Constraints between $\Delta\alpha_{\text{had}}(M_Z^2)$ and $(g_\mu - 2)_{\text{HVP}}$

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I discuss the possibility of estimating the shift $\delta\Delta\alpha(M_Z^2)$ on the running of the EM-coupling to the M_Z scale, induced by the discrepancy Δa_μ between two precise determinations of the hadronic vacuum polarization contribution to $g_\mu - 2$. It is shown that the size of Δa_μ implies rigorous bounds on the $\delta\Delta\alpha(M_Z^2)$ -shift. Any extra contribution to this minimal shift necessarily depends on the specific shape of the underlying spectral function $\frac{1}{\pi}\text{Im}\Delta(t)$ responsible for the Δa_μ -discrepancy. I show that, in the case of a quark model, the total $\Delta\alpha(M_Z^2)$ -shift remains small. I also discuss the scenario where $\frac{1}{\pi}\text{Im}\Delta(t)$ is a constant in a finite t -region and show that in this case, up to a $t_{\text{max}} \lesssim 5 \text{ GeV}^2$, the size of the total $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift, remains below or of the order of the present error value on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$.

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I. INTRODUCTION

Testing the EW-Standard Model requires knowledge of the running of the EM-coupling induced by the hadronic interactions, from the low-energy measured value $\alpha = 1/137.036\dots$ up to the scale of the M_Z -mass: $M_Z = 91.1876 \pm 0.0021 \text{ GeV}$. This running is encoded in a quantity called $\Delta\alpha_{\text{had}}(M_Z^2)$, which in terms of the hadronic spectral function $\frac{1}{\pi}\text{Im}\Pi_{\text{had}}(t)$ is given by the principal value integral:

$$\Delta\alpha_{\text{had}}(M_Z^2) = \text{PV} \int_{t_0=4m_\pi^2}^{\infty} \frac{dt}{t} \frac{M_Z^2}{M_Z^2 - t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad (1.1)$$

and contributes to the running of α at the M_Z -scale via the relation:

$$\alpha(M_Z^2) = \frac{\alpha}{1 - \Delta\alpha_{\text{had}}(M_Z^2) - \Delta\alpha_{\text{lep}}(M_Z^2)}, \quad (1.2)$$

where

$$\Delta\alpha_{\text{lep}}(M_Z^2) = (314.979 \pm 0.002) \times 10^{-4} \quad (1.3)$$

is the contribution due to the EM-couplings of the leptons [1,2].

The same hadronic spectral function governs the hadronic vacuum polarization (HVP) contribution to the muon

anomalous magnetic moment via the equivalent integral representations [3–7]:

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi} \int_{t_0}^{\infty} \frac{dt}{t} \int_0^1 dx \underbrace{\frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)}}_{K(t)} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad (1.4)$$

$$= -\frac{\alpha}{\pi} \int_0^1 dx (1-x) \Pi_{\text{had}}\left(\frac{x^2}{1-x} m_\mu^2\right), \quad (1.5)$$

where

$$-\Pi_{\text{had}}(Q^2) = \int_{t_0}^{\infty} \frac{dt}{t} \frac{Q^2}{Q^2 + t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t),$$

$$Q^2 \equiv \frac{x^2}{1-x} m_\mu^2. \quad (1.6)$$

is the hadronic self-energy in the Euclidean and the optical theorem relates the spectral function $\frac{1}{\pi}\text{Im}\Pi_{\text{had}}(t)$ to the observable one-photon annihilation cross section:

$$\sigma(t)_{e^+e^- \rightarrow \text{had}} \underset{m_e \rightarrow 0}{\sim} \frac{4\pi^2\alpha}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t). \quad (1.7)$$

This is the way that experimental data-driven determinations of $\Delta\alpha_{\text{had}}(M_Z^2)$ and a_μ^{HVP} have been obtained (see Refs. [8,9] and the recent review in Ref. [10]).

In Ref. [9], the contribution to $\Delta\alpha_{\text{had}}(M_Z^2)$ from the hadronic flavors induced by u , d , s , c , b quarks called $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, has been obtained¹ using experimental data up to

¹For other determinations of $\Delta\alpha_{\text{had}}(M_Z^2)$, all compatible within errors, see e.g., Table 6 in this reference and Ref. [10].

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$$\sqrt{t_{\max}} = 11.199 \text{ GeV}, \quad (1.8)$$

and pQCD beyond that energy with the overall result

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (276.09 \pm 1.12) \times 10^{-4}. \quad (1.9)$$

On the other hand, there is a recent LQCD determination of a_μ^{HVP} reported by the BMWc collaboration [11], which differs significantly from the data-driven results on $\sigma(t)_{e^+e^- \rightarrow \text{had}}$ quoted in Refs. [8,9]. To be precise, let me compare the lowest order HVP result of Ref. [9] (which has the largest discrepancy) with the one in Ref. [11] i.e.,

$$\begin{aligned} a_\mu^{\text{HVP}}(\text{KNT}) &= (692.78 \pm 2.42) \times 10^{-10}, \quad \text{versus} \\ a_\mu^{\text{HVP}}(\text{BMWc}) &= (708.7 \pm 5.3) \times 10^{-10}. \end{aligned} \quad (1.10)$$

This discrepancy

$$\begin{aligned} \Delta a_\mu &\equiv a_\mu^{\text{HVP}}(\text{BMWc}) - a_\mu^{\text{HVP}}(\text{KNT}) \\ &= (15.92 \pm 5.83) \times 10^{-10} \end{aligned} \quad (1.11)$$

of 2.7σ , has prompted a reexamination of the question raised sometime ago [12]:

At what precision is $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ constrained by an accurate determination of a_μ^{HVP} ?

Possible scenarios have been discussed in Refs. [13,14] that, though model dependent, suggest potential inconsistencies between the $a_\mu^{\text{HVP}}(\text{BMWc})$ result and the present value for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ in Eq. (1.9).²

The purpose of this note is to address the question above from a new perspective. In the next section, I discuss a model-independent analysis of the observables a_μ^{HVP} and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ within the Mellin-Barnes Representation framework. Section III discusses properties of the *first moment* related to these two observables. Rigorous bounds on the size of the shift implied on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ by the Δa_μ -discrepancy, *when only the value of Δa_μ is known*, are then derived in Sec. IV. Alternative phenomenological models to those discussed in Ref. [13] are then presented in Sec. V with conclusions in Sec. VI.

II. MELLIN-BARNES REPRESENTATIONS

Mellin-Barnes representations are particularly useful to obtain asymptotic expansions of functions. In the case of a_μ^{HVP} [16]:

$$\begin{aligned} a_\mu^{\text{HVP}} &= \left(\frac{\alpha}{\pi}\right) \frac{m_\mu^2}{t_0} \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{m_\mu^2}{t_0}\right)^{-s} \mathcal{F}(s) \mathcal{M}(s), \\ c_s &\equiv \text{Re}(s) \in]0, 1[, \end{aligned} \quad (2.1)$$

where $\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s)$, and

$$\mathcal{M}(s) = \int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t}{t_0}\right)^{s-1} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad (2.2)$$

is the Mellin transform of the spectral function. This representation has been proposed as a possible way to obtain successive approximations to a_μ^{HVP} from the knowledge of the moments $\mathcal{M}(s)$ at $s = 0, -1, -2, \dots$ and their extrapolation at all s -values using Mellin-Barnes approximants (see Refs. [17,18] and references therein).

There is also a similar representation for $\Delta\alpha_{\text{had}}(M_Z^2)$:

$$\begin{aligned} \Delta\alpha_{\text{had}}(M_Z^2) &= \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \int_{t_0}^{\infty} \frac{dt}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t) \left(\frac{t}{M_Z^2}\right)^{-s} \\ &\times \Gamma(s)\Gamma(1-s) \frac{\pi}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)}, \end{aligned} \quad (2.3)$$

which differs by the factor $\frac{\pi}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)}$ [19], when compared to the representation in the Euclidean:

$$\begin{aligned} \Delta\alpha_{\text{had}}(-M_Z^2) &= \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \int_{t_0}^{\infty} \frac{dt}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t) \left(\frac{t}{M_Z^2}\right)^{-s} \\ &\times \Gamma(s)\Gamma(1-s). \end{aligned} \quad (2.4)$$

Restricting the integration over the spectral function to an upper scale $t_{\max}$, as we shall later do, one has

$$\begin{aligned} \Delta\alpha_{\text{had}}^{(5)}(M_Z^2)_{\text{data}} &= \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{t_{\max}}{M_Z^2}\right)^{-s} \tilde{\mathcal{M}}(s) \\ &\times \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)} \pi, \end{aligned} \quad (2.5)$$

where $\tilde{\mathcal{M}}(s)$ is now the restricted Mellin transform

$$\tilde{\mathcal{M}}(s) = \int_{t_0}^{t_{\max}} \frac{dt}{t} \left(\frac{t}{t_{\max}}\right)^{-s} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t). \quad (2.6)$$

This representation governs the asymptotic expansion of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)_{\text{data}}$ in terms of the ratio $(\frac{t_{\max}}{M_Z^2})$. The series expansion in this ratio is fixed by the successive singularities at the left of the fundamental strip: $0 < \text{Re}c_s < 1$, i.e., at $s = 0, -1, -2, \dots$ of the integrand in the right-hand side (rhs) of Eq. (2.5), and the coefficients are proportional to the moments:

²See also Ref. [15] which has appeared after the completion of this work.

$$\int_{t_0}^{t_{\text{max}}} \frac{dt}{t} \left(\frac{t}{t_{\text{max}}} \right)^n \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad n = 0, 1, 2, \dots, \quad (2.7)$$

which give information about the spectral function $\frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t)$ in a different region to the one provided by the moments:

$$\int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t_0}{t} \right)^n \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad n = 1, 2, \dots, \quad (2.8)$$

which govern the $(g_\mu - 2)_{\text{had}}$ contribution [16–18]. Therefore, within this perspective, the possibility of relating the two observables $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ and $(g_\mu - 2)_{\text{had}}$ seems *a priori* a rather difficult task, unless of course one uses data or a model for the hadronic spectral function, or performs fully dedicated LQCD evaluations of both observables.

Moments like those in Eq. (2.7) can be related to the contour integrals (where Q^2 is now a complex variable) [20]:

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{|Q^2|=t_{\text{max}}} \frac{dQ^2}{Q^2} \left(\frac{Q^2}{t_{\text{max}}} \right)^n \Pi_{\text{had}}(Q^2) \\ &= (-1)^{n+1} \int_{t_0}^{t_{\text{max}}} \frac{dt}{t} \left(\frac{t}{t_{\text{max}}} \right)^n \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \end{aligned} \quad (2.9)$$

which define a particular case of finite energy sum rules. The experimental determination of the integrals on the rhs can thus be confronted, for sufficiently large t_{max} -values, to a theoretical calculation of the lhs using pQCD and the OPE at large- Q^2 values.

By contrast, the moments in Eq. (2.8) are related to derivatives of the self-energy function $\Pi_{\text{had}}(Q^2)$ at $Q^2 = 0$ [16]:

$$\begin{aligned} & \frac{(-1)^{n+1}}{(n+1)!} (t_0)^{n+1} \underbrace{\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}} \Pi(Q^2) \right)_{Q^2=0}}_{\text{LQCD}} \\ &= \underbrace{\int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t_0}{t} \right)^{1+n} \frac{1}{\pi} \text{Im}\Pi(t)}_{\text{Experiment}}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.10)$$

As discussed in Refs. [16–18] they provide excellent tests to confront LQCD evaluations of the lhs with experimental results, but they invoke a totally different Q^2 -region to the one which applies to Eq. (2.9).

III. THE FIRST MOMENT

I want to focus attention on the first moment $n = 0$ in Eq. (2.10):

$$-t_0 \frac{\partial \Pi_{\text{had}}(Q^2)}{\partial Q^2} \Big|_{Q^2=0} = \int_{t_0}^{\infty} \frac{dt}{t} \frac{t_0}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t). \quad (3.1)$$

A long time ago, the authors of Ref. [21] showed that there is an upper and lower bound for a_μ^{HVP} in terms of this moment, namely:

$$\begin{aligned} & \frac{\alpha}{\pi} \frac{1}{3} \frac{m_\mu^2}{t_0} \left(\int_{t_0}^{\infty} \frac{dt}{t} \frac{t_0}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t) \right) \left[1 - f\left(\frac{t_0}{m_\mu^2}\right) \right] \\ & < a_\mu^{\text{HVP}} < \frac{\alpha}{\pi} \frac{1}{3} \frac{m_\mu^2}{t_0} \left(\int_{t_0}^{\infty} \frac{dt}{t} \frac{t_0}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t) \right), \end{aligned} \quad (3.2)$$

where

$$f\left(\frac{t_0}{m_\mu^2}\right) = 3 \int_0^1 dx \frac{x^4}{x^2 + \frac{t_0}{m_\mu^2}(1-x)} = 0.36655, \quad \text{for } t_0 = 4m_{\pi^\pm}^2. \quad (3.3)$$

The bounds follow from the observation that the kernel $K(t)$ in Eq. (1.4) can be written as follows:

$$K(t) = \frac{m_\mu^2}{t} \int_0^1 dx \left[x^2 - \underbrace{\left(\frac{x^4}{x^2 + \frac{t}{m_\mu^2}(1-x)} \right)}_{\frac{1}{3}f\left(\frac{t}{m_\mu^2}\right)} \right], \quad (3.4)$$

and the fact that, within the t -integration range $t_0 < t < \infty$,

$$0 < f\left(\frac{t}{m_\mu^2}\right) < f\left(\frac{t_0}{m_\mu^2}\right). \quad (3.5)$$

In the language of the Mellin-Barnes representation of Sec. II, the upper bound in Eq. (3.2) corresponds to retaining only the leading term in the singular expansion of $\mathcal{F}(s)$, i.e., the contribution from the simple pole at $s = 0$. This contribution can be seen as induced by an effective local operator $\partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu}$ to which I shall later come back. The other terms of the singular expansion of $\mathcal{F}(s)$ give successive corrections to the upper-bound. The simple poles generate the moments in Eq. (2.10) for $n \geq 1$, which can also be seen as induced by local operators of higher and higher dimension. The double poles generate the log-weighted moments discussed in Ref. [16]:

$$\int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{t_0}{t} \right)^n \log \frac{t}{t_0} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad n = 1, 2, \dots, \quad (3.6)$$

and can be seen as generated by successive nonlocal operators. The lower bound in Eq. (3.2) is a rigorous bound to the contribution from the sum of all the terms of the singular expansion of $\mathcal{F}(s)$ beyond the leading one and, therefore, to the total effect of the underlying string of local and nonlocal operators which give rise to these terms.

The remarkable feature about the simplicity of the resulting lower bound in Eq (3.2) is that *it is proportional to the same leading moment which fixes the upper bound*. Therefore, this lower bound can also be seen as induced by an effective local operator of the type $\partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu}$ modulated, however, by a different coupling.

In the case of the electron $(g_e - 2)_{\text{HVP}}$ the bounds in Eq (3.2) are impressive. Using the experimental determination of the lowest HVP-moment³:

$$\int_{t_0}^{\infty} \frac{dt t_0}{t} \frac{1}{\pi} \text{Im} \Pi_{\text{had}}(t) = (0.7176 \pm 0.0026) \times 10^{-3}, \quad (3.7)$$

one finds:

$$1.8550 \times 10^{-12} < a_e^{\text{HVP}} < 1.8687 \times 10^{-12}, \quad (3.8)$$

which corresponds to an accuracy of 0.37%. I recall that the lowest order HVP experimental determination in the case of the electron anomaly gives [9]:

$$a_e^{\text{HVP}} = 1.8608(66) \times 10^{-12}. \quad (3.9)$$

The contribution from the HVP-moment in Eq. (3.7) to $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ in Eq. (1.1) is a small fraction of the total:

$$\begin{aligned} \Delta\alpha_{\text{had}}^{(5)}(M_Z^2) &= \underbrace{\int_{t_0}^{\infty} \frac{dt t_0}{t} \frac{1}{\pi} \text{Im} \Pi_{\text{had}}(t)}_{7.176 \times 10^{-4}} \\ &+ \text{PV} \int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{M_Z^2}{M_Z^2 - t} - \frac{t_0}{t} \right) \frac{1}{\pi} \text{Im} \Pi_{\text{had}}(t), \end{aligned} \quad (3.10)$$

but, as we shall see next, it gives the key to relate the two observables $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ and $(g_\mu - 2)_{\text{HVP}}$.

IV. THE EFFECTIVE OPERATOR $\partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu}$ AND RIGOROUS BOUNDS

The effective operator which governs the size of the slope at the origin of the photon self-energy, and therefore the first moment of the underlying spectral function, is the dimension $d = 6$ operator

$$\frac{1}{\Lambda^2} \partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu}. \quad (4.1)$$

For the HVP-contribution in particular we have

$$-\frac{\partial \Pi_{\text{had}}(Q^2)}{\partial Q^2} \Big|_{Q^2=0} = \frac{\alpha}{\pi} \frac{1}{\Lambda_{\text{had}}^2}, \quad (4.2)$$

³Private communication from the authors of Ref. [9].

and from Eq. (2.10)

$$\frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{had}}^2} = \int_{t_0}^{\infty} \frac{dt t_0}{t} \frac{1}{\pi} \text{Im} \Pi_{\text{had}}(t). \quad (4.3)$$

The experimental value of the scale Λ_{had} which follows from Eq. (3.7) is

$$\Lambda_{\text{had}} = (0.502 \pm 0.001) \text{ GeV}, \quad (4.4)$$

which, as expected, is a scale of the order of the ρ -mass.

The Δa_μ discrepancy in Eq. (1.11) can be interpreted as an excess, in some (so far) unknown t -range, of the underlying spectral function which results in the BMWc-evaluation, as compared to the spectral function which has been used in the KNT-evaluation i.e.,

$$\Delta a_\mu = \frac{\alpha}{\pi} \int_{t_0}^{\infty} \frac{dt}{t} K(t) \frac{1}{\pi} \text{Im} \Delta(t), \quad (4.5)$$

where

$$\begin{aligned} \frac{1}{\pi} \text{Im} \Delta(t) &\equiv \frac{1}{\pi} \text{Im} \Pi_{\text{BMWc}}(t) - \frac{1}{\pi} \text{Im} \Pi_{\text{KNT}}(t) \geq 0 \\ &\text{for } t_0 \leq t \leq \infty. \end{aligned} \quad (4.6)$$

This difference of spectral functions can be seen as generated, at least in part, by an effective local operator of the type in Eq. (4.1) which induces an effective moment

$$\frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2} = \int_{t_0}^{\infty} \frac{dt t_0}{t} \frac{1}{\pi} \text{Im} \Delta(t), \quad (4.7)$$

and in terms of which, the bounds in Eq. (3.2) when applied to Δa_μ , give

$$\frac{\alpha}{\pi} \frac{1}{3} \frac{m_\mu^2}{t_0} \left[1 - f\left(\frac{t_0}{m_\mu^2}\right) \right] \left(\frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2} \right) < \Delta a_\mu < \frac{\alpha}{\pi} \frac{1}{3} \frac{m_\mu^2}{t_0} \left(\frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2} \right). \quad (4.8)$$

Including the error of Δa_μ in Eq. (1.11) so as to take the largest and smallest values of the discrepancy, fixes the extreme values of the scale Λ_{eff}^2 to be within the limits:

$$\begin{aligned} \Lambda_{\text{eff}}^2 &= 19.89 \text{ GeV}^2 \text{ (from the upper bound)} \quad \text{and} \\ \Lambda_{\text{eff}}^2 &= 5.85 \text{ GeV}^2 \text{ (from the lower bound)}. \end{aligned} \quad (4.9)$$

These two scales fix the range of couplings of the effective operator which is at the origin, at least in part, of the anomaly discrepancy. Their effect on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ is to add a shift to the underlined term in the rhs of Eq. (3.10):

$$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) \Rightarrow \underbrace{\int_{t_0}^{\infty} \frac{dt}{t} \frac{t_0}{t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t)} + \frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2} + \text{PV} \int_{t_0}^{\infty} \frac{dt}{t} \left(\frac{M_Z^2}{M_Z^2 - t} - \frac{t_0}{t} \right) \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t) \quad (4.10)$$

$$= \underbrace{\frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2}}_{\text{shift}} + \text{PV} \int_{t_0}^{\infty} \frac{dt}{t} \frac{M_Z^2}{M_Z^2 - t} \frac{1}{\pi} \text{Im}\Pi_{\text{had}}(t), \quad (4.11)$$

with the shift constrained by the upper and lower bounds in Eq. (4.9) to be within the limits:

$$\frac{\alpha}{\pi} \frac{t_0}{19.89 \text{ GeV}^2} \leq \frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2} \leq \frac{\alpha}{\pi} \frac{t_0}{5.85 \text{ GeV}^2}. \quad (4.12)$$

$0.091 \times 10^{-4} \qquad \qquad \qquad 0.309 \times 10^{-4}$

We find, therefore, that the accepted range of the minimal shift on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ induced by the effective moment in Eq. (4.7) is rather small. In fact, the higher value is about four times smaller than the error 1.12×10^{-4} in the result in Eq. (1.9). In the absence of any information about the shape of the spectral function $\frac{1}{\pi} \text{Im}\Delta(t)$, this is as much as may be rigorously said about the size of the shift implied on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$.

Notice that as the discrepancy Δa_μ becomes smaller and smaller, Eq. (4.8) shows that Λ_{eff}^2 becomes larger and larger and, hence, the shift in Eq. (4.11) (the first term) smaller and smaller. By contrast, as the discrepancy Δa_μ becomes larger and larger Λ_{eff}^2 becomes smaller and smaller and, hence, the shift in Eq. (4.11) larger and larger.

Any extra contribution to the minimal shift in Eq. (4.12) is encoded in the difference

$$\text{PV} \int_{t_0}^{\infty} \frac{dt}{t} \frac{M_Z^2}{M_Z^2 - t} \frac{1}{\pi} \text{Im}\Delta(t) - \frac{\alpha}{\pi} \frac{t_0}{\Lambda_{\text{eff}}^2}, \quad (4.13)$$

with $\text{Im}\Delta(t)$ defined in Eq. (4.6), and its size depends on the specific shape of the underlying BMWc-spectral function.

V. PHENOMENOLOGICAL MODELS

A. The constituent chiral quark model

Models of $\frac{1}{\pi} \text{Im}\Delta(t)$ which lead to much larger shifts to $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ than the minimal bounds in Eq. (4.12) have been disused in Refs. [13,14]. However, it is also possible to construct models that do not give such large

FIG. 1. Shape of the difference of spectral functions in Eq. (5.3) plotted in 10^{-3} units.

shifts. The simple constituent chiral quark model (C χ QM)⁴ may be used to illustrate this. The hadronic spectral function in this model:

$$\frac{1}{\pi} \text{Im}\Pi_{\chi\text{QM}}(t, M) = \frac{\alpha}{\pi} \frac{5}{3} N_c \frac{1}{3} \left(1 + \frac{2M^2}{t} \right) \sqrt{1 - \frac{4M^2}{t}} \theta(t - 4M^2), \quad (5.1)$$

only depends on a M -mass parameter which can be adjusted so as to reproduce the central values of either the $a_\mu^{\text{HVP}}(\text{BMWc})$ or the $a_\mu^{\text{HVP}}(\text{KNT})$ determinations. These choices result in the values:

$$M_{\text{BMWc}} = 0.292 \text{ GeV} \text{ and } M_{\text{KNT}} = 0.296 \text{ GeV}, \quad (5.2)$$

and the relevant spectral function is then

$$\frac{1}{\pi} \text{Im}\Delta_{\text{model}}(t) = \frac{1}{\pi} \text{Im}\Pi_{\chi\text{QM}}(t, M_{\text{BMWc}}) - \frac{1}{\pi} \text{Im}\Pi_{\chi\text{QM}}(t, M_{\text{KNT}}), \quad (5.3)$$

which has a shape as shown (in 10^{-3} -units) in Fig. 1 (the cusp in the figure corresponds to the threshold opening at $4M_{\text{KNT}}^2 = 0.350 \text{ GeV}^2$). The shift induced on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ is then given by the integral

$$\text{PV} \int_{t_0=4m_\pi^2}^{\infty} \frac{dt}{t} \frac{M_Z^2}{M_Z^2 - t} \frac{1}{\pi} \text{Im}\Delta_{\text{model}}(t) = 0.42 \times 10^{-4}, \quad (5.4)$$

which, as expected, results in a number somewhat larger than the upper bound in Eq. (4.12).

⁴see Ref. [22] for details.

B. Beyond the minimal bounds

It would be nice to have some guidance to go beyond the minimal rigorous bounds previously discussed. In that respect I propose the following approach: it seems reasonable to assume that the region where the underlying spectral functions are responsible for the Δa_μ discrepancy is a finite region in the t -integration range i.e.,

$$\Delta a_\mu \equiv \frac{\alpha}{\pi} \int_{t_{\min}}^{t_{\max}} \frac{dt}{t} K(t) \frac{1}{\pi} \text{Im}\Delta(t), \quad \text{with } t_{\min} \geq t_0; \quad (5.5)$$

and $t_{\max} \ll M_Z^2$ because in any case both evaluations of the anomaly are using pQCD beyond a certain energy. The total shift on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ induced by the same spectral function is then given by the integral

$$\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \int_{t_{\min}}^{t_{\max}} \frac{dt}{t} \frac{M_Z^2}{M_Z^2 - t} \frac{1}{\pi} \text{Im}\Delta(t), \quad (5.6)$$

which is no longer a principal value integral. There is a Mellin-Barnes representation for this shift, similar to the one given in Eq. (2.5):

$$\begin{aligned} \delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) &= \frac{1}{2\pi i} \int_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{t_{\max}}{M_Z^2} \right)^{-s} \tilde{\mathcal{M}}(s) \\ &\times \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)} \pi, \end{aligned} \quad (5.7)$$

where $\tilde{\mathcal{M}}(s)$ is now the restricted Mellin transform of $\frac{1}{\pi} \text{Im}\Delta(t)$,

$$\tilde{\mathcal{M}}(s) = \int_{t_{\min}}^{t_{\max}} \frac{dt}{t} \left(\frac{t}{t_{\max}} \right)^{-s} \frac{1}{\pi} \text{Im}\Delta(t). \quad (5.8)$$

The leading contribution in the expansion which follows from Eq. (5.7) is precisely the one induced by the singularity of the integrand at $s = 0$ and, therefore,

$$\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \tilde{\mathcal{M}}(0) + \mathcal{O}\left(\frac{t_{\max}}{M_Z^2}\right), \quad (5.9)$$

with $\tilde{\mathcal{M}}(0)$ dominating largely the full contribution.

Let me now assume that $\frac{1}{\pi} \text{Im}\Delta(t)$ is a constant in the region $t_{\min} \leq t \leq t_{\max}$:

$$\frac{1}{\pi} \text{Im}\Delta(t) \equiv \beta \theta(t - t_{\min}) \theta(t_{\max} - t), \quad (5.10)$$

something which could well be the case if e.g., the Δa_μ discrepancy was due to a shift in a t -region of the spectrum that corresponds to an offset in its height. In this case

FIG. 2. Size of the $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift in 10^{-4} units as a function of the $t_{\max}$ -choice. The thick red curve shows the shift corresponding to the central value of the discrepancy Δa_μ ; the dashed curves the same shift when including the upper and lower errors on Δa_μ . The thick green line shows the present experimental error on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, also in 10^{-4} units.

$$\Delta a_\mu = \beta \frac{\alpha}{\pi} \int_{t_{\min}}^{t_{\max}} \frac{dt}{t} K(t) \quad \text{and} \quad \tilde{\mathcal{M}}(0) = \beta \log \frac{t_{\max}}{t_{\min}}, \quad (5.11)$$

which neglecting $\mathcal{O}(\frac{t_{\max}}{M_Z^2})$ terms in Eq. (5.9), results in a simple expression for the total $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift:

$$\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = \frac{\Delta a_\mu}{\frac{\alpha}{\pi} \int_{t_{\min}}^{t_{\max}} \frac{dt}{t} K(t)} \log \frac{t_{\max}}{t_{\min}}. \quad (5.12)$$

As a numerical illustration of this result, I show in Fig. 2 the size of the $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift as a function of $t_{\max}$ in the range $9m_\pi^2 \leq t_{\max} \leq 16 \text{ GeV}^2$ with $t_{\min} = 4m_\pi^2$. The thick red curve in the figure shows the $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift corresponding to the central value of the discrepancy Δa_μ in Eq. (1.11); the dashed curves the results of the same shift when including the upper and lower errors on Δa_μ . The thick green line shows the present experimental error of ± 1.12 on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ in Eq. (1.9) in 10^{-4} units. The plot in this figure shows that up to a $t_{\max} \lesssim 5 \text{ GeV}^2$, the size of the $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift, remains below or of the order of the present error value on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ (the thick green line).

VI. SUMMARY AND CONCLUSIONS

- (i) The bounds in Eq. (4.12) are rigorous minimal bounds on the $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift implied by the Δa_μ -discrepancy in Eq. (1.11). They do not assume any specific shape of energy dependence in the underlying spectral function $\frac{1}{\pi} \text{Im}\Delta(t)$ which is at the origin of the discrepancy. In the absence of any

information about the shape of $\frac{1}{\pi}\text{Im}\Delta(t)$ this is as much as can be said at present.

- (ii) The simple quark model discussed above illustrates the fact that it is possible to construct spectral functions which reproduce the two anomaly values $a_\mu(\text{BMWc})$ and $a_\mu(\text{KNT})$ and yet, their difference, induces a shift on $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ which although larger than the upper minimal bound in Eq. (4.12), is still much smaller than the present error in Eq. (1.9).
- (iii) Finally, I have discussed the scenario where the underlying spectral function $\frac{1}{\pi}\text{Im}\Delta(t)$ responsible for the Δa_μ -discrepancy could be constant in a finite t -range: from threshold $t_{\text{min}} = 4m_\pi^2$ up to an arbitrary $t_{\text{max}} \ll M_Z^2$ and show that in this case the

Δa_μ -discrepancy induces a $\delta\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ -shift which, as shown in Fig. 2, remains compatible with the present data-driven determination of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ up to $t_{\text{max}} \lesssim 5 \text{ GeV}^2$ -values.

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