Minimally modified A_4 Altarelli-Feruglio model for neutrino masses and mixings and its experimental consequences

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(Received 30 April 2020; accepted 13 July 2020; published 28 July 2020)

We present a simple modification of the Altarelli-Feruglio A_4 flavor symmetry model using a minimal number of parameters congruous with the symmetries of the original theory. The resulting model is consistent with all presently known data on neutrino masses and mixings. Furthermore, it makes testable tight predictions for future experiments aimed at improving the measurements of $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$, δ_{CP} and the neutrinoless double-beta decay parameter $|m_{ee}|$. Our model exploits the unique possibility of multiple allowed, yet qualitatively different, contractions of fields charged under the A_4 discrete symmetry.

DOI: [10.1103/PhysRevD.102.016023](https://doi.org/10.1103/PhysRevD.102.016023)

I. INTRODUCTION

Although SM has been remarkably consistent with experimental data, it is unable to explain several key issues such as the neutrino mass values and ordering, the number of fermion generations, and the observed values of neutrino mixing parameters. Several approaches have been proposed to determine the theory behind fermion mixing. One of the most promising is discrete flavor symmetry models, under which left- and right-handed fermions, as well as proposed new particles, transform as irreducible representations of some discrete symmetry group. The A_4 flavor symmetry model proposed by Altarelli and Feruglio [1] is a particularly interesting approach, being a minimal model which produces tri-bimaximal mixing that was considered at the time a good description of the neutrino mixing matrix.

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.1)$$

However, recent reactor neutrino experiments [2–4] have indicated that the U_{e3} component of the mixing matrix deviate sizably from zero with $|U_{e3}| \approx 0.15$ in violation of the tri-bimaximal pattern that requires $|U_{e3}| = 0$. Thus, A_4 is presently considered a good approximation upon which

to build in corrections. In this paper, by fully considering field contractions under A_4 symmetry group, we propose a simple modification to the Altarelli-Feruglio model that produce oscillation parameters that agree well with up-to-date experimental data, and discuss further implications.

II. ALTARELLI-FERUGLIO A_4 MODEL

The field content and the symmetry group assignments of the Altarelli-Feruglio model [1] are given in Table I. The leptons are assigned to the four irreducible representations of A_4 . A Majorana seesaw realization for neutrino masses arises, with RH neutrino field ν^c transforming as a triplet of A_4 . The A_4 flavor symmetry is broken by two flavon triplets φ_S , φ_T and by one or more singlets ξ . Two Higgs doublets $h_{u,d}$ invariant under A_4 are also introduced. A supersymmetric (SUSY) context is adopted, with the superpotential term for lepton masses given by:

$$\begin{aligned} \omega_l = & \frac{y_e}{\Lambda} (l\varphi_T)_1 h_d e^c + \frac{y_\mu}{\Lambda} (l\varphi_T)_{1'} h_d \mu^c + \frac{y_\tau}{\Lambda} (l\varphi_T)_{1''} h_d \tau^c \\ & + y(lh_u \nu^c)_1 + (x_A \xi + \tilde{x}_A \tilde{\xi})(\nu^c \nu^c)_1 \\ & + x_B (\varphi_S (\nu^c \nu^c)_{3_s})_1 + \text{H.c.} \end{aligned} \quad (2.1)$$

where Λ is the cutoff scale. Additional symmetries $U(1)_{FN}$, $U(1)_R$ are also implemented in the model. $U(1)_{FN}$, broken

TABLE I. Field content and symmetry group assignment of Altarelli A_4 model [1].

	l	e^c	μ^c	τ^c	ν^c	$h_{u,d}$	θ	φ_T	φ_S	$\xi/\tilde{\xi}$
A_4	3	1	1''	1'	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	ω^2	1	1	1	ω^2	ω^2
$U(1)_{FN}$	0	4	2	0	0	0	−1	0	0	0
$U(1)_R$	1	1	1	1	1	0	0	0	0	0

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by the A_4 singlet field θ , reproduces the observed hierarchy among m_e , m_μ , and m_τ . Its overall contribution to the superpotential charged lepton mass terms are implicitly assumed. $U(1)_R$, broken at low energy scale m_{SUSY} to discrete R-parity, is a common feature of supersymmetric formulations. A supplementary Z_3 symmetry is also implemented, restricting additional terms otherwise allowed by A_4 symmetry, such as the term obtained by exchanging $\varphi_T \leftrightarrow \varphi_S$.

In the model setup, it can be derived that the flavon fields naturally develop a vacuum expectation value (VEV) along the directions [5]:

$$\begin{aligned} \langle \varphi_T \rangle &= (\nu_T, 0, 0); & \langle \varphi_S \rangle &= (\nu_S, \nu_S, \nu_S); \\ \langle \xi \rangle &= v_\xi; & \langle \tilde{\xi} \rangle &= 0; \end{aligned} \quad (2.2)$$

With this vacuum alignment realized, the mass matrices of the charged lepton and neutrino sectors can be calculated after flavor and electroweak symmetry breaking. At leading order, working in the T -diagonal basis for the A_4 triplet representation, the charged lepton mass matrix is given by:

$$m_l = \frac{\nu_d \nu_T}{\Lambda} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \quad (2.3)$$

where the suppression coming from the breaking of $U(1)_{FN}$ is implicit. Similarly in the neutrino sector, the Dirac and Majorana masses after symmetry breaking are

$$\begin{aligned} m_D &= y v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ M_R &= \begin{pmatrix} A + 2B/3 & -B/3 & -B/3 \\ -B/3 & 2B/3 & A - B/3 \\ -B/3 & A - B/3 & 2B/3 \end{pmatrix} \end{aligned} \quad (2.4)$$

where

$$A \equiv x_A v_\xi, \quad B \equiv 3x_B v_S. \quad (2.5)$$

The light neutrino mass matrix is $m_\nu = (m_D)^T M_R^{-1} m_D$ with eigenvalues:

$$m_1 = \frac{y^2 \nu_u^2}{M_1}, \quad m_2 = \frac{y^2 \nu_u^2}{M_2}, \quad m_3 = \frac{y^2 \nu_u^2}{M_3} \quad (2.6)$$

where M_1, M_2, M_3 are eigenvalues of M_R given by:

$$M_1 = (A + B), \quad M_2 = A, \quad M_3 = (-A + B). \quad (2.7)$$

From the form of the mass eigenvalues, one can derive bounds on the lightest neutrino mass and the possible

values of the effective Majorana mass $|m_{ee}|$ [5]. For the normal hierarchy:

$$\begin{aligned} m_1 &\geq \sqrt{\frac{\Delta m_{\text{sol}}^2}{3}} \left(1 - \frac{4\sqrt{3}}{9} r + O(r^2) \right) \approx 0.004 \text{ eV} \\ m_1 &\leq \sqrt{\frac{\Delta m_{\text{sol}}^2}{3}} \left(1 + \frac{4\sqrt{3}}{9} r + O(r^2) \right) \approx 0.006 \text{ eV} \\ |m_{ee}| &\approx \frac{4}{3\sqrt{3}} \Delta m_{\text{sol}}^2 \approx 0.007 \text{ eV} \end{aligned} \quad (2.8)$$

And for inverse hierarchy:

$$\begin{aligned} m_3 &\geq \sqrt{\frac{\Delta m_{\text{atm}}^2}{8}} \left(1 - \frac{1}{6} r^2 + O(r^3) \right) \approx 0.017 \text{ eV} \\ |m_{ee}| &\geq \sqrt{\frac{\Delta m_{\text{atm}}^2}{8}} \approx 0.017 \text{ eV} \end{aligned} \quad (2.9)$$

where $\Delta m_{\text{sol}}^2 = \Delta m_{21}^2 = m_2^2 - m_1^2$, $\Delta m_{\text{atm}}^2 = \Delta m_{31}^2 = m_3^2 - m_1^2$, and $r = \Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$.

In the basis where the charged lepton matrix is diagonal, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix is just the unitary matrix U that diagonalizes the light neutrino mass matrix m_ν :

$$U^\dagger m_\nu U^* = \text{diag}(m_1, m_2, m_3) \quad (2.10)$$

For m_ν that can be calculated from m_D and M_R given in Eq. (2.4), U is simply:

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (2.11)$$

which is just the tri-bimaximal mixing matrix U_{TB} . The tri-bimaximal form is in reasonable agreement with observed neutrino mixing matrix and is frequently considered as a good first approximation in many flavor symmetry models. However, as noted above, recent reactor neutrino experiments [2–4] have indicated that the U_{e3} matrix element deviates sizably from zero $|U_{e3}| \approx 0.15$. Modified models that produce sizable corrections to the mixing parameters have been proposed, but often require more complicated models involving nonvanishing higher order contributions [6], soft A_4 symmetry breaking terms [7], misalignment of mass matrix elements [8], or larger symmetry groups than A_4 (e.g., T' , $\Delta(6 \times 10^2)$, $(Z_{18} \times Z_6) \rtimes S_3$, T_{13}) [9–12]. Here we propose a minimal modification to the Alteralli A_4 model by considering an additional Dirac mass term allowed by A_4 symmetry contractions that are also allowed by the other symmetries assumed in the model. The modified model retains the simplicity of the A_4 model,

and predicts neutrino masses and mixing parameters that are in good agreement with current experimental data.

III. MODIFIED ALTARELLI-FERUGLIO MODEL

In the Altarelli-Feruglio A_4 model, the Dirac mass is entirely specified by the standard model Yukawa term $y(lh_u\nu^c)_1$ with no flavon contraction modifications. A natural step to modify the existing model, is to replace the Dirac Yukawa term with terms involving flavons ξ and φ_S in parallel to the form of the Majorana terms:

$$\frac{y}{\Lambda}\xi(lh_u\nu^c)_1 + \frac{y'}{\Lambda}(\varphi_S(lh_u\nu^c)_3)_1. \quad (3.1)$$

The first term is modified by ξ , an A_4 singlet. For the second term, the A_4 contraction between l and ν^c is now another triplet that in turn contracts with the triplet φ_S field to form an A_4 invariant. Both ξ and φ_S have zero $U(1)_{FN}$ and $U(1)_R$ charges, preserving these symmetries even after they obtain vevs. The only symmetry that is violated by this modification in the original Altarelli-Feruglio model is the supplementary Z_3 symmetry, under which φ_S transforms as ω^2 , making the second term involving φ_S an ω^2 instead of a singlet under the Z_3 . However, since Z_3 is a supplementary symmetry introduced to restrict extra terms otherwise allowed by the A_4 symmetry, like the terms obtained by exchanging $\varphi_T \leftrightarrow \varphi_S$ and the term $\nu^c\nu^c$, we can modify it without affecting other elements of the model to allow this modification. This goal is accomplished by making the following change in Z_3 field assignment:

$$\begin{aligned} l &\rightarrow \omega^2 \\ e_R, \mu_R, \tau_R &\rightarrow \omega \end{aligned} \quad (3.2)$$

with all other Z_3 field assignments remaining the same. The new term in Eq. (3.1) now contracts to a Z_3 singlet, and all existing terms are still invariant, while additional terms such as those obtained by exchanging $\varphi_T \leftrightarrow \varphi_S$ and $\nu^c\nu^c$ are still not allowed by this new Z_3 symmetry.

As specified by the tensor product rules in Eq. (A6), there are two ways to contract two A_4 triplets l and ν^c to another triplet. The form of the projection matrix for each linearly independent contraction is given in Sec. A 3. Therefore, there could be two independent new terms in the form of Eq. (3.1), with different coupling constants. For simplicity of the model, in this paper we will only consider the term with the simpler project matrix:

$$\tau_{\mu\nu}^\rho = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]. \quad (3.3)$$

In the rest of the paper, we will show that this simple modification already predicts neutrino masses and oscillation parameters that fit well with current experimental data, and discuss additional implications of the model.

A. Parametrization of the model

With the additional Dirac triplet contraction term, the charged lepton mass matrix and the Majorana neutrino mass matrix remain the same, and the relationship specified by Eq. (2.7) still holds, but the Dirac neutrino mass matrix is now

$$m_D = \begin{pmatrix} a & -b & b \\ -b & b & a \\ b & a & -b \end{pmatrix} \quad (3.4)$$

where a, b is given by

$$a = yv_u v_\xi / \Lambda; \quad b = y'v_u v_S / \Lambda. \quad (3.5)$$

For convenience of analysis, we parametrize the model by defining complex dimensionless parameters H and η :

$$H = \frac{B}{A}, \quad \eta = \frac{b}{a} \quad (3.6)$$

so that

$$(m_\nu)_{ij} = (m_D^T M_R^{-1} m_D)_{ij} = k \Sigma_{ij} \quad (3.7)$$

where

$$k = \frac{y^2 v_u^2 v_\xi^2 / \Lambda^2}{x_A v_\xi \cdot 3(H^2 - 1)} \quad (3.8)$$

absorbs the dimension of the problem. As an overall factor in the mass matrix k does not affect the neutrino mixing matrix elements. The phase of k can be absorbed by phase redefinitions of the charged lepton fields, so it can be treated as a real parameter without loss of generality. The components of the matrix Σ are given in terms of the complex parameters H, η by

$$\begin{aligned} \Sigma_{11} &= 6\eta^2 + 6\eta^2 H + 2H + H^2 - 3 \\ \Sigma_{12} &= \Sigma_{21} = -3\eta^2 - 3\eta^2 H + 6\eta - H + H^2 \\ \Sigma_{13} &= \Sigma_{31} = -3\eta^2 - 3\eta^2 H - 6\eta - H + H^2 \\ \Sigma_{22} &= -3\eta^2 + 6\eta^2 H - 6\eta + 2H + H^2 \\ \Sigma_{23} &= \Sigma_{32} = 6\eta^2 - 3\eta^2 H - H + H^2 - 3 \\ \Sigma_{33} &= -3\eta^2 + 6\eta^2 H + 6\eta + 2H + H^2. \end{aligned} \quad (3.9)$$

B. Prediction for neutrino oscillation parameters

The light neutrino mass matrix m_ν is diagonalized by the unitary PMNS matrix U according to Eq. (2.10). U can be constructed from direct diagonalization of the Hermitian matrix $h = m_\nu m_\nu^\dagger$:

$$U^\dagger h U = \text{diag}(m_1^2, m_2^2, m_3^2). \quad (3.10)$$

Following the framework of calculating oscillation parameters of a generalized neutrino mass matrix by Adhikary *et al.* [13], the row-wise elements of U are given in terms of the elements of the h and its eigenvalues m_i^2 :

$$U_{\text{PMNS}} = P_\phi \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} P_M \quad (3.12)$$

where

$$P_\phi = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \quad (3.13)$$

contains the unphysical phases that can be rotated away by phase redefinitions of the charged lepton fields, and

$$P_M = \begin{pmatrix} e^{\frac{i\alpha_M}{2}} & 0 & 0 \\ 0 & e^{\frac{i\beta_M}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.14)$$

where N_i is the normalization constant. Following the PDG convention [14], the PMNS matrix U can be parametrized as:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} P_M \quad (3.12)$$

contains two Majorana phases α_M and β_M . The mixing angles θ_{12} , θ_{23} , and θ_{13} can be expressed in terms of the elements of U as

$$s_{12}^2 = \frac{|U_{12}|^2}{1 - |U_{13}|^2}, \quad s_{23}^2 = \frac{|U_{23}|^2}{1 - |U_{13}|^2}, \quad s_{13}^2 = |U_{13}|^2. \quad (3.15)$$

The Dirac phase δ is obtained from the phase redefinition independent quantity $h_{12}h_{23}h_{31}$ through

$$\delta = \sin^{-1} \left(\frac{8 \text{Im}(h_{12}h_{23}h_{31})}{P} \right) \quad (3.16)$$

where P is

$$P = (m_2^2 - m_1^2)(m_3^2 - m_2^2)(m_3^2 - m_1^2) \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13}. \quad (3.17)$$

Replacing the elements of h and the m_i 's with H and η through Eq. (3.9), the mixing angles and the Dirac phase are all ultimately expressed in terms of the modified model parameters.

To compare with the most recent experiment data on neutrino mixing parameters [15], we fit the modified A_4 model to the experimental data by minimizing the following χ^2 function [16]:

$$\chi^2 = \sum_i \frac{(\rho_i - \rho_i^0)^2}{\sigma_i^2}, \quad (3.18)$$

where ρ_i^0 is the data of the i th experimental observable, and σ_i is the corresponding 1σ error. ρ_i is the model prediction

for the i th observable. The experimental values for all the observables fitted are given in Table II. We fit the six observables to the four free parameters¹ of our modified model (i.e., two degrees of freedom). We obtain in the normal hierarchy (NH) $\chi_{\min}^2 \approx 1.45$, indicating a good description of the NH data. For inverse hierarchy (IH) we obtain $\chi_{\min}^2 \approx 2.57$, which is somewhat less robust than in the NH case and is consistent with other recent finds that give slight preference in favor of the normal hierarchy over the inverse hierarchy [15].

¹The six observables are those listed in Table II, and the four free parameters are the real and imaginary parts of the complex parameters H and η [see Eq. (3.6)].

TABLE II. Neutrino oscillation parameters determined from global analysis of experimental data [15].

Parameter	Best fit $\pm 1\sigma$	2σ range	3σ range
$\Delta m_{21}^2 [10^{-5} \text{ eV}^2]$	$7.55^{+0.20}_{-0.16}$	7.20–7.94	7.05–8.14
$ \Delta m_{31}^2 [10^{-3} \text{ eV}^2]$ (NH)	2.50 ± 0.03	2.44–2.57	2.41–2.60
$ \Delta m_{31}^2 [10^{-3} \text{ eV}^2]$ (IH)	$2.42^{+0.03}_{-0.04}$	2.34–2.47	2.31–2.51
$\sin^2 \theta_{12}/10^{-1}$	$3.20^{+0.20}_{-0.16}$	2.89–3.59	2.73–3.79
$\sin^2 \theta_{23}/10^{-1}$ (NH)	$5.47^{+0.20}_{-0.30}$	4.67–5.83	4.45–5.99
$\sin^2 \theta_{23}/10^{-1}$ (IH)	$5.51^{+0.18}_{-0.30}$	4.91–5.84	4.53–5.98
$\sin^2 \theta_{13}/10^{-2}$ (NH)	$2.160^{+0.083}_{-0.069}$	2.03–2.34	1.96–2.41
$\sin^2 \theta_{13}/10^{-2}$ (IH)	$2.220^{+0.074}_{-0.076}$	2.07–2.36	1.99–2.44
δ/π (NH)	$1.21^{+0.21}_{-0.15}$	1.01–1.75	0.87–1.94
δ/π (IH)	$1.56^{+0.13}_{-0.15}$	1.27–1.82	1.12–1.94

Figure 1 shows the allowed regions of model parameters $|H|$, ϕ_H , $|\eta|$, ϕ_η at 1σ , 2σ , and 3σ C.L., defined as contours in $\Delta\chi^2$ with respect to $\chi^2_{\min}$. Here $|z|$ and ϕ_z are the amplitude and phase respectively of the complex

parameter z . The best fit values for $|H|$, ϕ_H , $|\eta|$, and ϕ_η are $(0.967, 1.934\pi, 0.113, 1.275\pi)$ in NH and $(1.063, 0.660\pi, 0.086, 0.212\pi)$ in IH.

In Figs. 2 and 3, the predictions for the oscillation parameters and neutrino mass squared differences are presented for both NH and IH cases. In NH, the best fit values of $\sin\theta_{23}$ and $\sin\theta_{13}$ are $(0.555, 0.0216)$, in good agreement with experimental values of $(0.547^{+0.020}_{-0.030}, 0.0216^{+0.0083}_{-0.0069})$. The best fit value of δ is 1.32π , slightly above the experimental value at 1.21π , but well within the 1σ range. The model prediction for $\sin\theta_{12}$ is narrowly centered around 0.340, at the upper edge of the 1σ range of the experimental value centered at 0.320. This is a sharp prediction of the model that can be tested as experimental uncertainty on the mixing angles narrows. The mass squared differences Δm_{21}^2 and Δm_{31}^2 are also fitted and show excellent agreement with the data. The predictions in the IH case are similar, with the predicted value of $\sin\theta_{23}$ and δ lying on the lower edge of the 1σ range of the global fit to experimental values. Thus, the modified Altarelli-Feruglio model with the additional contraction term in Eq. (3.1) produces deviation from the tri-bimaximal matrix

FIG. 1. The allowed region of the model parameters $|H|$, ϕ_H , $|\eta|$, ϕ_η at 3σ C.L. in both NH and IH cases. Here the C.L. ranges are defined as contours in Δ^2 with respect to $\chi^2_{\min}$. The best fit values with $\chi^2 = \chi^2_{\min}$ are indicated by the red dots. $\chi^2_{\min} \approx 1.45$ in NH and $\chi^2_{\min} \approx 2.57$ in IH.

FIG. 2. The allowed region of the oscillation parameters predicted by the model at 1σ (green), 2σ (light blue), 3σ (dark blue) C.L. in the normal hierarchy case. The best fit of the model is indicated by the black x. For comparison, the experimental best fit data and 1σ range are indicated by red dot and vertical/horizontal bars. The best fit gives $\chi^2_{\min} \approx 1.45$, indicating good agreement with current data with nonzero θ_{13} and Dirac phase δ , improving from original tri-bimaximal mixing model.

that is in good agreement with recent experimental data on neutrino oscillation parameters with nonzero θ_{13} and Dirac phase δ .

C. Prediction for J and effective Majorana mass

In Fig. 4 we present the additional predictions of the modified model on CP -violation Jarlskog parameter J , and the effective Majorana mass $|m_{ee}|$, characterizing $0\nu\beta\beta$

decay. The Jarlskog invariant is given by the phase redefinition invariant quantity

$$J = \text{Im}\{U_{11}U_{22}U_{12}^*U_{21}^*\} = s_{12}c_{13}^2s_{12}c_{12}s_{23}c_{23}\sin\delta, \quad (3.19)$$

and the effective Majorana mass is given by

$$|m_{ee}| = \left| \sum_i U_{1i}^2 m_i \right| = |c_{12}^2 c_{13}^2 m_1 e^{i\alpha_M} + s_{12}^2 c_{13}^2 m_2 e^{i\beta_M} + s_{13}^2 m_3 e^{-i2\delta}| = |(m_\nu)_{11}|. \quad (3.20)$$

J is completely specified by the mixing angles and the Dirac phase discussed in the previous section. The effective Majorana mass contains the Majorana phases α_M and β_M . Our obtained 3σ range of values for $|m_{ee}|$ is $4.26 \text{ meV} \sim 8.13 \text{ meV}$ in NH, compared to $|m_{ee}| \approx 7 \text{ meV}$ in the original Altarelli-Feruglio model; and $15.0 \text{ meV} \sim 95.6 \text{ meV}$ in IH, compared to $|m_{ee}| \geq 17 \text{ meV}$. The effective Majorana mass has the upper bound of

$|m_{ee}| \leq 160 \text{ meV}$, corresponding to $T_{1/2}^{0\nu\beta\beta}(^{136}\text{Xe}) \geq 1.1 \times 10^{26} \text{ yr}$ at 90% C.L., which follows from the data of the KamLAND-Zen experiment [17]. For IH, the model predicted $|m_{ee}|$ range is within the sensitivity of near future $0\nu\beta\beta$ decay experiments. For example, the bolometric CUORE experiment, using ^{130}Te , has a sensitivity of $|m_{ee}| \leq 50 \text{ meV}$, which corresponds to $T_{1/2}^{0\nu\beta\beta}(^{130}\text{Te}) \geq 10^{26} \text{ yr}$ [18]. There are also planned ton-scale next-to-next

FIG. 3. The allowed region of the oscillation parameters predicted by the model at 1σ (green), 2σ (light blue), 3σ (dark blue) C.L. in the inverse hierarchy case. The best fit of the model is indicated by the black x. For comparison, the experimental best fit data and 1σ range are indicated by red dot and vertical/horizontal bars. The best fit gives $\chi^2_{\min} \approx 2.57$, a better description than the original tri-bimaximal model, but less compatible with experimental data than in the NH case.

generation $0\nu\beta\beta$ experiments using ^{136}Xe [19,20] and ^{76}Ge [21,22] that can reach a sensitivity of $-|m_{ee}| \sim 12\text{--}30$ meV, corresponding to $T_{1/2}^{0\nu\beta\beta} \geq 10^{27}$ yr [23].

However, this sensitivity is still outside the range of model predicted $|m_{ee}|$ values in NH, which is too small to be tested in the immediate next-generation $0\nu\beta\beta$ experiments.

FIG. 4. The allowed region of the Jarlskog invariant (J) and effective Majorana neutrino mass ($|m_{ee}|$), predicted by the model at 1σ (green), 2σ (light blue), 3σ (dark blue) C.L. for both NH and IH. The best fit of the model is indicated by the black x. The current upper bound of $|m_{ee}| = 0.16$ eV is indicated by the red dotted vertical line. The predicted $|m_{ee}|$ range in IH can be tested with near-future $0\nu\beta\beta$ experiments, but the NH predictions are too small for planned detector sensitivities.

IV. CONCLUSION

In summary, we proposed a simple modification to the Altarelli-Feruglio A_4 flavor symmetry model that retains the minimal structure of the original model, while predicting oscillation parameters in much better agreements with current experimental data. The best model-data agreement is found in the normal hierarchy case, with $\chi^2_{\min} \approx 1.45$, consistent with recent data analyses in favor of the normal hierarchy over the inverse hierarchy [15]. The prediction of $\sin \theta_{12}$ narrowly centered around 0.34 can be tested in the future as experimental uncertainty on the mixing angles narrows in the next round of experiments. The prediction of the modified model on CP -violating parameters and effective Majorana mass are also investigated. The predicted range of $|m_{ee}|$ in the IH case are within the sensitivity of planned detectors, whereas the NH predicted values are still too small to be tested in the near future. In the process of studying the modified model, we systematically worked out the A_4 field contraction projection matrices in the different cases of Dirac ($\chi^\dagger \tau \psi$) vs Majorana ($\chi^T \tau \psi$) fields and S - vs T -diagonal bases. All cases are listed in the Appendix for future reference.

ACKNOWLEDGMENTS

We wish to thank B. Shakya for discussions. J.D.W. is supported in part by DOE Grant No. de-sc0007859. M. L. and J. D. W. are also supported in part by the Leinweber Center for Theoretical Physics (LCTP).

APPENDIX: A_4 GROUP CONTRACTIONS

1. Basics of the A_4 group

A_4 is the group of even permutation of 4 elements, or geometrically, the symmetry group of a tetrahedron. It has $(4!)/2 = 12$ elements, which can all be generated by two basic elements S and T given in permutation form by $S = (4321)$ and $T = (2314)$. S and T satisfy the properties

$$S^2 = T^3 = (ST)^3 = 1 \quad (\text{A1})$$

which specify the presentation of the group.

By looking at the equivalence classes of the A_4 group and the characters of elements in each class, one can derive that A_4 has four inequivalent irreducible representations: three singlet representations 1, $1'$, and $1''$ and one triplet representation 3. In the singlet representations, the generator S and T are given by

$$\begin{aligned} 1: S &= 1 & T &= 1 \\ 1': S &= 1 & T &= e^{i2\pi/3} \equiv \omega \\ 1'': S &= 1 & T &= e^{i4\pi/3} \equiv \omega^2. \end{aligned} \quad (\text{A2})$$

In the triplet representation, in the basis where S is diagonal, the generators are given by

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (\text{A3})$$

In the T diagonal basis the generators of the triplet representation are

$$\begin{aligned} S &= VS'V^\dagger = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \\ T &= VT'V^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \end{aligned} \quad (\text{A4})$$

with the unitary transformation matrix V given by

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \omega^2 & \omega \\ 0 & \omega & \omega^2 \end{pmatrix}. \quad (\text{A5})$$

A comprehensive review of the properties of the A_4 group and other non-Abelian discrete symmetries can be found in Ref. [24].

2. Tensor product and field contraction

The tensor product rule for A_4 representations are given by

$$\begin{aligned} 1 \times 1 &= 1; & 1 \times 1' &= 1'; & 1 \times 1'' &= 1''; \\ 1' \times 1' &= 1''; & 1' \times 1'' &= 1; & 1'' \times 1'' &= 1'; \\ 1/1/1'' \times 3 &= 3 \times 1/1'/1'' = 3; \\ 3 \times 3 &= 1 + 1' + 1'' + 3_1 + 3_2. \end{aligned} \quad (\text{A6})$$

In the context of A_4 flavor symmetry models, the tensor product rules specify the contraction rules of two particle fields χ and ψ under A_4 :

$$\begin{aligned} \chi^\dagger \tau \psi &= \eta \quad (\text{Dirac Fields}) \\ \chi^T \tau \psi &= \eta \quad (\text{Majorana Fields}) \end{aligned} \quad (\text{A7})$$

where η is the result of the contraction and is a particular A_4 representation specified by the tensor product rules. τ is the projection matrix that determines the components of η in terms of the components of χ and ψ . Thus, it is important for A_4 flavor symmetry models to know the form of the

projection matrix τ in each tensor product combination, which can be different in S -diagonal or T -diagonal basis, and for Dirac vs Majorana fields. The forms of the projection matrices for each of these different scenarios are summarized in the next section.

3. Form of projection matrix

Here we summarize the form of the projection matrix τ for Dirac ($\chi^\dagger \tau \psi$) vs Majorana ($\chi^T \tau \psi$) fields, in S -diagonal or T -diagonal basis.

a. Only singlets

The contractions involving only singlet representations 1, $1'$, and $1''$ are trivial and have $\tau = 1$ for all cases of Dirac vs Majorana and S -diagonal vs T -diagonal.

b. Singlet and triplet

For the trivial contractions $1 \times 3 = 3$ and $3 \times 1 = 3$ where a singlet 1 is involved, $\tau = I_3$, the 3×3 identity matrix in all cases.

For $1' \times 3 = 3$, we have

$$\begin{array}{cc} \text{Dirac}(\chi^\dagger \tau \psi) & \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; & \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \quad (\text{A8})$$

where τ_S and τ_T denote the projection matrix in S and T -diagonal bases respectively.

$3 \times 1' = 3$:

$$\begin{array}{cc} \text{Dirac}(\chi^\dagger \tau \psi) & \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; & \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \quad (\text{A9})$$

$1'' \times 3 = 3$:

$$\begin{array}{cc} \text{Dirac}(\chi^\dagger \tau \psi) & \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; & \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad (\text{A10})$$

$3 \times 1'' = 3$:

$$\begin{array}{cc} \text{Dirac}(\chi^\dagger \tau \psi) & \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; & \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} & \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad (\text{A11})$$

c. Only triplets

Here we list the form of the project matrices for all triplet contractions. In the cases where two triplets contract to another triplet, there are two linearly independent contractions with different $\tau_{\mu\nu}^\rho$.

$3 \times 3 = 1$:

$$\begin{array}{c} \text{Dirac}(\chi^\dagger \tau \psi) \\ \tau_S = \tau_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \end{array} \quad \begin{array}{c} \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \quad (\text{A12})$$

$3 \times 3 = 1'$:

$$\begin{array}{c} \text{Dirac}(\chi^\dagger \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \end{array} \quad \begin{array}{c} \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \tau_T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad (\text{A13})$$

$3 \times 3 = 1''$:

$$\begin{array}{c} \text{Dirac}(\chi^\dagger \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \end{array} \quad \begin{array}{c} \text{Majorana}(\chi^T \tau \psi) \\ \tau_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \tau_T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \quad (\text{A14})$$

$3 \times 3 = 3_1$:

$$\begin{array}{c} \text{Dirac}(\chi^\dagger \tau \psi) \\ \tau_{S\mu\nu}^\rho = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ \tau_{T\mu\nu}^\rho = \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \right] \end{array}$$

$$\begin{array}{c} \text{Majorana}(\chi^T \tau \psi) \\ \tau_{S\mu\nu}^\rho = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ \tau_{T\mu\nu}^\rho = \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right] \end{array} \quad (\text{A15})$$

$$3 \times 3 = 3_2:$$

$$\begin{aligned}
 & \text{Dirac}(\chi^\dagger \tau \psi) & \text{Majorana}(\chi^T \tau \psi) \\
 \tau_{S\mu\nu}^\rho &= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] & \tau_{S\mu\nu}^\rho = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
 \tau_{T\mu\nu}^\rho &= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] & \tau_{T\mu\nu}^\rho = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]
 \end{aligned} \tag{A16}$$

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