Excitations of elementary fermions in gauge Higgs theories

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Static quark-antiquark states in QCD, at finite quark separation, have a spectrum of metastable states corresponding to stringlike excitations of the gauge field. In this article I suggest that there may also exist an excitation spectrum of heavy fermions in some gauge Higgs theories deep in the Higgs phase. In this situation there are no color electric flux tubes connecting quarks with antiquarks. There may, nonetheless, exist stable excitations of the bosonic fields surrounding an isolated fermion, below the particle production threshold. I present numerical evidence indicating the existence of such excitations in an SU(3) gauge Higgs theory, with the scalar field in the fundamental representation of the gauge group.

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I. INTRODUCTION

It has long been known that static quark-antiquark states in QCD have a spectrum of stringlike excitations of the color electric field joining the quarks. In a system with light dynamical quarks these excitations are of course only metastable, due to string breaking, and indeed the light quark-antiquark states themselves have a spectrum of metastable excitations, lying on linear Regge trajectories. We expect the same phenomena in the confinement phase of a gauge Higgs theory, with the scalar field in the fundamental representation of the gauge group.

In the Higgs phase of a gauge Higgs theory, however, there are no color electric flux tubes and therefore no spectrum of string excitations associated with an isolated fermion. I will argue in this article, however, that there may still exist excited states of isolated fermions, corresponding to a spectrum of excitations of the surrounding gauge and Higgs fields. Lattice Monte Carlo evidence for such a possibility is presented below, in an SU(3) gauge Higgs theory with a unimodular scalar field in the fundamental representation of the gauge group.

II. VARIETIES OF CONFINEMENT

Let us begin with ordinary QCD, and ask the question: what is the binding energy of a hadron, e.g., the J/ψ ? Of course it is impractical to address this question experimentally. Any attempt to “ionize” a quarkonium state will

just result in more hadrons, rather than a well separated pair of color-charged particles. Nevertheless, there exist states in the physical Hilbert space which correspond to precisely that latter situation. For massive, static quarks, such states have the form,

$$\Psi_V(R) = \bar{q}^a(\mathbf{x}) V^{ab}(\mathbf{x}, \mathbf{y}; U) q^b(\mathbf{y}) \Psi_0, \quad (1)$$

where $\bar{q}$, q are quark/antiquark operators transforming in the fundamental representation of the gauge group, a , b are color indices, and Ψ_0 is the vacuum state. The V operator transforms under a gauge transformation like a Wilson line running between points $\mathbf{x}$ and $\mathbf{y}$, and depends *only* on the gauge field U_i (we assume throughout a lattice regularization), and not on the quark fields or any other matter fields. In QCD, a state of this kind might represent a quark of electric charge $+2/3$, some long distance away from an antiquark of electric charge $-2/3$, with no other electric charge in the region. Of course such a state would not persist for long and would soon decay into a set of integer charged hadrons, as indicated in Fig. 1. But the point is that states with a large separation between a quark-antiquark pair, unscreened by any other matter fields, do exist in the Hilbert space. We are interested in how the energy of this subclass of states varies with quark separation.

Let $E_V(R)$, with $R = |\mathbf{x} - \mathbf{y}|$, be the expectation value of the energy of the state Ψ_V above the vacuum energy. We say that the binding energy of a $q\bar{q}$ state is infinite, or that QCD has the property of “separation-of-charge” (S_c) confinement [1], if and only if,

$$\lim_{R \rightarrow \infty} E_V(R) = \infty, \quad (2)$$

for any choice of the operator $V(\mathbf{x}, \mathbf{y}; U)$, again with the important restriction that V is a functional of the gauge field

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FIG. 1. Decay of a state with widely separated quark-antiquark color charges and fractional electric charge into a set of color neutral hadrons of integer electric charge. The property of S_c confinement is related to the energy of the color charge separated state $\Psi_V(R)$, in the limit of color charge separation $R \rightarrow \infty$.

only. Of course it is always possible to choose a particular V such that this condition is satisfied; an example is a Wilson line running between $\mathbf{x}$ and $\mathbf{y}$, in which case (2) holds even in a nonconfining theory such as QED. The S_c condition requires that the condition is satisfied for *every* V .

Now suppose, instead of QCD, we consider a pure gauge theory with only massive, static quarks at points $\mathbf{x}, \mathbf{y}$. Then at any R there is a spectrum of energy eigenstates,

$$\Psi_n(R) = \bar{q}^a(\mathbf{x}) V_n^{ab}(\mathbf{x}, \mathbf{y}; U) q^b(\mathbf{y}) \Psi_0, \quad (3)$$

which correspond to the ground and excited states of a color electric flux tube, running between points $\mathbf{x}, \mathbf{y}$. Such a spectrum has in fact been observed in lattice Monte Carlo simulations [2].¹ Of course states whose excitation energy exceeds the mass of a glueball cannot be energy eigenstates, since they can emit a glueball and fall into a lower energy state. But below this limit the spectrum is stable. In QCD there are no stable flux tube states for separations R greater than some critical distance, due to string breaking. Nevertheless, some of the Ψ_n may still exist as metastable states in the theory. In fact, in QCD, this is exactly the case for the resonances which lie on linear Regge trajectories. These are not stable states, of course, but rather correspond to metastable excitations of the color electric flux tube.

Let us now consider gauge Higgs theories, with the Higgs scalar in the fundamental representation of the gauge group. As was emphasized recently by Matsuyama and myself in Ref. [4], such theories have at least two distinct phases: a confinement phase, with the property of S_c confinement defined above, and a Higgs phase in which the S_c confinement property is lost, and the global Z_N center subgroup of the local $SU(N)$ gauge group is spontaneously broken. The Higgs phase turns out to be closely analogous to a spin glass phase, and there is a gauge invariant order parameter for the confinement-to-Higgs transition, which is a direct translation, from condensed

¹For a Nambu-Goto string stretched between two fixed points, the excitation spectrum was derived by Arvis [3].

matter to a gauge theory context, of the Edwards-Anderson order parameter for spin glass transitions. The asymptotic spectrum of the Higgs phase still consists of color singlets, along the lines discussed by Fröhlich *et al.* [5] and 't Hooft [6] (see also Maas *et al.* [7]). We refer to this property as “color” (C) confinement; it is much weaker than the S_c confinement condition. Thus the Higgs phase is distinguished from the confinement phase both by symmetry, and by type of confinement. The two phases are not, however, necessarily separated by some line of nonanalyticity in the free energy [8,9]. In this sense the transition may be analogous to a Kertesz line [10].

III. PSEUDOMATTER FIELDS

In discussing the spectrum of elementary fermions in the Higgs phase, the concept of a “pseudomatter” operator will be crucial. A pseudomatter operator $\rho^a(\mathbf{x}; U)$ is a nonlocal functional of the gauge field which transforms, at point $\mathbf{x}$, like a matter field in the *fundamental* representation of the gauge group, with the important exception that, like the gauge field itself, it is insensitive to transformations in the global center subgroup of the gauge group. Hence a pseudomatter operator transforms in the same way under gauge transformations $g(\mathbf{x})$ and $zg(\mathbf{x})$, where z is an element of the global center subgroup. The simplest example of a pseudomatter operator, in an infinite volume, comes from the Abelian theory,

$$\rho(\mathbf{x}; A) = \exp \left[-i \frac{e}{4\pi} \int d^3z A_i(z) \frac{\partial}{\partial z_i} \frac{1}{|\mathbf{x} - \mathbf{z}|} \right]. \quad (4)$$

Let us consider gauge transformations $g(\mathbf{x}) = e^{i\theta(\mathbf{x})}$, and we separate out the zero mode $\theta(\mathbf{x}) = \theta_0 + \tilde{\theta}(\mathbf{x})$. It is easy to verify that under such transformations,

$$\rho(\mathbf{x}; g \circ A) = e^{i\tilde{\theta}(\mathbf{x})} \rho(\mathbf{x}; A). \quad (5)$$

Using this pseudomatter field, one may construct physical states corresponding to an isolated point charge,

$$\Psi' = \rho^\dagger(\mathbf{x}; A) \psi(\mathbf{x}) \Psi_0. \quad (6)$$

This construction is very well known and was first introduced by Dirac [11]. Note that while the operator $\rho^\dagger(\mathbf{x}; A) \psi(\mathbf{x})$ is invariant under local gauge transformations, it still transforms under global $U(1)$ transformations. This is the hallmark of an operator which can create a physical state associated with a definite isolated charge, given that the ground state is itself an eigenstate of zero charge.

Note also that the gauge transformation defined by $g_C(\mathbf{x}; A) = \rho^\dagger(\mathbf{x}; A)$ is precisely the transformation to Coulomb gauge, so in that gauge $\Psi' = \psi(\mathbf{x}) \Psi_0$. The fact that a gauge choice defines a set of pseudomatter operators

is quite general and is not restricted to the Abelian theory. Let $g_F^{ab}(\mathbf{x}; U)$ be the transformation to a physical gauge defined by some condition $F(U) = 0$ imposed on spacelike links in an $SU(N)$ gauge theory. Then we may always express $g_F^\dagger(\mathbf{x}; U)$ at any point $\mathbf{x}$ in terms of its eigenvectors (enumerated by the index n),

$$g_F^{\dagger an}(\mathbf{x}; U) = u_{(n)}^a(\mathbf{x}; U), \quad u_{(m)}^{\dagger a}(\mathbf{x}; U) u_{(n)}^a(\mathbf{x}; U) = \delta_{mn}. \quad (7)$$

Now let g be any infinitesimal gauge transformation. Then g_F must have the property,

$$g_F^\dagger(\mathbf{x}; g \circ U) = g(\mathbf{x}) g_F^\dagger(\mathbf{x}; U), \quad (8)$$

which means that

$$u_{(n)}(\mathbf{x}; g \circ U) = g(\mathbf{x}) u_{(n)}(\mathbf{x}; U), \quad (9)$$

from which we conclude that $u_{(n)}(\mathbf{x}; U)$ is a pseudomatter field. This observation can be turned around: from a set of N orthogonal pseudomatter fields, with orthogonality defined by

$$\sum_{\mathbf{x}} \rho_n^{\dagger a}(\mathbf{x}; U) \rho_m^a(\mathbf{x}; U) = \delta_{nm}, \quad (10)$$

it is possible to construct another set of pseudomatter fields $u_{(n)}^a(\mathbf{x}; U)$ which define a gauge choice, i.e., a transformation g_F to some physical gauge. This is the logic of the Laplacian gauge introduced by Vink and Wiese in [12], and the procedure for constructing the $u_{(n)}(\mathbf{x}; U)$ from a set of pseudomatter operators $\rho_n(\mathbf{x}; U)$ is outlined in that reference.

The eigenstates $\zeta_n(\mathbf{x}; U)$ of the covariant lattice Laplacian operator,

$$(-D_i D_i)_{xy}^{ab} \zeta_n^b(\mathbf{y}; U) = \lambda_n \zeta_n^a(\mathbf{x}; U), \quad (11)$$

where

$$\begin{aligned} & (-D_i D_i)_{xy}^{ab} \\ &= \sum_{k=1}^3 \left[2\delta^{ab} \delta_{xy} - U_k^{ab}(\mathbf{x}) \delta_{y, \mathbf{x}+\hat{k}} - U_k^{\dagger ab}(\mathbf{x}-\hat{k}) \delta_{y, \mathbf{x}-\hat{k}} \right], \end{aligned} \quad (12)$$

are all examples of pseudomatter fields and will be especially important here. Once again, these fields transform like matter in the fundamental representation of the gauge group, apart from their invariance under the global Z_N subgroup of the $SU(N)$ gauge group. The ‘‘Laplacian Landau gauge’’ introduced by Vink and Wiese [12] made use of the low-lying eigenstates of the Laplacian operator in

four Euclidean dimensions. In the next section we will be concerned with the low-lying eigenstates of the three-dimensional lattice Laplacian operator (12), defined at fixed time on a $D = 4$ dimensional lattice.²

It was shown in [4] that in the spin glass (i.e., Higgs) phase of the gauge Higgs theory, it is always possible to find a physical gauge defined by $F(U) = 0$ such that $\langle \phi \rangle$ is nonzero, i.e.,

$$\langle g_F(\mathbf{x}; U) \phi(\mathbf{x}) \rangle \neq 0. \quad (13)$$

A corollary is the loss of S_c confinement in the Higgs phase. If $g_F(\mathbf{x}; U)$ is the gauge transformation to a gauge in which $\langle \phi \rangle$ is nonzero (and g_F , as just pointed out, can always be decomposed into a set of pseudomatter fields), then one may choose

$$V^{ab}(\mathbf{x}, \mathbf{y}; U) = g_F^{\dagger ac}(\mathbf{x}; U) g_F^{cb}(\mathbf{y}; U), \quad (14)$$

and show that $E_V(R)$ has a finite limit at $R \rightarrow \infty$. Conversely, in the phase of unbroken global Z_N gauge symmetry, and assuming the absence of a massless phase, we must have $E_V(R) \rightarrow \infty$, i.e., S_c confinement, in the same limit. For details, cf. [4]. Physical quark-antiquark states in the confined phase, with finite energy in the $R \rightarrow \infty$ limit, are created by operators such as

$$Q(\mathbf{x}, \mathbf{y}) = [\bar{q}^a(\mathbf{x}) \phi^a(\mathbf{x})] \times [\phi^{\dagger b}(\mathbf{y}) q^b(\mathbf{y})], \quad (15)$$

which can be thought of as creating two color neutral quark-scalar bound states.

A. Pseudomatter in finite volumes

One cannot create a single electric charge in a finite volume with periodic boundary conditions. The reason that the construction (4) does not work in a finite volume, in ordinary QED, is that the equation $-\nabla^2 D(\mathbf{z}) = \delta^3(\mathbf{x} - \mathbf{z})$ is not soluble in a finite periodic volume. Instead one must create $\pm$ charges in pairs, as in (1), with

$$\begin{aligned} V(\mathbf{x}, \mathbf{y}; A) &= \exp \left[-ie \int d^3 z A_i(\mathbf{z}) \frac{\partial}{\partial z_i} D(\mathbf{z}) \right] \\ &\quad - \nabla^2 D(\mathbf{z}) = \delta^3(\mathbf{z} - \mathbf{x}) - \delta^3(\mathbf{z} - \mathbf{y}). \end{aligned} \quad (16)$$

Likewise, in the non-Abelian case, the eigenstates $\zeta_n(\mathbf{x}; U)$ are determined only up to a global gauge-invariant phase. Unless these operators occur in pairs such that the global phases cancel, i.e., $\zeta_n^a(\mathbf{x}; U) \zeta_n^{\dagger b}(\mathbf{y}; U)$, they will vanish in expectation values due to wild fluctuations in the global phase.

²These eigenstates are computed numerically via the Arnoldi algorithm, as implemented in the ARPACK software package (<https://www.caam.rice.edu/software/ARPACK/>).

In general, in a finite volume, we consider in the Higgs phase operators of the form,

$$V^{ab}(\mathbf{x}, \mathbf{y}; U) = \sum_n c_n \rho_n^a(\mathbf{x}; U) \rho_n^{\dagger b}(\mathbf{y}; U), \quad (17)$$

and consider taking the $R = |\mathbf{x} - \mathbf{y}| \rightarrow \infty$ limit (along with the infinite volume limit). Then instead of (13), the criterion for spontaneous breaking of global Z_N gauge symmetry is the existence of a finite limit in the correlator,

$$\lim_{R \rightarrow \infty} |\langle \phi^{\dagger a}(\mathbf{x}) V^{ab}(\mathbf{x}, \mathbf{y}; U) \phi^b(\mathbf{y}) \rangle| > 0, \quad (18)$$

for some V . In the Higgs phase, there will always exist a transformation $g_F(\mathbf{x}; U)$ to some F -gauge, such that this criterion is satisfied by the V operator in (14).³

IV. EXCITATIONS OF FERMIONS

I now put forward the conjecture that just as there is a set of metastable states (3) in the confined phase at fixed R , so there is also a spectrum of excitations of a static fermion-antifermion system in the Higgs phase, at least for some gauge Higgs theories, with a finite energy above the ground state out to $R \rightarrow \infty$. The term “ground state” now refers not to the vacuum, but to the minimal energy state containing a static fermion-antifermion pair. It is supposed that this gap in energy is too small to be explained simply by the presence of additional vector or Higgs bosons. I support this conjecture with an example. The model is SU(3) lattice gauge theory with a standard Wilson action and a unimodular $[\phi^\dagger(x)\phi(x) = 1]$ Higgs field in the fundamental representation,

$$S = -\frac{\beta}{3} \sum_{\text{plaq}} \text{ReTr}[U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x)] - \gamma \sum_{x, \mu} \text{Re}[\phi^\dagger(x) U_\mu(x) \phi(x + \hat{\mu})]. \quad (19)$$

The methods of Ref. [4] can be used to determine the transition between the S_c confining and the spin glass (Higgs) phases. In this article I will work at the Wilson coupling $\beta = 5.5$ and a variety of γ . At this β value

³There is an alternative approach to estimating the left-hand side of (13) in some F -gauge, which is generally employed in computer simulations, via computation of the quantity,

$$\frac{1}{V} \left\langle \sum_{t=1}^{L_t} \left| \sum_x g_F(\mathbf{x}; U) \phi(\mathbf{x}, t) \right| \right\rangle,$$

on a $V = L^3 \times L_t$ lattice volume, and extrapolation to infinite spatial volume. The modulus of the sum over $\mathbf{x}$ is used to eliminate the ambiguity with respect to any remnant global symmetry transformations in the F -gauge.

the extrapolation method of [4] yields an estimate of $\gamma = 1.35(3)$ at the transition.

Now consider, at each $R = |\mathbf{x} - \mathbf{y}|$, the following set of four (in general nonorthogonal) states:

$$\Phi_n(R) = Q_n(R) \Psi_0, \quad (20)$$

where, for $n = 1, 2, 3$,

$$Q_n(R) = [\bar{q}^a(\mathbf{x}) \zeta_n^a(\mathbf{x}; U)] \times [\zeta_n^{\dagger b}(\mathbf{y}; U) q^b(\mathbf{y})], \quad (21)$$

and

$$Q_4(R) = [\bar{q}^a(\mathbf{x}) \phi^a(\mathbf{x})] \times [\phi^{\dagger b}(\mathbf{y}) q^b(\mathbf{y})], \quad (22)$$

where the $\zeta_n(\mathbf{x}; U)$ are pseudomatter operators corresponding to eigenstates of the lattice Laplacian with the three largest eigenvalues λ_n . Of course the Φ_n states all have the form (1), with

$$V_n^{ab}(\mathbf{x}, \mathbf{y}; U) = \zeta_n^a(\mathbf{x}; U) \zeta_n^{\dagger b}(\mathbf{y}; U) \quad (n = 1, 2, 3) \\ V_4^{ab}(\mathbf{x}, \mathbf{y}; \phi) = \phi^a(\mathbf{x}) \phi^{\dagger b}(\mathbf{y}). \quad (23)$$

As already mentioned, the Higgs and confinement phases are distinguished by the spontaneous breaking of the global Z_3 center subgroup of SU(3) gauge symmetry in the Higgs phase, and what this implies is that in the unbroken, S_c confinement phase, $\Phi_4(R)$ is orthogonal to the other three states in the $R \rightarrow \infty$ limit. The reason is that the operator $\bar{q}^a(\mathbf{x}) \phi^a(\mathbf{x})$ is invariant under all gauge transformations, while $\bar{q}^a(\mathbf{x}) \zeta_n^a(\mathbf{x}; U)$ transforms under the global Z_N subgroup, since $\bar{q}$ transforms under this symmetry, while $\zeta_n^a(\mathbf{x}; U)$ does not. This implies the orthogonality just stated, providing the vacuum itself is invariant under the global Z_3 gauge symmetry.

In order to compute energy expectation values $E_n^\Phi(R)$ corresponding to the $\Phi_n(R)$ states, we begin from the Euclidean time identity,

$$\langle Q_m^\dagger(R, t) Q_n(R, 0) \rangle = \langle \Phi_m(R) | e^{-(H - \mathcal{E}_0)t} | \Phi_n(R) \rangle, \quad (24)$$

where $\mathcal{E}_0$ is the vacuum energy, and $Q(R, t)$ indicates that the operator $Q(R)$ is evaluated at time t . Then we see that the energy of state $\Phi_n(R)$ above the vacuum energy is given by

$$E_n^\Phi(R) = - \left[\frac{d}{dt} \log \langle Q_m^\dagger(R, t) Q_n(R, 0) \rangle \right]_{t=0}, \quad (25)$$

and the appropriately normalized overlap of states Φ_m, Φ_n is

$$\begin{aligned}
o_{mn}(R) &= \frac{\langle \Phi_m | \Phi_n \rangle}{\sqrt{\langle \Phi_m | \Phi_m \rangle \langle \Phi_n | \Phi_n \rangle}} \\
&= \frac{\langle Q_m^\dagger(R, 0) Q_n(R, 0) \rangle}{\{\langle Q_m^\dagger(R, 0) Q_m(R, 0) \rangle \langle Q_n^\dagger(R, 0) Q_n(R, 0) \rangle\}^{1/2}}.
\end{aligned} \quad (26)$$

This may be generalized. We define

$$E_n^\Phi(R, T) = - \left[\frac{d}{dt} \log \langle Q_n^\dagger(R, t) Q_n(R, 0) \rangle \right]_{t=T}, \quad (27)$$

and

$$o_{mn}(R, T) = \frac{\langle Q_m^\dagger(R, T) Q_n(R, 0) \rangle}{\{\langle Q_m^\dagger(R, T) Q_m(R, 0) \rangle \langle Q_n^\dagger(R, T) Q_n(R, 0) \rangle\}^{1/2}}. \quad (28)$$

These can be interpreted as the energies and the overlaps of states obtained by evolving the Φ_n for a Euclidean time interval $T/2$, i.e., $\Phi_n(R, T/2) = \exp[-HT/2]\Phi_n(R)$, followed by normalization.

With discretized time on a hypercubic lattice, the logarithmic time derivative must be replaced by the corresponding lattice expression,

$$E_n^\Phi(R, T) = - \log \left[\frac{\langle Q_n^\dagger(R, T) Q_n(R, 0) \rangle}{\langle Q_n^\dagger(R, T-1) Q_n(R, 0) \rangle} \right]. \quad (29)$$

For T an odd integer, this is interpreted as the energy expectation value (minus the vacuum energy) of a state evolved for $(T-1)/2$ units of Euclidean time. The $Q^\dagger Q$ correlators are computed on the lattice as follows: define a timelike Wilson line,

$$P(\mathbf{x}, t, T) = U_0(\mathbf{x}, t) U_0(\mathbf{x}, t+1) \dots U_0(\mathbf{x}, T-1). \quad (30)$$

Then, after integrating out the static fermions, and discarding, since we are only interested in the energy due to the dynamical fields, an irrelevant quark mass (hopping parameter) factor,

$$\begin{aligned}
\langle Q^\dagger(R, T) Q(R, 0) \rangle &= \langle \text{Tr}[V_i^\dagger(\mathbf{x}, \mathbf{y}, U(t+T)) P^\dagger(\mathbf{x}, t, T) \\
&\quad \times V_j(\mathbf{x}, \mathbf{y}; U(t)) P(\mathbf{y}, t, T)] \rangle.
\end{aligned} \quad (31)$$

In the numerical calculation of this quantity we average over all $\mathbf{x}, \mathbf{y}$ with fixed $R = |\mathbf{x} - \mathbf{y}|$.

The overlap $o_{41}(R)$ between the state $\Phi_4(R)$ constructed with the Higgs field, and the state $\Phi_1(R)$ built with the ζ_1 pseudomatter field, is displayed in Fig. 2(a) in the confined phase, at $\beta = 5.5$, $\gamma = 0.5$. We see that this overlap tends rapidly to zero as $R \rightarrow \infty$, as required by the invariance of the vacuum, in the confined phase, under global Z_3 gauge

transformations. This global subgroup of the gauge symmetry is broken in the Higgs phase, so that $\Phi_1(R)$ and $\Phi_4(R)$ are not necessarily orthogonal in the $R \rightarrow \infty$ limit. That is what we see in Fig. 2(b), with data obtained in the Higgs phase, at $\beta = 5.5$, $\gamma = 3.5$, where the overlap between these states is quite large. It is also found in the confinement phase, in Fig. 2(c), that the energy of the quark-pseudomatter state $E_1^\Phi(R)$ rises linearly with R , consistent with S_c confinement. The energy of $E_4^\Phi(R)$ is almost R independent at $R > 1$, which reflects the fact that $\Phi_4(R)$ consists of a noninteracting pair of color singlet (quark-Higgs) objects. In the Higgs phase the energies of both the Φ_1 and Φ_4 states are nearly R -independent, as seen in Fig. 2(d).⁴

Now let $\tau = \exp(-H)$ be the operator corresponding to the lattice transfer matrix, and we would like to calculate the eigenstates and eigenvalues of this operator in the Higgs phase, in the four-dimensional subspace of Hilbert space spanned by the nonorthogonal set of states $\{\Phi_n\}$. We define the 4×4 matrices $[T], [O]$ whose matrix elements are

$$\begin{aligned}
[T]_{mn} &= \langle \Phi_m | e^{-(H-\mathcal{E}_0)} | \Phi_n \rangle \\
&= \langle Q_m^\dagger(R, 1) Q_n(R, 0) \rangle \\
[O]_{mn} &= o_{mn},
\end{aligned} \quad (32)$$

respectively. The eigenstates and eigenvalues of $\mathcal{T} = \tau e^{\mathcal{E}_0}$ in the subspace are obtained by solving the generalized eigenvalue problem,

$$[T] \mathbf{v}^{(n)} = \lambda_n [O] \mathbf{v}^{(n)}, \quad (33)$$

and we have energies above the vacuum energy $\mathcal{E}_0$, given by

$$E_n(R) = -\log(\lambda_n), \quad (34)$$

and ordered such that E_n increases with n , corresponding to eigenstates in the subspace,

$$\Psi_n(R) = \sum_{i=1}^4 v_i^{(n)} \Phi_i(R). \quad (35)$$

⁴It should be noted that the property of S_c confinement in the confinement phase implies that the energies $E_\Phi(R)$ of states $\Phi_{1-3}(R)$ diverge to infinity as $R \rightarrow \infty$. But it is not necessarily true that the energies of these particular states have a finite limit as $R \rightarrow \infty$ everywhere in the Higgs phase, although this finite limit is in fact seen for $\beta = 5.5$ at $\gamma > 1.4$. While there must always exist, everywhere in the Higgs phase, finite energy states corresponding to isolated (i.e., $R \rightarrow \infty$) fermions, these need not be the $n = 1, 2, 3$ states listed in (20), which correspond to a particular choice of the V operator. For a further discussion of this point, cf. [4].

FIG. 2. Contrasting properties of pseudomatter states in the confinement and Higgs phases of an SU(3) gauge Higgs theory. (a) Overlap vs R of normalized fermion-antifermion states using pseudomatter (Φ_1) and the Higgs field (Φ_4) states in the confined phase, at $\beta = 5.5$, $\gamma = 0.5$. (b) Same as subfigure (a), but in the Higgs phase at $\beta = 5.5$, $\gamma = 3.5$. (c) Energy expectation value $E_\Phi(R)$ vs separation R of the Φ_1 and Φ_4 states in the confined phase, $\beta = 5.5$, $\gamma = 0.5$. (d) Same as subfigure (c), but in the Higgs phase at $\beta = 5.5$, $\gamma = 3.5$.

Likewise we consider evolving the states Ψ_n in Euclidean time,

$$\begin{aligned} \mathcal{T}_{nn}(R, T) &= \langle \Psi_n | e^{-(H-\mathcal{E}_0)T} | \Psi_n \rangle \\ &= v_i^{*(n)} \langle \Phi_i | e^{-(H-\mathcal{E}_0)T} | \Phi_j \rangle v_j^{(n)} \\ &= v_i^{*(n)} \langle Q_i^\dagger(R, T) Q_j(R, 0) \rangle v_j^{(n)}, \end{aligned} \quad (36)$$

and compute

$$E_n(R, T) = -\log \left[\frac{\mathcal{T}_{nn}(R, T)}{\mathcal{T}_{nn}(R, T-1)} \right]. \quad (37)$$

This can be regarded (for T an odd integer) as the energy expectation value of state $\Psi_n(R)$ which has evolved for $(T-1)/2$ units of Euclidean time. Of course (36) generalizes to off diagonal elements $\mathcal{T}_{mn}$ in an obvious way.

There are several possibilities, for each Ψ_n :

- (1) $\Psi_n(R)$ is an exact eigenstate of the transfer matrix in the full Hilbert space. Then $E_n(R) = E_n(R, T)$ is time independent. This situation is rather unlikely.
- (2) $\Psi_n(R)$ has a substantial overlap with the true ground state and therefore evolves steadily, in Euclidean time, towards that ground state. Then $E_n(R, T)$ drops rapidly to the lowest possible energy of the static quark-antiquark system with increasing T .
- (3) $\Psi_n(R)$ has very little overlap with the ground state and rapidly evolves in Euclidean time to a stable or metastable excited state. Then $E_n(R, T)$ converges to a value which is almost constant, over some range of Euclidean time, above the ground state energy. This is the interesting situation.

Figure 3 displays energies $E_n(R, T)$ for $n = 1, 2$ corresponding to Euclidean time evolution of states $\Psi_{1,2}$, with T ranging from 4 to 12 for the $n = 1$ state, and 4 to 10 for the $n = 2$ state, with (off axis) charge separations $R = |\mathbf{x} - \mathbf{y}| \leq 10$, having components $|x_i - y_i| \leq 6$ in the three spatial directions. The energies are computed in the

FIG. 3. Energies $E_n(R, T)$, defined in (37), of states Ψ_1, Ψ_2 after evolution for a period of $(T-1)/2$ units of Euclidean time, at lattice couplings $\beta = 5.5, \gamma = 3.5$. Note the energy gap, which persists out to the largest T values shown, of $E_2(R, T) - E_1(R, T) \approx 0.2$ in lattice units.

Higgs phase at $\beta = 5.5, \gamma = 3.5$, on a lattice volume of $14^3 \times 32$. Data are obtained from 220 lattices separated by 100 Monte Carlo update sweeps. For both $n = 1, 2$ there seems to be a rapid convergence with increasing T to the ground ($n = 1$) and an excited ($n = 2$) state energy, respectively, separated by an energy gap of ≈ 0.2 in units of inverse lattice spacing. Thus the third possibility, of the three just enumerated, appears to be realized in this theory for the $n = 2$ state. That is the main result, offered in support of the conjecture that there might exist excited states of noncomposite fermions in some gauge Higgs theories. Data for $E_n(R, T)$ for the $n = 3, 4$ states are very noisy, and no conclusions can be drawn about them at this stage. For $n = 2$, the data become rather noisy for $T > 10$; nevertheless the data at $T = 12$ simply fluctuate around the value obtained at lower T . In Figs. 4(a) and 4(b) we display separately the data for $E_1(R, T)$ and $E_2(R, T)$, including the data at $T = 1, 2$, and (for $n = 2$) the noisy data at $T = 12$. Note that while the $\Psi_{1,2}$ are clearly not energy eigenstates, they converge rapidly in Euclidean time to stable states already at $T = 4$.

Of course the $\Psi_{1,2}$ states are orthogonal by construction. But in principle this orthogonality need not persist under Euclidean time (as opposed to real time) evolution, beyond $T = 1$. However, the rapid convergence of $\Psi_{1,2}$ to states with differing energies implies the near-orthogonality of the two states under Euclidean time evolution. In fact the overlap corresponding to off diagonal matrix elements,

$$\begin{aligned} \mathcal{O}(R, T) &= \frac{\langle \Psi_1 | e^{-HT} | \Psi_2 \rangle}{\sqrt{\langle \Psi_1 | e^{-HT} | \Psi_1 \rangle \langle \Psi_2 | e^{-HT} | \Psi_2 \rangle}} \\ &= \frac{\mathcal{T}_{12}(R, T)}{\sqrt{\mathcal{T}_{11}(R, T) \mathcal{T}_{22}(R, T)}}, \end{aligned} \quad (38)$$

FIG. 4. Energies (a) $E_1(R, T)$, and (b) $E_2(R, T)$ at $\beta = 5.5, \gamma = 3.5$, showing the convergence from $T = 1$ to $T = 14$ ($n = 1$) and $T = 12$ ($n = 2$). Neither Ψ_1 nor Ψ_2 is an energy eigenstate, but both appear to rapidly converge towards different eigenstates after a short evolution in Euclidean time.

can be calculated for any R, T . This has the interpretation of an overlap between states obtained from $\Psi_{1,2}$ evolved for $T/2$ units of Euclidean time, and then normalized. The result, for $T = 2, 4, 8, 10$ (again at $\beta = 5.5, \gamma = 3.5$) is shown in Fig. 5. It is clear that the states obtained from evolving Ψ_1, Ψ_2 in Euclidean time are very nearly orthogonal, as we had already deduced.

However, there is still the possibility that the energy gap seen in Fig. 3 is not really due to an excitation of the gauge field surrounding the static fermions, but is rather due to some low momentum particle excitation, e.g., a massive vector boson, or some other particle state. This would be the case in ordinary QED, where any excited state of a static dipole simply consists of the dipole field plus photons. In order to rule this out, it is necessary to compare the excitation energy of an excited state with the mass of the vector boson in the Higgs phase. In this connection it is important to observe that the particle spectrum of an $SU(3)$

FIG. 5. The overlap $\mathcal{O}(R, T)$ between states $\Psi_1(R)$ and $\Psi_2(R)$ which are evolved (and then normalized) for $T/2$ units of Euclidean time. $\mathcal{O}(R, T) = 0$ by construction at $T = 0, 1$, but the Euclidean time-evolved states are seen to remain approximately orthogonal for $R > 2$ at larger T .

gauge Higgs theory is not necessarily the spectrum that might be expected perturbatively, for reasons that have been discussed at length by Maas *et al.* [7]. Briefly, if one follows the approach of Fröhlich *et al.* [5] and 't Hooft [6], reasoning that particles in the asymptotic spectrum are created by local gauge invariant operators or, more precisely, that they show up as poles in the correlation functions of such local operators, then the correspondence between the perturbative and the actual spectrum in the electroweak theory is to some extent coincidental, a consequence of the approximate SU(2) custodial symmetry, and does not extend to higher gauge groups.

The spectrum of an SU(3) gauge Higgs theory in the Higgs phase was determined by lattice simulations in [13], in a theory with somewhat different lattice couplings, and with a fluctuating (rather than unimodular) scalar field in the fundamental representation. It was found that the lightest state was a 1^{--} vector meson, whose mass could also be determined analytically from correlators of the gauge-invariant operator,

$$\phi^\dagger(\mathbf{x}) U_k(\mathbf{x}) \phi(\mathbf{x} + \hat{k}). \quad (39)$$

Assuming that such an operator also creates the lightest state in the version of SU(3) gauge Higgs theory under consideration here, I have estimated the mass of this lightest state via the standard procedure of projecting, at fixed time, to the zero momentum component,

$$Q_k(t) = \sum_{\mathbf{x}} \phi^\dagger(\mathbf{x}, t) U_k(\mathbf{x}, t) \phi(\mathbf{x} + \hat{k}, t), \quad (40)$$

and then computing the Euclidean time correlation function of the zero momentum operators,

FIG. 6. The Euclidean time correlator $G(T)$ [see Eq. (41)], shown on a log scale, associated with a zero-momentum vector meson state. The couplings are $\beta = 5.5, \gamma = 3.5$. The straight line is a best fit through the three data points at $T > 0$ and yields an estimate of the vector boson mass of 1.30(1) in lattice units.

$$G(T) = \frac{\langle \sum_{k=1}^3 Q_k^\dagger(T) Q_k(0) \rangle_{conn}}{\langle \sum_{k=1}^3 Q_k^\dagger(0) Q_k(0) \rangle_{conn}}, \quad (41)$$

where $\langle \dots \rangle_{conn}$ denotes the connected correlator. The data found on a 14^4 lattice volume, at $\beta = 5.5, \gamma = 3.5$, are plotted in Fig. 6 and corresponds to a vector boson mass of 1.30(1) in lattice units, which is more than 6 times larger than the energy gap of approximately 0.2 in lattice units seen in Fig. 3. The conclusion is that the energy gap between the ground and first excited states of the fermion-antifermion system cannot be interpreted as a threshold for production of a vector meson in a fermion-antifermion background.

One can also compute the time correlator of a gauge invariant, but spatially nonlocal operator, constructed by

FIG. 7. The overlap o_{14} [from Eq. (26)] vs R at fixed $\beta = 5.5$ and a selection of γ values, where o_{41} is the overlap between the normalized Φ_1 and Φ_4 states, which use pseudomatter and the Higgs field respectively to enforce gauge invariance. Note that $\gamma = 1.3$ is either within or very near the confined phase, while all other γ values are in the Higgs phase.

FIG. 8. Same as Fig. 3, but at smaller γ values, still at $\beta = 5.5$. (a) $\gamma = 2.15$; (b) $\gamma = 1.8$; (c) $\gamma = 1.4$, which is just above the transition to the confined phase. Note that the data for $n = 2$, for data drawn from a fixed number (220) of configurations, become noisier at smaller values of T , as γ approaches the transition. At $\gamma = 1.4$ we cannot draw any conclusions about the large T convergence of the $n = 2$ data.

replacing the Higgs field with an eigenstate of the covariant Laplacian operator, i.e.,

$$\mathcal{Q}'_k(t) = \sum_{\mathbf{x}} \zeta_1^\dagger(\mathbf{x}, t; U) U_k(\mathbf{x}, t) \zeta_1(\mathbf{x} + \hat{k}, t; U). \quad (42)$$

The corresponding time correlation function is almost indistinguishable from $G(T)$ in (41), and the mass is in agreement, within error bars, with the vector boson mass extracted from the data in Fig. 6. Note that if one used different Laplacian eigenstates on the right and left side of U_k in Eq. (42), the time correlator would vanish, due to wild fluctuations in the global phase of $\zeta_n(\mathbf{x}; U)$.⁵

It is natural to ask what happens as γ approaches the Higgs to confinement (spin glass to symmetric) transition at $\gamma \approx 1.35$, $\beta = 5.5$. As one might expect, the overlap ρ_{14} between the normalized Φ_1 and Φ_4 states, built from pseudomatter and the Higgs field respectively, drops steadily as γ is reduced, as seen in Fig. 7, and is consistent with zero at large R at $\gamma = 1.3$, which is either within or very near the confined phase.

As γ is reduced below $\gamma = 3.5$ in the Higgs phase at $\beta = 5.5$, both the ground and excited state energies gradually rise, but an energy gap remains. The data, for the fixed number of 220 lattices which were used at each γ , become noisier for $E_2(R, T)$ at small values of T as γ is reduced. These tendencies are illustrated in Fig. 8 for $E_n(R, T)$, again at $\beta = 5.5$ and $\gamma = 2.15, 1.8, 1.4$. The data for E_2 at $T = 10$, for $\gamma = 2.15$, and at $T = 8$ for $\gamma = 1.8$ have obviously quite a lot of statistical error, which presumably could be reduced by increasing statistics. Still, the existence of an energy gap between the $n = 1$ and $n = 2$ levels appears to persist, at least at these lower γ values. At a still lower value of $\gamma = 1.4$, which is quite close to the Higgs/confinement transition, the data so far obtained for $E_2(R, T)$ are very noisy beyond $T = 4$, and it is not possible to make any statement about convergence to a stable excitation.

V. CONCLUSIONS

I have constructed four gauge-invariant states for a static fermion-antifermion pair, at each fermion pair separation R , by combining the fermion operators with Higgs and pseudomatter operators. Working in the SU(3) gauge Higgs theory of Eq. (19) at lattice couplings $\beta = 5.5$ and $\gamma = 3.5$, it is found that one of these states (Ψ_1) converges rapidly, under Euclidean time evolution, to the ground state, while the Ψ_2 state also rapidly converges, but to a state with an energy above the ground state. Of course Ψ_1 and Ψ_2 are orthogonal by construction, but it appears that they remain almost orthogonal upon evolution in Euclidean time T . What is significant is that the energy gap between the $n = 1$ and $n = 2$ states is nearly T independent (for $T \geq 4$) and only weakly dependent on the fermion-antifermion separation R . The gap also seems to be small compared to the

⁵One could construct gauge-invariant states containing two Laplacian eigenstates and two vector bosons, which would be independent of the global phases. The energy calculation would then require computation of a time correlation function of four vector boson operators, which I have not attempted.

mass of the lowest lying particle excitation, assuming (based on the results of [13]) that the vector boson created by the operator (39) is the lightest particle. This gap, below the vector meson threshold, indicates the existence of at least one gauge + Higgs field excitation of the fermion-antifermion system which cannot be readily interpreted as a fermion-antifermion ground state plus an additional particle. It appears instead to be a stable excitation of the bosonic fields surrounding each of the elementary fermions. Since the gap, judging from Fig. 3, appears to have a finite limit at $R \rightarrow \infty$, this is a physical excitation which is relevant to fermion-antifermion pairs at infinite separation.

Of course these are only some first results indicating excitations of elementary fermions, and there are many open questions. First, it would be helpful to have a more systematic examination of the particle spectrum of the action (19), to confirm that the vector boson associated with

the operator (39) is in fact the lightest particle in the spectrum, and to map out the magnitude of the excitation gap throughout the Higgs/spin glass phase of the phase diagram. One would also like to generalize the action beyond special case of a unimodular Higgs field. Secondly, we would like to know whether there are additional excitations of the fermion-antifermion system beyond the one found here; perhaps this could be studied with a larger basis of Ψ states, and much improved statistics. Finally, it would be very interesting to know whether any of this is relevant to the electroweak theory or to phenomenology beyond the Standard Model. We reserve these questions for later investigation.

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