Gauge technique approximation to the $\pi\gamma$ production and the pion transition form factor

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(Received 7 June 2019; revised 17 June 2020; accepted 6 July 2020; published 31 July 2020)

The pion transition form factor $G(q^2)$ is computed on the entire domain of spacelike and timelike momenta using a quantum field theory continuum approach. In an analytical continuation of the function $G(Q^2)$ we utilized the gauge technique with the quark propagator determined from Minkowski space solution of QCD Dyson-Schwinger equations. The scale is set up by the phenomena of dynamical chiral symmetry breaking, which is a striking feature of low energy QCD.

DOI: [10.1103/PhysRevD.102.014049](https://doi.org/10.1103/PhysRevD.102.014049)

I. INTRODUCTION

Transition meson form factors carry nontrivial information about the interaction of quarks inside of mesons. Among them, the pion transition form factor $G(Q_1^2, Q_2^2)$ describes $2\gamma^* \rightarrow \pi$ transition, turns to be particularly useful means testing of nonperturbative techniques in QCD. Aforementioned double variable function $G(Q^2 = Q_1^2, Q_2^2)$, where Q_i are the momenta of the virtual photons, is measured in more involved process $e^+e^- \rightarrow e^+e^-\pi_0$ and kinematically selected events provide the single variable pion transition form factor $G(Q^2)$ in the limit of nearly on-shell photon $Q_2^2 = 0$. The form factor $G(Q^2)$ at the domain of spacelike momentum $Q^2 > 0$ has been recently extracted from the data by the Belle Collaboration [1], earlier by the BABAR [2], CELLO [3] and CLEO [4] collaborations. In the timelike domain of the four momentum ($q^2 > 0; Q^2 \rightarrow -q^2$), the form factor $G^2(q^2)$ appears linearly in the cross section of the annihilation process $e^+e^- \rightarrow \pi^0\gamma$ with one of the outgoing photon perfectly on-shell. Ignoring the interference with other sources of 3γ , the pion form factor $G(q^2)$ has been extracted from the data collected by CMD2/SND collaboration [5].

Using predictions of perturbative QCD [6–8] the transition pion form factor is absolutely normalized at asymptotic momentum transfer $Q \gg m_\pi$ with simple functional form:

$$G(Q^2) \rightarrow \frac{4\pi^2 f_\pi}{Q^2}. \quad (1.1)$$

The validity of perturbative QCD represented here by Eq. (1.1) has been experimentally tested at momenta $Q < 6$ GeV. Owing the known discrepancy between the data collected by BABAR and Belle Collaborations, only the later being with an agreement with the Eq. (1.1) at $Q_0^2 \simeq 20 \text{ GeV}^2$, where Q_0 is the energy scale of perturbative QCD limit. The BABAR data overestimated the perturbative asymptotic of Eq. (1.1), provoking discussion at that time [9,10] that the BABAR sample of data does not accurately represent the pion transition form factor, including also arguments coming from the prompt use of Dyson-Schwinger equations [11].

In this paper we contribute toward understanding of the pion transition form factor by using the existing Dyson-Schwinger equations (DSEs) calculation [12], which has the capacity to provide nonperturbative results not only for the Euclidean momentum (spacelike), but on the entire domain of Minkowski space momentum (we use the Minkowski metric convention $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, thus q^2 is positive for the timelike momentum, while capital Q is used for the spacelike domain of momentum, where $q^2 < 0; Q^2 = -q^2$ and the sign is omitted when writing the argument in the function G as a part of convention). The DSEs are traditional tool for such calculations, they have the potential to provide nonperturbative results in infrared momentum domain, where the QCD coupling is strong, while smoothly approaching the perturbative limit when momenta are asymptotically large $Q^2 \gg \Lambda_{\text{QCD}}^2$. Using the formalism of QCD DSEs, chiral symmetry breaking and its connection with meson spectra is well understood today [13–18]. The electromagnetic form-factors of pion and kaon [19–22] and meson transition form factors [23–26] were studied in a unified picture of Euclidean space QCD DSEs.

The development of nonperturbative tools which incorporates confinement and naturally avoid the use of on-shell

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quark spinors is one of the aims of the presented paper. A confining solution for the quark propagator as obtained in minimal symmetry preserving approximation of QCD DSEs studied in the paper [12] is used to construct two photon gauge vertex. For this purpose we use the gauge technique developed in [27–31] and employed in practice in [12,22,32–34].

The paper is organized as the following: Next two sections describe the equations needed for calculation of the pion transition form factor. Particularly, the gauge technique is reviewed in the Sec. III. The main result, the integral expression relating the quark propagator with the pion transition amplitude is presented in the Sec. IV. Numerical results are presented in the Sec. V and we conclude in Sec. VI. Some subtleties relating with the extraction of spectral representation from the quark gap equation are described in the Appendix.

II. EQUATIONS FOR TRANSITION FORM FACTOR

The two photons pion transition, which determines the function $G(q^2)$ is computed from

$$M_{\mu\nu}(k, l) = T_{\mu\nu}(k, l) + T_{\nu\mu}(k, l), \quad (2.1)$$

where the pion's momentum $P = k + l$, k and l are the photon momenta and two variable form factor is defined as

$$T_{\mu\nu}(k, l) = \frac{e^2}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} k^\alpha l^\beta G(k^2, l^2), \quad (2.2)$$

$$M_{\mu\nu}(k, l) = i \int \frac{d^4 q}{(2\pi)^4} \Gamma_\pi(q_1, q_2) G_{\mu\nu}(q_1, q_2; k, l), \quad (2.3)$$

where $\Gamma_\pi(q_1, q_2)$ is the pion Bethe-Salpeter vertex function for which purpose we use the solution of the pionic Bethe-Salpeter equation in ladder-rainbow approximation as obtained in [12]. The model is described in the Appendix for purpose of completeness.

The two Lorentz indices function $G_{\mu\nu}(q_1, q_2; k, l)$ in the Eq. (2.3) is nonamputated fully dressed quark-antiquark-two gamma vertex function in notation where the variables q_1 and q_2 stand for the constituent (anti)quark momenta. The function $G_{\mu\nu}$ consistent with rainbow-ladder approximation can be written uniquely as the decomposition of the quark propagators S and the proper quark-photon vertices Γ

$$\begin{aligned} G_{\mu\nu}(q_1, q_2; k, l) &= S(q_1) \Gamma_\mu(q_1, q_1 - k) \\ &\times S(q) \Gamma_\nu(q_2 - k, q_2) S(q_2) \\ &+ (k \rightarrow l), \end{aligned} \quad (2.4)$$

with momenta labeling that keeps $q_1 = q_2 + k + l$. Putting the Eq. (2.4) into (2.3) one gets the integral over the three quark propagators with three vertices placed among

them. Diagrammatically thus they consist the well-known triangle diagram, which represents a common starting approximation [23,26].

To get a single variable pion transition form factor, the kinematics constrains read

$$l^2 = 0, \quad P^2 = k^2 + 2kl = m_\pi^2, \quad (2.5)$$

albeit we will keep both photons off-shell during the derivation here.

In order to express $G_{\mu\nu}$ we employ a gauge technique where we use confining solution for the quark propagator as obtained in S [12]. Recall, the obtained quark propagator is known in the entire Minkowski space and the absence of on-shell pole is a striking and important property of the solution.

Perturbative and nonperturbative parts unavoidably matches in any formula for exclusive process in QCD. At this point recall here the well-known QCD factorization formula for the transition form factor, which has led to the Eq. (1.1). It has the following structure:

$$G(Q^2) = 4\pi^2 f_\pi \int_0^1 dx T_H(x, Q^2, \alpha(\xi), \xi) \phi_\pi(x, \xi). \quad (2.6)$$

It contains a nonperturbative part: the pion distribution function ϕ , which can be expressed as the light cone projection of Bethe-Salpeter amplitude [8,23] and the hard part of the parton scattering T is evaluated perturbatively within the use of on-shell quark spinors.

On the other hand, the phenomena of confinement in QCD can be mathematically formulated to the statement that there are no free on-shell quarks in Nature. In other words, the on-shell spinors could not be an adequate description for the confined quark modes inside hadrons. In this respect, we elucidate the sensitivity of the $\gamma^* \gamma \rightarrow \pi^0$ transition form factor to the behavior of the quark propagator S in the presence of analytical confinement.

III. GAUGE TECHNIQUE

The proliferation of Nakanishi's perturbation theory integral representation from scalar theory [35] to vertices including fermions began with the paper [36]. Note, the generalization to the Abelian quark-photon vertex attracted attention already more than two decades ago [37]. Since till today, the numerical realization of such a program is still not yet completed, we begin with the gauge technique approximation, which presumably will be a part of more sophisticated but not yet completed integral representation for this vertex.

The gauge technique is a well-established method [27–31,37,38] of solving Dyson-Schwinger equations in gauge theories. The main idea is construct an Ansatz that expresses a given vertex as functional of the propagator spectral function, such that the corresponding

Ward-Takahashi [39–41] or Slavnov-Taylor identity is automatically satisfied. In our case, the gauge technique consists of writing a solution for the double vector unamputated vertex in the following form

$$G^{\mu\nu}(q_1, q_2; k, l) = \int_{\Gamma} dx \rho(x) \frac{1}{q_1 - x} \left[\gamma^{\mu} \frac{1}{q_1 - \not{k} - x} \gamma^{\nu} + \gamma^{\nu} \frac{1}{q_2 + \not{k} - x} \gamma^{\mu} \right] \frac{1}{q_2 - x}, \quad (3.1)$$

where we used a generalized spectral representation for the quark propagator

$$S(p) = \int_{\Gamma} dx \frac{\rho(x)}{(\not{p} - x)}. \quad (3.2)$$

Actually it is easy to show, that $G^{\mu\nu}$ given by the Eq. (3.1) satisfies the Ward-Takahashi identity:

$$k_{\mu} G^{\mu\nu}(p', p; k, l) = G_{\nu}(p', p + k) - G_{\nu}(p' - k, p), \quad (3.3)$$

where

$$G^{\mu}(p, q) = \int_{\Gamma} dx \rho(x) \frac{1}{\not{p} - x} \gamma^{\mu} \frac{1}{\not{q} - x}, \quad (3.4)$$

which is the one body form of gauge technique approximation to the dressed (untruncated) quark-photon vertex ($G = S\Gamma S$). This is established again that the vertex (3.4) satisfies the Ward-Takahashi identity:

$$(p + q)_{\mu} G^{\mu}(p, q) = S(p) - S(q), \quad (3.5)$$

for a propagator satisfying (3.2). We use the function ρ as obtained in the paper [12] and we describe some details of its extraction in the Appendix here.

The main drawback of the aforementioned approach is that it leaves the transverse part of the vertex undetermined. Attempts to solve the problem have been done by using a wider set of the Euclidean space DSEs [19,42–44] or by resorting the transverse Ward identities [45–50].

IV. METHOD OF EVALUATION

To make the rainbow-ladder truncation of DSEs a meaningful approximation of QCD a common practice used for instance in [12–17,51–55] is to solve the quark gap equations simultaneously within the Bethe-Salpeter equation for meson. The same was done for the model used herein in the paper [12] for the first time and the fitting procedure was just repeated for our purpose here. The model parameters were varied in order to meet pionic observable: the pion mass and pionic weak decay constant. For this purpose the pion bound state vertex function was obtained via solution of the Bethe-Salpeter equation:

$$\Gamma_{\pi}(P, p) = i \int \frac{d^4 k}{(2\pi)^4} \gamma_{\mu} S_q(k_+) \Gamma_{\pi}(P, k) S_q(k_-) \times \gamma_{\nu} \left[-g^{\mu\nu} V_{\nu}(p - k) - 4/3 \xi g^2 \frac{L^{\mu\nu}(p - k)}{(p - k)^2} \right], \quad (4.1)$$

where for the sake of consistency, the Bethe-Salpeter and the quark Dyson-Schwinger equation must have identical kernels. Here we use the model [12], which has the simple functional form

$$V_{\nu}(q) = \int d\omega \frac{\rho(\omega)}{q^2 - \omega + i\epsilon} \\ \rho_{\nu}(\omega) = c_{\nu} [\delta(\omega - m_g^2) - \delta(\omega - m_L^2)], \quad (4.2)$$

with the product of $SU(3)$ color matrices $\lambda_a \lambda_a$ absorbed in the constant c_{ν} and where $L^{\mu\nu}$ stands for the longitudinal projector $L^{\mu\nu}(q) = q^{\mu} q^{\nu} / q^2$.

In the Eq. (4.1) P is the total momentum of meson satisfying $P^2 = M^2$, $M = 140$ MeV for the ground state and the arguments in the quark propagator are $k_{\pm} = k \pm P/2$. The DSE/BSE system provides the solution with small error $h \simeq 5.010^{-5}$ defined by (A14) for the quark propagator calculated in the gauge $(g\xi)^2/(4\pi)^2 = 0.18$ and the kernel couplings (4.2) $c_{\nu}/(4\pi)^2 = -1.8$ and $m_g^2/m_L^2 = 2/7.5$ where m_g in physical units is $m_g = 556$ MeV.

The pion BSE vertex function reads

$$\Gamma_{\pi}(P, p) = \gamma_5 (\Gamma_A(P, p) + \not{p} \Gamma_B(P, p) + \not{P} \Gamma_C(P, p) + [\not{p}, \not{P}] \Gamma_D(P, p)), \quad (4.3)$$

where all Γ_X was used to determine the pion mass and where we skip the isospin matrix notation, which is not necessary in the given truncation of the DSE system. The evaluation of BSE and decay constant was performed in standard manner and reader can find an appropriate eigenvalue method solution in [16] and f_{π} was calculated according to Eq. (33) in [13], representing codes used for purpose of this study are available at [56].

The gauge-independence of physical quantities is essentially assumed, and we have used a nonzero gauge fixing parameter ξ to accelerate numerical convergence of equations. Due to the required stability of all equations a single gauge $(g\xi)^2/(4\pi)^2 = 0.18$ was finally chosen at the end of iteration process. The numerical convergence is not better for presented model in Landau gauge (the gauge almost exclusively chosen in other studies). At this stage we do not attempt to search for solutions at other individual gauges and leave this task for a future exercise.

In what follows we describe approximations made that allow us to write the pion transition entirely in terms of the

quark propagator. First of all, let us note that the production of the vector strangeonium represented by ϕ meson peak is suppressed by two orders when compared the ω/ρ peak in experimental $\pi\gamma$ cross section [5] and therefore we restrict to two flavors $f = u, d$ in its isospin (equal mass) limit.

In the chiral limit, which we subsequently employ exclusively, the following Goldberger-Treiman-like identity [57–59]:

$$\Gamma_A(0, p) = \frac{B(p)}{f_\pi} \quad (4.4)$$

exhibits the equivalence between the one-body quark and pseudoscalar two-body problem in QCD.

The renormalized quark function B obtained from the Eq. (A2) satisfies the dispersion relation

$$\begin{aligned} B(p) &= m_q + \int d\omega \frac{\rho_B(\omega)}{p^2 - \omega + i\epsilon} \\ &= m_q(\zeta) + \int d\omega \frac{\rho_B(\omega)}{p^2 - \omega + i\epsilon} \\ &\quad - \int d\omega \frac{\rho_B(\omega)}{\zeta - \omega + i\epsilon}, \end{aligned} \quad (4.5)$$

where ρ_B is the imaginary part of self-energy Σ_B and can be easily extracted from the solution of DSE. In this study we replace the pion vertex function in the expression for transition form factor (2.3) by what B could be in the chiral limit

$$\Gamma_\pi(p, P) = \gamma_5 \frac{1}{\mathcal{N}} \int d\omega \frac{\rho_B(\omega)}{p^2 - \omega + i\epsilon}, \quad (4.6)$$

where $\mathcal{N}$ is the normalization of the BSE vertex, satisfying approximately $\mathcal{N} = f_\pi$, with its exact value dictated by the canonical normalization of the BSE vertex [60].

Substituting (4.6) and gauge technique integral representation into (2.3), one gets the following expression for the double variable pion transition form factor $\epsilon_{\mu\alpha\beta} k^\alpha l^\beta G(k^2, l^2) \simeq I_{\mu\nu}(k, l)$

$$\begin{aligned} I_{\mu\nu}(k, l) &= \text{Tr} \int d\omega \int_\Gamma d\sqrt{s} \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma_5 \rho_B(\omega)}{p^2 - \omega + i\epsilon} \\ &\quad \times \frac{\not{p} + \not{P}/2 + \sqrt{s}}{(p + P/2)^2 - s + i\epsilon} \left[\gamma_\mu \frac{\not{p} + (\not{k} - \not{l})/2 + \sqrt{s}}{(p - \frac{k-l}{2})^2 - s + i\epsilon} \gamma_\nu \right. \\ &\quad \left. + \gamma_\nu \frac{\not{p} + (\not{l} - \not{k})/2 + \sqrt{s}}{(p - \frac{l-k}{2})^2 - s + i\epsilon} \gamma_\mu \right] \\ &\quad \times \frac{\not{p} - \not{P}/2 + \sqrt{s}}{(p - P/2)^2 - s + i\epsilon} \rho(\sqrt{s}); \end{aligned} \quad (4.7)$$

where the integral over the variable ω is due to the Eq. (4.6) and the integral over the variable $\sqrt{s}$ is due to the Eq. (3.1). Two weight functions ρ and $\rho_B(\omega) = -\Im B(\omega)/\pi$ are not independent, since related through the definition (A1).

In what follow we integrate over the momentum analytically and within the help (3.2) we rewrite the integral (4.7) equivalently into the expression where we integrate over the known and regular function: the quark propagator. For this purpose we match all denominators in Eq. (4.7) by using the Feynman parametrization. Let us use the variables x, y to match the quark propagators denominator as expressed in spectral representations. Then, using the Feynman variable z , the obtained result we further match together with $p^2 - \omega$ in the denominator. After a usual shift of the momentum, which complete the square, we perform the Wick rotation and momentum integration. We get the following result (again in Minkowski space):

$$\begin{aligned} I_{\mu\nu} &= \frac{1}{(4\pi)^2} \int d\omega \rho_B(\omega) \int_\Gamma d\sqrt{s} \sqrt{s} \rho(\sqrt{s}) \int_0^1 dx dy dz \frac{-i4\epsilon_{\alpha\beta\mu\nu} k^\alpha l^\beta y z^2}{[f(k, l; x, y, z) - sz - \omega(1-z)]^2} - k \rightarrow l, \\ f(k, l; x, y, z) &= \frac{P^2}{4} yz + r^2(1-y)z - z^2 \left[\frac{P^2}{4} (1-2x^2)y^2 + \frac{P \cdot r}{2} (1-2x)y(1-y) + \frac{r^2}{4} (1-y)^2 \right], \end{aligned} \quad (4.8)$$

where we have labeled $r = k - l$. After factorization of the variable z^2 out of the numerator, then according to (3.2), one can recognize that the integrand together with the factor $\int d\sqrt{s} s$ is once differentiated quark propagator, which is further integrated over the auxiliary variables x, y, z . The expression (4.8) is exact and if needed it can be straightforwardly used

in order to get the result for the pion transition form factor.

In order to reduce number of auxiliary integrals, one can get very accurate expression by neglecting the small term $z^2 m_\pi^2 (1-2x)^2 y^2 / 4$, which includes the pion mass in the denominator of Eq. (4.8). Doing this explicitly and integrating over the variable x one gets

$$\begin{aligned}
I_{\mu\nu} &= \frac{1}{(4\pi)^2} \Sigma_{\pm} \int d\omega \rho_B(\omega) \int_{\Gamma} d\sqrt{s} \sqrt{s} \rho(\sqrt{s}) \int_0^1 dy dz \frac{-i4\epsilon_{\alpha\beta\mu\nu} k^{\alpha} l^{\beta}}{(1-y)z[t_{\pm} - s + i\epsilon]} - k \rightarrow l \\
&= \frac{1}{(4\pi)^2} \Sigma_{\pm} \int d\omega \rho_B(\omega) \int_0^1 dy dz \frac{-i4\epsilon_{\alpha\beta\mu\nu} k^{\alpha} l^{\beta}}{(1-y)z} S_s(t_{\pm}) - k \rightarrow l \\
t_{\pm} &= P^2/4y + r^2(1-y) - z \left(\pm \frac{P \cdot r}{2} y(1-y) + \frac{r^2}{4} (1-y)^2 \right) - \frac{\omega(1-z)}{z},
\end{aligned} \tag{4.9}$$

where the variable $\sqrt{s}$ was absorbed into the usual definition of the scalar part of the quark propagator (A1).

Taking $l^2 = 0$, the pion transition form factor finally reads

$$\begin{aligned}
G(k^2) &= F(k, l) + F(l, k) \\
F(k, l) &= \frac{\text{const}}{(4\pi)^2} \Sigma_{\pm} \int d\omega \rho_B(\omega) \int_0^1 \frac{dy dz}{(1-y)z} S_s(t_{\pm}), \tag{4.10}
\end{aligned}$$

where one has to take $P \cdot r = k^2$ and $r^2 = k^2$ in the function $F(k, l)$ in the expression (4.10) and one should keep in mind that $P \cdot r = -k^2$, when evaluating the function $F(l, k)$.

V. DISCUSSION OF SOLUTION

We begin with discussion of the spacelike asymptotic behavior of the pion transition form factor $G(Q^2)$, which has curious dependence on the interaction between quarks. It was shown in [11], that in the case of the contact interaction typical for quarks in Nambu-Jona-Lasinio models, the asymptotic of $G(Q^2)$ differs from (1.1) such that

$$G(Q^2) \simeq \frac{2M^2}{Q^2} \ln^2(Q/M), \tag{5.1}$$

where M is constant constituent quark mass obtained within the contact interaction.

Interesting asymptotic form was obtained [61] for the double variable transition form factor in the symmetric limit

$$Q^2 G(Q^2, Q^2) \simeq \frac{2}{3} Q^2 G(Q^2, 0) = \frac{2}{3} Q^2 G(Q^2), \tag{5.2}$$

which tells us, that up to the prefactor, the same asymptotic behavior persist for the double variable transition form factor as well.

From the equations (1.1), (5.1) a specific logarithmic dependence of $Q^2 G(Q^2)$ seems to be governed by modeled interaction as follows: a weaker interaction between the quarks at large relative momentum a smaller power of $\log$ (anomalous dimension) one gets. In our simple model, the interquark interaction consists from the interaction softer than QCD and from the purely longitudinal term.

The solution is shown in the Fig. 1, where the function $Q^2 G(Q^2)$ is displayed. As we can see, the presence of longitudinal mode is ignored and the expected behavior is confirmed here. The pion transition form factor does not reach its QCD perturbative asymptotic, but instead, the function $Q^2 G(Q^2)$ shows up the maximum at few GeV, where it starts to decrease with negative anomalous dimension.

A similar asymptotic form is observed for the large timelike momenta as well, however it happens at much larger asymptotic momentum $q_o \simeq 10 \text{ GeV}$ as is shown in the Fig. 2. Before the asymptotic is reached, the function $q^2 G(q) \rightarrow \text{const}$ reach the flat plateau with its detailed shape arises due to the interference of all components entering the form factor.

We also show the graph of square of the pion transition form factor against the energy $\sqrt{s}$ in the Fig. 3, and compare to the extracted data from the SND cross section $\sigma_{\pi, \gamma}$. Considering the fact that SND data are far from the threshold $E_{th} = m_{\pi}$ and the properly normalized ratio $G(q)/G(0)$ is extracted from the experiments by using:

$$G(q^2)/G(0) = \frac{\sigma_{\pi, \gamma} 12\pi^2 f_{\pi}^2}{\alpha_{QED}^3} \tag{5.3}$$

FIG. 1. Pion transition form factor $G(Q)$ for the spacelike momentum Q calculated as described in the text: the quantity $(Q^2 G(Q^2)/G(0))$ is shown. Each line labeled by I, II, and III use a different interpolator for the pion vertex function described in the text.

FIG. 2. Pion transition form factor $G(q^2)$ for the timelike argument $s = q^2$ as described in the text: the quantity $(q^2 G(q^2))/G(0)$ is shown.

FIG. 3. Square of the pion transition form factor $G(s)$ (thick) and $\Re$ and $\Im$ parts of $G(s)/G(0)$ (thin lines) shown for the timelike argument s and compared to SND data.

with α_{QED} being the fine structure constant and $\sigma_{\pi,\gamma}$ is the total cross section of the process $e^+e^- \rightarrow \pi^0\gamma$. In Fig. 3 we also show the imaginary and the real parts of form factors. To guide the reader eyes, note that those lines optically vanishing at zero momenta stand for the imaginary parts.

A single line labeled by “I” corresponds to the form factor as obtained from the Eq. (4.10) in all presented figures. There is a cusp—the remnant of standard threshold enhancement—located at $\sqrt{s} \simeq 500$ MeV. The position of this cusp is dictated by the shape of the quark propagator and it appears at value $2M(0)$, where $M(0)$ is the quark constituent mass. We expect the cusp effect could be even more pronounced for heavier quarks due to the smaller washout of the perturbative pole in the quark propagator. More generally, the quark quasithreshold cusp must be a universal part of many other measured cross sections.

Since it is large enough, observed theoretical cusp should be already part of the observed structure in $\pi\gamma$ cross section. In the case studied it should be located near of inside the peak of ω and ρ resonances. Since the obtained cusp is obtained below ρ meson mass, it is likely that the amount of generated dynamical mass is slightly underestimated in our simple model. When rescaling all dimension-full quantities in (4.2) by hand, such that we move the position of the cusp into the $\rho - \omega$ mesons centered mass, then instead of 140 MeV light pion one gets $m_\pi = 200$ MeV afterwards. In other words, the pion gets heavier compared to other scales in theory, e.g., to the vector meson masses, which is likely what happen in our Schwinger-Dyson equations model. We therefore expect the cusp should move into higher scale in a more realistic model (say, with a more realistic interacting kernel, if one does need to go beyond LRA at all).

The suggestion has been raised [34], that the gauge technique alone can provide a more significant portion of the $\rho - \omega$ peak. This claim is unsupported by more self-consistent approach presented here. Actually, even if we would use the quark propagator with the real pole presented in and enhance the pseudothreshold cusp into the ordinary threshold, then its absolute value is far not enough to describe entire experimental ρ meson peak. Some additional important transverse pieces of the quark-photon vertex are needed for a more correct description.

To see other possible effects, we have checked changes in $G(q^2)$ behavior, which stem from different extrapolations used for the pion Bethe-Salpeter vertex function. For this purpose we consider the interpolator suggested in [58], which takes the form

$$\Gamma_\pi(p, P) = \gamma_5 \frac{1}{\mathcal{N}} \int_{-1}^1 du \frac{\rho_5(z)}{(k + uP^2/2)^2 - \Lambda_\pi^2 + i\epsilon}. \quad (5.4)$$

Furthermore, we have used more general two variable integral representation for the pion vertex function:

$$\Gamma_\pi(p, P) = \gamma_5 \frac{1}{\mathcal{N}} \int d\omega \int_{-1}^1 dz \frac{\rho_5(\omega, u)}{(k + uP^2/2)^2 - \omega + i\epsilon}, \quad (5.5)$$

which arises when one studies the integral representation for the pseudoscalar BSE in the Minkowski space self-consistently [36].

Then our derived formula for the pion transition form factor can be written as:

$$G(k^2, l^2) = F(k, l) + F(l, k);$$

$$F(k, l) = \text{const} \int d\omega du \rho_5(\omega, u) \times \int_0^1 \frac{dy dz}{(1-y)zP \cdot r} [S_s(p_+^2) - S_s(p_-^2)]; \quad (5.6)$$

$$p_{\pm}^2 = r^2(1-y) + \frac{r^2}{4}(1-y)^2z + \frac{P^2}{4}y + \frac{P^2}{4}u^2\frac{1-z}{z} + \omega\frac{1-z}{z} - \frac{P \cdot r}{2}(1-y)[\pm yz + (1-z)u] \quad (5.7)$$

valid for both cases (5.4) as well as (5.5). Note that the representation (5.4) is a special case of the second (5.5), where simplified version of the Bethe-Salpeter weight function includes the delta function $\delta(u - \Lambda_{\pi})$ as an assumption.

The results for the choice $\rho_5(z) = (1 - z^2)$ are labeled as the model II. and added in figures for comparison. Obviously a large enhancement of the cusp in the timelike region is the main feature of presented approximation. The approximation made does not affect the decreasing character of the asymptotic behavior in the ultraviolet spacelike as seen Figs. 1 and 4. A similar statement is valid for behaviour of G in the timelike domain of momenta, which can be seen in the Fig. 2.

It is interesting that almost the same pion form factor $G(Q^2)$ is achieved with a more primitive interpolator:

$$\Gamma_{\pi}(p, P) = \gamma_5 \frac{1}{\mathcal{N}} \frac{1}{k^2 - \Lambda_{\pi}^2 + i\epsilon}, \quad (5.8)$$

for which the formula (5.6) applies as well. The lines, which represent the pion transition form factor evaluated with use of the interpolator (5.8), are labeled by III in all presented figures. For better comparison of all form factors, the constant Λ_{π} is taken as $\Lambda_{\pi} = m_g$ and we have used a rescaled quark propagator as discussed above.

As seen in the Fig. 3 the cusps are similarly and largely pronounced for interpolators II (5.5) and III (5.8), mainly due to the fact that the pion vertex has similar unphysical singularity in both cases. Even so, all obtained results are

FIG. 4. Pion transition form factor $G(Q)$ for spacelike Q as described in the text: the quantity $\Gamma_{\gamma\gamma}(Q) = (\pi/4)\alpha_{QED}m_{\pi}^3G^2(Q^2)$ is shown. Form factors are normalized such that the pion width is reproduced $\Gamma_{\gamma\gamma}(0) = 7.23$ eV.

far from revealing structure of vector resonances observed in the experiment, which further support the aforementioned above: to reveal the form factor structure at resonances, one needs to incorporate transverse vertices.

The pion transition form factor $G(Q^2)$ in approximation I is compared to the approximation II in the Fig. (4). The difference is attributed to a different asymptotic of approximated Bethe-Salpeter pion vertex function. In accordance to observation made in [24], a softer interaction provides anomalously decreasing form factor for a large Q . In this analysis we do not optimize the model by further improving the pion vertex function fit and we let the error freely propagate to higher Q .

VI. CONCLUSIONS AND PROSPECTS

We completed a computation of the pion transition form factor $G(q^2)$ in two flavor QCD. Theoretical Nakanishi parametrization for correlators allows us to compute $G(q^2)$ on the entire domain of spacelike as well as of timelike momenta in Minkowski space. The later represents the analytical continuation of the former and numerical evaluation of the form factor G for the timelike arguments appears for the first time in the literature. All required elements, e.g., the quark propagator within its spectral function are determined by the solution of QCD's Dyson-Schwinger equations obtained in the rainbow-ladder truncation, the leading order in a systematic and symmetry preserving approximation scheme. New developed technique based on the subtraction at the timelike domain of momenta was the important step to accomplish desired analytical continuation of QCD quark propagator.

A cusp observed at square of momentum $q^2 \simeq 4(B(0))^2$ is a universally predicted as quasithreshold of confined quarks. Observed cusp in the square of the pion transition form factor is about one or two order of magnitudes smaller than ω/ρ peak observed in the experiment [5]. A missing amount of the function $G(q^2)$ in resonance region needs an inclusion of important transverse components form factors in the quark-photon vertex. The formalism developed herein represents small step toward our better understanding of how confinement mechanism can be incorporated in unified picture of hadron production form factors in the entire Minkowski space.

ACKNOWLEDGMENTS

I am grateful to Petr Bydžovský and Daniel Gazda for their help with the manuscript.

APPENDIX: QUARK GAP EQUATION IN MINKOWSKI SPACE-METHOD AND PROPERTIES OF SOLUTION

Only very recently, an easy way to get rid of resisting numeric was recognized in [12]. Since the knowledge how to obtain spectral functions in a nonperturbative framework

of Dyson-Schwinger equations does not belong to a conventional wisdom, we describe the method in details in this Appendix.

Neglecting the weak interaction, the quark propagator is completely characterized by two scalar functions A and B (or alternatively S_v and S_s):

$$S(q) = \not{q} S_v(q) + 1 S_s(q) = [A(q)\not{q} - B(q)]^{-1}, \quad (\text{A1})$$

where the renormalization scheme independent dynamical mass function is conventionally defined as $M = B/A$. The quark propagator is obtained from the solution of Dyson-Schwinger equation

$$S^{-1}(q) = \not{q} - m_q - \Sigma(q),$$

$$\Sigma(q) = i \int \frac{d^4 k}{(2\pi)^4} \frac{\lambda^a}{2} \gamma_\mu S(k) \frac{\lambda^a}{2} \Gamma_{qg}^\nu(k, q) D_{\mu\nu}(k - q) \quad (\text{A2})$$

where the product of the gluon propagator D and the quark-gluon vertex Γ_{qg} in the exact expression (A2) was approximated by the interaction kernel (4.2), such that the self-energy is

$$\Sigma(k) = -i \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \gamma^\mu V_\nu(k - q) - i \frac{4\xi g^2}{3} \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \gamma_\nu \frac{L^{\mu\nu}(k - q)}{(k - q)^2}. \quad (\text{A3})$$

Note, albeit the first is finite and does not require renormalization, this is still the strength of the kernel V which is responsible for the mass generation of quarks in the model [12]. The second term in Eq. (A2) is a pure gauge term, i.e., we neglect the momentum dependence of the vertex in our approximation.

While the observable, like meson masses and cross sections, must be gauge fixing independent, the analytical properties of Green's functions do depend on the gauge fixing. Facing the necessity of truncation of infinity system of Dyson-Schwinger equations, the proper choice of gauge fixing has nontrivial influence on convergence of numerical solution. Due to its own importance, a new procedure of solving the gap equation (A2) working for a broad range of gauge fixing parameter ξ is described in the rest of this Appendix.

As the first step, the dimensional regularization was used to express the self-energy as a sum $\Sigma(p, \mu) = \not{p}\Sigma(p^2, \mu) + \Sigma_B(p^2, \mu)$, each scalar function Σ_A and Σ_B is then represented by relatively simple integral over the spectral quark functions σ_v and σ_s , which appear in the representation for the quark propagator:

$$S(k) = \int_\Gamma dx \frac{\rho(x)}{(\not{k} - x)} = \int_0^\infty da \frac{\not{k}\sigma_v(a) + \sigma_s(a)}{p^2 - a + i\epsilon}. \quad (\text{A4})$$

where the latter integral is taken over the real positive domain.

For a small coupling one can find the solution by solving the system of equations numerically in manner described for instance in the review [62]. The obtained solution exhibit a presence of a delta function in the spectrum in this soft coupling limit, which corresponds to on shell mass and proper definition of the propagator residuum at that point. However, when coupling increases, the residuum vanishes and the resulting system of equations provide the only stable symmetry preserving solution: $\sigma_{v,s} = 0$, in which we are not interested here. The nontrivial chiral symmetry breaking familiar from many DSEs studies does not occur within the use of numerical search [62].

It turns out, it is very useful to add zero in an identical form, i.e., subtract the equation for A and B functions with itself at some arbitrary timelike scale ζ . Thus one gets

$$B(p^2) = B(\zeta) + \Sigma_B(p^2) - \Sigma_B(\zeta),$$

$$A(p^2) = A(\zeta) - \Sigma_A(p^2) + \Sigma_A(\zeta). \quad (\text{A5})$$

and only after this crucial step one compares with the assumed integral representation (A4). With the additional assumption that the only singularity of the propagator is a cut at the entire timelike real axis of momenta, one can write down the desired set of equations for spectral functions:

$$\sigma_s(s) = -\frac{1}{\pi} \frac{\Im B(s) R_D(s) + \Re B(s) I_D(s)}{R_D^2(s) + I_D^2(s)}, \quad (\text{A6})$$

where $s = p^2 > 0$ in our metric, and where the functions R_D and I_D stand for the square of the real and the imaginary part of the function $sA^2(s) - B^2(s)$, i.e.,

$$R_D(s) = s[\Re A(s)]^2 - s[\Im A(s)]^2 - \Re B(s)^2 + [\Im B(s)]^2$$

$$I_D(s) = 2s\Re A(s)\Im A(s) + 2\Re B(s)\Im B(s), \quad (\text{A7})$$

Similarly for the function σ_v one gets

$$\sigma_v(s) = -\frac{1}{\pi} \frac{\Im A(s) I_D(s) - \Re A(s) R_D(s)}{R_D^2(s) + I_D^2(s)}. \quad (\text{A8})$$

First let us note, that since the Eqs. (A5) represent the expressions for the propagator renormalized in the momentum renormalization scheme, the functions A and B do not depend on the renormalization scale μ anymore, and as usually, the regularization scheme can be forgotten from this point. However, since we were subtracting at timelike momentum values rather than at spacelike one, the functions $A(\zeta)$, $B(\zeta)$ ($S_v(\zeta)$, $S_s(\zeta)$) are complex valued and while their real parts can be regarded as a renormalization condition, the angles defined as $\theta_{s,v} = \Re S_{s,v}(x) / \Im S_{s,v}(x)$ are unique for fixed values of three numbers: x , $\Re A(x)$, $\Re B(x)$. The angles

$\theta_{s,v}$ are uniquely determined by Eq. (A6) noting that $\Im\Sigma_{s,v}(p^2) = -\pi\sigma_{s,v}$ and

$$\Re S_{s,v}(s) = P \cdot \int_0^\infty da \frac{\sigma_{s,v}(a)}{p^2 - a} \quad (\text{A9})$$

It is a matter of fact, that the Eqs. (A6) and (A8) provide numerically convergent solution even for very incorrectly imposed values of angles $\theta_{s,v}(\zeta)$. To get rid of infinite number of fake solutions, one needs to implement the condition (A9) into the numerical search from very beginning.

In order to get the solution, we stay with the method of iterations and in addition to making some initial guess of continuous functions $\sigma_{v,s}$, we choose also some initial values for four numbers $\Re A(\zeta)$, $\Im A(\zeta)$ and $\Re B(\zeta)$, $\Im B(\zeta)$ and solve the system by iterations. If the system of equations (A6), (A8) is numerically convergent, then one changes gradually the imaginary parts $\Im A(\zeta)$ and $\Im B(\zeta)$ and look for a new solutions with desired analyticity improved.

For this purpose we also construct the auxiliary functions

$$L_s = P \cdot \int_0^\infty da \frac{\sigma_s(a)}{p^2 - a}$$

$$R_s = \frac{-\Im B(s)I_D(s) + \Re B(s)R_D(s)}{R_D^2(s) + I_D^2(s)}, \quad (\text{A10})$$

as well as we construct another pair of functions:

$$L_v = P \cdot \int_0^\infty da \frac{\sigma_v(a)}{p^2 - a}$$

$$R_v = \frac{-\Re A(s)R_D(s) + \Im A(s)I_D(s)}{R_D^2(s) + I_D^2(s)}. \quad (\text{A11})$$

The final search for functions $\Im A(\zeta)$ and $\Im B(\zeta)$ must yield

$$L_v(s) = R_v(s) = \frac{1}{4p^2} \Re \text{Tr} \not{p} S(s) = \Re S_v(s). \quad (\text{A12})$$

Therefore we check the equalities:

$$L_{s,v}(s) = R_{s,v}(s) \quad (\text{A13})$$

by evaluation the following error norms

$$h(\sigma_v, \Im A) = \frac{\int (L_v(s) - R_v(s))^2 ds}{\int (L_v(s) + R_v(s))^2 ds},$$

$$h(\sigma_s, \Im B) = \frac{\int (L_s(s) - R_s(s))^2 ds}{\int (L_s(s) + R_s(s))^2 ds}, \quad (\text{A14})$$

for a given choice of angles $\theta_{v,s}(\zeta)$ at each individual step of iteration cycle. Then the functions σ_v and σ_s , which are

extracted during the iteration process, turns to be true only in the limit $h \rightarrow 0$. In order to control the numeric, in addition to Eqs. (A10), (A11) and, (A13), another error functions were used. They determine the quality of iterations itself.

Thus there are basically two iteration cycles, the first, the external one, is used to find a correct value of angles $\theta_{v,s}(\zeta)$, the second cycle, say the inner one iterates the solution for a fixed angles. The latter must provides vanishing error and for a reasonably value of gauge fixing parameter, only (≈ 30) inner iterations are needed to provide the stable solution. This is necessary for meaningful evaluation of the errors (A14) and correct value of angles $\theta_{s,v}$ can be identified with reasonable precision. In our case, a typical value $h \simeq 10^{-3}$ is achieved for hundred of integration mesh points, while for instance $h \simeq 5 \cdot 10^{-5}$ for 4000 integration points.

Just for explanatory purpose of this Appendix we do not require the propagator in (A2) to provide correct physical pion properties and we vary the parameter ξ alone. For clarity let us note, that the effective parameters of our model in Eq. (A2) do depend on the gauge fixing parameter implicitly (since the DSE kernels are dressed n-point functions). Therefore numerical results presented here should be regarded as different toy models, rather than a single model at different gauges. The mass was fixed such that $B(\zeta = 0.5) = m_g$ and $A(\zeta = 0.5) = 2$ at fixed timelike scale $\zeta = 0.5m_g^2$ and $m_g = 1$, $m_L = 2$, $V_v/(4\pi)^2 = -2$ (presented in units where $m_g = 1$). Solutions of the gap equation for spectral functions σ_v (v) and σ_s (s) are shown for various parameters $(g\xi)^2/(4\pi)^2 = 10^{-8}, 0.25; 0.5$ in Fig. 5 and in Fig. 6.

The reader can find similarity between the procedure described above and the momentum renormalization scheme applied at timelike scale. Likely it can be used in such extent, however stressed here, that the method

FIG. 5. Spectral functions σ_v and σ_s shown for various parameters $(g\xi)^2/(4\pi)^2$ as described in Sec. IV.

FIG. 6. The same as in the previous figure.

adopted here is not applied for purpose of removing UV infinities. The renormalization constants stay real, as they were identified by imposing renormalization conditions at the spacelike t'Hooft renormalization scale μ .

According to the hypothesis of the quark confinement, spectral functions violate the axiom of positivity. Furthermore, spectral functions share other important property—they do not have thresholds and they do not contain on mass shell delta function peak in the spectrum. The absence of the production threshold in the propagator is consistent with the Wilsonian area law [63] which in principle could tell us an equivalent information about presence of confinement without studying particular correlator function.

The absence of a delta function in the spectrum is an obvious and striking fact and it is equivalent to the absence of particle pole in the propagator. It automatically implies the absence of free moving quarks in the spectrum. We can see a certain peak formation in both Figs. 6 and 5, probably best pronounced for Landau gauge and less pronounced for larger ξ gauges. Thus further questions arise e.g., what qualitatively specify the confinement, what kind of behavior can be identified from the spectrum and how the spectrum changes in deconfinement transitions [64–67].

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