Q - \alpha)$. The global conformal Ward identities on the correlation functions of N Whittaker operators follow from the commutation relations $[L_n, V_{\alpha,\beta}(z_i)]$ for $n = -1, 0, 1$:

$$\sum_{i=1}^N \frac{\partial}{\partial z_i} = 0, \quad \sum_{i=1}^N z_i \frac{\partial}{\partial z_i} + \Delta_i + \beta_i \frac{\partial}{\partial \beta_i} = 0, \quad \sum_{i=1}^N z_i^2 \frac{\partial}{\partial z_i} + 2z_i \Delta_i + 2z_i \beta_i \frac{\partial}{\partial \beta_i} + 2\beta_i(Q - \alpha_i) = 0. \quad (83)$$

Note that these expressions reduce to the well-known formulas involving primary operators if we take $\beta_i = 0$.

For the confluent Heun equation (6), the relevant conformal block has three insertions of primary operators and one with a nontrivial Whittaker operator:

$$\langle V_{(2,1)}(z_0) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \Pi_{\Delta} V_{\alpha_3,\beta}(z_3) \rangle = \mathcal{G}_b(z_i; \alpha_i, \Delta, \beta). \quad (84)$$

Using the global conformal Ward identities to solve for $\partial_{z_i} \mathcal{G}_b$ and setting $z_3 = 0$, $z_2 = 1$, and $z_1 = \infty$, we find that the null vector condition is

$$\left(\frac{1}{b^2} \frac{\partial^2}{\partial z_0^2} - \left(\frac{1}{z} + \frac{1}{z-1} \right) \frac{\partial}{\partial z_0} + \frac{\Delta_1}{z_0^2} + \frac{\Delta_2}{(z_0-1)^2} + \frac{\Delta_3 - \Delta_{(2,1)} - \Delta_1 - \Delta_2}{z_0(z_0-1)} + \frac{2\beta(Q-\alpha)}{z_0^3} - \frac{\beta^2}{z_0^4} + \frac{1}{z_0(z_0-1)} \beta \frac{\partial}{\partial \beta} \right) \mathcal{G}_b = 0. \quad (85)$$

The semiclassical limit is obtained through the scaling:

$$\alpha_i = \frac{\eta_i}{b}, \quad \beta = \frac{t_0}{2b}, \quad \Delta = \frac{\delta}{b^2}. \quad (86)$$

In this limit, the three insertions at $0, 1, \infty$ become “heavy” and set the background over which the “light” operator $V_{(2,1)}(z_0)$ will induce fluctuations. Assuming exponentiation, the four-point function (84) should factorize as

$$\langle V_{(2,1)}(z_0) V_{\alpha_1}(\infty) V_{\alpha_2}(1) \Pi_{\Delta} V_{\alpha_3,\beta}(0) \rangle|_{b \rightarrow 0} \simeq \psi(z_0; \delta_i, \delta; t_0) \exp\left(\frac{1}{b^2} \mathcal{B}(\delta_i, \delta; t_0)\right), \quad (87)$$

where $\mathcal{B}$ is the semiclassical confluent conformal block of the first kind, defined by analogy with $\mathcal{F}$ above. Setting $z_0 = t_0/z$, we have for $\tilde{\psi} = z\psi$, as $b \rightarrow 0$,

$$\frac{\partial^2}{\partial z^2} \tilde{\psi} + \left(-\frac{1}{4} + \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-t_0)^2} + \frac{1-\eta_3}{z} + \frac{\delta-\delta_2-\delta_1}{z(z-t_0)} \right) \tilde{\psi} = 0, \quad (88)$$

with

$$\delta_i = \eta_i(1 - \eta_i), \quad i = 1, 2, 3; \quad \delta = \eta_3(1 - \eta_3) + t_0 \frac{\partial}{\partial t_0} \mathcal{B}. \quad (89)$$

The last equality for δ is required by the projection operator Π_Δ .

Equation (88) can be cast in the canonical confluent Heun form (6) if we write it in terms of

$$y(z) = z^{-\eta_1} (z - t_0)^{-\eta_2} \tilde{\psi}(z) \quad (90)$$

and set

$$\eta_1 = \frac{1 - \theta_0}{2}, \quad \eta_2 = \frac{1 - \theta_{t_0}}{2}, \quad \eta_3 = 1 - \frac{\theta_\infty}{2}, \quad (91)$$

$$t_0 c_{t_0} = -t_0 \frac{\partial \mathcal{B}}{\partial t_0} - (\eta_1 + \eta_2 + \eta_3 - 1)(\eta_1 + \eta_2 - \eta_3) = -t_0 \frac{\partial \mathcal{B}}{\partial t_0} + \frac{(\theta_\infty - 1)^2}{4} - \frac{(\theta_0 + \theta_{t_0} - 1)^2}{4}. \quad (92)$$

Comparing with the expression (67a) above, we can compute $\mathcal{B}$:

$$\mathcal{B}(\theta_\infty; \sigma; \theta_t; t_0) = \frac{(\theta_\infty - 1)^2 - (\sigma - 1)^2}{4} \log t_0 - k_1 t_0 - \frac{k_2}{2} t_0^2 - \dots - \frac{k_n}{n} t_0^n + \dots, \quad (93)$$

with k_n given as (67b) and (67c). This expression for the irregular classical conformal block can be confronted with the $b \rightarrow 0$ limit of the confluent conformal block (47),

$$\tilde{\mathcal{B}}(\delta_i, \delta; t) = \lim_{b \rightarrow 0} b^2 \log \mathcal{B}(P_\infty; P; \frac{P_t}{P_0}; t/b), \quad P_i = \frac{\theta_i}{2b}, \quad P = \frac{\sigma - 1}{2b}, \quad (94)$$

which yields

$$\tilde{\mathcal{B}}(\delta_i, \delta; t) = \frac{(\sigma - 1)^2 - \theta_0^2 - \theta_t^2 + 1}{4} \log t - k_1 t - \frac{k_2}{2} t^2 - \dots - \frac{k_n}{n} t^n + \dots, \quad (95)$$

with the difference in the leading term stemming from the different normalization conditions on the Whittaker vector $V_{\alpha, \beta}(0)$. The expansion (47) requires $\langle \Delta | \alpha, \beta \rangle = 1$ in the semiclassical limit, whereas (93) is computed entirely from $c = 1$ blocks.

A. The radial equation

After this long exposition we can turn to the radial equation (4). The Teukolsky master equation for the radial part is

$$\Delta^{-s} \frac{d}{dr} \left(\Delta^{s+1} \frac{dR(r)}{dr} \right) + \left(\frac{K^2(r) - 2is(r - M)K(r)}{\Delta} + 4is\omega r - {}_s\lambda_{\ell, m} - a^2\omega^2 + 2am\omega \right) R(r) = 0, \quad (96)$$

where

$$K(r) = (r^2 + a^2)\omega - am, \quad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-). \quad (97)$$

In order to bring it to our canonical form, let us define

$$\theta_- = s - i \frac{\omega - m\Omega_-}{2\pi T_-}, \quad \theta_+ = s + i \frac{\omega - m\Omega_+}{2\pi T_+}, \quad \theta_\infty = 2s - 4iM\omega, \quad (98)$$

$$2\pi T_\pm = \frac{r_+ - r_-}{4Mr_\pm}, \quad \Omega_\pm = \frac{a}{2Mr_\pm}. \quad (99)$$

By changing variables

$$R(r) = (r - r_-)^{-(\theta_- + s)/2} (r - r_+)^{-(\theta_+ + s)/2} y(r), \quad z = 2i\omega(r - r_-), \quad (100)$$

we arrive at

$$\frac{d^2 y}{dz^2} + \left[\frac{1 - \theta_-}{z} + \frac{1 - \theta_+}{z - z_0} \right] \frac{dy}{dz} + \left[-\frac{1}{4} + \frac{\theta_\infty}{2z} - \frac{z_0 c_{z_0}}{z(z - z_0)} \right] y(z) = 0, \quad (101)$$

with

$$z_0 = 2i(r_+ - r_-)\omega, \quad z_0 c_{z_0} = {}_s\lambda_{\ell, m} + 2s + 2i(1 - 2s)M\omega - is(r_+ - r_-)\omega - (2r_+ + r_-)r_+\omega^2. \quad (102)$$

We note the following relations:

$$\theta_- + \theta_+ + \theta_\infty = 4s, \quad \theta_- + \theta_+ - \theta_\infty = 4i(r_+ + r_-)\omega, \quad (103)$$

The relation between the accessory parameter and the semiclassical confluent conformal block allows us to interpret the single monodromy parameters θ_i as Liouville momenta. The Regge-Okamoto symmetry of the confluent conformal block [38]

$$\mathcal{B}\left(P_\infty; P; \frac{P_t}{P_0}; t\right) = t^{\frac{1}{2}(P_\infty^2 - (P_0 + P_t)^2)} e^{\frac{1}{2}\Pi t} \mathcal{B}\left(P_\infty - 2\Pi; P; \frac{P_t - \Pi}{P_0 - \Pi}; t\right), \quad \Pi = \frac{1}{2}(P_0 + P_t + P_\infty), \quad (104)$$

when applied to the radial equation (99), due to the relations (103), allows for the following association with the Liouville momenta of the primary insertions at $z = 0, t_0$. From the assignment (94) and the relation (103), we have $\Pi = s/b$ and the shifted momenta

$$P_+ \equiv P_t - \Pi = \frac{i}{4\pi b} \frac{\omega - m\Omega_+}{T_+}, \quad P_- \equiv P_0 - \Pi = -\frac{i}{4\pi b} \frac{\omega - m\Omega_-}{T_-}, \quad P_\infty - 2\Pi = -\frac{4iM\omega}{b}, \quad (105)$$

which, just as the analogue in the scalar case in five dimensions [29], has the interpretation of entropy influx at the horizons given a quanta of energy ω and angular momentum m . If we take b to be purely imaginary, these are real numbers for real ω . Note that these expressions make sense only in the $b \rightarrow 0$ limit.

In order to phrase the quantization condition for the radial equation in terms of monodromy data, in terms of $y(z)$ the radial boundary conditions of purely ingoing wave at infinity ($r \rightarrow \infty$) and purely outgoing at the horizon ($r \rightarrow r_+$) are [43]

$$y(z) = \begin{cases} e^{\frac{1}{2}z}(1 + \mathcal{O}(z^{-1})), & z \rightarrow +i\infty; \\ 1 + \mathcal{O}(z - z_0), & z \rightarrow z_0, \end{cases} \quad (106)$$

the relative normalizations can be worked out but are not relevant to this problem. We note that the first condition is rather at $z \rightarrow +i\infty$ due to the relation between it and the radial coordinate $z = 2i\omega(r_+ - r_-)$, assuming $\Re\omega > 0$. Because of the Stokes phenomenon, we can only guarantee that the solution of the radial equation will display the behavior (106) in a sector containing the ray $\arg z = \pi/2$.

We see that the solutions from the first row of the fundamental matrix (10) of the matrix system $y_1(z)$ and $y_2(z)$ satisfying (14) correspond to (non-normalized) Jost functions for the scattering problem: purely ingoing waves at $z = +i\infty$ and outgoing at $z = z_0$. We need some care to translate this to conditions on the connection matrices, because the usual parametrization for the monodromy matrices, given in Sec. II E, assumes that M_∞ is diagonal. The conditions (106) ask that one compares the Frobenius basis at $z = t$ —in which the monodromy matrix M_t is diagonal, with the “Jost” basis at infinity where the boundary conditions for the fundamental matrix are given by (14). This basis is sometimes called “Floquet,” or a path-multiplicative basis [44]. In this basis, the monodromy matrix at infinity is given by (20).

In order to show that this basis does not change under the isomonodromy flow, consider the Schlesinger equations (30). By them, the quantities

$$\text{Tr}(A_0 + A_t) = \hat{\theta}_0 + \hat{\theta}_t, \quad \text{Tr}[\sigma_3(A_0 + A_t)] = -\hat{\theta}_\infty \quad (107)$$

are isomonodromy invariants. Therefore, the diagonal elements of $A_0 + A_t$ are invariant under the flow. These

elements set the asymptotic form of the solution at $z = \infty$ (in $\mathcal{S}_2$) to be (14). Write

$$\Phi_k(z) = \left(1 + \frac{B_1}{z} + \dots\right) z^{B_0} e^{\frac{1}{2}\sigma_3 z} \quad (108)$$

for the solution of (9) near $z = \infty$. The existence of this limit requires that the matrix B_0 is diagonal. The subleading term gives

$$B_0 + \frac{1}{2}[B_1, \sigma_3] = A_0 + A_t. \quad (109)$$

Since the diagonal terms of $[B_1, \sigma_3]$ vanish, the off-diagonal elements of $A_0 + A_t$ alter only the subleading $\mathcal{O}(1/z)$ terms of $\Phi_k(z)$, and then preserve the asymptotic form of the wave function.

The condition that the connection matrix between $z = +i\infty$ and $z = z_0$ is lower triangular can be read from the explicit representation (57),

$$\kappa = \frac{\Gamma^2(1 - \sigma) \Gamma(1 + \frac{1}{2}(\sigma - \theta_\infty)) \Gamma(-\frac{1}{2}(\theta_+ + \theta_- - \sigma)) \Gamma(-\frac{1}{2}(\theta_+ - \theta_- - \sigma))}{\Gamma^2(1 + \sigma) \Gamma(1 - \frac{1}{2}(\sigma - \theta_\infty)) \Gamma(-\frac{1}{2}(\theta_+ + \theta_- + \sigma)) \Gamma(-\frac{1}{2}(\theta_+ - \theta_- + \sigma))}. \quad (110)$$

Comparing to the analogue problem of finding quasinormal modes in Kerr-AdS₅ black holes [15], the usefulness of this expression is somewhat wanting. The lack of natural small parameters makes it difficult to study radial eigenmodes analytically. They can, however, be studied numerically. The isomonodromy method will most likely not be as fast as Leaver’s method [43], but, on the other hand, we have more control on the analyticity of the functions involved. The investigation is under way and will be reported elsewhere.

V. DISCUSSION

In this work we considered the Teukolsky master equation eigenvalue problem tackled by the isomonodromy method. We have seen from a more general perspective the relationship between the ensuing confluent Heun equations and confluent conformal blocks, built on Whittaker modules. The mixture of classical complex analysis, integrable systems (through Riemann-Hilbert problems), and conformal blocks has drawn some attention of late [45,46], and we have found in this paper that the eigenvalue problems for both the angular and the radial equations can be cast in terms of monodromy data and solved by expansions of the Painlevé V τ function.

Using the Painlevé V small isomonodromic time expansion [25,38], we derived expansions for the spheroidal harmonic angular eigenvalue in terms of the frequency. We have verified heuristically the exponentiation property for semiclassical confluent conformal blocks (of the first type as defined in [32]) and used then to rederive the small t expansion of the composite monodromy parameter σ . In turn, this allowed us to interpret the radial equation as the null condition of a composition of two primary operators at the radial positions of the inner and outer horizons and a Whittaker operator seated at radial infinity. Curiously, the primary operators can be seen to have real Liouville momenta, in a “unitary” description of sorts.

Using the relation between the accessory parameter and the zero of the isomonodromic tau function, we have an effective way to compute the monodromy parameters—and

thus the connection matrix—of the solutions of the confluent differential equation (6). This gives an effective algorithm to compute scattering data and solve the eigenvalue problem which is procedural. We are currently investigating methods to efficiently compute the quasinormal modes in the notoriously hard quasiextremal regime ($r_- \rightarrow r_+$). While in all probability the method will not be as fast as existing numerical methods for computing quasinormal modes (see [7]), the analytical properties of monodromy parameters make a precision study amenable, as well as enhancements in precision.

The basic ingredients involved in the analysis are the monodromy parameters and their relation to primary/Whittaker operators of a conformal field theory (CFT). We have found for both the radial and angular equations that these monodromy parameters are associated with CFTs which can be considered unitary, and in the radial equation the Liouville momentum of the operator at the outer horizon is proportional to the entropy intake. The angular eigenvalue condition has again the interpretation of an equilibrium condition between the “angular” and “radial” systems, just as the lore in [15]. It is tempting to try to interpret (105) as small perturbations on a “macroscopic” black hole state—the CFT vacuum in this case—and by composing these perturbations arrive at some macroscopic state corresponding to a different black hole. Given the $c = 1$ interpretation of the process, we can perhaps count the difference in the number of states using known facts about the representation of Virasoro algebra. We leave these as enticing prospects for future work.

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