Parity and CP operations for Majorana neutrinos

Kazuo Fujikawa

Interdisciplinary Theoretical and Mathematical Sciences Program, RIKEN, Wako 351-0198, Japan

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The parity transformation law of the fermion field $\psi(x)$ is usually defined by the “ γ^0 parity” $\psi^P(t, -\vec{x}) = \gamma^0 \psi(t, -\vec{x})$ with eigenvalues ± 1 , while the “ $i\gamma^0$ parity” $\psi^P(t, -\vec{x}) = i\gamma^0 \psi(t, -\vec{x})$ is required for the Majorana fermion. The compatibility issues of these two parity laws arise in generic fermion number violating theories where a general class of Majorana fermions appear. In the case of Majorana neutrinos constructed from chiral neutrinos in an extension of the Standard Model, the Majorana neutrinos can be characterized by CP symmetry although C and P are separately broken. It is then shown that either choice of the parity operation, γ^0 or $i\gamma^0$, in the level of the starting fermions gives rise to the consistent and physically equivalent descriptions of emergent Majorana neutrinos both for Weinberg’s model of neutrinos and for a general class of seesaw models. The mechanism of this equivalence is that the Majorana neutrino constructed from a chiral neutrino, which satisfies the classical Majorana condition $\psi(x) = C\overline{\psi(x)}^T$, allows the phase freedom $\psi(x) = e^{i\alpha} \nu_L(x) + e^{-i\alpha} C\overline{\nu_L(x)}^T$ with $\alpha = 0$ or $\pi/4$. This accounts for the phase coming from the different definitions of parity for $\nu_L(x)$ and ensures the consistent definitions of CP symmetry $(CP)\psi(x)(CP)^\dagger = \pm i\gamma^0 \psi(t, -\vec{x})$.

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I. INTRODUCTION

The parity symmetry, which is a disconnected component of the Lorentz transformation, is defined for the Dirac fermion with real m

$$S = \int d^4x \overline{\psi(x)} [i\gamma^\mu \partial_\mu - m] \psi(x) \quad (1)$$

by the substitution rule [1]

$$\psi(x) \rightarrow \gamma^0 \psi(t, -\vec{x}) = \psi^P(t, -\vec{x}), \quad \psi^P(t, -\vec{x}) \rightarrow \psi(x), \quad (2)$$

which is the symmetry of (1). The above parity symmetry P is understood physically as the mirror symmetry, and thus good parity implies left-right symmetry. The charge conjugation symmetry of the Dirac fermion is defined by the representation theory of the Clifford algebra using $C = i\gamma^2 \gamma^0$ in the convention of [1] by

$$\psi(x) \rightarrow C\overline{\psi(x)}^T = \psi^c(x), \quad \psi^c(x) \rightarrow C\overline{\psi^c(x)} = \psi(x), \quad (3)$$

which is the symmetry of the action (1). The combination of C and P then gives

$$\begin{aligned} \psi(x) &\rightarrow \psi^{cP}(t, -\vec{x}) \equiv (\psi^c)^P(t, -\vec{x}) \\ &= C\overline{\gamma^0 \psi(t, -\vec{x})} = -\gamma^0 \psi^c(t, -\vec{x}), \\ \psi(x) &\rightarrow \psi^{pc}(t, -\vec{x}) \equiv (\psi^P)^c(t, -\vec{x}) \\ &= \gamma^0 C\overline{\psi(t, -\vec{x})} = \gamma^0 \psi^c(t, -\vec{x}), \end{aligned} \quad (4)$$

which is also the symmetry of the action (1). As for the ordering of the operation, we have $C\psi(x)C^\dagger = C\overline{\psi(x)}^T$ and $P\psi(x)P^\dagger = \gamma^0 \psi(t, -\vec{x})$ in a formal operator notation, and thus

$$\begin{aligned} PC\psi(x)C^\dagger P^\dagger &= PC\overline{\psi(x)}^T P^\dagger = C\overline{\gamma^0 \psi(t, -\vec{x})}^T \\ &= -\gamma^0 \psi^c(t, -\vec{x}) = \psi^{cP}(t, -\vec{x}), \\ CP\psi(x)P^\dagger C^\dagger &= C\gamma^0 \psi(t, -\vec{x}) C^\dagger = \gamma^0 C\overline{\psi(t, -\vec{x})}^T \\ &= \gamma^0 \psi^c(t, -\vec{x}) = \psi^{pc}(t, -\vec{x}). \end{aligned} \quad (5)$$

This rule shows that the order of C and P is important when one considers CP , although the action $S = \int d^4x \mathcal{L}$, which is bilinear in the fermion field and is invariant for either way of the combination of C and P if it is invariant for one of the orders. As is well known, the intrinsic parity of the Dirac fermion $\psi^P(t, -\vec{x}) = \gamma^0 \psi(t, -\vec{x})$ and the intrinsic parity of its antifermion $(\psi^c)^P(t, -\vec{x}) = -\gamma^0 \psi^c(t, -\vec{x})$ have opposite

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signatures and it is physically important to define the parity of a composite boson such as the charmonium.

On the other hand, the original Majorana fermion is defined by the same action as (1) but with purely imaginary Dirac gamma matrices γ^μ [2]. Then the Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi(x) = 0 \quad (6)$$

is a real differential equation, and one can impose the reality condition on the solution

$$\psi(x)^* = \psi(x), \quad (7)$$

which implies the self-conjugate under the charge conjugation.¹ The conventional parity transformation $\psi(x) \rightarrow \psi^P(t, -\vec{x}) = \gamma^0 \psi(t, -\vec{x})$ cannot maintain the reality condition (7) for the purely imaginary γ^0 . Thus the “ $i\gamma^0$ parity,”

$$\psi(x) \rightarrow \psi^P(t, -\vec{x}) = i\gamma^0 \psi(t, -\vec{x}), \quad (8)$$

is chosen as a natural parity transformation rule for the Majorana fermion [2,3].

In the generic representation of the Dirac matrices [1], the “ $i\gamma^0$ parity” satisfies the condition

$$i\gamma^0 \psi(t, -\vec{x}) = C \overline{i\gamma^0 \psi(t, -\vec{x})}^T \quad (9)$$

for the field which satisfies the classical Majorana condition

$$\psi(x) = C \overline{\psi(x)}^T, \quad (10)$$

and thus $i\gamma^0$ parity is a natural choice of the parity for the Majorana fermion in this generic representation also. The choice $\psi^P(t, -\vec{x}) = -i\gamma^0 \psi(t, -\vec{x})$ also satisfies the condition (9), but we choose the convention $\psi^P(t, -\vec{x}) = i\gamma^0 \psi(t, -\vec{x})$ in (8) as a primary definition. For consistency, we assign the $i\gamma^0$ -parity convention to charged leptons also when we assign $i\gamma^0$ parity to neutrinos in an extension of the Standard Model, although this extra phase is cancelled in the lepton number conserving terms in the charged lepton sector. See [4] for the phase freedom of parity operation.

The combination of C and P for the Majorana fermion then becomes

$$\begin{aligned} \psi(x) &\rightarrow \psi^{CP}(t, -\vec{x}) = C \overline{\psi^P(t, -\vec{x})} = C \overline{i\gamma^0 \psi(t, -\vec{x})} \\ &= i\gamma^0 \psi^c(t, -\vec{x}), \\ \psi(x) &\rightarrow \psi^{PC}(t, -\vec{x}) = i\gamma^0 C \overline{\psi(t, -\vec{x})} = i\gamma^0 \psi^c(t, -\vec{x}), \end{aligned} \quad (11)$$

which is also the symmetry of the action (1). In the present case, CP operation does not depend on the ordering of C

and P . Also, the parity $\psi^P(t, -\vec{x}) = i\gamma^0 \psi(t, -\vec{x})$ of a fermion and the parity of an antifermion $\psi^{cP}(t, -\vec{x}) = i\gamma^0 \psi^c(t, -\vec{x})$ have symmetric forms, which is natural since we do not distinguish the particle and its antiparticle in the case of the Majorana fermion.

These two choices of parity, γ^0 and $i\gamma^0$, are equivalent for the Dirac fermion with the fermion number $U(1)$ freedom since $i\gamma^0$ is regarded as a composition of γ^0 and $U(1)$ transformations, but their physical equivalence in generic theories with fermion number nonconservation is not obvious. One may rather suspect that the common γ^0 parity is inconsistent in theories where the Majorana fermion appears. In the following sections, we are going to show that these two different choices of the parity, γ^0 and $i\gamma^0$, give the physically equivalent descriptions of Majorana neutrinos using Weinberg’s model [5] in an extension of the Standard Model (SM). Namely, we show that the different definitions of parity in the level of starting fermions in Weinberg’s model lead to the consistent and physically equivalent descriptions of emergent Majorana neutrinos using the CP symmetry to characterize Majorana neutrinos. The same statement applies to a general class of seesaw models of Majorana neutrinos, which is shown in the Appendix. It appears that the conventional γ^0 parity is often used in the phenomenological analyses of the models with Majorana neutrinos [6], while theoretically the use of $i\gamma^0$ parity is natural. An explicit demonstration of the physical equivalence of the two different definitions of parity in the analysis of Majorana neutrinos in an extension of the SM and the explanation of its basic mechanism will thus be practically useful.

Technically, we employ the characterization of Majorana fermions constructed from chiral fermions by the CP symmetry as a basic means to discuss this issue. The characterization of the Majorana fermion such as $\psi(x) = \nu_L(x) + C \overline{\nu_L}^T(x)$ by the CP symmetry has been suggested in [7] since this field has no symmetry under P nor C but can have a good symmetry under the CP symmetry, as will be discussed later. In this sense, our characterization of the Majorana neutrino in an extension of the Standard Model by the CP symmetry is very close to the characterization of the Weyl neutrino by the CP symmetry in SM. It should be noted that our use of the CP symmetry for the Majorana fermion, which is constructed from chiral fermions, is very different from the proposal of the use of CP or ultimately CPT (which is defined as a product of CP and the time reversal T) symmetries in the definition of general Majorana neutrinos by taking into account of the possible symmetry breaking by weak interactions, as was discussed, for example, in [8].

II. WEINBERG’S MODEL OF MASSIVE MAJORANA NEUTRINOS

Weinberg’s model of massive Majorana neutrinos in an extension of the Standard Model is defined in terms of

¹The pure imaginary condition $\psi^*(x) = -\psi(x)$ is also allowed, but we take (7) as the primary definition in the present paper.

chiral fermions [5]. It is known that Weinberg's model of Majorana neutrinos represent the essential aspects of the general models of Majorana neutrinos such as the seesaw models [6,9–11]. We are going to compare the emergent Majorana neutrinos when we use the two different definitions of parity operation for starting fermions in Weinberg's model.

A. The $i\gamma^0$ parity

We define the charge conjugation and parity operations of chiral fermions by the chiral projection of the transformation rules of the Dirac fermion using the operator notation

$$\begin{aligned} C\nu_L(x)C^\dagger &= \overline{C\nu_R(x)}^T, \\ C\nu_R(x)C^\dagger &= \overline{C\nu_L(x)}^T, \\ \mathcal{P}\nu_L(x)\mathcal{P}^\dagger &= i\gamma^0\nu_R(t, -\vec{x}), \\ \mathcal{P}\nu_R(x)\mathcal{P}^\dagger &= i\gamma^0\nu_L(t, -\vec{x}), \\ (\mathcal{PC})\nu_L(x)(\mathcal{PC})^\dagger &= i\gamma^0\overline{C\nu_L(t, -\vec{x})}^T, \\ (\mathcal{PC})\nu_R(x)(\mathcal{PC})^\dagger &= i\gamma^0\overline{C\nu_R(t, -\vec{x})}^T, \end{aligned} \quad (12)$$

where we adopt the $i\gamma^0$ parity $\mathcal{P}\nu(x)\mathcal{P}^\dagger = i\gamma^0\nu(t, -\vec{x})$ for a Dirac fermion with

$$\nu_{R,L}(x) = \left(\frac{1 \pm \gamma_5}{2}\right)\nu(x). \quad (13)$$

The relations (12) fix the notational conventions of the transformation laws of chiral fermions. We are not assuming that the actual massive neutrinos are Dirac fermions, although we use the notations $\nu_{L,R}(x)$ for simplicity. These rules extracted from the Dirac fermion are mathematically consistent and the symmetries of the action

$$\begin{aligned} S = \int d^4x \{ & \overline{\nu_L}(x)i\gamma^\mu\partial_\mu\nu_L(x) + \overline{\nu_R}(x)i\gamma^\mu\partial_\mu\nu_R(x) \\ & - m\overline{\nu_L}(x)\nu_R(x) - m\overline{\nu_R}(x)\nu_L(x) \}. \end{aligned} \quad (14)$$

Physically, parity is defined as the mirror symmetry and one can check if the given Lagrangian is parity preserving or not using these rules. Good P naturally implies left-right symmetry, and P is represented in the form of a doublet representation $\{\nu_R(x), \nu_L(x)\}$. The doublet representation of the charge conjugation C in (12) is related to the absence of the Majorana-Weyl fermion in $d = 4$ dimensions, which is a consequence of the representation theory of the Clifford algebra; intuitively, the absence of the Majorana-Weyl fermion is related to the fact that the charge conjugation inevitably changes the signature of γ_5 in $d = 4$, namely, $\gamma_5 \rightarrow -\gamma_5$ and thus $\nu_{L,R}(x) \rightarrow \overline{C\nu_{R,L}(x)}^T$ as in the first line of (12). If one uses the charge conjugation operation other than (12), one would have a potential danger of spoiling the

condition of the absence of Majorana-Weyl fermions in $d = 4$ [7]. The definitions of C and P transformation rules of chiral fermions (12) are highly unique in this sense. See [12] for the problematic aspects of the “pseudo C symmetry” for the chiral fermion, which is often used in the analysis of seesaw models. See also (63) later.

Weinberg's model of Majorana neutrinos is defined by an effective Hermitian Lagrangian [5]

$$\mathcal{L} = \overline{\nu_L}(x)i\partial\!\!\!/ \nu_L(x) - (1/2)\{\nu_L^T(x)Cm_L\nu_L(x) + \text{H.c.}\} \quad (15)$$

with an arbitrary 3×3 symmetric complex mass matrix m_L ; m_L is symmetric because of the symmetry properties of the matrix C and fermion fields $\nu_L(x)$. This Lagrangian contains only the left-handed chiral components and is not invariant under C nor P in (12). Under the CP transformation in (12), the action defined by $\mathcal{L}$ (15) is transformed as

$$\begin{aligned} \int d^4x \mathcal{L} \rightarrow \int d^4x \{ & \overline{\nu_L}(x)i\partial\!\!\!/ \nu_L(x) \\ & - (1/2)[\nu_L^T(x)Cm_L^\dagger\nu_L(x) + \text{H.c.}] \} \end{aligned} \quad (16)$$

and thus $\mathcal{L}$ is CP invariant (in terms of $i\gamma^0$ parity) if the symmetric mass matrix is real

$$m_L^\dagger = m_L. \quad (17)$$

To be precise, we use this CP invariance condition for the diagonalized mass eigenvalues, while leaving the bare parameter m_L to be general and not necessarily real. After the diagonalization of the symmetric complex mass matrix by the 3×3 Autonne-Takagi factorization using a unitary U [13,14]

$$U^T m_L U = M, \quad (18)$$

with a real positive 3×3 diagonal matrix M , we define

$$\nu_L(x) = U\tilde{\nu}_L(x), \quad (19)$$

and thus transfer the possible CP breaking contained in U to the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix which contains a mixing matrix coming from the charged lepton sector also in an extension of the Standard Model. We then have a Hermitian Lagrangian [suppressing the tilde symbol of $\tilde{\nu}_L(x)$]

$$\begin{aligned} \mathcal{L} &= \overline{\nu_L}(x)i\partial\!\!\!/ \nu_L(x) - (1/2)\{\nu_L^T(x)CM\nu_L(x) + \text{H.c.}\} \\ &= (1/2)\{\bar{\psi}(x)i\partial\!\!\!/ \psi(x) - \bar{\psi}(x)M\psi(x)\} \end{aligned} \quad (20)$$

where we defined

$$\psi(x) \equiv \nu_L(x) + C\overline{\nu_L}^T(x). \quad (21)$$

The field $\psi(x)$ satisfies the classical Majorana condition

$$\psi(x) = C\overline{\psi(x)}^T \quad (22)$$

identically, regardless of the choice of $\nu_L(x)$ in the sense that the condition (22) is satisfied even if one replaces $\nu_L(x)$ by an arbitrary chiral $N_L(x)$. In the present approach where the charge conjugation C is not necessarily a good symmetry, we define the Majorana fermion by (22) together with the Dirac equation $[i\partial - M]\psi(x) = 0$. The condition (22), which is fixed by the analysis of the Clifford algebra, is referred to as the classical Majorana condition in the present paper.

The transformation (19) belongs to a canonical transformation, which preserves the form of the kinetic term in the Lagrangian and thus preserves the canonical anti-commutation relations [15–17]. The well-known Kobayashi-Maskawa analysis is also an example of the use of the canonical transformation [18]. In the canonical transformation, we apply the transformation rules of discrete symmetries in (12) to the new variables every time after the canonical transformation. We apply the discrete symmetries in (12) to the old variables also. Thus the discrete symmetries applied to the old variables do not generate the discrete symmetries of the new variables in general [17]. This procedure is the same as in the Kobayashi-Maskawa analysis. The Lagrangian in (20), which contains only the left-handed chiral fermions, is thus not invariant under C nor P but invariant under the CP transformation since M is real and diagonal. See the CP invariance condition (17) when we define CP in terms of $i\gamma^0$ parity.

In the present case (20), C and P are thus not specified for the field ν_L , but CP symmetry, $(\mathcal{P}C)\nu_L(x)(\mathcal{P}C)^\dagger = i\gamma^0 C\overline{\nu_L(t, -\vec{x})}^T$, in (12) is well defined. The chiral fermion $\nu_L(x)$ appearing in $\psi(x)$, of which mass term (the dimension 5 operator) is generated by a renormalization group flow starting with the massless Weyl fermion in an extension of the Standard Model [5], for example, has a well-defined CP after the mass diagonalization just as the starting massless Weyl fermion in the SM.

We thus naturally characterize the Majorana fermion (21) by the CP symmetry

$$\begin{aligned} (\mathcal{P}C)[\nu_L(x) + C\overline{\nu_L}^T(x)](\mathcal{P}C)^\dagger \\ = i\gamma^0 [C\overline{\nu_L}^T(t, -\vec{x}) + \nu_L(t, -\vec{x})], \end{aligned} \quad (23)$$

namely,

$$(\mathcal{P}C)\psi(x)(\mathcal{P}C)^\dagger = i\gamma^0 C\overline{\psi(t, -\vec{x})}^T = i\gamma^0 \psi(t, -\vec{x}). \quad (24)$$

The first equality in (24) implies the operator relation while the second equality in (24) implies the classical Majorana condition (22), which holds identically in the sense that (22) holds irrespective of the choice of $\nu_L(x)$. Note that the

CP relation $\psi(x) \rightarrow i\gamma^0 \psi(t, -\vec{x})$ in (24), which is an exact symmetry of (20), preserves the classical Majorana identity (22)

$$i\gamma^0 \psi(t, -\vec{x}) = C i\gamma^0 \overline{\psi(t, -\vec{x})}^T, \quad (25)$$

namely, the CP transformation and the classical Majorana condition are consistent. This is an analog of the crucial consistency of parity operation in (7) and (8).

The chiral component $\nu_L(x)$ of $\psi(x)$ describes the weak interaction in an extension of the Standard Model

$$\int d^4x [(g/\sqrt{2})\bar{l}_L(x)\gamma^\mu W_\mu(x)U_{\text{PMNS}}\nu_L(x) + \text{H.c.}] \quad (26)$$

perfectly well, since the conventional C and P are broken in the parity-violating weak interaction and thus the specification of C and P for $\nu_L(x)$ in (26) is not required, analogously to the Weyl neutrino in the SM. In the present formulation, we use the transformation rules of C , P , and CP in (12) for the chiral components of both charged leptons and neutrinos in a uniform manner, although some of them are broken. We recall that $i\gamma^0$ -parity rules are applied to charged leptons also when we define $i\gamma^0$ parity for neutrinos.

The CP symmetry breaking is described by the PMNS matrix U_{PMNS} when combined with the CP symmetry properties of the charged lepton $l_L(x)$ and the chiral component $\nu_L(x)$ of the neutrino in (26). The absence of the $U(1)$ phase freedom of $\nu_L(x)$ in (20), namely, the lepton number nonconservation, is important to count an increase in the number of possible CP violating phases in U_{PMNS} ; for example, a model with two generations of leptons can have CP violation [19]. The entire weak interaction is thus described by the chiral component $\nu_L(x)$ using its CP property. The Majorana neutrinos characterized by the CP symmetry retain certain information of their original chiral contents; for example, the parity by itself is not specified in the present characterization of Majorana neutrinos (the parity is not specified but the CP parity is specified). The fact that the neutrino is a chiral projection of a Majorana fermion is assured by $\nu_L(x) = [(1 - \gamma_5)/2]\psi(x)$ and (24) which contains the classical identity (22).

If one wishes to define the parity and charge conjugation operators valid for the Majorana neutrinos in the present formulation, it is possible to define a deformed symmetry generated by [7,17]

$$C_M = 1, \quad \mathcal{P}_M = \mathcal{P}C, \quad (27)$$

which is the good symmetry of (20), and the Majorana field $\psi(x)$ is transformed as

$$C_M \psi(x) C_M^\dagger = \psi(x), \quad \mathcal{P}_M \psi(x) \mathcal{P}_M^\dagger = i\gamma^0 \psi(t, -\vec{x}). \quad (28)$$

The nontrivial part of this deformation is the CP symmetry and, in this sense, this deformation is essentially equivalent to the formulation of the Majorana neutrinos with the CP symmetry described above. The classical Majorana condition $\psi(x) = C\overline{\psi(x)}^T$, which determines whether a given fermion is a Majorana fermion or not, carries the same physical information as the trivial operation $C_M\psi(x)C_M^\dagger = \psi(x)$ applied to the fermion $\psi(x)$, which is assumed to be a Majorana fermion $\psi(x) = C\overline{\psi(x)}^T$.

The formulation (28) leads to a formal enhancement of discrete symmetries in (20) by assigning C and P to the chiral component $\nu_L(x) = (\frac{1-\gamma_5}{2})\psi(x)$,

$$\begin{aligned} C_M\nu_L(x)C_M^\dagger &= \nu_L(x) = C\overline{\nu_L(x)}^T, \\ \mathcal{P}_M\nu_L(x)\mathcal{P}_M^\dagger &= i\gamma^0 C\overline{\nu_L(t, -\vec{x})}^T = i\gamma^0\nu_R(t, -\vec{x}), \end{aligned} \quad (29)$$

where we defined $\nu_R(x) \equiv (\frac{1+\gamma_5}{2})\psi(x) = C\overline{\nu_L(x)}^T$, and

$$C_M\nu_R(x)C_M^\dagger = C\overline{\nu_L(x)}^T, \quad \mathcal{P}_M\nu_R(x)\mathcal{P}_M^\dagger = i\gamma^0\nu_L(t, -\vec{x}). \quad (30)$$

These transformation rules are mathematically consistent and imply perfect left-right symmetry expected for a Majorana fermion $\psi = \nu_L + \nu_R$.

These rules (29) and (30) may be compared to the rules in (12). The physical degrees of freedom of a Majorana fermion $\psi = \nu_L + \nu_R$ are the same as either a left-handed chiral fermion or a right-handed chiral fermion but not both, contrary to the case of a Dirac fermion in (12) where the left- and right-handed components are independent. If one measures the left-handed projection of the Majorana fermion ψ , for example, one obtains the chiral freedom ν_L and simultaneously the information of ν_R also. Physically, this introduction of C_M and $\mathcal{P}_M$ does not add new ingredients to the analysis of weak interactions [7].

B. The γ^0 parity

We now examine the same Weinberg's model using the γ^0 -parity operation of chiral fermions defined by the chiral projection of the transformation rules of the Dirac fermion, which are written in the operator notation

$$\begin{aligned} C\nu_L(x)C^\dagger &= C\overline{\nu_R(x)}^T, \\ C\nu_R(x)C^\dagger &= C\overline{\nu_L(x)}^T, \\ \mathcal{P}\nu_L(x)\mathcal{P}^\dagger &= \gamma^0\nu_R(t, -\vec{x}), \\ \mathcal{P}\nu_R(x)\mathcal{P}^\dagger &= \gamma^0\nu_L(t, -\vec{x}), \\ (\mathcal{P}C)\nu_L(x)(\mathcal{P}C)^\dagger &= -\gamma^0 C\overline{\nu_L(t, -\vec{x})}^T, \\ (\mathcal{P}C)\nu_R(x)(\mathcal{P}C)^\dagger &= -\gamma^0 C\overline{\nu_R(t, -\vec{x})}^T, \end{aligned} \quad (31)$$

where

$$\nu_{R,L}(x) = \left(\frac{1 \pm \gamma_5}{2}\right)\nu(x) \quad (32)$$

instead of (12). Note the appearance of the minus sign in the last two relations in (31); namely, we adopt the convention of $\psi^{CP}(t, -\vec{x})$ in (4). These rules extracted from the Dirac fermion are mathematically consistent and are symmetries of the action (14).

Physically, parity is defined as the mirror symmetry. Good P naturally implies left-right symmetry, and P is represented in the form of a doublet representation $\{\nu_R(x), \nu_L(x)\}$. The doublet representation of the charge conjugation C in (31) is the same as the charge conjugation C in (12) and it is related to the absence of the Majorana-Weyl fermion in $d = 4$ [7]. See [12] for the problematic aspects of the “pseudo C symmetry” for the chiral fermion, which is often used in the analysis of seesaw models. See also (63) later.

Weinberg's model of Majorana neutrinos is defined by the same effective Hermitian Lagrangian as (20)

$$\begin{aligned} \mathcal{L} &= \overline{\nu_L}(x)i\not\partial\nu_L(x) - (1/2)\{\nu_L^T(x)Cm_L\nu_L(x) \\ &\quad + \overline{\nu_L(x)}Cm_L^\dagger\overline{\nu_L(x)}^T\}, \end{aligned} \quad (33)$$

with an arbitrary 3×3 symmetric complex mass matrix m_L . We want to show that the same form of the starting Lagrangian when analyzed with the γ^0 parity leads to the logically consistent and physically equivalent predictions as with the $i\gamma^0$ parity.

This Lagrangian (33) contains only the left-handed chiral components and is not invariant under C nor P in (31). Under the CP transformation in (31), the action defined by $\mathcal{L}$ (33) is transformed to

$$\begin{aligned} \int d^4x \{ &\overline{\nu_L}(x)i\not\partial\nu_L(x) - (1/2)[-\nu_L^T(x)Cm_L^\dagger\nu_L(x) \\ &- \overline{\nu_L(x)}Cm_L\nu_L(x)^T]\}, \end{aligned} \quad (34)$$

and thus $\mathcal{L}$ is CP invariant if the symmetric mass matrix is pure imaginary

$$m_L^\dagger = -m_L. \quad (35)$$

To be precise, we use this CP invariance condition for the diagonalized mass eigenvalues, while leaving the bare parameter m_L to be general and not necessarily pure imaginary. We diagonalize the symmetric complex mass matrix by the 3×3 Autonne-Takagi factorization using a unitary U' [13,14]

$$(U')^T m_L U' = iM, \quad (36)$$

with a real and positive 3×3 diagonal matrix M to ensure that the Lagrangian after mass diagonalization is CP invariant in accord with the condition (35). We then define

$$\nu_L(x) = U' \tilde{\nu}_L(x). \quad (37)$$

Note that the unitary U' in (36) is related to U in (18) by

$$U' = U e^{i\pi/4} \quad (38)$$

for a given m_L . Note that the Autonne-Takagi factorization is very different from the conventional diagonalization of a Hermitian matrix by a unitary transformation; the Autonne-Takagi factorization basically gives rise to characteristic values but the phase freedom of the diagonal elements is still left free, and the specific phase convention in (36) is chosen to define the CP invariant Lagrangian (using γ^0 parity) after the mass diagonalization.

After the mass diagonalization described above, we have a Hermitian Lagrangian [suppressing the tilde symbol of $\tilde{\nu}_L(x)$]

$$\begin{aligned} \mathcal{L} &= \bar{\nu}_L(x) i \not{\partial} \nu_L(x) - (i/2) \{ \nu_L^T(x) C M \nu_L(x) \\ &\quad - \overline{\nu_L(x)} C M \overline{\nu_L(x)}^T \}, \\ &= (1/2) \{ \bar{\psi}(x) i \not{\partial} \psi(x) - \bar{\psi}(x) M \psi(x) \}, \end{aligned} \quad (39)$$

where we defined

$$\psi(x) \equiv e^{i\pi/4} \nu_L(x) + e^{-i\pi/4} C \overline{\nu_L(x)}^T. \quad (40)$$

The Hermitian Lagrangian (39) is confirmed to be invariant under CP transformation in (31) since real symmetric M is chosen to satisfy (35), and the effects of possible CP breaking contained in U' are transferred to the PMNS mixing matrix which contains a mixing matrix coming from the charged lepton sector also in an extension of the Standard Model. The field $\psi(x)$ in (40) satisfies the classical Majorana condition,

$$C \overline{\psi(x)}^T = \psi(x), \quad (41)$$

identically regardless of the choice of $\nu_L(x)$.

The transformation (37) belongs to a canonical transformation that preserves the form of the kinetic term in the Lagrangian and thus preserves the canonical anticommutation relations. In the canonical transformation, we apply the transformation rules of discrete symmetries in (31) to the new variables every time after the canonical transformation [15–17]. The Hermitian Lagrangian (39) is thus not invariant under C nor P in (31) but invariant under the CP transformation since M is real and symmetric, as we have already noted; C and P in (31) are not assigned to the field ν_L , but CP symmetry $(\mathcal{P}C)\nu_L(x)(\mathcal{P}C)^\dagger = -\gamma^0 C \overline{\nu_L(t, -\vec{x})}^T$ in (31) is well defined. We thus naturally characterize the Majorana fermion (40) by the CP symmetry

$$\begin{aligned} &(\mathcal{P}C)(e^{i\pi/4} \nu_L(x) + e^{-i\pi/4} C \overline{\nu_L(x)}^T)(\mathcal{P}C)^\dagger \\ &= -i\gamma^0 [e^{-i\pi/4} C \overline{\nu_L(t, -\vec{x})}^T + e^{i\pi/4} \nu_L(t, -\vec{x})], \end{aligned} \quad (42)$$

namely,

$$(\mathcal{P}C)\psi(x)(\mathcal{P}C)^\dagger = -i\gamma^0 C \overline{\psi(t, -\vec{x})}^T = -i\gamma^0 \psi(t, -\vec{x}). \quad (43)$$

The first equality in (43) implies the operator relation while the second equality in (43) implies the classical Majorana condition (41), which holds identically in the sense that (41) holds irrespective of the choice of $\nu_L(x)$.

Note that the operator relation in (43) has a signature opposite to the relation $(\mathcal{P}C)\psi(x)(\mathcal{P}C)^\dagger = i\gamma^0 \psi(t, -\vec{x})$ in (24), but it preserves the classical Majorana identity (41)

$$-i\gamma^0 \psi(t, -\vec{x}) = C \overline{-i\gamma^0 \psi(t, -\vec{x})}^T, \quad (44)$$

after the CP transformation. The CP symmetry and the classical Majorana condition are consistent. When one defines the fermions $\psi(x)$ in terms of chiral fermions, which satisfy the classical Majorana condition $\psi(x) = C \overline{\psi(x)}^T$ identically, and if the action for the chiral fermions which determine Majorana fermions is invariant under the CP symmetry as in (20) and (39), the consistency condition is always satisfied: Namely, the CP transform of Majorana fermions $\psi(x)$ satisfy the classical Majorana condition in the form either (25) or (44).

The chiral component of the neutrino $\nu_L(x)$ describes the weak interaction in an extension of the Standard Model, which was originally defined in terms of the left-handed gauge eigenstates before the diagonalization of the neutrino masses,

$$\begin{aligned} &\int d^4x [(g/\sqrt{2}) \bar{l}_L(x) \gamma^\mu W_\mu(x) U'_{\text{PMNS}} \nu_L(x) + \text{H.c.}] \\ &= \int d^4x [(g/\sqrt{2}) \bar{l}_L(x) \gamma^\mu W_\mu(x) U_{\text{PMNS}} \hat{\nu}_L(x) + \text{H.c.}], \end{aligned} \quad (45)$$

where we introduced an auxiliary chiral variable defined by

$$\hat{\nu}_L(x) \equiv e^{i\pi/4} \nu_L(x), \quad (46)$$

for which the Majorana neutrino (40) is written as

$$\psi(x) = \hat{\nu}_L(x) + C \overline{\hat{\nu}_L(x)}^T, \quad (47)$$

and many equations below are simplified. The weak mixing matrix U'_{PMNS} is defined by U' in (38), and thus $U'_{\text{PMNS}} = U_{\text{PMNS}} e^{i\pi/4}$, which leads to $U'_{\text{PMNS}} \nu_L(x) = U_{\text{PMNS}} \hat{\nu}_L(x)$ in (45). The mixing matrix U_{PMNS} is defined in (26). The C and P symmetries are not defined for $\hat{\nu}_L(x)$ without the right-handed components, but these

symmetries are broken in the chiral weak interactions and thus need not be specified.

The CP transformation law is

$$\begin{aligned}\hat{\nu}_L(x) &= \frac{(1 - \gamma_5)}{2} \psi(x) \rightarrow \frac{(1 - \gamma_5)}{2} (-i\gamma^0) \psi(t, -\vec{x}) \\ &= -i\gamma^0 \hat{C} \bar{\nu}_L^T(t, -\vec{x}),\end{aligned}\quad (48)$$

which contains an extra $U(1)$ phase $i = e^{i\pi/2}$ compared to the CP transformation law of $\nu_L(x)$ in (31). This CP transformation law is also directly obtained from $\nu_L(x) \rightarrow -\gamma^0 \hat{C} \bar{\nu}_L^T(t, -\vec{x})$ in (31) and the definition $\nu_L(x) = e^{-i\pi/4} \hat{\nu}_L(x)$. In accord with the transformation (48), it is convenient to formally assign the CP transformation law to the charged lepton,

$$l_L(x) \rightarrow -i\gamma^0 \hat{C} \bar{l}_L^T(t, -\vec{x}), \quad (49)$$

instead of the original, $l_L(x) \rightarrow -\gamma^0 \bar{C} l_L^T(t, -\vec{x})$, in the analysis of CP breaking using the pair of fields $\{l_L(x), \hat{\nu}_L(x)\}$. This formal replacement of CP law for the charged lepton does not change the CP analysis by (45), since the fermion number preserving charged-lepton sector after the mass diagonalization is invariant even when the extra overall $U(1)$ phase i is added to the transformation law. Also, (45) is invariant under the CP laws thus defined if one sets $U_{\text{PMNS}} = 1$, and thus the modified CP laws are consistent to analyze the CP breaking induced by U_{PMNS} .

In this setting, the analysis of CP symmetry breaking is the same as the CP analysis with $i\gamma^0$ parity (26) except for the replacement $i\gamma^0 \rightarrow -i\gamma^0$ in the CP transformation laws, which does not change physics. The absence of the fermion number symmetry in the neutrino sector (39) accounts for the possible increase of the CP -violating phases for the Majorana neutrinos [19].

Alternatively, one may consider the replacement in (45)

$$\begin{aligned}&\int d^4x [(g/\sqrt{2}) \bar{l}_L(x) \gamma^\mu W_\mu(x) U'_{\text{PMNS}} \nu_L(x) + \text{H.c.}] \\ &\rightarrow \int d^4x [(g/\sqrt{2}) \bar{l}_L(x) \gamma^\mu W_\mu(x) U_{\text{PMNS}} \nu_L(x) + \text{H.c.}]\end{aligned}\quad (50)$$

The chiral fermion $\nu_L(x)$ together with $U'_{\text{PMNS}} = U_{\text{PMNS}} e^{i\pi/4}$ describe weak interactions. But the overall $U(1)$ phase $e^{i\pi/4}$ in U'_{PMNS} is absorbed in the redefinition of the charged lepton fields, $\bar{l}_L(x) e^{i\pi/4} \rightarrow \bar{l}_L(x)$, and we obtain the last expression of (50) in the analysis of CP breaking. In this setting, we use the transformation rules of C , P (γ^0 parity) and CP for the chiral components of charged leptons and neutrinos in a uniform manner, in particular, the CP transformation

$$l_L(x) \rightarrow -\gamma^0 \bar{C} l_L^T(t, -\vec{x}), \quad \nu_L(x) \rightarrow -\gamma^0 \bar{\nu}_L^T(t, -\vec{x}), \quad (51)$$

although some of them are broken. The absence of the fermion number symmetry in the neutrino sector (39) accounts the possible increase of the CP -violating phases with the Majorana neutrinos [19]. This formulation (50) appears to be a natural one in the actual analysis of the CP symmetry with the γ^0 parity. Physically, this formulation is equivalent to the case of the $i\gamma^0$ -parity convention in (26) in the analysis of CP breaking induced by U_{PMNS} .

The entire weak interaction is thus described by the chiral component $\nu_L(x)$ using its CP property defined by the γ^0 parity. The Majorana neutrinos characterized by the CP symmetry retain certain information from their original chiral contents; for example, the parity by itself is not specified in the present characterization of Majorana neutrinos (P parity is not specified, but CP parity is specified instead). The fact that the neutrino is a chiral projection of the Majorana fermion is assured by $\hat{\nu}_L(x) = [(1 - \gamma_5)/2] \psi(x) = e^{i\pi/4} \nu_L(x)$ and (43), which contains the classical Majorana identity (41).

If one wishes to define the parity and charge conjugation operators valid for the Majorana neutrino (40) in the present formulation, it is possible to define a deformed symmetry generated by [7,17]

$$C_M = 1, \quad \mathcal{P}_M = \mathcal{P}C, \quad (52)$$

which is a symmetry of (39) and

$$C_M \psi(x) C_M^\dagger = \psi(x), \quad \mathcal{P}_M \psi(x) \mathcal{P}_M^\dagger = -i\gamma^0 \psi(t, -\vec{x}). \quad (53)$$

The nontrivial part of this deformation is the CP symmetry; in this sense, this deformation is essentially equivalent to the formulation of the Majorana neutrino with $\mathcal{P}C = \mathcal{P}_M C_M$ described above. The classical Majorana condition $\psi(x) = \bar{C} \psi(x)^T$, which determines if a given fermion is a Majorana fermion or not, carries the same physical information as the trivial operation $C_M \psi(x) C_M^\dagger = \psi(x)$ applied to the fermion $\psi(x)$ which is assumed to be the Majorana fermion $\psi(x) = \bar{C} \psi(x)^T$.

The formulation (53) with a deformed symmetry leads to a formal enhancement of discrete symmetries in (39) by assigning C and P to the chiral component $\hat{\nu}_L(x) = \frac{(1 - \gamma_5)}{2} \psi(x) = e^{i\pi/4} \nu_L(x)$ in (47),

$$\begin{aligned}C_M \hat{\nu}_L(x) C_M^\dagger &= \hat{\nu}_L(x) = \bar{C} \bar{\nu}_R^T(x), \\ \mathcal{P}_M \hat{\nu}_L(x) \mathcal{P}_M^\dagger &= -i\gamma^0 \hat{\nu}_R(t, -\vec{x}),\end{aligned}\quad (54)$$

where we defined $\hat{\nu}_R(x) \equiv \frac{(1 + \gamma_5)}{2} \psi(x) = \bar{C} \bar{\nu}_L^T(x)$, and

$$\mathcal{C}_M \hat{\nu}_R(x) \mathcal{C}_M^\dagger = \overline{\mathcal{C} \hat{\nu}_L}^T(x), \quad \mathcal{P}_M \hat{\nu}_R(x) \mathcal{P}_M^\dagger = -i\gamma^0 \hat{\nu}_L(t, -\vec{x}). \quad (55)$$

These transformation rules are mathematically consistent and imply perfect left-right symmetry expected for a Majorana fermion $\psi = \hat{\nu}_L + \hat{\nu}_R$.

These rules, (54) and (55), may be compared to the rules in (12); $i\gamma^0$ in (12) is replaced by $-i\gamma^0$ in (54) and (55), but they are physically equivalent. The physical degrees of freedom of a Majorana fermion $\psi = \hat{\nu}_L + \hat{\nu}_R$ are the same as either a left-handed chiral fermion or a right-handed chiral fermion but not both, which is contrary to the case of a Dirac fermion in (12) where the left- and right-handed components are independent. If one measures the left-handed projection of the Majorana fermion ψ , for example, one obtains the chiral freedom $\hat{\nu}_L$ and simultaneously the information of $\hat{\nu}_R$ also. Physically, this introduction of $\mathcal{C}_M$ and $\mathcal{P}_M$ does not add new ingredients to the analysis of weak interactions [7].

III. DISCUSSION AND CONCLUSION

We have demonstrated the physical equivalence of γ^0 parity and $i\gamma^0$ parity in the description of emergent Majorana neutrinos formed from chiral fermions using Weinberg's model of neutrinos in an extension of the Standard Model. In this analysis, the characterization of the emergent Majorana neutrinos by the CP symmetry plays an important role. We can describe the model for two different definitions of parity by the same physical parameters M and U_{PMNS} , which are fixed by experiments, using the chiral fermion $\nu_L(x)$ or $\hat{\nu}_L(x)$, which is defined as a chiral component such as $\hat{\nu}_L(x) = [(1 - \gamma_5)/2]\psi(x)$ of the Majorana field $\psi(x)$ in (47). The analysis of CP breaking using the two different definitions of parity is thus equivalent. It is known that Weinberg's model of Majorana neutrinos represent the essential aspects of the general models of Majorana neutrinos such as the seesaw models [6,9–11], where two classes of massive Majorana neutrinos generally appear and both of them may be characterized by the CP symmetry. Our analysis thus justifies the use of either γ^0 parity or $i\gamma^0$ parity in the analysis of the Majorana neutrinos in an extension of the SM. In the Appendix, an explicit analysis of seesaw models is presented.

Theoretically, the two different parity operations are equivalent in the Standard Model where the fermion number is conserved. The Standard Model deformed by the lepton number violating neutrino mass term is still characterized by the discrete symmetries C , P , and CP in (12) or (31) after the diagonalization of the mass term by a canonical transformation. Neither C nor P is a good symmetry, but the combined CP symmetry can be always chosen to be a good symmetry after the mass diagonalization using the specific property of the Autonne-Takagi factorization. The CP transformation of the emergent

Majorana fermions thus defined is consistent with the classical Majorana condition, as in (25) and (44). The definition of Majorana fermions formed from chiral fermions by the CP symmetry works for either form of parity operation by resolving the strictures implied by the original Majorana conditions on the parity operation in (7) and (8).

It is also possible to reformulate the Majorana fermions defined by the CP symmetry using a formally deformed symmetry formed by $\mathcal{C}_M$ and $\mathcal{P}_M$ in (27) or in (52), and it illustrates some of the naively expected properties of the Majorana fermion and its chiral projections, as in (28) and (53). It is assuring that the “parity” represented by $\mathcal{P}_M$ is always the $\pm i\gamma^0$ parity to be consistent with the classical Majorana condition, irrespective of the definitions of parity for the starting fermions. It has been noted, however, that this formulation with the deformed symmetry does not add new physical ingredients beyond the original definition of Majorana neutrinos by the CP symmetry [7].

In connection with the deformed symmetry generated by $\mathcal{C}_M$ and $\mathcal{P}_M$ in (28), one may consider another approach that may presumably be more common in neutrino physics including the seesaw models [6,9–11]. One may start with a simple diagonalization of the given Lagrangian (15) without asking C and P properties of the chiral fermion $\nu_L(x)$ by obtaining the Lagrangian (20),

$$\begin{aligned} \mathcal{L} &= \overline{\nu_L}(x) i \not{\partial} \nu_L(x) - (1/2) \{ \nu_L^T(x) C M \nu_L(x) + \text{H.c.} \}, \\ &= (1/2) \{ \bar{\psi}(x) i \not{\partial} \psi(x) - \bar{\psi}(x) M \psi(x) \}, \end{aligned} \quad (56)$$

with

$$\psi(x) = \nu_L(x) + C \overline{\nu_L}^T(x), \quad (57)$$

which satisfies the classical Majorana condition $\psi(x) = C \overline{\psi(x)}^T$ identically. It is known that the essence of various seesaw models is covered by the above model (56). After the considerations of several physical consistency conditions in [11], for example, one arrives at the CP transformation law

$$(\tilde{\mathcal{P}} \tilde{\mathcal{C}}) \psi(x) (\tilde{\mathcal{P}} \tilde{\mathcal{C}})^\dagger = i\gamma^0 \psi(t, -\vec{x}), \quad (58)$$

which agrees with our identification (24). In the common treatment of Majorana neutrinos in seesaw models in an extension of the Standard Model, it is customary to introduce a new charge conjugation law for Majorana neutrinos in (57) [6,9–11]

$$\tilde{\mathcal{C}} \psi(x) \tilde{\mathcal{C}}^\dagger = \psi(x), \quad (59)$$

with

$$\tilde{\mathcal{C}} \nu_L(x) \tilde{\mathcal{C}}^\dagger = C \overline{\nu_L(x)}^T, \quad (60)$$

which has been named as the pseudo C symmetry in [12]. This rule may be compared to the standard C symmetry $C\nu_L(x)C^\dagger = \overline{C\nu_R(x)}^T$ in (12). It is also possible to define the pseudo P symmetry by [7]

$$\tilde{P}\psi(x)\tilde{P}^\dagger = i\gamma^0\psi(t, -\vec{x}), \quad (61)$$

with

$$\tilde{P}\nu_L(x)\tilde{P}^\dagger = i\gamma^0\nu_L(t, -\vec{x}), \quad (62)$$

which may be compared to the standard parity $\mathcal{P}\nu_L(x)\mathcal{P}^\dagger = i\gamma^0\nu_R(t, -\vec{x})$ in (12). One can confirm that the CP operator defined by $\tilde{P}\tilde{C}$ generates the desired CP transformation in (58) and $\nu_L(x) \rightarrow i\gamma^0\tilde{C}\nu_L(t, -\vec{x})^T$. It may appear that $\tilde{C}$ and $\tilde{P}$ thus defined provide alternatives to $\mathcal{C}_M$ and $\mathcal{P}_M$ in (28).

A drawback to the pseudo C and P operators defined above is that they are not operatorially consistent by noting $\nu_L(x) = (\frac{1-\gamma_5}{2})\nu_L(x)$ [7,12,20],

$$\begin{aligned} \tilde{C}\nu_L(x)\tilde{C}^\dagger &= \left(\frac{1-\gamma_5}{2}\right)\tilde{C}\nu_L(x)\tilde{C}^\dagger \\ &= \left(\frac{1-\gamma_5}{2}\right)\overline{C\nu_L(x)}^T = 0, \\ \tilde{P}\nu_L(x)\tilde{P}^\dagger &= \left(\frac{1-\gamma_5}{2}\right)\tilde{P}\nu_L(x)\tilde{P}^\dagger \\ &= \left(\frac{1-\gamma_5}{2}\right)i\gamma^0\nu_L(t, -\vec{x}) = 0, \end{aligned} \quad (63)$$

since $\overline{C\nu_L(x)}^T$ and $i\gamma^0\nu_L(t, -\vec{x})$ are both right handed. The pseudo C and P symmetries cannot be substitutes for $\mathcal{C}_M$ and $\mathcal{P}_M$ in (28). It has been emphasized in [7] that essentially the entire physics of Majorana neutrinos in an extension of the SM is described by the CP symmetry without referring to C and P separately and thus the above operatorial inconsistency of pseudo C and P symmetry operators has not been recognized in the past practical analyses of Majorana neutrinos [6,9–11], as long as the basic CP transformation law (58) is correctly chosen.

Finally, we make a brief comment on the parity operation for the “elementary” Majorana fermion in (7) and (8), which motivated us to define the $i\gamma^0$ parity but has not played a direct role in the present analysis. It is instructive to recall the construction of two Majorana fermions with a degenerate mass from a Dirac fermion to understand the role of parity in Majorana fermions. One may start with

$$\begin{aligned} \mathcal{L} &= \int d^4x \overline{\psi_D(x)} [i\gamma^\mu \partial_\mu - m] \psi_D(x), \\ &= \int d^4x \overline{\psi_+(x)} [i\gamma^\mu \partial_\mu - m] \psi_+(x) \\ &\quad + \int d^4x \overline{\psi_-(x)} [i\gamma^\mu \partial_\mu - m] \psi_-(x), \end{aligned} \quad (64)$$

with

$$\begin{aligned} \psi_+ &= \frac{1}{\sqrt{2}} [\psi_D(x) + \overline{C\psi_D(x)}^T], \\ \psi_- &= \frac{1}{\sqrt{2}} [\psi_D(x) - \overline{C\psi_D(x)}^T], \end{aligned} \quad (65)$$

which satisfy the classical Majorana condition

$$\overline{C\psi_+(x)}^T = \psi_+(x), \quad \overline{C\psi_-(x)}^T = -\psi_-(x), \quad (66)$$

and the charge conjugation

$$C\psi_+(x)C^\dagger = \psi_+(x), \quad C\psi_-(x)C^\dagger = -\psi_-(x), \quad (67)$$

with $C\psi_D(x)C^\dagger = \overline{C\psi_D(x)}^T = \psi_D^c(x)$ and $C\psi_D^c(x)C^\dagger = \psi_D(x)$. The classical Majorana condition and the charge conjugation condition agree in this special case. Under the charge conjugation, the action (64) is invariant.

If one adopts the $i\gamma^0$ parity $\mathcal{P}\psi_D(x)\mathcal{P}^\dagger = i\gamma^0\psi_D(t, -\vec{x})$, one finds

$$\begin{aligned} \mathcal{P}\psi_+(x)\mathcal{P}^\dagger &= i\gamma^0\psi_+(t, -\vec{x}), \\ \mathcal{P}\psi_-(x)\mathcal{P}^\dagger &= i\gamma^0\psi_-(t, -\vec{x}), \\ \mathcal{P}C\psi_+(x)C^\dagger\mathcal{P}^\dagger &= i\gamma^0\psi_+(t, -\vec{x}), \\ \mathcal{P}C\psi_-(x)C^\dagger\mathcal{P}^\dagger &= -i\gamma^0\psi_-(t, -\vec{x}), \end{aligned} \quad (68)$$

while if one adopts the γ^0 parity $\mathcal{P}\psi_D(x)\mathcal{P}^\dagger = \gamma^0\psi_D(t, -\vec{x})$, one finds

$$\begin{aligned} \mathcal{P}\psi_+(x)\mathcal{P}^\dagger &= \gamma^0\psi_-(t, -\vec{x}), \\ \mathcal{P}\psi_-(x)\mathcal{P}^\dagger &= \gamma^0\psi_+(t, -\vec{x}), \\ \mathcal{P}C\psi_+(x)C^\dagger\mathcal{P}^\dagger &= -\gamma^0\psi_-(t, -\vec{x}), \\ \mathcal{P}C\psi_-(x)C^\dagger\mathcal{P}^\dagger &= \gamma^0\psi_+(t, -\vec{x}). \end{aligned} \quad (69)$$

The action (64) is invariant under both (68) and (69). But the difference is that the $i\gamma^0$ parity (68) maintains the two Lagrangians of Majorana fermions $\psi_+(x)$ and $\psi_-(x)$ separately invariant in the action (64), while the γ^0 parity (69) interchanges the two Lagrangians of Majorana fermions in the action (64). It is interesting that the γ^0 parity is represented as a doublet representation in both the decomposition of a Dirac fermion $\psi_D(x)$ into two Majorana fermions as in (69)

$$\psi_D(x) = \psi_+(x) + \psi_-(x) \quad (70)$$

and the chiral decomposition as in (31)

$$\psi_D(x) = \psi_L(x) + \psi_R(x). \quad (71)$$

If one would extract the first action for the Majorana fermion $\psi_+(x)$ or the second action for the Majorana fermion $\psi_-(x)$ in the second line of (64) as a mathematical model of a Majorana fermion, one would have to adopt $i\gamma^0$ parity. In such a case, one would need to give a physical meaning to the imaginary parity eigenvalues $\pm i$.²

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APPENDIX: COMMENT ON THE SPONTANEOUS PARITY BREAKING

The following issue was raised by an anonymous referee: the two different definitions of parity are obviously equivalent for Dirac fermions. One may then expect that the equivalence of these two parity operations is more transparent in the left-right symmetric models for generating Majorana neutrinos, for example, the model by Mohapatra and Senjanovic [21].

We here show that the model [21], which is regarded as one of the possible versions of the seesaw model, has common features and no essential differences from other seesaw models, whose physics aspects are described by Weinberg's model of Majorana neutrinos we use in the body of the paper. The analysis in this Appendix is thus regarded as an extension of the analysis in the body of the present paper to an explicit analysis of a class of seesaw models.

They start with a left-right symmetric model following Pati and Salam [21]. They then introduce a spontaneous parity breaking, which breaks the fermion number symmetry simultaneously. In the starting theory without the fermion number breaking one can use either γ^0 parity or $i\gamma^0$ parity for all the fermions. When one adds the fermion number violating Higgs term, these two parity operations are not apparently equivalent. Nevertheless, one can make the fermion number violating term invariant under either γ^0 parity or $i\gamma^0$ parity by suitably assigning the parity transformation laws to the extra Higgs field which breaks the fermion number and parity spontaneously.

In this setting and after the spontaneous symmetry breaking, they obtain the neutrino sector (we write only the relevant terms in our notation) [21]

²It is customary to assign real eigenvalues ± 1 to the parity operation and formulate the parity selection rules based on them in the case of parity conserving interactions, such as in the charmonium phenomenology. I thank J. Arafune for a helpful discussion on parity operation. See also [3].

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \frac{1}{2} \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R \\ & - \frac{1}{2} \overline{\psi}_L \begin{pmatrix} m_N & m_e \\ m_e & 0 \end{pmatrix} \psi_R + \text{H.c.}, \end{aligned} \quad (A1)$$

where the large mass parameter m_N arises from the spontaneous symmetry breaking and

$$\psi_L = \begin{pmatrix} C\overline{\nu}_R^T \\ \nu_L \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \nu_R \\ C\overline{\nu}_L^T \end{pmatrix}. \quad (A2)$$

After the Autonne-Takagi factorization with real positive M_1 and M_2 ,

$$U^T \begin{pmatrix} m_N & m_e \\ m_e & 0 \end{pmatrix} U = \begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix}, \quad (A3)$$

with a unitary U and

$$\tilde{\psi}_R = U\psi_R, \quad \tilde{\psi}_L = U^*\psi_L, \quad (A4)$$

the above Lagrangian is written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \overline{\tilde{\psi}}_L i\gamma^\mu \partial_\mu \tilde{\psi}_L + \frac{1}{2} \overline{\tilde{\psi}}_R i\gamma^\mu \partial_\mu \tilde{\psi}_R \\ & - \frac{1}{2} \overline{\tilde{\psi}}_L \begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix} \tilde{\psi}_R + \text{H.c.}, \\ = & \overline{\tilde{\nu}}_R i\gamma^\mu \partial_\mu \tilde{\nu}_R - \frac{1}{2} \tilde{\nu}_R^T C M_1 \tilde{\nu}_R + h.c. + \overline{\tilde{\nu}}_L i\gamma^\mu \partial_\mu \tilde{\nu}_L \\ & + \frac{1}{2} \tilde{\nu}_L^T C M_2 \tilde{\nu}_L + \text{H.c.}, \\ = & \frac{1}{2} \overline{\tilde{\psi}}_1 [i\gamma^\mu \partial_\mu - M_1] \tilde{\psi}_1 + \frac{1}{2} \overline{\tilde{\psi}}_2 [i\gamma^\mu \partial_\mu - M_2] \tilde{\psi}_2, \end{aligned} \quad (A5)$$

where we defined

$$\psi_1 = \tilde{\nu}_R + C\overline{\tilde{\nu}}_R^T, \quad \psi_2 = \tilde{\nu}_L - C\overline{\tilde{\nu}}_L^T, \quad (A6)$$

which satisfy the Majorana conditions (to be definite we assume $M_1 \gg M_2$, namely, $M_1 \sim m_N$ and $M_2 \sim m_e^2/m_N$)

$$\begin{aligned} \psi_1(x) &= C\overline{\psi}_1^T(x), \quad [i\gamma^\mu \partial_\mu - M_1]\psi_1(x) = 0, \\ \psi_2(x) &= -C\overline{\psi}_2^T(x), \quad [i\gamma^\mu \partial_\mu - M_2]\psi_2(x) = 0. \end{aligned} \quad (A7)$$

The present scheme is valid for a model with 3 generations using 6×6 Autonne-Takagi matrix U and 3×3 diagonal mass matrices M_1 and M_2 [17,21], although we use the notation of a single generation model in this Appendix.

We examine the C , P , and CP properties of Majorana neutrinos thus defined (by omitting the tilde symbols of $\tilde{\nu}_L$ and $\tilde{\nu}_R$, for notational simplicity).

(i) γ^0 parity:

$$\begin{aligned}
 C\psi_1(x)C^\dagger &= \nu_L(x) + C\bar{\nu}_L^T(x), \\
 \mathcal{P}\psi_1(x)\mathcal{P}^\dagger &= \gamma^0[\nu_L(t, -\vec{x}) - C\bar{\nu}_L^T(t, -\vec{x})], \\
 \mathcal{PC}\psi_1(x)C^\dagger\mathcal{P}^\dagger &= \gamma^0[\nu_R(t, -\vec{x}) - C\bar{\nu}_R^T(t, -\vec{x})], \\
 C\psi_2(x)C^\dagger &= -\nu_R(x) + C\bar{\nu}_R^T(x), \\
 \mathcal{P}\psi_2(x)\mathcal{P}^\dagger &= \gamma^0[\nu_R(t, -\vec{x}) + C\bar{\nu}_R^T(t, -\vec{x})], \\
 \mathcal{PC}\psi_2(x)C^\dagger\mathcal{P}^\dagger &= -\gamma^0[\nu_L(t, -\vec{x}) + C\bar{\nu}_L^T(t, -\vec{x})]. \quad (\text{A8})
 \end{aligned}$$

We observe that none of C , P , and CP defined in terms of γ^0 parity is the good symmetry of Majorana neutrinos. We next examine

(ii) $i\gamma^0$ parity:

$$\begin{aligned}
 C\psi_1(x)C^\dagger &= \nu_L(x) + C\bar{\nu}_L^T(x), \\
 \mathcal{P}\psi_1(x)\mathcal{P}^\dagger &= i\gamma^0[\nu_L(t, -\vec{x}) + C\bar{\nu}_L^T(t, -\vec{x})], \\
 \mathcal{PC}\psi_1(x)C^\dagger\mathcal{P}^\dagger &= i\gamma^0[\nu_R(t, -\vec{x}) + C\bar{\nu}_R^T(t, -\vec{x})] \\
 &= i\gamma^0\psi_1(t, -\vec{x}), \\
 C\psi_2(x)C^\dagger &= -\nu_R(x) + C\bar{\nu}_R^T(x), \\
 \mathcal{P}\psi_2(x)\mathcal{P}^\dagger &= i\gamma^0[\nu_R(t, -\vec{x}) - C\bar{\nu}_R^T(t, -\vec{x})], \\
 \mathcal{PC}\psi_2(x)C^\dagger\mathcal{P}^\dagger &= -i\gamma^0[\nu_L(t, -\vec{x}) - C\bar{\nu}_L^T(t, -\vec{x})] \\
 &= -i\gamma^0\psi_2(t, -\vec{x}), \quad (\text{A9})
 \end{aligned}$$

namely, only the CP symmetry in terms of $i\gamma^0$ parity is a good symmetry, $\mathcal{PC}\psi_1(x)C^\dagger\mathcal{P}^\dagger = i\gamma^0\psi_1(t, -\vec{x})$ and $\mathcal{PC}\psi_2(x)C^\dagger\mathcal{P}^\dagger = -i\gamma^0\psi_2(t, -\vec{x})$.

What we learn from the model with spontaneous parity and fermion number breaking is summarized as follows:

- (1) We can define CP for the neutrinos only if we use $i\gamma^0$ parity, and C and P are all broken in the neutrino sector, just as the case of the Weyl neutrino in SM. CP thus defined is consistent with the Majorana condition (A7).
- (2) We can thus analyze the CP breaking arising from PMNS matrix using $i\gamma^0$ parity (not γ^0 parity) in the present model (or seesaw models). This analysis is essentially the same as the analysis of Weinberg's models with $i\gamma^0$ parity in Sec. II A of the body of the present paper and also the analysis of seesaw models in [7].

One may next ask if the crucial difficulty (A8) of the conventional γ^0 parity, which has been used by many authors in the past, is resolved? This is the main subject of Sec. II B in the body of the present paper. In the present context of a seesaw model, the essential idea is to use the Autonne-Takagi factorization in the form [instead of (A3)] with real positive M_1 and M_2

$$U'^T \begin{pmatrix} m_N & m_e \\ m_e & 0 \end{pmatrix} U' = (-i) \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \quad (\text{A10})$$

with a unitary U' and

$$\tilde{\psi}_R = U' \psi_R, \quad \tilde{\psi}_L = U'^* \psi_L, \quad (\text{A11})$$

the above Lagrangian (A1) is written as

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2} \bar{\tilde{\psi}}_L i\gamma^\mu \partial_\mu \tilde{\psi}_L + \frac{1}{2} \bar{\tilde{\psi}}_R i\gamma^\mu \partial_\mu \tilde{\psi}_R \\
 &\quad + \frac{i}{2} \bar{\tilde{\psi}}_L \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tilde{\psi}_R + \text{H.c.}, \\
 &= \bar{\tilde{\nu}}_R i\gamma^\mu \partial_\mu \tilde{\nu}_R + \frac{i}{2} \tilde{\nu}_R^T C M_1 \tilde{\nu}_R - \frac{i}{2} \bar{\tilde{\nu}}_R C M_1 \tilde{\nu}_R^T, \\
 &\quad + \bar{\tilde{\nu}}_L i\gamma^\mu \partial_\mu \tilde{\nu}_L - \frac{i}{2} \tilde{\nu}_L^T C M_2 \tilde{\nu}_L + \frac{i}{2} \bar{\tilde{\nu}}_L C M_2 \tilde{\nu}_L^T, \\
 &= \frac{1}{2} \bar{\tilde{\psi}}_1 [i\gamma^\mu \partial_\mu - M_1] \tilde{\psi}_1 + \frac{1}{2} \bar{\tilde{\psi}}_2 [i\gamma^\mu \partial_\mu - M_2] \tilde{\psi}_2, \quad (\text{A12})
 \end{aligned}$$

where we defined

$$\begin{aligned}
 \psi_1 &= e^{i\pi/4} \tilde{\nu}_R - e^{-i\pi/4} C\bar{\tilde{\nu}}_R^T, \\
 \psi_2 &= e^{i\pi/4} \tilde{\nu}_L + e^{-i\pi/4} C\bar{\tilde{\nu}}_L^T, \quad (\text{A13})
 \end{aligned}$$

which satisfy the Majorana conditions (to be definite we assume $M_1 \gg M_2$, namely, $M_1 \sim m_N$ and $M_2 \sim m_e^2/m_N$)

$$\begin{aligned}
 \psi_1(x) &= -C\bar{\psi}_1^T(x), \quad [i\gamma^\mu \partial_\mu - M_1]\psi_1(x) = 0, \\
 \psi_2(x) &= C\bar{\psi}_2^T(x), \quad [i\gamma^\mu \partial_\mu - M_2]\psi_2(x) = 0. \quad (\text{A14})
 \end{aligned}$$

Note that the Autonne-Takagi factorization is very different from the conventional diagonalization of a hermitian matrix by a unitary transformation; the Autonne-Takagi factorization basically gives rise to characteristic values, but the phase freedom of the diagonal elements is still left free, and the specific phase convention in (A10) is chosen to define the CP invariant Lagrangian (using γ^0 parity) after the mass diagonalization. In this choice of factorization it is crucial that the classical Majorana condition $\psi_2(x) = C\bar{\psi}_2^T(x)$ in (A14), for example, is valid for any choice of the phase $\psi_2 = e^{i\alpha} \tilde{\nu}_L + e^{-i\alpha} C\bar{\tilde{\nu}}_L^T$ in (A13).

We now examine the C , P , and CP properties of Majorana neutrinos thus defined in (A13) (by omitting the tilde symbols of $\tilde{\nu}_L$ and $\tilde{\nu}_R$, for notational simplicity).

(i) γ^0 parity:

$$\begin{aligned}
C\psi_1(x)C^\dagger &= -e^{-i\pi/4}\nu_L(x) + e^{i\pi/4}C\bar{\nu}_L^T(x) = i\psi_2(x), \\
P\psi_1(x)P^\dagger &= \gamma^0[e^{i\pi/4}\nu_L(t, -\vec{x}) + e^{-i\pi/4}C\bar{\nu}_L^T(t, -\vec{x})] = \gamma^0\psi_2(t, -\vec{x}), \\
PC\psi_1(x)C^\dagger P^\dagger &= i\gamma^0[e^{i\pi/4}\nu_R(t, -\vec{x}) - e^{-i\pi/4}C\bar{\nu}_R^T(t, -\vec{x})] = i\gamma^0\psi_1(t, -\vec{x}), \\
C\psi_2(x)C^\dagger &= e^{-i\pi/4}\nu_R(x) + e^{i\pi/4}C\bar{\nu}_R^T(x) = -i\psi_1(x), \\
P\psi_2(x)P^\dagger &= \gamma^0[e^{i\pi/4}\nu_R(t, -\vec{x}) - e^{-i\pi/4}C\bar{\nu}_R^T(t, -\vec{x})] = \gamma^0\psi_1(t, -\vec{x}), \\
PC\psi_2(x)C^\dagger P^\dagger &= -i\gamma^0[e^{i\pi/4}\nu_L(t, -\vec{x}) + e^{-i\pi/4}C\bar{\nu}_L^T(t, -\vec{x})] = -i\gamma^0\psi_2(t, -\vec{x}).
\end{aligned} \tag{A15}$$

We observe that CP defined in terms of γ^0 parity is the good symmetry of Majorana neutrinos, $PC\psi_1(x)C^\dagger P^\dagger = i\gamma^0\psi_1(t, -\vec{x})$ and $PC\psi_2(x)C^\dagger P^\dagger = -i\gamma^0\psi_2(t, -\vec{x})$, although C and P are separately broken for $M_1 \neq M_2$. The CP thus defined is consistent with the classical Majorana condition (A14), and thus we can formulate the weak interactions of Majorana neutrinos consistently using γ^0 parity.

We next examine (ii) $i\gamma^0$ parity:

$$\begin{aligned}
C\psi_1(x)C^\dagger &= -e^{-i\pi/4}\nu_L(x) + e^{i\pi/4}C\bar{\nu}_L^T(x) = i\psi_2(x), \\
P\psi_1(x)P^\dagger &= i\gamma^0[e^{i\pi/4}\nu_L(t, -\vec{x}) - e^{-i\pi/4}C\bar{\nu}_L^T(t, -\vec{x})], \\
PC\psi_1(x)C^\dagger P^\dagger &= -\gamma^0[e^{i\pi/4}\nu_R(t, -\vec{x}) + e^{-i\pi/4}C\bar{\nu}_R^T(t, -\vec{x})], \\
C\psi_2(x)C^\dagger &= e^{-i\pi/4}\nu_R(x) + e^{i\pi/4}C\bar{\nu}_R^T(x) = -i\psi_1(x), \\
P\psi_2(x)P^\dagger &= i\gamma^0[e^{i\pi/4}\nu_R(t, -\vec{x}) + e^{-i\pi/4}C\bar{\nu}_R^T(t, -\vec{x})], \\
PC\psi_2(x)C^\dagger P^\dagger &= \gamma^0[e^{i\pi/4}\nu_L(t, -\vec{x}) - e^{-i\pi/4}C\bar{\nu}_L^T(t, -\vec{x})],
\end{aligned} \tag{A16}$$

namely, neither C , P , nor CP in terms of $i\gamma^0$ parity is a good symmetry for $M_1 \neq M_2$. The $i\gamma^0$ parity is not consistently formulated with the present choice of Autonne-Takagi factorization.

One may thus conclude that a general class of seesaw models for Majorana neutrinos is formulated using either $i\gamma^0$ parity or γ^0 parity with the consistent good CP symmetry if one chooses the suitable forms of the Autonne-Takagi factorization of the mass matrix, in agreement with the analysis of Weinberg's model in the body of the present paper.

Incidentally, how to define the consistent nontrivial C for Majorana neutrinos is another interesting issue in the seesaw models, as is mentioned in Sec. III. The nontrivial charge conjugation C for the Majorana is apparently broken in the above formulation of the seesaw model for $M_1 \neq M_2$ as in (A8), (A9), (A15), and (A16). The possible resolution of this issue has been discussed in detail in [7,12,20].

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- [1] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw Hill, New York, 1965).
 - [2] E. Majorana, *Nuovo Cimento* **14**, 171 (1937).
 - [3] B. Kayser, *Phys. Rev. D* **26**, 1662 (1982), and references therein.
 - [4] S. Weinberg, *The Quantum Theory of Fields I* (Cambridge University Press, Cambridge, England, 1995).
 - [5] S. Weinberg, *Phys. Rev. Lett.* **43**, 1566 (1979).
 - [6] Zhi-zhong Xing, Neutrino physics, in *Proceedings of the 1st Asia-Europe-Pacific School of High-Energy Physics (AEP-SHEP)*, 2012, published in CERN Yellow Report CERN-2014-001,177-217, [arXiv:1406.7739](#).
 - [7] K. Fujikawa, *Eur. Phys. J. C* **80**, 285 (2020).
 - [8] B. Kayser and A. S. Goldhaber, *Phys. Rev. D* **28**, 2341 (1983).
 - [9] M. Fukugita and T. Yanagida, *Physics of Neutrinos and Application to Astrophysics* (Springer, Berlin Heidelberg, 2002).
 - [10] C. Giunti and C. W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford University Press, Oxford, 2007).
 - [11] S. Bilenky, *Introduction to the Physics of Massive and Mixed Neutrinos*, *Lect. Notes. Phys.* Vol. 817 (Springer, Berlin Heidelberg, 2010).
 - [12] K. Fujikawa and A. Tureanu, *Eur. Phys. J. C* **79**, 752 (2019).
 - [13] L. Autonne, *Ann. Univ. Lyon* **38**, 1 (1915).
 - [14] T. Takagi, *Jpn. J. Math.* **1**, 83 (1925).
 - [15] W. Pauli, *Nuovo Cimento* **6**, 204 (1957).
 - [16] F. Gursey, *Nuovo Cimento* **7**, 411 (1958).
 - [17] K. Fujikawa, *Phys. Lett. B* **789**, 76 (2019).
 - [18] M. Kobayashi and T. Maskawa, *Prog. Theor. Phys.* **49**, 652 (1973).
 - [19] S. M. Bilenky, J. Hosek, and S. T. Petcov, *Phys. Lett. B* **94**, 495 (1980); M. Doi, T. Kotani, H. Nishiura, K. Okuda, and E. Takasugi, *Phys. Lett.* **102B**, 323 (1981); J. Schechter and J. W. F. Valle, *Phys. Rev. D* **23**, 1666 (1981).
 - [20] K. Fujikawa and A. Tureanu, *Phys. Lett. B* **767**, 199 (2017); *Phys. Rev. D* **97**, 055042 (2018).
 - [21] R. N. Mohapatra and G. Senjanovic, *Phys. Rev. Lett.* **44**, 912 (1980).