Study of the sensitivity to CKM angle γ under simultaneous determination from multiple B meson decay modes

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Several methods exist to measure CP violation observables related to the CKM unitarity triangle angle γ using B meson decays. These observables are different for every B meson decay considered, although the information they contain on γ is encoded in a similar way for all of them. This paper describes a strategy for a simultaneous measurement of γ using several B meson decays that takes into account possible correlations between them based on the methodologies described in [J. Garra Tico, [arXiv:1804.05597](https://arxiv.org/abs/1804.05597)]. Sensitivity studies demonstrate that the simultaneous analysis of several B meson decay modes results in smaller uncertainties and improved statistical behavior compared to a combination of standalone measurements.

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I. INTRODUCTION

The angle $\gamma \equiv \arg(-\frac{V_{ub}V_{cb}^*}{V_{ud}V_{cd}^*})$ of the Cabibbo-Kobayashi-Maskawa (CKM) unitarity triangle can be measured using tree-level B meson decays that involve interference between $b \rightarrow u$ and $b \rightarrow c$ quark transitions. Time-integrated measurements can be made to measure γ using decays of the type $B \rightarrow DX$, where D represents an admixture of the flavor eigenstates D^0 and $\bar{D}^0$ and X a final state containing one or more kaons or pions. Examples include $B^\pm \rightarrow D^{(*)}K^{(*)\pm}$, $B^0 \rightarrow DK^{*0}$, and $B^\pm \rightarrow D^{(*)}\pi^\pm$ decays. Alongside this, a time-dependent approach can be employed to measure γ from decays such as $B_s^0 \rightarrow D_s^- K^+$. More details on the extraction of the CKM angle γ can be found in Refs. [1–3].

Several methods can be utilized to measure different CP violation observables in these decays [4–10]. The measurements are then typically used to place tree-level constraints on γ without the need for any theoretical input. The current world average value of $\gamma = (71.1^{+4.6}_{-5.3})^\circ$ [2,11] is dominated by measurements from the LHCb experiment [12,13].

This paper presents results for a simultaneous approach to constrain γ from multiple B meson decays [14], allowing for the treatment of experimental candidates reconstructed under different decay hypotheses and for the

straightforward determination of correlations between systematic uncertainties. The technique employs a reduced set of CP violation parameters and is applicable to all possible measurement approaches. The results presented in this paper make a comparison between the simultaneous method and the traditional approach of fitting for each decay mode independently.

II. SIMULTANEOUS APPROACH FOR TIME-INTEGRATED MEASUREMENTS WITH $B \rightarrow DX$ DECAYS

A. Admixture coefficients $z_\pm^m$

As described above, several B meson decays produce admixtures of neutral D mesons that involve γ . In this paper, $|D^0\rangle$ and $|\bar{D}^0\rangle$ are used to denote the flavor eigenstates of neutral D mesons; $|D_+\rangle$ represents the D meson produced in B^+ or B^0 meson decays and $|D_-\rangle$ the D meson produced in B^- or $\bar{B}^0$ decays.

In general, one can write

$$\begin{aligned} |D_-^m\rangle &\sim |D^0\rangle + z_-^m |\bar{D}^0\rangle \\ |D_+^m\rangle &\sim |\bar{D}^0\rangle + z_+^m |D^0\rangle \end{aligned} \Rightarrow \begin{cases} A_-^m \sim A_D + z_-^m A_{\bar{D}} \\ A_+^m \sim A_{\bar{D}} + z_+^m A_D, \end{cases} \quad (1)$$

where m denotes the B decay mode under consideration. The amplitudes $A_D = \langle f | \mathcal{H} | D^0 \rangle$, $A_{\bar{D}} = \langle f | \mathcal{H} | \bar{D}^0 \rangle$, and $A_\pm^m = \langle f | \mathcal{H} | D_\pm^m \rangle$ define the D meson decay to a final state f . The complex coefficients $z_\pm^m$ are specific to each B decay, and are typically expressed in either Cartesian ($x_\pm^m, y_\pm^m$) or polar (r_m, δ_m, γ) coordinates as

$$z_\pm^m = x_\pm^m + iy_\pm^m = r_m e^{i\delta_m} e^{\pm i\gamma}, \quad (2)$$

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where all parameters with subscript or superscript m are specific to a particular B meson decay. It is apparent that r_m and δ_m represent the ratio of amplitude magnitudes and their strong phase difference. Using the definition

$$z_m = r_m e^{i\delta_m}, \quad (3)$$

leads to

$$z_{\pm}^m = z_m e^{\pm i\gamma}, \quad (4)$$

and reveals an invariant for each B meson decay,

$$\frac{z_+^m}{z_-^m} = e^{2i\gamma} \Rightarrow \gamma = \frac{1}{2} \arg \left(\frac{z_+^m}{z_-^m} \right). \quad (5)$$

If both $z_{\pm}^m$ coefficients are multiplied by any complex coefficient ξ , the result will contain exactly the same information on γ . In particular, it is always possible to relate the $z_{\pm}^m$ coefficients for channel m to those for $B^{\pm} \rightarrow DK^{\pm}$ decays, denoted $z_{\pm}$:

$$z_{\pm}^m = \xi_m z_{\pm}, \quad (6)$$

where

$$\xi_m = \frac{z_m}{z_{DK}}. \quad (7)$$

By definition, from Eqs. (3) and (7), the ξ_m coefficients do not depend on γ and can, therefore, be considered as nuisance parameters.

When global averages are extracted for γ [1], many of the different input measurements depend on the hadronic unknowns, r_m and δ_m , as well as γ . Subsequently, it is not valid to simply average the various measurements of γ . Instead, individual measurements extract values of $z_{\pm}^m$ which are used as inputs to a global combination. Thus, using the ξ_m coefficients to perform a simultaneous fit for the Cartesian parameters in N distinct B meson decay modes, reduces the number of independent parameters in the fit from $4N$ to $4 + 2(N - 1)$: 4 parameters for $B^{\pm} \rightarrow DK^{\pm}$, and 2 for each of the other decays. Although this is only one more parameter than a simultaneous fit for the $1 + 2N$ polar coordinates (γ, r_m, δ_m), it has the advantage that the real and imaginary components of $z_{\pm}$ and ξ_m are expected to exhibit Gaussian behavior and can be used in conjunction with other orthogonal measurements in a global combination.

B. The η function

In order to simplify later notation, it is useful to define the η function as

$$\eta(a, b, \kappa) = |a|^2 + |b|^2 + 2\kappa \text{Re}(a^* b), \quad (8)$$

where $a, b \in \mathbb{C}$ and $\kappa \in \mathbb{R}$. This function is symmetric with respect to the exchange of a and b [$\eta(a, b, \kappa) = \eta(b, a, \kappa)$] and scales as $\eta(a, b, \kappa) = |a|^2 \eta(1, \frac{b}{a}, \kappa)$. The κ coefficient, commonly known as the *coherence factor*, indicates the fraction of coherent sum that contributes to η ,

$$\eta(a, b, \kappa) = \kappa |a + b|^2 + (1 - \kappa)(|a|^2 + |b|^2). \quad (9)$$

In this paper, when the coherence factor argument is omitted, it should be assumed that it is implicit and, if one of the complex arguments is omitted, it should be assumed that it is 1 [for example, $\eta(a) \equiv \eta(a, 1, \kappa)$].

C. Signal amplitude

The probability distribution function for B mesons to decay via a particular decay mode m is proportional to the squared amplitude $|A_{\pm}^m|^2$ integrated over the phase space of the final state particles in the B decay,

$$p_{\pm}^m \sim \int d\mathcal{P}_B |A_{\pm}^m|^2. \quad (10)$$

For a specific B decay mode, defining A_c as the decay amplitude corresponding to a $b \rightarrow c$ transition and $A_u e^{\pm i\gamma}$ as the decay amplitude corresponding to a $b \rightarrow u$ transition leads to

$$A_- \sim A_c A_D + A_u e^{-i\gamma} A_{\bar{D}}, \quad (11)$$

$$A_+ \sim A_c A_{\bar{D}} + A_u e^{+i\gamma} A_D. \quad (12)$$

For a 2-body B meson decay, such as $B^{\pm} \rightarrow DK^{\pm}$, the amplitudes A_c and A_u are simply the magnitude of the transition amplitude integrated over the phase space, and one can write

$$A_- \sim A_D + z_- A_{\bar{D}}, \quad (13)$$

$$A_+ \sim A_{\bar{D}} + z_+ A_D, \quad (14)$$

where

$$z_{\pm} = \frac{A_u}{A_c} e^{\pm i\gamma}. \quad (15)$$

This implies that, for 2-body decays,

$$p_- \sim \eta(A_D, z_- A_{\bar{D}}), \quad (16)$$

$$p_+ \sim \eta(A_{\bar{D}}, z_+ A_D). \quad (17)$$

In the case of a multibody B meson decay with 3 or more particles in the final state, such as $B^0 \rightarrow DK\pi$, the amplitude $|A_{\pm}^m|^2$ may be integrated over a reduced part

of the B decay phase space, for example around the $K^{*0}(892)$ resonance for $B^0 \rightarrow DK\pi$.

By squaring the modulus of expressions (13) and (14) and defining

$$N_{\alpha\beta} = \int d\mathcal{P}_B A_\alpha^* A_\beta, \quad (18)$$

$$X_{\alpha\beta} = \frac{N_{\alpha\beta}}{\sqrt{N_{\alpha\alpha}N_{\beta\beta}}}, \quad (19)$$

one can write

$$p_- \sim |A_D|^2 + \frac{N_{uu}}{N_{cc}} |A_{\bar{D}}|^2 + 2|X_{cu}| \text{Re} \left(\sqrt{\frac{N_{uu}}{N_{cc}}} \frac{X_{cu}}{|X_{cu}|} e^{-i\gamma} A_D^* A_{\bar{D}} \right), \quad (20)$$

$$p_+ \sim |A_{\bar{D}}|^2 + \frac{N_{uu}}{N_{cc}} |A_D|^2 + 2|X_{cu}| \text{Re} \left(\sqrt{\frac{N_{uu}}{N_{cc}}} \frac{X_{cu}}{|X_{cu}|} e^{+i\gamma} A_D^* A_{\bar{D}} \right). \quad (21)$$

It should be noted that $|X_{\alpha\beta}| \leq 1$, because of the Cauchy-Schwarz inequality. Defining

$$\kappa = |X_{cu}|, \quad (22)$$

$$r = \sqrt{\frac{N_{uu}}{N_{cc}}}, \quad (23)$$

$$e^{i\delta} = \frac{X_{cu}}{|X_{cu}|}, \quad (24)$$

$$z = re^{i\delta}, \quad (25)$$

$$z_{\pm} = ze^{\pm i\gamma}, \quad (26)$$

the signal amplitude probability distribution for B meson decay mode m can then be expressed as

$$p_-^m \sim \eta(A_D, \xi_m z_- A_{\bar{D}}, \kappa_m) = |A_D|^2 + |\xi_m z_-|^2 |A_{\bar{D}}|^2 + 2\kappa_m \text{Re}(\xi_m z_- A_D^* A_{\bar{D}}), \quad (27)$$

$$p_+^m \sim \eta(A_{\bar{D}}, \xi_m z_+ A_D, \kappa_m) = |A_{\bar{D}}|^2 + |\xi_m z_+|^2 |A_D|^2 + 2\kappa_m \text{Re}(\xi_m z_+ A_D^* A_{\bar{D}}). \quad (28)$$

These expressions describe the physics of the neutral D meson admixture that leads to the different CP observables used to measure γ , but they are not specific to any measurement method.

III. SPECIFIC FORMALISM FOR ESTABLISHED METHODOLOGIES

A. Decays to CP eigenstates

This method (commonly referred to as the GLW method [4,5]) uses two sets of final states from the D meson decay. The first are those states that are accessible from only one of the D meson flavor eigenstates, either $|D^0\rangle$ or $|\bar{D}^0\rangle$, such that $A_f^D = \langle f_D | \mathcal{H} | D^0 \rangle$, $A_{\bar{f}}^{\bar{D}} = \langle f_{\bar{D}} | \mathcal{H} | \bar{D}^0 \rangle$, and $\langle f_{\bar{D}} | \mathcal{H} | D^0 \rangle = \langle f_D | \mathcal{H} | \bar{D}^0 \rangle = 0$. The second are those states that are accessible from one of the CP eigenstates $|D_{\pm}^{CP}\rangle$, such that $A_{\pm}^{CP} = \langle f_{\pm} | \mathcal{H} | D_{\pm}^{CP} \rangle$ and $\langle f_{\mp} | \mathcal{H} | D_{\pm}^{CP} \rangle = 0$.

The observables of interest for a given B decay mode m are ratios and asymmetries that can be used to constrain $z_{\pm}$ and ξ_m . For example, for $B^{\pm} \rightarrow D\pi^{\pm}$ decays,

$$R_{CP}^{\pm D\pi} = \frac{\Gamma(B^- \rightarrow D_{\pm}^{CP} \pi^-) + \Gamma(B^+ \rightarrow D_{\pm}^{CP} \pi^+)}{\Gamma(B^- \rightarrow D^0 \pi^-) + \Gamma(B^+ \rightarrow \bar{D}^0 \pi^+)} = \frac{1}{2} \left| \frac{A_{\pm}^{CP}}{A_f^D} \right|^2 \frac{\eta(\pm \xi_{D\pi} z_-) + \eta(\pm \xi_{D\pi} z_+)}{2}, \quad (29)$$

$$A_{CP}^{\pm D\pi} = \frac{\Gamma(B^- \rightarrow D_{\pm}^{CP} \pi^-) - \Gamma(B^+ \rightarrow D_{\pm}^{CP} \pi^+)}{\Gamma(B^- \rightarrow D_{\pm}^{CP} \pi^-) + \Gamma(B^+ \rightarrow D_{\pm}^{CP} \pi^+)} = \frac{\eta(\pm \xi_{D\pi} z_-) - \eta(\pm \xi_{D\pi} z_+)}{\eta(\pm \xi_{D\pi} z_-) + \eta(\pm \xi_{D\pi} z_+)}. \quad (30)$$

B. Decays to Cabibbo-favored and doubly-Cabibbo-suppressed final states

This method (commonly referred to as the ADS method [6,7]) uses final states that are accessible from both neutral D meson flavor eigenstates, enhancing the possible CP asymmetry by considering the interference between a favored B meson decay followed by a doubly CKM-suppressed D decay, and a suppressed B meson decay followed by a CKM-favored D decay. For the example of $B^{\pm} \rightarrow D\pi^{\pm}$ decays, with the convention $CP|D^0\rangle = |\bar{D}^0\rangle$, and assuming no direct CP violation in the D decay,

$$\Gamma_{\text{fav}}^{\pm} = \Gamma(B^{\pm} \rightarrow D_{\text{fav}} \pi^{\pm}) = |A_{\bar{D}}^{\pm} A_f^D|^2 \eta(1, \rho \xi_{D\pi} z_{\pm}), \quad (31)$$

$$\Gamma_{\text{sup}}^{\pm} = \Gamma(B^{\pm} \rightarrow D_{\text{sup}} \pi^{\pm}) = |A_{\bar{D}}^{\pm} A_f^D|^2 \eta(\rho, \xi_{D\pi} z_{\pm}), \quad (32)$$

where $\rho = \frac{A_f^{\bar{D}}}{A_f^D}$, and the subscripts “sup” and “fav” refer to the suppressed and favored decay modes of the produced D meson, respectively.

The CP observables of interest are

$$R_{\text{ADS}}^{\pm D\pi} = \frac{\Gamma_{\text{sup}}^{\pm}}{\Gamma_{\text{fav}}^{\pm}} = \frac{\eta(\rho, \xi_{D\pi} z_{\pm})}{\eta(1, \rho \xi_{D\pi} z_{\pm})}. \quad (33)$$

Other ratios and asymmetries are also commonly used,

$$R_{\text{ADS}}^{D\pi} = \frac{\Gamma_{\text{sup}}^- + \Gamma_{\text{sup}}^+}{\Gamma_{\text{fav}}^- + \Gamma_{\text{fav}}^+} = \frac{\eta(\rho, \xi_{D\pi} z_-) + \eta(\rho, \xi_{D\pi} z_+)}{\eta(1, \rho \xi_{D\pi} z_-) + \eta(1, \rho \xi_{D\pi} z_+)}, \quad (34)$$

$$A_{\text{ADS}}^{\text{sup} D\pi} = \frac{\Gamma_{\text{sup}}^- - \Gamma_{\text{sup}}^+}{\Gamma_{\text{sup}}^- + \Gamma_{\text{sup}}^+} = \frac{\eta(\rho, \xi_{D\pi} z_-) - \eta(\rho, \xi_{D\pi} z_+)}{\eta(\rho, \xi_{D\pi} z_-) + \eta(\rho, \xi_{D\pi} z_+)}, \quad (35)$$

$$A_{\text{ADS}}^{\text{fav} D\pi} = \frac{\Gamma_{\text{fav}}^- - \Gamma_{\text{fav}}^+}{\Gamma_{\text{fav}}^- + \Gamma_{\text{fav}}^+} = \frac{\eta(1, \rho \xi_{D\pi} z_-) - \eta(1, \rho \xi_{D\pi} z_+)}{\eta(1, \rho \xi_{D\pi} z_-) + \eta(1, \rho \xi_{D\pi} z_+)}, \quad (36)$$

again showing the relationship between the observables and $\xi_{D\pi}$ and $z_{\pm}$.

C. Decays to multi-body self-conjugate final states

This method (commonly referred to as the BPGGSZ method [8,9,15]) uses D meson decays to three or more final state particles that can be accessed from both $|D^0\rangle$ or $|\bar{D}^0\rangle$ (e.g., $D^0 \rightarrow K_S^0 \pi^+ \pi^-$). In contrast to the GLW or ADS approaches, this method does not involve intermediate observables, and the goal is to fit for ξ_m and $z_{\pm}$ directly, using Eqs. (27), (28).

D. Extension to time-dependent measurements

The time evolution of the amplitude of the decay $B_s^0 \rightarrow D_s^{\mp} K^{\pm}$ is governed by the equation

$$A_{D_s^{\mp}}^{B_s^0}(t) = A_{D_s^{\mp}}^{B_s^0}[g_+(t) + \lambda_{\pm} g_-(t)], \quad (37)$$

where t is the B_s^0 meson lifetime and $g_{\pm}(t)$ are functions that describe the mixing of the B_s^0 meson flavor eigenstates. The $\lambda_{\pm}$ parameters can be expressed as

$$\lambda_{\pm} = \xi_m z_{\pm} e^{\pm i \phi_q}, \quad (38)$$

where ϕ_q is the weak phase arising in the interference between decay and mixing of the initial state neutral B -meson. In this case m specifically denotes $B_s^0 \rightarrow D_s^{\mp} K^{\pm}$ and $\phi_q = \phi_s$.

Existing measurements of γ in decays with B meson mixing introduce intermediate observables instead of targeting the $z_{\pm}^m$ parameters themselves. In these cases, the squared amplitude of the time-dependent probability distribution function is expressed as

$$e^{-\Gamma t} \frac{1 + |\lambda_-|^2}{2} \left[\cosh(x\Gamma t) + \underbrace{\frac{1 - |\lambda_-|^2}{1 + |\lambda_-|^2}}_{C_f} \cos(y\Gamma t) - \underbrace{\frac{-2\text{Re}(\lambda_-)}{1 + |\lambda_-|^2}}_{A_f^{\Delta\Gamma}} \sinh(x\Gamma t) - \underbrace{\frac{2\text{Im}(\lambda_-)}{1 + |\lambda_-|^2}}_{S_f} \sin(y\Gamma t) \right], \quad (39)$$

where x and y are parameters describing the B meson mixing and Γ is its decay rate; there is a similar expression

for the conjugated amplitude. Although this approach [10] is observable based, it is possible to include the C_f , $A_f^{\Delta\Gamma}$, and S_f parameters, and equivalent parameters for the conjugate decay, in a simultaneous fit that shares the $z_{\pm}$ parameters with other B decays and only introduces ξ_m as a new coefficient.

IV. SENSITIVITY STUDIES

Monte Carlo simulation is used to evaluate the impact of applying a simultaneous measurement technique on the sensitivity to $z_{\pm}$, and subsequently the weak phase γ , using the BPGGSZ approach. A simplified model description of the D meson decay amplitude over its phase space is implemented [16] in order to generate events. This has been cross-checked with a slightly more sophisticated model [17,18] and gives identical results. Two thousand signal-only pseudoexperiments are generated, with each pseudoexperiment including five B meson decay modes $B^{\pm} \rightarrow DK^{\pm}$, $B^0 \rightarrow DK^{*0}$, $B^{\pm} \rightarrow D^{*0} K^{\pm}$, $B^{\pm} \rightarrow DK^{*\pm}$, and $B^{\pm} \rightarrow D\pi^{\pm}$, where the D decays into the final state $K_S^0 \pi^+ \pi^-$. Experimental effects such as variation of the efficiency across the phase space, variation in the amplitude model, background contributions and momentum resolution have not been considered in this paper. It is expected that inclusion of these effects, incorporating cross-feed and correlations between decay modes, will have a small impact on the overall conclusions drawn by this study. Ensembles of pseudoexperiments are generated with sample sizes equivalent to 3 fb^{-1} and 9 fb^{-1} of LHCb data, representing the milestones reached after Run 1 and Run 2 of the LHC, respectively. World average values [2,11] are used for the hadronic parameters r_m and δ_m in each B meson decay mode, with $\gamma = 70^\circ$ in all cases. Since current experimental constraints on the admixture coefficients for $B^{\pm} \rightarrow D\pi^{\pm}$ decays are rather loose [12], values of $r_{D\pi} = 0.005$ and $\delta_{D\pi} = 300^\circ$ are used in the $B^{\pm} \rightarrow D\pi^{\pm}$ mode.

A standalone fit to each pseudoexperiment with each separate decay mode is used as a reference. In this case each decay mode is used to independently determine $z_{\pm}^m$, resulting in a total of $4N$ parameters. Several simultaneous fits are performed, progressively adding each decay mode to the set of decays considered. In this case, $z_{\pm} = x_{\pm} + iy_{\pm}$ and the appropriate ξ_m parameters are determined, resulting in a total of $2(N+1)$ parameters. The obtained statistical uncertainties on these parameters, i.e., the width of the distribution of fitted values across the ensemble, assuming 3 fb^{-1} of LHCb data, are summarized in Tables I and II. For all of these parameters, when performing the simultaneous fit, the results exhibit unbiased Gaussian behavior and the uncertainty estimates provide the appropriate coverage. This is not the case for all parameters when performing the standalone fits, particularly in the $B^{\pm} \rightarrow D\pi^{\pm}$ and $B^{\pm} \rightarrow DK^{*\pm}$ decay modes. In these cases, when fitting the standalone parameters $z_{\pm}^m$ (for $m = D\pi^{\pm}$ or $m = DK^{*\pm}$),

TABLE I. Expected statistical uncertainties of the Cartesian parameters when fitting using the simultaneous method, adding successive decay modes, assuming 3 fb^{-1} of LHCb data.

Channel(s)	$B^\pm \rightarrow DK^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}, B^\pm \rightarrow D^{*0}K^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}, B^\pm \rightarrow D^{*0}K^\pm, B^\pm \rightarrow DK^{*\pm}$
σ_{x_-}	0.0193 ± 0.0003	0.0199 ± 0.0003	0.0197 ± 0.0003	0.0191 ± 0.0003	0.0192 ± 0.0003
σ_{y_-}	0.0220 ± 0.0004	0.0222 ± 0.0004	0.0214 ± 0.0003	0.0206 ± 0.0003	0.0210 ± 0.0004
σ_{x_+}	0.0200 ± 0.0003	0.0202 ± 0.0003	0.0195 ± 0.0003	0.0192 ± 0.0003	0.0194 ± 0.0003
σ_{y_+}	0.0214 ± 0.0004	0.0218 ± 0.0004	0.0210 ± 0.0003	0.0204 ± 0.0003	0.0209 ± 0.0004
$\sigma_{\text{Re}(\xi_{D\pi})}$	...	0.0420 ± 0.0008	0.0418 ± 0.0007	0.0419 ± 0.0008	0.0419 ± 0.0007
$\sigma_{\text{Im}(\xi_{D\pi})}$	...	0.0392 ± 0.0007	0.0395 ± 0.0007	0.0392 ± 0.0007	0.0394 ± 0.0007
$\sigma_{\text{Re}(\xi_{DK^{*0}})}$	...	...	0.9011 ± 0.0176	0.9045 ± 0.0176	0.9003 ± 0.0176
$\sigma_{\text{Im}(\xi_{DK^{*0}})}$	...	...	0.8557 ± 0.0161	0.8571 ± 0.0160	0.8562 ± 0.0162
$\sigma_{\text{Re}(\xi_{D^*K})}$	...	...	...	0.5189 ± 0.0090	0.5173 ± 0.0087
$\sigma_{\text{Im}(\xi_{D^*K})}$	...	...	...	0.5351 ± 0.0098	0.5356 ± 0.0100
$\sigma_{\text{Re}(\xi_{DK^*})}$	...	...	...	...	0.6847 ± 0.0116
$\sigma_{\text{Im}(\xi_{DK^*})}$	...	...	...	...	0.7115 ± 0.0128

TABLE II. Expected statistical uncertainties of the Cartesian parameters when fitting each decay mode with the standalone fit method, assuming 3 fb^{-1} of LHCb data.

Channel	$B^\pm \rightarrow DK^\pm$	$B^\pm \rightarrow D\pi^\pm$	$B^0 \rightarrow DK^{*0}$	$B^\pm \rightarrow D^{*0}K^\pm$	$B^\pm \rightarrow DK^{*\pm}$
$\sigma_{x_-}^m$	0.0193 ± 0.0003	0.0050 ± 0.0001	0.1112 ± 0.0019	0.0636 ± 0.0011	0.0900 ± 0.0015
$\sigma_{y_-}^m$	0.0220 ± 0.0004	0.0056 ± 0.0001	0.1134 ± 0.0020	0.0668 ± 0.0012	0.1017 ± 0.0018
$\sigma_{x_+}^m$	0.0200 ± 0.0003	0.0054 ± 0.0001	0.1064 ± 0.0020	0.0687 ± 0.0013	0.0841 ± 0.0014
$\sigma_{y_+}^m$	0.0214 ± 0.0004	0.0055 ± 0.0001	0.1112 ± 0.0021	0.0769 ± 0.0014	0.0911 ± 0.0015

the results do not demonstrate Gaussian behavior, with biases as large as 40% of the statistical uncertainty. The reason for this in the $B^\pm \rightarrow DK^{*\pm}$ mode is that the expected number of events is very low (~ 90 at 3 fb^{-1}), therefore the D decay phase space is very sparsely populated and these fits are far from the Gaussian regime. In the $B^\pm \rightarrow D\pi^\pm$ case, where r_m is very small, the Cartesian parameters $z_\pm^m$ are very close to zero, which impacts on fit stability and reliable error estimation of the polar parameters. Both of these issues are resolved when reparametrizing the problem in terms of ξ using the simultaneous fit methodology outlined here.

Using the pseudoexperiment sizes considered in this study, the inclusion of $B^\pm \rightarrow D\pi^\pm$ decays does not improve the sensitivity to γ , mainly because the size of the interference in this mode is very small ($r_{D\pi} \approx 0.005$). However, inclusion of this high statistics mode is very important when fitting experimental data because it can be used to determine the variation in candidate reconstruction and selection efficiency across the $D \rightarrow K_S^0 \pi^+ \pi^-$ phase space [17]. These studies show that inclusion of this decay mode is statistically robust when using the simultaneous parametrization, which is not the case when treating it as an independent mode. Furthermore, the addition of each other decay mode contributes to a smaller uncertainty on $z_\pm$.

After the simultaneous and standalone fits above have been performed, the fitted values and full covariance matrix for each pseudoexperiment are used to determine a value

FIG. 1. Progression of the expected statistical uncertainty on CKM angle γ when incorporating additional decay modes using the simultaneous approach (dashed lines) and combining the standalone measurements (solid lines) with pseudoexperiment sample sizes corresponding to Run 1 (3 fb^{-1} , blue) and Run 2 (9 fb^{-1} , red) of LHCb data.

TABLE III. Expected statistical uncertainty on CKM angle γ from the simultaneous (top part) and standalone (bottom part) fit methods when incorporating additional decay modes.

Channel(s)		$B^\pm \rightarrow DK^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}, B^\pm \rightarrow D^{*0}K^\pm$	$B^\pm \rightarrow DK^\pm, B^\pm \rightarrow D\pi^\pm, B^0 \rightarrow DK^{*0}, B^\pm \rightarrow D^{*0}K^\pm, B^\pm \rightarrow DK^{*\pm}$
Simultaneous	Run 1	$(8.73 \pm 0.16)^\circ$	$(8.71 \pm 0.15)^\circ$	$(8.27 \pm 0.15)^\circ$	$(7.97 \pm 0.15)^\circ$	$(7.89 \pm 0.13)^\circ$
	Run 2	$(4.05 \pm 0.07)^\circ$	$(4.04 \pm 0.08)^\circ$	$(3.67 \pm 0.06)^\circ$	$(3.51 \pm 0.06)^\circ$	$(3.50 \pm 0.06)^\circ$
Standalone	Run 1	$(8.73 \pm 0.16)^\circ$	$(8.68 \pm 0.15)^\circ$	$(8.31 \pm 0.15)^\circ$	$(8.16 \pm 0.15)^\circ$	$(8.47 \pm 0.13)^\circ$
	Run 2	$(4.05 \pm 0.07)^\circ$	$(4.05 \pm 0.08)^\circ$	$(3.69 \pm 0.06)^\circ$	$(3.53 \pm 0.06)^\circ$	$(3.64 \pm 0.07)^\circ$

for the CP -violating weak phase γ . The corresponding results for the expected sensitivity when progressively adding decay modes are shown in Table III and Fig. 1. It can be seen that the simultaneous method provides a marginal gain in sensitivity over the standalone method, although its main advantage, as highlighted above, is its statistical robustness. Moreover, when determining the sensitivity to γ using the standalone method, the pull distributions for the hadronic ratio parameters, r_m , show large biases and widths far from unity. Given that the uncertainty on γ is inversely proportional to the central value of r_m this results in inaccurate estimations of the uncertainty on γ when using the standalone method. Consequently, the quoted values for the expected precision on γ from the standalone fits actually fluctuate around their unknown true values, and should be interpreted accordingly. This can cause behavior such as the uncertainty appearing to increase when adding more measurements, which occurs in Fig. 1, although it is typically mitigated as the sample sizes increase.

Figure 2 (left) shows a comparison between the distribution of fitted γ values for the two methods when all five decay modes are included and a sample size corresponding

to 3 fb^{-1} of LHCb data is used. In a small fraction of cases for the standalone pseudoexperiments ($\sim 3\%$) the fit fails to converge whereas for the simultaneous extraction this has not been found to happen. The distributions shown in Fig. 2 (left) have 0.1% (3.5%) of events in underflow or overflow bins for the simultaneous (standalone) extractions. This demonstrates that the rate of outliers seems to be much higher in the standalone case. The right-hand figure shows, per pseudoexperiment, the difference between the uncertainty on γ determined from the standalone method and from the simultaneous method. On average, there is a small (0.3°) gain when fitting simultaneously, but the difference can be as large as $2\text{--}3^\circ$ in either direction. However, as previously stated, the evaluated uncertainties for the standalone method are less reliable for the $B^\pm \rightarrow D\pi^\pm$ and $B^\pm \rightarrow DK^{*\pm}$ decay modes. This is highlighted by further analysis of the far reaching tails in Fig. 2 which reveals that for events with $\sigma_{\gamma(\text{stand})} - \sigma_{\gamma(\text{sim})} < 0.5^\circ$ or $\sigma_{\gamma(\text{stand})} - \sigma_{\gamma(\text{sim})} > 1^\circ$ the width of the distribution of values for γ is 8.5° (14°) for the simultaneous (standalone) extraction. Furthermore, for these events the normalized pull distributions have a mean of -0.03 ± 0.05 (-0.05 ± 0.06) and a width of 1.09 ± 0.04 (1.40 ± 0.04) for the simultaneous

FIG. 2. Left: Comparison between the simultaneous approach (blue) and combination of standalone measurements (red) for the value of γ determined with pseudoexperiments corresponding to a sample size of 3 fb^{-1} of LHCb data. The solid lines show a Gaussian fit to each distribution, with the mean and width of the fitted Gaussian shown in the top left and top right, respectively. Right: The difference of the fitted uncertainty on the CKM angle γ obtained from pseudoexperiments when using the simultaneous approach and a combination of standalone measurements.

(standalone) extraction demonstrating the less reliable error estimation in the standalone case.

It should be noted that the statistical precision on γ obtained in any future experimental analyses employing this technique will depend on the exact results obtained for $z_{\pm}$ and ξ_m from data. Nevertheless, the technique provides a rigorous and straightforward treatment of the uncertainties, allowing for simple inclusion of correlations between various decay modes. The studies presented here demonstrate good statistical behavior and a modest improvement in sensitivity when using the reparametrization of Eq. (6) in signal-only simulations. It may be that the improvement is larger when also including experimental backgrounds, efficiency variations and systematic effects.

V. CONCLUSIONS

An approach to simultaneously measuring the CP parameters $z_{\pm}$ sensitive to the CKM angle γ in multiple B meson decays has been presented. The formalism reduces the number of free parameters, allows for the

consideration of experimentally reconstructed decays that are signal in one case but background in another, and allows for a common treatment of systematic uncertainties. Sensitivity studies show that including additional B meson decay modes contributes marginally to a smaller uncertainty on $z_{\pm}$ but has considerably safer statistical behavior, and thus has the potential to offer an improvement compared to a combination of standalone measurements. This is likely to be further enhanced when systematic uncertainties and their correlations are also considered, particularly for large future data sets.

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- [1] P. A. Zyla *et al.* (Particle Data Group), Review of particle physics, *Prog. Theor. Exp. Phys.* **(2020)**, 083C01.
 - [2] Y. Amhis *et al.* (Heavy Flavor Averaging Group), Averages of b -hadron, c -hadron, and τ -lepton properties as of 2018, [arXiv:1909.12524](https://arxiv.org/abs/1909.12524), updated results and plots available at <https://hflav.web.cern.ch>.
 - [3] T. Gershon and V. V. Gligorov, CP violation in the B system, *Rep. Prog. Phys.* **80**, 046201 (2017).
 - [4] M. Gronau and D. London, How to determine all the angles of the unitarity triangle from $B_d^0 \rightarrow DK_S$ and $B_s^0 \rightarrow D\phi$, *Phys. Lett. B* **253**, 483 (1991).
 - [5] M. Gronau and D. Wyler, On determining a weak phase from CP asymmetries in charged B decays, *Phys. Lett. B* **265**, 172 (1991).
 - [6] D. Atwood, I. Dunietz, and A. Soni, Enhanced CP Violation with $B \rightarrow KD^0(\bar{D}^0)$ Modes and Extraction of the CKM Angle Gamma, *Phys. Rev. Lett.* **78**, 3257 (1997).
 - [7] D. Atwood, I. Dunietz, and A. Soni, Improved methods for observing CP violation in $B^{\pm} \rightarrow K^{\pm}D$ and measuring the CKM phase gamma, *Phys. Rev. D* **63**, 036005 (2001).
 - [8] A. Giri, Y. Grossman, A. Soffer, and J. Zupan, Determining γ using $B^{\pm} \rightarrow DK^{\pm}$ with multibody D decays, *Phys. Rev. D* **68**, 054018 (2003).
 - [9] A. Bondar, in Proceedings of BINP Special Analysis Meeting on Dalitz Analysis (to be published).
 - [10] R. Fleischer, New strategies to obtain insights into CP violation through $B_s \rightarrow D_s^{\pm}K^{\mp}$, $D_s^{*\pm}K^{\mp}$ and $B_d \rightarrow D^{\pm}\pi^{\mp}$, $D^{*\pm}\pi^{\mp}$ decays, *Nucl. Phys.* **B671**, 459 (2003).
 - [11] M. Tanabashi *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **98**, 030001 (2018).
 - [12] R. Aaij *et al.* (LHCb Collaboration), Measurement of the CKM angle γ from a combination of LHCb results, *J. High Energy Phys.* **12** (2016) 087.
 - [13] R. Aaij *et al.* (LHCb Collaboration), Update of the LHCb combination of the CKM angle γ , Report No. LHCb-CONF-2018-002 (2018).
 - [14] J. Garra Tico, A strategy for a simultaneous measurement of CP violation parameters related to the CKM angle γ in multiple B meson decay channels, [arXiv:1804.05597](https://arxiv.org/abs/1804.05597).
 - [15] A. Poluektov *et al.* (Belle collaboration), Measurement of ϕ_3 with Dalitz plot analysis of $B^{\pm} \rightarrow D^{(*)}K^{\pm}$ decay, *Phys. Rev. D* **70**, 072003 (2004).
 - [16] P. del Amo Sanchez *et al.* (BABAR Collaboration), Measurement of D^0 - $\bar{D}^0$ Mixing Parameters Using $D^0 \rightarrow K_S^0\pi^+\pi^-$ and $D^0 \rightarrow K_S^0K^+K^-$ Decays, *Phys. Rev. Lett.* **105**, 081803 (2010).
 - [17] R. Aaij *et al.* (LHCb Collaboration), Measurement of CP violation and constraints on the CKM angle γ in $B^{\pm} \rightarrow DK^{\pm}$ with $D \rightarrow K_S^0\pi^+\pi^-$ decays, *Nucl. Phys.* **B888**, 169 (2014).
 - [18] J. Garra Tico, The cfit fitting package, <http://www.github.com/cfit>.