Geometry of entanglement

Andrea Prudenziati

*International Institute of Physics, Universidade Federal do Rio Grande do Norte,
Campus Universitario, Lagoa Nova, Natal-RN 59078-970, Brazil*

(Received 2 August 2019; accepted 13 August 2020; published 31 August 2020)

In the context of the surface-state correspondence we propose the geodesic curvature of a convex curve as a local measure of factorization of the dual Conformal Field Theory state. Its integral will be interpreted as computing the total bipartite entanglement among degrees of freedom with support on the chosen domain. We will derive results through application of the Gauss-Bonnet theorem and show quantitative agreement with computations using the Multiscale Entanglement Renormalization Ansatz tensor network and the formalism of entanglement density.

DOI: [10.1103/PhysRevD.102.046020](https://doi.org/10.1103/PhysRevD.102.046020)

I. INTRODUCTION AND MOTIVATIONS

Since the proposal from Ryu and Takayanagi to compute entanglement entropy holographically [1–3], an impressive effort has been done to further develop the relation between entanglement and gravity. A few relevant, cherry picked examples are [4–8]. The results achieved however mostly suffer, and sometimes exploit, the same common issue that the holographic map is nonlocal. Generally speaking it is only sufficiently close to the boundary that we can restore with good approximation a local map between the entanglement content of a state and the dual geometry.

A step forward in this direction has been done with the conjectural surface-state correspondence [9], where CFT states $|\psi\rangle_\Sigma$ are associated to generic convex surfaces Σ embedded in the holographic space. These states can in principle be produced by unitary nonlocal operations to the boundary state, so that evolution along the radial bulk direction translates into a purely CFT flux. Unfortunately, with perhaps the only notable exception of constant-radius surfaces that we will discuss later on, we do not have a recipe for constructing $|\psi\rangle_\Sigma$. Nonetheless the simple fact that the Ryu-Takayanagi formula applies also in this context still allows to extract information about the entanglement content of these states; moreover, at least in principle, using such surfaces we can restore the quasilocal to local identification between entanglement and geometry at any chosen position in the bulk, the nonlocal part having been pushed entirely into the CFT. A geometrical representation of this flux has been proposed in [10].

A different, and apparently unrelated, issue in quantum information is to define measures of entanglement for multipartite states and/or with infinite dimensionality. Entanglement entropy for instance just captures the entanglement content across the boundary of a certain entangling region. For a review on general entanglement measures see for example [11].

In the present paper we will attempt both a connection and a step forward in the resolution of these two problems. The aim will be to propose a holographic measure of bipartite entanglement, for a generic 2d CFT state ρ_γ dual to a convex curve γ , as an integral of a local geometrical quantity on γ . The process is summarized as follows: given a small interval $\gamma_A \subset \gamma$ with $\gamma_A = \gamma_{A_1} \cup \gamma_{A_2}$, we begin by choosing a quantum information measure $K(\rho_{\gamma_A} \| \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}})$ of how much the reduced density matrix $\rho_{\gamma_A} = \text{Tr}_{\gamma_A^c} \rho_\gamma$ is far from being factorized as $\rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}$; a discussion on possible choices of K will follow. $K(\rho_{\gamma_A} \| \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}})$ then depends, among other things, on the interval size $\Delta\tau$. We define its second derivative in $\Delta\tau$ local factorizability and write it as $J_K(\gamma)$; see (1). We will show that, once integrated along γ , it provides a measure of the total amount of bipartite entanglement of ρ_γ inside the integral domain of the dual curve. We claim that $J_K(\gamma)$ can be represented holographically, at least for some choices of measure K , as local geometrical quantities on γ , and we will present an example of this. Then, inverting the process, we will pick a very specific quantity, the geodesic curvature of γ and, by studying its dual quantum information properties, we will show that it can be interpreted as representing the holographic dual of $J_K(\gamma)$ for some unknown choice of K . Following the quantum information argument the integral of the geodesic curvature along some interval on γ should then represent a holographic measure of bipartite entanglement of ρ_γ there.

With the help of the Gauss-Bonnet theorem we will transform this line integral into a surface integral, compute

*andrea.prudenziati@gmail.com

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examples and test our physical interpretation against equivalent computations in the MERA tensor network and entanglement density formalisms.

When the domain of interest coincides with the total curve, a measure of total bipartite entanglement for the state is then available, and explicit cases will also be discussed.

A final note is that throughout the paper we will consider a holographic description of entanglement at the classical level in the bulk coupling constant G_N . This will have an impact on the physical interpretation and results of our analysis. Some discussion on quantum corrections can be found in the last section.

II. LOCAL FACTORIZABILITY

We restrict to three bulk and two boundary dimensions. We will work with a generic spacelike slice M_{sl} of the bulk space M^b but for a few explicit examples where we pick $M_{sl} = M_{t=\text{const}}^b$.

We first set our problem by defining what we mean by local factorizability of the state holographically dual to the convex curve. We will consider for the moment only pure states, $\rho_\gamma = (|\psi\rangle\langle\psi|)_\gamma$, represented by closed and topologically trivial curves $\gamma \subset M_{sl}$. Density matrices will be briefly discussed in the context of a Bañados, Teitelboim, and Zanelli (BTZ) black hole.

Consider a connected closed and finite region γ_A on γ of length $2\Delta\tau$.¹ According to the surface-state correspondence we can associate to γ_A the reduced density matrix $\rho_{\gamma_A} = \text{Tr}_{\gamma_A^c}(|\psi\rangle\langle\psi|)_\gamma$. We then divide γ_A into two closed and connected intervals of equal length $\Delta\tau$ such that $\gamma_A = \gamma_{A_1} \cup \gamma_{A_2}$, with intersection point $p = \gamma_{A_1} \cap \gamma_{A_2}$. Now consider any measure $K(\rho\|\sigma)$ of distinguishability between the states ρ and σ and define the local factorizability of the state $|\psi\rangle_\gamma$ in p with respect to the measure K , or $J_K(\gamma)|_p^A$, as

$$J_K(\gamma)|_p^A := \frac{\partial^2}{\partial(\Delta\tau)^2} K(\rho_{\gamma_A} \| \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}). \quad (1)$$

The above definition is purely quantum mechanical, with the only holographic connection being the dependence of $K(\rho_{\gamma_A} \| \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}})$ on the size $\Delta\tau$ of the intervals on the dual curve. This is analogous, for instance, to the dependence of the entanglement entropy on the size of the entanglement region, which is also a distance at the boundary of the holographic space. Equation (1) can be applied to generic finite regions on the curve γ but in the following we will mostly be interested in the infinitesimal case $\Delta\tau \ll 1$; when it is so we will drop the subscript A and define local factorizability as the leading order term in $\Delta\tau$ of (1).

¹The variable τ will be the proper length along the curve normalized such that $\|\partial_\tau\gamma\| = 1$.

There are various choices for measures of distance between quantum states, and to some of them holographic dual quantities have been associated in the literature. Examples are the Fisher information metric, Bures distance and relative entropy, see [12–17]. If we had knowledge on the way to compute the state ρ_γ given a generic convex curve γ then these proposals would allow a holographic characterization of (1). Unfortunately in general we do not know how to construct the holographic map $\gamma \rightarrow \rho_\gamma$ so some alternative path should be followed. The key insight here is the assumed validity of the Ryu-Takayanagi formula in this context. The reason behind this assumption is that the machinery used to prove the Ryu-Takayanagi formula for boundary states goes through, at least in principle, unaffected when picking generic convex surfaces in the holographic space [9,18]. So, even without knowledge on ρ_γ , we can still extract information on its entanglement content by computing the entanglement entropy for any interval γ_A of choice, by measuring the length of the geodesic whose end points coincide with $\partial\gamma_A$. Based on this let us choose $K(\rho\|\sigma) = S(\rho\|\sigma)$, the latter being the relative entropy:

$$S(\rho\|\sigma) := \text{Tr}(\rho \log \rho - \rho \log \sigma)$$

then it is easily verified that

$$S(\rho_A \| \rho_{A_1} \otimes \rho_{A_2}) = I(A_1 : A_2),$$

with $I(A_1 : A_2) = S(A_1) + S(A_2) - S(A_1 A_2)$ being the mutual information and $S(A)$ the entanglement entropy. As discussed we can evaluate this quantity holographically for the case of interest by considering the Ryu-Takayanagi surfaces anchored to γ_{A_i} .² In the Appendix we computed the mutual information around p for small intervals and then applied the double derivative from the definition (1) to obtain, at first order in the interval size $\Delta\tau$,³

$$J_S(\gamma)|_p = \partial_{\Delta\tau}^2 I|_p = \frac{3\Delta\tau}{4G_N} (\partial_\tau \gamma^\theta \partial_\tau \gamma^\sigma \partial_\nu g_{\theta\sigma} k^\nu) \Big|_{\tau_p} + O(\Delta\tau^3). \quad (2)$$

²In the paper [16] Fisher information metric was defined as the second order variation of relative entropy in some deformation parameter λ when considering the CFT ground state ($\lambda = 0$) and some slightly higher energy state (at finite λ). If we could generalize the argument here with base state ρ_A and deformed state $\rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}$, and express λ as $\lambda(\Delta\tau)$, then we would have some holographic representation of $J_F(\gamma)|_p$ with F the Fisher information metric.

³We write $\tau(p) = \tau_p$. Moreover note that γ here is called $\tilde{\gamma}$ in the Appendix.

The expression (2) is interesting for different reasons. First of all it depends on the vector k^μ , whose norm is called the geodesic curvature and which is defined as⁴

$$k^\mu(\gamma) := (\nabla_\tau \partial_\tau \gamma^\mu)_{TM_{sl}}, \quad \|\partial_\tau \gamma\| = 1, \quad (3)$$

with the obvious consequence that $k^\mu(\gamma = \text{geodesic}) = 0$ so that $J_S(\gamma = \text{geodesic}) = 0$. Geometrically we can simply visualize this: if a curve is a geodesic on a finite interval γ_B , then there the curve coincides with the Ryu-Takayanagi surface for any interval $\gamma_A \subset \gamma_B$, so the holographic mutual information of the dual state trivially vanishes, the state is exactly factorized as $\rho_{\gamma_A} = \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}$ and $J_S(\gamma)|_{p \in \gamma_A} = 0$. In fact the property that $\rho_{\gamma_A} = \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}$ when the dual curve γ is a geodesic along some segment means that $K(\rho_{\gamma_A} \parallel \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}) = 0$ for *any* choice of K , which means that local factorizability always vanishes there. This implies that the holographic dual to (1), for whatever choice of measure K , should always be proportional to a geometrical quantity bound to vanish when evaluated along a geodesic, for instance some power of the geodesic curvature.

The double derivative in (1) can roughly be interpreted as the quantum information equivalent of the “geodesic” equation for a generic curve (3). In fact, without the second order derivative in (1), the local factorizability would vanish as $\sim \Delta\tau^3$ for $\Delta\tau \rightarrow 0$, much more severely than in (2); one power of $\Delta\tau$ comes from the integration domain and two powers from the geometric property that a generic curve is infinitesimally different from its tangent geodesic in p only at second order in the distance from p . Finally, if we consider as an example the Poincaré metric of AdS_3 with radius L , then

$$\partial_\nu g_{\theta\sigma} k^\nu \sim -2 \frac{L^2}{z^3} k^z, \quad (4)$$

which sets the relative weight of $J_S(\gamma)|_p$ compared to curves with different bulk embedding, depending on the radial position of $\gamma(p)$.

III. DEFINITION OF $J(\gamma)$

The scaling result (4) shows that local factorizability with measure the relative entropy (2) does weight the entanglement between ρ_{A_1} and ρ_{A_2} with a scaling factor of $\sim z^{-3}$. Following the MERA representation of anti-de Sitter (AdS) we interpret the AdS radius as some scale of entanglement after a coarse graining procedure has been implemented; see Sec. IV B with related references. However the MERA tensor network weights entanglement

at different length scales equally, so to have agreement with this counting we would like to get rid of the additional factor (4). Analogously we can count entanglement between degrees of freedom at different length scales using the formalism of entanglement density (see Sec. IV C); also here the qubit pairs are identically weighted independently of their distance. Thus inspired by the holographic computation of local factorizability using relative entropy (2), we would like to study a “simpler” but similar geometrical quantity:

$$J(\gamma)|_p := \frac{L}{4G_N} k(\gamma)|_{\tau_p} \Delta\tau. \quad (5)$$

This definition is purely geometrical in nature but eventually we would like to interpret it as the holographic dual of local factorizability for the state $|\psi\rangle_\gamma$ based on some unknown measure X of state distinguishability, so that $J(\gamma) = J_X(\gamma)$. The definition is with respect to some infinitesimal interval of size $2\Delta\tau$ centered at τ_p . Above we have set $k(\gamma)$ to be the norm of $k^\mu(\gamma)$ from (3) which is called the geodesic curvature of γ . In the following sections we will provide evidence for this claim, for the moment we simply emphasize that this definition retains the following good properties:

- (i) $J(\gamma)|_p = 0$ when γ coincides with a geodesic on $\gamma_B \supset \gamma_A$, with $p \in \gamma_A$, which implies (from the discussion on holographic mutual information) that $\rho_{\gamma_A} = \rho_{\gamma_{A_1}} \otimes \rho_{\gamma_{A_2}}$ there.
- (ii) It is positive definite in any point p for generic convex curves.
- (iii) It is integrable along γ in τ_p by replacing $\Delta\tau \rightarrow d\tau$.
- (iv) It is dimensionless.

Notice that (5) differs substantially from (2) as it is a more “democratic” measure of local factorizability at different AdS_3 length scales, not being weighted by the radial depth in the bulk space.

Additivity of some measure E is defined as $E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2)$. In holography we can consider two bulk spaces M_1 and M_2 dual to the states ρ_1 and ρ_2 and obtain the bulk space M dual to $\rho = \rho_1 \otimes \rho_2$ as the disconnected union $M = M_1 \cup M_2$. Then it is trivially verified that

- (i) $J(\gamma)$ is additive.

Finally let us discuss local operations (LO), as a good entanglement measure should not increase under LO.⁵ We first choose a CFT unitary transformation $U(\gamma_A)$ that has domain of application on γ_A and acts as $U(\gamma_A)|\psi\rangle_\gamma = |\psi\rangle_{\gamma'}$, with $\gamma' - \gamma := \delta\gamma$ and $\delta\gamma \neq 0$ only inside γ_A . As the boundary conditions of the deformation are $\delta\gamma|_{\partial\gamma_A} = 0$, the Ryu-Takayanagi formula tells us that $S(\gamma_A) = S(\gamma'_A)$ so

⁴In the bulk M^b we have $k_c^\mu(\gamma) := \nabla_\tau \partial_\tau \gamma^\mu$ whose norm is related to the norms of the projection over the tangent plane to the spacelike slice M_{sl} in the chosen point, $k^\mu(\gamma) := (\nabla_\tau \partial_\tau \gamma^\mu)_{TM_{sl}}$ and the normal part, $k_n^\mu(\gamma) := (\nabla_\tau \partial_\tau \gamma^\mu)_{NM_{sl}}$ by $k_c = \sqrt{k^2 + k_n^2}$.

⁵the measure should also not increase under classical communications, although how to describe them in the holographic setting is quite unclear.

FIG. 1. Part (a) is a graphic representation of the (discretized) integral on τ_p of $J(\gamma)$, while part (b) shows the vanishing of (bulk classical) mutual information for noncontiguous intervals.

that we do not modify the entanglement entropy of the reduced density matrix ρ_{γ_A} under the action of $U(\gamma_A)$, and we may call the latter LO for the subsystem γ_A . Our formalism is consistent with this identification because if we apply these LO on γ_{A_1} and γ_{A_2} they transform γ there but keep fixed the boundary points at $\partial\gamma_{A_1}$ and $\partial\gamma_{A_2}$. Then (for notational simplicity we skip below the subscript TM_{sl})

$$\begin{aligned} |k(\gamma') - k(\gamma)|_{\tau_p} &= \|\nabla_\tau \partial_\tau \gamma'\| - \|\nabla_\tau \partial_\tau \gamma\|_{\tau_p} \leq \|\nabla_\tau \partial_\tau \delta\gamma\|_{\tau_p} \\ &= \|\partial_\tau^2 \delta\gamma + \Gamma_{\mu\nu} \partial_\tau \delta\gamma_\mu \partial_\tau \delta\gamma_\nu\|_{\tau_p} = 0, \end{aligned}$$

where we used the fact that the derivatives are expressed as incremental differences in $\Delta\tau \rightarrow d\tau$ of $\delta\gamma$, which means evaluated at the boundary points of $\partial\gamma_{A_1}$ and $\partial\gamma_{A_2}$. We conclude that

- (i) $J(\gamma)$ does not vary under LO.⁶

IV. APPLICATIONS

To study the properties of $J(\gamma)$ we will consider some generic state $|\psi\rangle_\gamma$ and integrate (5) in τ_p along some region γ_A , having done the substitution $\Delta\tau \rightarrow d\tau$:

$$J(\gamma_A) := \int_{\gamma_A} J(\gamma)|_{\tau_p} = \frac{L}{4G_N} \int_{\gamma_A} k d\tau. \quad (6)$$

In this way γ_A is divided into infinitely many subregions A_i of size $d\tau$ with boundary points τ_{p_i} and $\tau_{p_{i+1}}$. The integral on τ_p will then add up all the local factorizability values measuring how much the state $|\psi\rangle_\gamma$ is far from being factorized across any couple of infinitesimal contiguous subregions A_i, A_{i+1} , see Fig. 1(a).

Two observations generalize this result: first of all if we want to describe factorization of the state between two noncontiguous subregions at the *classical* level in the bulk coupling constant G_N , for instance A_i, A_{i+2} by measuring

⁶For finite intervals we should either require additional boundary conditions on $\delta\gamma$ at $\partial\gamma_A$ as part of the definition of LO, if we want to preserve $J(\gamma)$, or alternatively accept that local factorizability can increase under the more general definition of LO.

the mutual information $I(A_i : A_{i+2})$, this is trivially zero. The reason being that the configuration of the two disconnected geodesics with boundary ∂A_i and ∂A_{i+2} is favored in this case (of regions of identical size) over the case of the two disconnected geodesics with boundary points $\partial(A_i \cup A_{i+1} \cup A_{i+2})$ and ∂A_{i+1} , see Fig. 1(b). So factorization along noncontiguous regions is described holographically by quantum corrections in the bulk and we expect this to hold true even when using different entanglement measures than the relative entropy. Because of this we should not worry about the distinction between a state that is factorized only between all the contiguous couples of A_i s and one that is factorized along *any* couple, if we stick with a classical bulk regime. This will be important when we will compare results with the entanglement density formalism in Sec. IV C.

Moreover rescaling $\tau \rightarrow \lambda\tau$ changes the interval size of the subregions A_i as $d\tau \rightarrow \lambda d\tau$, so that the above discussion holds true for two-parties factorization of any two unions of λ contiguous subregions, $\cup_{i=j}^{j+\lambda} A_i$ and $\cup_{i=j+\lambda+1}^{j+2\lambda+1} A_i$. This observation will be relevant for the comparison with results from the MERA tensor network in Sec. IV B.

Given the above discussion the integral (6) is then interpreted as a measure of the total amount of two-parties factorization of $|\psi\rangle_\gamma$, alternatively referred to as bipartite entanglement, evaluated over all couples of regions with support on γ_A . Seen from a lattice point of view this is a sum of entanglement from all the couples of degrees of freedom. The holographic dual description is just the measure of how much the curve differs from being a geodesic along γ_A .

What we cannot deduce however is for $J(\gamma)$ to be a measure of *total* factorization, that is how much $\rho_{\cup_i A_i}$ differs from $\rho_{A_1} \otimes \rho_{A_2} \otimes \dots \otimes \rho_{A_n}$. Even in a three-parties state two-parties entanglement is not enough to reconstruct the general three-parties entanglement; a simple example of this is the three qubits entangled GHZ state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ whose reduced density matrix by tracing out any of the three qubits is *not* entangled in the remaining two qubits. For general multipartite states the situation becomes even more involved so that it is indeed hopeless to imply that a state is totally factorized, based on

two-parties factorization alone, and in particular does not make (6) a measure of the total amount of entanglement of the state.

A. Entanglement below a geodesic and the Gauss-Bonnet theorem

We compute here the difference in J between two states that share the same smooth curve γ^c but for a certain region A (for simplicity here connected but this can be straightforwardly generalized) where one state follows a smooth curve γ_A and the other $\tilde{\gamma}_A$. For generality we also include the jump angles as in Fig. 2, which are nothing but local divergences of the geodesic curvature:

$$\begin{aligned} \frac{4G_N}{L} (J(\gamma^c \cup \tilde{\gamma}_A) - J(\gamma^c \cup \gamma_A)) \\ = \tilde{\alpha}_i + \tilde{\alpha}_f + \int_{\gamma^c \cup \tilde{\gamma}_A} k - \left(\alpha_i + \alpha_f + \int_{\gamma^c \cup \gamma_A} k \right) \\ = \beta_i + \beta_f - 2\pi + \int_{\partial M} k, \end{aligned} \quad (7)$$

where we have denoted $\partial M = (-\gamma_A) \cup \tilde{\gamma}_A$, while $\beta = \pi - (\alpha - \tilde{\alpha})$ are the jump angles of the piecewise smooth curve ∂M . Following Fig. 2 then

$$J(\gamma^c \cup \tilde{\gamma}_A) - J(\gamma^c \cup \gamma_A) = J(\partial M) - 2\pi \frac{L}{4G_N}. \quad (8)$$

An important result that we will use extensively in the following is the Gauss-Bonnet theorem that states

$$\int_{\partial M} k + \sum_i \alpha_i + \int_M \frac{R}{2} = 2\pi\chi, \quad (9)$$

where χ is the topological Euler number of M and R is the Ricci scalar. It then follows that⁷

$$\begin{aligned} J(\gamma^c \cup \tilde{\gamma}_A) - J(\gamma^c \cup \gamma_A) &= \frac{L}{4G_N} \left(-\int_M \frac{R}{2} + 2\pi(\chi - 1) \right) \\ &= -\frac{L}{4G_N} \int_M \frac{R}{2}. \end{aligned} \quad (10)$$

This interesting result is interpreted to measure the difference in total bipartite entanglement between the states $|\psi\rangle_{\gamma^c \cup \tilde{\gamma}_A}$ and $|\psi\rangle_{\gamma^c \cup \gamma_A}$.

As a nice application of this formula let us consider the case where γ_A is a geodesic with boundary conditions $\partial\gamma_A = \partial\tilde{\gamma}_A$. In this case the state $|\psi\rangle_{\gamma^c \cup \gamma_A}$ is completely

⁷The formula below applies also to the case of A being made by n disconnected intervals, with M being the corresponding n disconnected regions, simply replacing $2\pi(\chi - 1) \rightarrow 2\pi(\chi - n) = 4\pi(1 - n)$, because of the additional n factors of 2π from (7) and the Euler number definition.

FIG. 2. Difference in integrated local factorizability between two curves, jump angles are taken in the counterclockwise direction.

factorized on γ_A so no local entanglement should be present there and indeed $\int_{\gamma_A} k = 0$ in this case. Then (10) transforms into

$$\begin{aligned} J^{\text{tot}}(\tilde{\gamma}_A) &:= J(\gamma^c \cup \tilde{\gamma}_A) - J(\gamma^c \cup \gamma_A)|_{\gamma_A = \text{RT curve}} \\ &= -\frac{L}{4G_N} \int_M \frac{R}{2}. \end{aligned} \quad (11)$$

In this case we indeed interpret $J^{\text{tot}}_{\tilde{\gamma}_A}$ as a measure of the total amount of two-parties local entanglement on $\tilde{\gamma}_A$.

We can compute (11) for the simple case of $\tilde{\gamma}$ being the infinitesimal boundary cutoff state $z = \epsilon$ in the Poincaré AdS_3 coordinates at $t = \text{const}$ of Sec. IV D 2, and $\tilde{\gamma}_A$ a subregion of length l . The geodesic γ_A then is a semicircle and M the region inside, which makes the computation and the result remarkably simple:

$$\begin{aligned} J^{\text{tot}}(z = \epsilon, x \in [-l/2, l/2]) \\ = -\frac{L}{4G_N} \int_M \frac{R}{2} \\ = \frac{L}{4G_N} \int_{\delta}^{l/2} dr \int_{\epsilon/r}^{\pi-\epsilon/r} d\phi \frac{L^2}{r \sin^2 \phi} \frac{1}{L^2} \xrightarrow{[\epsilon, \delta \rightarrow 0]} \frac{c}{6} \frac{l}{\epsilon}, \end{aligned} \quad (12)$$

where we set the temporary cutoff δ for the size of A to be $l \gg \delta \gg \epsilon$ and we have used the famous relation $\frac{1}{G_N} = \frac{2c}{3L}$.

Interesting also is the computation for finite boundary cutoff $z = z_c$; as the geodesics now is no longer a half circle we find it easier to switch back to the line integral formalism. We parametrize the $z = z_c$ curve as $x(\tau) = \frac{z_c}{L} \tau + \text{const}$ and compute the geodesic curvature along to be simply $k = 1/L$, to obtain

$$\begin{aligned} J^{\text{tot}}(z = z_c, x \in [-l/2, l/2]) \\ = \frac{L}{4G_N} \left(\beta_i + \beta_f - 2\pi + \int_{\partial M} k \right) \\ = \frac{L}{4G_N} \int_{\tau_i}^{\tau_f} d\tau \frac{1}{L} = \frac{c}{6} \frac{l}{z_c}, \end{aligned} \quad (13)$$

which is a deceivingly simple generalization of (12).

In the following two subsections we will test our conjectural interpretation of (11) by comparing the results (12) and (13) with two different approaches for estimating

bipartite entanglement inside some boundary AdS_3 region: the MERA tensor network and entanglement density formalism.

B. Comparison with MERA

MERA [19] is a tensor network that represents some given state, for instance an n qubits chain, by an iterative application of tensor disentanglers acting on nearby qubits followed by coarse graining. This layered network has been proposed to represent a discretized version of the holographic bulk space [20], with the iterative direction being interpreted as the additional bulk dimension. As each disentangling operation at a given scale u acts by killing entanglement between two coarse grained degrees of freedom at the same scale u , by counting the minimal total number of such operators in the network surrounding the entangling region A , we should estimate the total entanglement of A with its complementary, which is the entanglement entropy of A . Indeed, if we start with A made of $\frac{l}{\epsilon}$ qubits (here we imagine a qubit for each lattice site of size ϵ), at the level u this number has been reduced by coarse graining to be $\frac{l}{\epsilon}2^{-u}$, $u \geq 0$. The number of disentanglers is then half of the number of qubits at level u , where u is related to the z radius in Poincaré coordinates by $z = \epsilon 2^u$. Thus the total number of disentanglers surrounding A can be represented as a “geodesic” curve in the MERA network, attached to the boundary of A , which counts the entanglement as the total number of intersections with disentangler tensors⁸:

$$\#\text{disent} \approx 2 \int_0^{\log_2 l/\epsilon} du \frac{1}{2} = \frac{1}{\log 2} \log \left(\frac{l}{\epsilon} \right). \quad (14)$$

This indeed coincides with the Ryu-Takayanagi formula if we weight the entanglement contribution of each tensor contraction by a factor $\log 2 \frac{\epsilon}{3}$.

This simple interpretation of entanglement across A can be refined to obtain (12). We proposed this formula as a measure of the total amount of two-parties entanglement among regions of any (identical) size living in A , which from the MERA perspective should then correspond to the *total* number of disentanglers below the Ryu-Takayanagi curve in the network. Including the $\log 2 \frac{\epsilon}{3}$ weight from above this is given by

$$\begin{aligned} \text{tot 2-ent in } A &\approx \log 2 \frac{c}{3} \int_0^{\log_2 l/\epsilon} du \frac{1}{2} 2^{-u} \\ &= \frac{c}{6\epsilon} l + O(1), \end{aligned} \quad (15)$$

which again agrees with (12).

⁸The factor of 2 comes from the two sides of the geodesic while the factor 1/2 from the relative counting of disentanglers compared to the level.

The generalization for a curve at $z = z_c$ is trivially achieved by starting at level $u = 0$ from $\frac{l}{z_c}$ qubits, and so replacing $\epsilon \rightarrow z_c$ in Eq. (15) to nicely match (13).

C. Comparison with entanglement density

Entanglement density $n(\delta, \xi)$ was first introduced in [21] and is defined in a lattice theory as the number of entangled pairs centered at ξ and separated by a distance δ . A simple calculation performed in [21] shows that the entanglement entropy is related to the entanglement density as $n(\delta, \xi) = \frac{1}{8} \partial_\xi^2 S(\delta, \xi) - \frac{1}{2} \partial_\delta^2 S(\delta, \xi)$. In the specific case of the AdS_3 boundary state $\gamma = \{z = \epsilon\}$ entanglement entropy has the simple value $S(\delta, \xi) = S(\delta) = \frac{\epsilon}{3} \log(\frac{\delta}{\epsilon})$, so that

$$n(\delta) = \frac{c}{6} \frac{1}{\delta^2}. \quad (16)$$

Note that the radial cutoff decouples and the entanglement density is cutoff independent.

To compute the total two-parties entanglement below the Ryu-Takayanagi curve for a boundary entangling region of size l (for simplicity centered at the origin), we should then count all such entangled pairs that fit inside the entangling region, with the cutoff ϵ introducing a minimal distance for each couple. This is easily computed as

$$\begin{aligned} \text{tot 2-ent in } A &\approx 2 \int_0^{l/2-\epsilon/2} d\xi \int_\epsilon^{l-2\xi} d\delta n(\delta) \\ &= \frac{c}{6\epsilon} l - \frac{c}{6} \log \left(\frac{l}{\epsilon} \right) + O(1). \end{aligned} \quad (17)$$

This result agrees with (12) in the leading term thus supporting our interpretation of the integral (11). The additional subleading term in (17), which is nothing but minus a half of the corresponding entanglement entropy, is probably due to some under counting of entangled pairs near the boundary of the Ryu-Takayanagi surface, which this simple evaluation missed.

More interesting is the comparison for a curve at finite cutoff z_c , because the Ryu-Takayanagi formula in this case predicts the entanglement entropy to be⁹

$$S(l)_{z_c} = \frac{c}{3} \text{ArcSech} \left(\frac{2z_c}{\sqrt{l^2 + 4z_c^2}} \right). \quad (18)$$

The entanglement density can be defined also for finite cutoff, generalizing the original computation in [21],

⁹It is perhaps interesting to note that the form (18) for entanglement entropy can be reduced to the usual logarithmic scaling if we introduce an effective cutoff $\tilde{z}_c(l, z_c) = \frac{z_c l^2}{l^2 - z_c^2}$ such that $S(l)_{\tilde{z}_c(l, z_c)} = \frac{\epsilon}{3} \log(\frac{l}{\tilde{z}_c})$. This may be relevant in the ensuing discussion at the end of this section.

to obtain $n(\delta, \xi - z_c/2, z_c) + n(\delta, \xi + z_c/2, z_c) = \frac{1}{4} \partial_\xi^2 S(\delta, \xi)_{z_c} - \partial_\delta^2 S(\delta, \xi)_{z_c}$. Translational invariance simplifies the relation to $n(\delta, z_c) = -\frac{1}{2} \partial_\delta^2 S(\delta)_{z_c}$ and using (18) we have

$$n(\delta, z_c) = \frac{c\delta}{6(4z_c^2 + \delta^2)^{\frac{3}{2}}}, \quad (19)$$

with the cutoff now appearing inside the entanglement density. Repeating the computation (17) we then obtain

$$\begin{aligned} \text{tot 2-ent in } A|_{z_c} &\approx 2 \int_0^{l/2 - z_c/2} d\xi \int_{z_c}^{l-2\xi} d\delta n(\delta, z_c) \\ &= \frac{1}{\sqrt{5}} \frac{c}{6} \frac{l}{z_c} - \sqrt{5} \frac{c}{6} \log \left(\frac{l + \sqrt{l^2 + 4z_c^2}}{z_c} \right) \\ &\quad + O(1). \end{aligned} \quad (20)$$

This time we do not have agreement with (13) in the leading term because of the multiplying factor of $1/\sqrt{5}$. We notice a couple of points: first even with this slight mismatch it is indeed remarkable that the scaling remains linear in l/z_c , considering all the path followed. From the CFT point of view this last computation is probably oversimplified, the state dual to the finite $z = z_c$ curve having been recently identified with the vacuum of the $\text{T}\bar{\text{T}}$ -deformed theory with deformation parameter $\mu(z_c)$ [22,23]. Thus we expect to need a more refined counting of entanglement couples to match our proposal, probably a slightly different definition of entanglement density starting from entanglement entropy, which is the link between the CFT interpretation and the holographic data. It is perhaps important to stress that, as in the more general case of surface-state correspondence, the Ryu-Takayanagi formula is still believed to hold for $\text{T}\bar{\text{T}}$ -deformed theories, from both the holographic replica trick argument of [18] and explicit computations [24,25].

D. Total (two-parties) entanglement of a state

Following these ideas we are naturally led to consider Eq. (11) for a space filling region. What happens to the geodesic γ_A , which bounds M together with $\tilde{\gamma}_A$, depends on the bulk geometry. For example it vanishes in AdS_3 but wraps the horizon for a BTZ black hole. The simple geometrical computation of $J_{\tilde{\gamma}}$ now upgrades its physical interpretation to a measure of the total two-parties entanglement of the state $|\psi\rangle_{\tilde{\gamma}}$.

1. AdS_3 in global coordinates

We consider here $\tilde{\gamma}$ to be the curve at constant $\rho = \rho_{\text{UV}} \gg 1$ in global AdS_3 coordinates at $t = \text{const}$:

$$ds^2 = L^2(d\rho^2 + \sinh^2 \rho d\phi^2).$$

The corresponding geodesic γ_A vanishes as $\tilde{\gamma}_A$ extends to include the full ϕ -domain, so that $J_{\tilde{\gamma}}$ becomes a measure of total two-parties entanglement for the state $|\psi\rangle_{\rho=\rho_{\text{UV}}}$. We can compute the integral (11) to obtain

$$\begin{aligned} J^{\text{tot}}(\rho = \rho_{\text{UV}}) &= -\frac{L}{4G_N} \int_0^{\rho_{\text{UV}}} d\rho \int_0^{2\pi} d\phi \sqrt{g} \frac{R}{2} \\ &= \frac{\pi L}{2G_N} \int_0^{\rho_{\text{UV}}} d\rho \sinh \rho \\ &\xrightarrow{\rho_{\text{UV}} \rightarrow \infty} \frac{\pi L}{4G_N} e^{\rho_{\text{UV}}} = \frac{c}{6} \pi e^{\rho_{\text{UV}}}. \end{aligned} \quad (21)$$

Note that this is the same result we would get by simply computing

$$\begin{aligned} \frac{\text{length}(\partial \text{AdS}_3|_{t=0})}{4G_N} &= \frac{1}{4G_N} \int_0^{2\pi} d\phi \sqrt{g_{\phi\phi}} \xrightarrow{\rho_{\text{UV}} \rightarrow \infty} \frac{\pi L}{4G_N} e^{\rho_{\text{UV}}} \\ &= \frac{c}{6} \pi e^{\rho_{\text{UV}}} \approx N_{\text{dof}}. \end{aligned}$$

The last equivalence with N_{dof} comes from the idea that a gravitational theory contains as many physical degrees of freedom as the size of its boundary in Plank units. Indeed also a quantity that scales as $\sim c \frac{l}{\epsilon}$ for a finite region l and cutoff ϵ should be seen as proportional to the number of local degrees of freedom. So in general we may argue that

$$J^{\text{tot}}(\tilde{\gamma}_A) \approx \text{total 2-entanglement in } \tilde{\gamma}_A \approx N_{\text{dof}} \text{ in } \tilde{\gamma}_A. \quad (22)$$

Perhaps it would be interesting to exploit further this relation in the context of black holes and entanglement production by Hawking radiation, which is indeed an example of two-parties entanglement.

2. AdS_3 in Poincaré coordinates

Similarly to the previous case we still consider AdS_3 at $t = \text{const}$ but this time the cutoff is in Poincaré coordinates¹⁰:

$$ds^2 = \frac{L^2}{z^2} (dz^2 + dx^2)$$

and $\tilde{\gamma}$ corresponds to $z = \epsilon \ll 1$. Then introducing a large IR cutoff $C \gg 1$ on the x coordinate,

$$\begin{aligned} J^{\text{tot}}(z = \epsilon) &= -\frac{L}{4G_N} \int_\epsilon^\infty dz \int_{C/2}^{C/2} dx \sqrt{g} \frac{R}{2} \\ &= \frac{CL}{4G_N} \int_\epsilon^\infty dz \frac{1}{z^2} = \frac{CL}{4G_N} \frac{1}{\epsilon}. \end{aligned} \quad (23)$$

¹⁰That covers only one half of the total AdS_3 space.

3. BTZ black hole

The final simple example is a BTZ black hole at constant time:

$$ds^2 = \frac{L^2 r^2}{(r^2 - r_-^2)(r^2 - r_+^2)} dr^2 + r^2 d\phi^2,$$

and for simplicity we pick zero angular momentum or $r_- = 0$. Here the CFT states dual to generic curves are not pure, so all the physical interpretation we have attached loses much of its meaning, starting with the discussion on mutual information and factorizability, as the procedure of constructing a reduced density matrix mixes up entanglement effects with the entropy of the original density matrix we have started with. Nonetheless it is easy to compute J^{tot} also in this case so we will just do it.

The geodesic γ_A now wraps the horizon at $r = r_+$ for A space filling. So our integral (11) will span all the region in between the horizon and the curve $\tilde{\gamma}$ that we choose to be at $r = r_{\text{UV}} \gg 1$. We obtain

$$\begin{aligned} J^{\text{tot}}(r = r_{\text{UV}}) &= -\frac{L}{4G_N} \int_{r_+}^{r_{\text{UV}}} dr \int_0^{2\pi} d\phi \sqrt{g} \frac{R}{2} \\ &= \frac{\pi}{2G_N} \int_{r_+}^{r_{\text{UV}}} dr \frac{r}{\sqrt{r^2 - r_+^2}} \\ &= \frac{\pi}{2G_N} \sqrt{r_{\text{UV}}^2 - r_+^2}. \end{aligned} \quad (24)$$

V. CONCLUSIONS AND FUTURE WORK

We have discussed a geometric local measure of factorization of CFT states dual to some convex curve as the geodesic curvature of such a curve. This geometric measure $J(\gamma)$ is argued to have its dual counterpart as the second derivative in the interval size of some quantum information distance between a two-parties entangled state and its factorized version. Inspiration has come from the holographic computation of mutual information for small intervals, by using the relative entropy as a distance between states. Quantum properties of a “good” entanglement measure have been studied, and the physical interpretation of the integral $J(\gamma_A) = \int_{\gamma_A} J(\gamma)|_{\tau_p}$ as a measure of bipartite entanglement within γ_A discussed. The use of the Gauss-Bonnet theorem permits a simple computation of such a quantity and the case of a boundary state has been compared with results from the entanglement density and MERA tensor network formalisms. As the physical interpretation of the latter is quite clear, agreement with the result confirms our derivation. Finally some simple examples of total bipartite entanglement for the full state have been computed, using the boundary curve of AdS_3 at two different cutoffs and the BTZ black hole, where the entanglement interpretation becomes less transparent.

There are some main points that deserve further study in the future.

A. Bulk reconstruction

In this paper we pushed a little bit forward the geometry-to-entanglement map by identifying local factorizability of generic CFT states dual to convex curves with the local geodesic curvature of the curve. This association has the merit of being local, but the demerit of being based on some unknown state-measure and nonlocal CFT transformation from the usual boundary state to our curve state. In other words we have moved all the nonlocality to the CFT side. I believe this to be a good starting point to better understand the bulk reconstruction problem, see for instance [26–31].

B. Covariantization of MERA

The choice to work on a constant time slice of the bulk space for computing explicit examples has been done mostly in order to obtain a simple comparison with MERA and entanglement density. However the geometric results are perfectly covariant, for any spacelike slice M_{sl} , and we see no reason for not attaching to these the analogous physical meaning in terms of factorizability and entanglement. Hopefully this can be a guideline in fully understanding how to covariantize tensor networks, see for example [29] in this context.

C. Tripartite entanglement and beyond

An obvious question is if it exists a tripartite entanglement version of our measure, or higher generalizations. A possible starting point would be to use relative entropy for multipartite states, $S(\rho_{A_1 \cup A_2 \cup A_3} \| \rho_{A_1} \otimes \rho_{A_2} \otimes \rho_{A_3}) = S(A_1) + S(A_2) + S(A_3) - S(A_1 \cup A_2 \cup A_3)$, repeat the holographic computation in the Appendix and use the result as a hint towards a possible geometric measure.

D. Higher dimensionality

The generalization to three and four boundary dimensions is an interesting but complicated geometrical problem, mostly because a higher dimensional surface has “more ways” to differ from an extremal one than a curve to differ from a geodesic. Analogously we would need a better understanding of these additional terms from the point of view of the CFT state. Nonetheless the main idea goes through, that an extremal surface is dual to a locally factorized state as the mutual information vanishes.

E. Relation with differential entropy

An interesting geometric result from [32] that received a quantum information theoretic interpretation in [33] is that the length of a convex curve γ can be expressed as the integral of the difference between the entanglement entropies of certain configurations of boundary intervals $I(x)$, centered at x :

$$\frac{\text{lenth}(\gamma)}{4G_N} = \int (S(I(x)) - S(I(x) \cap I(x - dx))). \quad (25)$$

The shape of γ in (25) determines the choice of sizes for $I(x)$ such that the corresponding Ryu-Takayanagi curves nicely cancel each other but for the infinitesimal part tangent to γ , which builds up the result.

Equation (25), whose right-hand side received the name of differential entropy, has obvious similarities with (6). Thus it is intriguing to imagine some connection between the two results: on one side a surface-state interpretation of (25) as some measure associated with the state $|\psi\rangle_\gamma$, on the other a boundary-CFT information theoretic understanding of (6).

F. Computing complexity

As much as counting MERA disentangles below a geodesic, bounded by an interval $\tilde{\gamma}_A$, should be interpreted as a measure of bipartite entanglement for $\rho_{\tilde{\gamma}_A}$, adding to the counting the number of coarse graining operations as well should produce a viable definition for complexity¹¹ for $\rho_{\tilde{\gamma}_A}$, with reference state the locally geodesic state ρ_{γ_A} . Analogously, by extending the region to include the full curve, we have a measure of complexity for $|\psi\rangle_\gamma$, with reference the boundary state with no real entanglement dual to a point, see [9,29]. As there is a coarse graining operation for each disentangler in the discrete MERA network and if we associate the same complexity weight for both operations, from the geometric point of view just by multiplying by two our result (11) we would have a complexity measure. This idea resembles, and generalizes to subsystems of a large class of CFT states, two other famous proposals for holographic complexity measures [34,35].

G. Quantum corrections

To generalize our discussion to quantum bulk corrections is far from obvious but indeed a very interesting goal. Quantum corrections to entanglement entropy have been discussed in various papers, see for example [36–40].

The first thing that we notice is that the result of vanishing mutual information for a geodesic state goes through at least to order $O(1)$ in G_N . This because at this order quantum corrections to the entanglement entropy have been shown to assume the following structure [39]:

$$S_{\text{quantum}} = S_{\text{bulk}} + \frac{\delta A}{4G_N} + \langle \Delta S_{\text{Wald-like}} \rangle + S_{\text{counter}}, \quad (26)$$

with S_{bulk} referring to the bulk entanglement entropy with bulk entangling region given by the inside of the Ryu-Takayanagi curve and boundary entangling region, while the remaining terms are line integrals. Then, as the

Ryu-Takayanagi curve coincides with the entangling region for a geodesic state, the latter terms cancel each other in computing the mutual information analogously to the classical part, while S_{bulk} trivially vanishes because its entangling region squeezes to zero. Because of this we still expect a geodesic state to be locally factorized in all the couples of subregions, either contiguous or not, at least up to order $O(G_N^0)$. What instead will not work is all the discussion on how measuring entanglement among any two contiguous regions implies $J(\gamma)$ to be a measure of entanglement along noncontiguous subregions as well. Because of this, disagreement with the entanglement density formalism is expected to show up when considering quantum bulk corrections.

ACKNOWLEDGMENTS

I would like to thank Rafael Chavez for discussion, Tadashi Takayanagi for email correspondence and Aditya Mehra, Dmitry Melnikov, Filiberto Ares, Giancarlo Camilo, Máté Lencsés and Thiago Fleury for attending a seminar presenting a first draft of the paper and providing interesting feedback. This work has been done under financial support from the Brazilian ministries MCTI and Ministério da Educação e Cultura (MEC).

APPENDIX: MUTUAL INFORMATION AND GEODESIC CURVATURE

We consider the setup of Fig. 3 to compute the mutual information $I_{\tilde{\gamma}}(A_1 : A_2)$ for a state $|\psi\rangle_{\tilde{\gamma}}$ and two infinitesimal intervals A_1, A_2 with $\partial A_1 = \{\tau_1, \tau_p\}$ and $\partial A_2 = \{\tau_p, \tau_2\}$. The proper distance $d\tau^2 = d\xi^a d\xi^b \delta_{ab}$ for a locally flat metric parametrizes both the curve $\tilde{\gamma}$ and the nearby Ryu-Takayanagi geodesics $\gamma_{A_1}, \gamma_{A_2}$ and γ for the regions considered here. The definitions for the displacement vectors are

$$\gamma_{A_1} = \gamma + \delta\gamma_{A_1} \quad \gamma_{A_2} = \gamma + \delta\gamma_{A_2} \quad \tilde{\gamma} = \gamma + \delta\gamma \quad (A1)$$

and their boundary conditions:

$$\begin{aligned} \delta\gamma(\tau_1) = \delta\gamma(\tau_2) = \delta\gamma_{A_1}(\tau_1) = \delta\gamma_{A_2}(\tau_2) = 0 \\ \delta\gamma_{A_1}(\tau_p) = \delta\gamma_{A_2}(\tau_p) = \delta\gamma(\tau_p). \end{aligned} \quad (A2)$$

FIG. 3. Holographic mutual information $I_{\tilde{\gamma}}(A_1 : A_2)$.

¹¹I thank Giancarlo Camilo for proposing the idea.

The mutual information, from now on shortly indicated as I , is holographically computed as ($c = \hbar = 1$):

$$\begin{aligned} I &= S_{\tilde{\gamma}}(A_1) + S_{\tilde{\gamma}}(A_2) - S_{\tilde{\gamma}}(A_1 A_2) \\ &= \frac{1}{4G_N} \int_{\tau_1}^{\tau_p} \sqrt{\partial\gamma_{A_1}^\mu \partial\gamma_{A_1}^\nu g_{\mu\nu}} d\tau + \frac{1}{4G_N} \int_{\tau_p}^{\tau_2} \sqrt{\partial\gamma_{A_2}^\mu \partial\gamma_{A_2}^\nu g_{\mu\nu}} d\tau \\ &\quad - \frac{1}{4G_N} \int_{\tau_1}^{\tau_2} \sqrt{\partial\gamma^\mu \partial\gamma^\nu g_{\mu\nu}} d\tau \end{aligned} \quad (\text{A3})$$

and we denote $\partial = \frac{\partial}{\partial\tau}$. We first rescale τ so that $\partial\gamma^\mu \partial\gamma^\nu g_{\mu\nu} = v = 1$; next we simplify the remaining two integrands expanding at first order in $\delta\gamma_{A_1}$ and $\delta\gamma_{A_2}$. This simplifies the mutual information to

$$\begin{aligned} 4G_N I &= \int_{\tau_1}^{\tau_p} \partial\delta\gamma_{A_1}^\mu \partial\gamma^\nu g_{\mu\nu} d\tau \\ &\quad + \int_{\tau_p}^{\tau_2} \partial\delta\gamma_{A_2}^\mu \partial\gamma^\nu g_{\mu\nu} d\tau + O(\delta\gamma_{A_1, A_2}^2). \end{aligned} \quad (\text{A4})$$

We now perform integration by parts and use the boundary conditions (A2) to obtain

$$\begin{aligned} 4G_N I &= - \int_{\tau_1}^{\tau_p} d\tau (\delta\gamma_{A_1}^\mu \partial^2 \gamma^\nu g_{\mu\nu} + \delta\gamma_{A_1}^\mu \partial\gamma^\nu \partial g_{\mu\nu}) \\ &\quad - \int_{\tau_p}^{\tau_2} d\tau (\delta\gamma_{A_2}^\mu \partial^2 \gamma^\nu g_{\mu\nu} + \delta\gamma_{A_2}^\mu \partial\gamma^\nu \partial g_{\mu\nu}). \end{aligned} \quad (\text{A5})$$

Using the geodesic equation for γ ,

$$\partial^2 \gamma^\mu + \Gamma_{\rho\sigma}^\mu \partial\gamma^\rho \partial\gamma^\sigma = 0,$$

and the metricity for the metric $\nabla_\tau g_{\mu\nu} = 0$ the expression further transforms to

$$\begin{aligned} 4G_N I &= - \int_{\tau_1}^{\tau_p} d\tau \delta\gamma_{A_1}^\mu \partial\gamma^\rho \partial\gamma^\sigma \Gamma_{\rho\mu}^\nu g_{\nu\sigma} \\ &\quad - \int_{\tau_p}^{\tau_2} d\tau \delta\gamma_{A_2}^\mu \partial\gamma^\rho \partial\gamma^\sigma \Gamma_{\rho\mu}^\nu g_{\nu\sigma}. \end{aligned} \quad (\text{A6})$$

Now we Taylor expand $\delta\gamma_{A_1}(\tau)$ to second order in $\tau - \tau_1$:

$$\begin{aligned} \delta\gamma_{A_1}(\tau) &= \delta\gamma_{A_1}|_{\tau_1} + (\tau - \tau_1) \partial\delta\gamma_{A_1}|_{\tau_1} + \frac{(\tau - \tau_1)^2}{2} \partial^2 \delta\gamma_{A_1}|_{\tau_1} \\ &\quad + O((\tau - \tau_1)^3) \end{aligned}$$

and similarly $\delta\gamma_{A_2}(\tau)$ to second order in $\tau - \tau_2$. The constant term vanishes because of (A2) while the second derivative can be transformed to a first derivative by application of the formula that constrains the infinitesimal displacement between two nearby geodesics, here γ and either $\gamma_{A_1(A_2)}$,

$$\begin{aligned} \partial^2 \delta\gamma_{A_1(A_2)}^\mu &= - \frac{\Gamma_{\nu\lambda}^\mu}{\partial\gamma^\rho} \delta\gamma_{A_1(A_2)}^\rho \partial\gamma^\nu \partial\gamma^\lambda \\ &\quad - 2\Gamma_{\nu\lambda}^\mu \partial\gamma^\nu \partial\delta\gamma_{A_1(A_2)}^\lambda, \end{aligned} \quad (\text{A7})$$

and (A2). Doing so we obtain

$$\delta\gamma_{A_1}^\mu(\tau) = (\tau - \tau_1) \partial\delta\gamma_{A_1}^\mu|_{\tau_1} (\delta_\rho^\mu - (\tau - \tau_1) (\Gamma_{\nu\rho}^\mu \partial\gamma^\nu)_{\tau_1}) \quad (\text{A8})$$

and analogously for $\delta\gamma_{A_2}$ around τ_2 . Once more we impose (A2)

$$\begin{aligned} \delta\gamma_{A_1}^\mu(\tau_p) &= (\tau_p - \tau_1) \partial\delta\gamma_{A_1}^\mu|_{\tau_1} (\delta_\rho^\mu - (\tau_p - \tau_1) (\Gamma_{\nu\rho}^\mu \partial\gamma^\nu)_{\tau_1}) \\ &= \delta\gamma^\mu(\tau_p) \end{aligned}$$

with a similar equation for $\delta\gamma_{A_2}$ replacing $\tau_1 \leftrightarrow \tau_2$. This permits us to solve for $\partial\delta\gamma_{A_1}|_{\tau_1}$ (respectively $\partial\delta\gamma_{A_2}|_{\tau_2}$) by inverting the matrix in between brackets above at first order in $(\tau_p - \tau_1)$ [respectively $(\tau_p - \tau_2)$]:

$$\partial\delta\gamma_{A_1}^\mu|_{\tau_1} = \frac{\delta\gamma^\rho|_{\tau_p}}{(\tau_p - \tau_1)} (\delta_\rho^\mu + (\tau_p - \tau_1) (\Gamma_{\nu\rho}^\mu \partial\gamma^\nu)_{\tau_1}) \quad (\text{A9})$$

and similarly for $\partial\delta\gamma_{A_2}|_{\tau_2}$. Then we insert these results inside (A8) [and its counterpart for $\delta\gamma_{A_1}^\mu(\tau)$] so that

$$\delta\gamma_{A_1}^\mu(\tau) = \frac{(\tau - \tau_1)}{(\tau_p - \tau_1)} (\delta_\lambda^\mu - (\tau - \tau_p) (\Gamma_{\nu\lambda}^\mu \partial\gamma^\nu)_{\tau_1}) \delta\gamma^\lambda|_{\tau_p}$$

and again a similar story for $(A_1, \tau_1) \leftrightarrow (A_2, \tau_2)$. Inserting back into the expression for the mutual information (A6) we have

$$\begin{aligned} 4G_N I &= - \int_{\tau_1}^{\tau_p} d\tau \frac{(\tau - \tau_1)}{(\tau_p - \tau_1)} (\delta_\lambda^\mu - (\tau - \tau_p) (\Gamma_{\nu\lambda}^\mu \partial\gamma^\nu)_{\tau_1}) \delta\gamma^\lambda|_{\tau_p} \\ &\quad \times \partial\gamma^\rho \partial\gamma^\sigma \Gamma_{\rho\mu}^\nu g_{\nu\sigma} \\ &\quad - \int_{\tau_p}^{\tau_2} d\tau \frac{(\tau - \tau_2)}{(\tau_p - \tau_2)} (\delta_\lambda^\mu - (\tau - \tau_p) (\Gamma_{\nu\lambda}^\mu \partial\gamma^\nu)_{\tau_2}) \delta\gamma^\lambda|_{\tau_p} \\ &\quad \times \partial\gamma^\rho \partial\gamma^\sigma \Gamma_{\rho\mu}^\nu g_{\nu\sigma}. \end{aligned} \quad (\text{A10})$$

The next step is to expand $\delta\gamma^\lambda|_{\tau_p}$ around $\delta\gamma^\lambda|_{\tau_1}$ and $\delta\gamma^\lambda|_{\tau_2}$ in the two integrands, again at second order in the τ variation. The key point here is that the displacement $\delta\gamma$ is not between two geodesics but between a geodesic γ and a nearby generic convex curve $\tilde{\gamma}$, so Eq. (A7) acquires a new term which is the vector k_c^μ defined as $k_c^\mu(\tilde{\gamma}) = \nabla_\tau \partial_\tau \tilde{\gamma}^\mu$:

$$\partial^2 \delta\gamma^\mu = -\partial_\rho \Gamma_{\nu\lambda}^\mu \delta\gamma^\rho \partial\gamma^\nu \partial\gamma^\lambda - 2\Gamma_{\nu\lambda}^\mu \partial\gamma^\nu \partial\delta\gamma^\lambda + k_c^\mu(\tilde{\gamma}). \quad (\text{A11})$$

A few remarks: first everything in the above equation is along γ but k_c^μ that is evaluated on $\tilde{\gamma}$. Second there should be a normalization $\|\partial\tilde{\gamma}\|^2$ multiplying $k_c^\mu(\tilde{\gamma})$, as the definition

assumes a unit tangent vector; nonetheless the normalization is $\|\partial\tilde{\gamma}\| = 1 + O(\delta\gamma)$, so being k_c already of order $O(\delta\gamma)$ we can discard this correction.

We then essentially repeat the previous steps for the evaluation of the second order expansion of $\delta\gamma^\lambda(\tau_p)$; first we expand it at second order around τ_1 (respectively τ_2) and use the boundary conditions $\delta\gamma|_{\tau_2} = 0$ (respectively $\delta\gamma|_{\tau_1} = 0$) from (A2) to fix the constant term and the derivatives obtaining

$$\delta\gamma|_{\tau_1} = 0 \quad \partial\delta\gamma|_{\tau_1} = -\frac{1}{2}(\tau_2 - \tau_1)\partial^2\delta\gamma|_{\tau_1}^{TM_{sl}} \quad \partial^2\delta\gamma|_{\tau_1}^{NM_{sl}} = 0 \quad (\text{A12})$$

(and similarly in τ_2), where we have indicated as TM_{sl} and NM_{sl} the projection along the tangent and the normal to the spacelike slice M_{sl} to which $\tilde{\gamma}$ belongs. After simple algebra it produces

$$\delta\gamma(\tau_p) = \frac{1}{2}(\tau_p - \tau_1)(\tau_p - \tau_2)\partial^2\delta\gamma|_{\tau_1}^{TM_{sl}} \quad (\text{A13})$$

together with the equivalent equation from $\partial^2\delta\gamma|_{\tau_1} \leftrightarrow \partial^2\delta\gamma|_{\tau_2}$ to be used in the other integrand.¹² With this expression at hand we can use (A11) in τ_1 (respectively τ_2) plus (A12) to obtain an equation for the second derivative¹³:

$$\begin{aligned} \partial^2\delta\gamma^\mu|_{\tau_1}^{TM_{sl}}(\delta_\mu^\nu - (\Gamma_{\nu\lambda}^\mu\partial\gamma^\lambda)_{\tau_1}(\tau_2 - \tau_1)) \\ = k_c^\mu(\tilde{\gamma})|_{\tau_1}^{TM_{sl}} = k^\mu(\tilde{\gamma})|_{\tau_1} \end{aligned} \quad (\text{A14})$$

¹²Incidentally $\partial^2\delta\gamma|_{\tau_1} = \partial^2\delta\gamma|_{\tau_2}$ because of the form of (A13) and (A2), but we will still keep the two indices for clarity.

¹³See footnote 5 for reference on the definition of the various k_s .

that inverted at first order and inserted into (A13) produces

$$\begin{aligned} \delta\gamma^\mu(\tau_p) &= \frac{1}{2}(\tau_p - \tau_1)(\tau_p - \tau_2)k^\mu(\tilde{\gamma})|_{\tau_1} \\ &\times (\delta_\mu^\nu + (\Gamma_{\nu\lambda}^\mu\partial\gamma^\lambda)_{\tau_1}(\tau_2 - \tau_1)) \end{aligned} \quad (\text{A15})$$

and again we have a similar equation by exchanging $\tau_1 \leftrightarrow \tau_2$. Plugging (A15) inside (A10) gives, at second order in the τ variation,

$$\begin{aligned} 4G_N I &= \int_{\tau_1}^{\tau_p} d\tau \frac{(\tau - \tau_1)(\tau_2 - \tau_p)}{2} \partial\gamma^\theta \partial\gamma^\sigma \Gamma_{\theta\mu}^\xi g_{\xi\sigma} k^\mu(\tilde{\gamma})|_{\tau_1} \\ &\times (\delta_\nu^\mu + [(\tau_p - \tau) + (\tau_2 - \tau_1)](\Gamma_{\rho\nu}^\mu\partial\gamma^\rho)_{\tau_1}) \\ &+ \int_{\tau_p}^{\tau_2} d\tau \frac{(\tau_2 - \tau)(\tau_p - \tau_1)}{2} \partial\gamma^\theta \partial\gamma^\sigma \Gamma_{\theta\mu}^\xi g_{\xi\sigma} k^\mu(\tilde{\gamma})|_{\tau_2} \\ &\times (\delta_\nu^\mu + [(\tau_p - \tau) + (\tau_1 - \tau_2)](\Gamma_{\rho\nu}^\mu\partial\gamma^\rho)_{\tau_2}). \end{aligned} \quad (\text{A16})$$

Finally we fix $\Delta\tau = (\tau_p - \tau_1) = (\tau_2 - \tau_p)$, compute the double derivative $\partial_{\Delta\tau}^2 I$ and expand at first order in $\Delta\tau$. Rewriting

$$\partial\gamma^\theta \partial\gamma^\sigma \Gamma_{\theta\mu}^\xi g_{\xi\sigma} k^\mu(\tilde{\gamma}) = \partial\gamma^\theta \partial\gamma^\sigma \partial_\nu g_{\theta\sigma} k^\nu(\tilde{\gamma}) \quad (\text{A17})$$

we finally obtain

$$\begin{aligned} J_S(\tilde{\gamma})|_{\tau_p} &= \partial_{\Delta\tau}^2 I|_{\tau_p} \\ &= \frac{3\Delta\tau}{4G_N} (\partial\tilde{\gamma}^\theta \partial\tilde{\gamma}^\sigma \partial_\nu g_{\theta\sigma} k^\nu)|_{\tau_p} + O(\Delta\tau^3). \end{aligned} \quad (\text{A18})$$

Note that formally the term $\partial\gamma^\theta \partial\gamma^\sigma \partial_\nu g_{\theta\sigma}$ from (A17) should be computed along the geodesic γ , but at leading order in $\Delta\tau$ this just coincides with the same quantity along the curve $\tilde{\gamma}$, so this justifies Eq. (A18) where everything is along $\tilde{\gamma}$.

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