Photon directional profile from stimulated decay of axion clouds with arbitrary momentum distributions

Liang Chen^{*} and Thomas W. Kephart[†]

Department of Physics and Astronomy, Vanderbilt University, Nashville, Tennessee 37235, USA

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We model clusters of axions with spherically symmetric spatial but arbitrary momentum distributions and study the directional profile of photons produced in their evolution through spontaneous and stimulated decay of axions via the process $a \rightarrow \gamma\gamma$. Several specific examples are presented.

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I. INTRODUCTION

The strong interactions conserve CP invariance to a high degree of accuracy, but the Lagrangian for QCD

$$\mathcal{L} = \frac{1}{4}F^2 + \theta F\tilde{F} + m_F\bar{\psi}\psi$$

appears to have two allowed sources of CP violation. One is the instant term $\theta F\tilde{F}$, and the other is the fermion mass matrix m_F . The full contribution can be written $\bar{\theta} = \theta + \arg[\det(m_F)]$ and compared with the best experimental limit from the neutron electric dipole moment [1] is $\bar{\theta} \leq 10^{-10}$ rad. To avoid such a fine-tuning, Peccei and Quinn [2,3] proposed a solution to this so-called strong CP problem where $\bar{\theta}$ is promoted to a field a , the axions, whose potential when minimized relaxes the field (and affectively $\bar{\theta}$) to zero. While this axion has a kilo-electron-volt mass [4,5] and interacts too strongly to be allowed by experiment, other “invisible axion” variants are lighter and allowed [6].

Invisible axions are weakly coupled light pseudoscalars that are perfect dark matter candidates. Current experiments limit the axions to the mass range 10^{-3} to 10^{-5} eV. They are nonrelativistic if produced at the electroweak (EW) phase transition and are unlike other particles in this mass range such as neutrinos which were produced and thermalized in the big bang. The lower limit of the axion mass range is due to the fact that if they were lighter they would be so abundant that they would overclose the Universe. Details can be found in [7–9]. Early reviews of axions and

the strong CP problem and axions in cosmology and astrophysics are in Refs. [10–13].

The physics of the axion is similar to the π^0 in that they are both pseudoscalars and can decay to two photons through the one loop triangle anomaly diagram related to breaking the axial $U(1)$ global symmetry of massless QCD. The coupling of the axion field $a(x)$ to photons is

$$\mathcal{L}_{a\rightarrow\gamma\gamma} = \frac{\alpha K}{8\pi F_a} a(x) F\tilde{F},$$

where α is the fine structure constant, K is an $O(1)$ model dependent constant, and F_a is the axion decay constant that can be related to the pion mass and decay constant through $m_a F_a \approx m_\pi F_\pi$ where m_a is the axion mass. The axion lifetime scales with the π^0 lifetime

$$\tau_a \approx \left(\frac{m_\pi}{m_a}\right)^5 \tau_{\pi^0} \sim 10^8 \left(\frac{\text{eV}}{m_a}\right)^5 \text{ G yr},$$

so axions are very long lived compared to the age of the Universe. Since axions are long lived and copiously produced at rest in the EW phase transition, they are candidates for cold dark matter and can be the source of early universe density perturbation that grows in a way consistent with observations of the cosmic microwave background (CMB), which is unlike the results for free streaming hot dark matter perturbation that do not grow at small length scales.

Axions in vacuum can undergo spontaneous decay $a \rightarrow \gamma\gamma$, and if their density is high enough, stimulated decay can cause lasing to commence and rapidly deplete the axion number density, while the photon number grows exponentially. Sufficiently dense regions of axions can release enough monochromatic electromagnetic energy for possible detection [14,15]. The case of lasing axions for a spherical symmetric distribution, in both momentum and coordinate space, has been considered in [14–16]. However, recent studies of axions as dark matter suggest

^{*}bqipd@pm.me

[†]tom.kephart@gmail.com

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they may be in several other phase space configurations. For example, they may fall along caustics in galaxies [17,18], or they may be produced by superradiance, a Penrose type of process, around black holes [19] and fill a hydrogenlike orbit with quantum numbers $(n, l, m) = (2, 1, 1)$. (Since axions are bosons, there is no fundamental limit on their numbers in this state.) For these reasons we have decided to begin an exploration of the general case of lasing for arbitrary axion distributions in terms of a spherical harmonic expansion.

The axion mass range can be expanded somewhat by fine-tuning, but this begins to defeat the purpose of introducing the axions in the first place to avoid fine-tuning $\bar{\theta}$. The range can also be loosened by considering axionlike particles not necessarily designed to solve the strong CP problem. The analysis we present here can be applied to any of these cases as long as the axion or axionlike particles have a dominant decay mode to two photons.

In this paper we focus on arbitrary momentum distributions while keeping the coordinate space distribution spherically symmetric. In particular, we write a general expansion of momentum modes in spherical harmonics. Many of the algebraic details are relegated to the appendices. Once we have derived the general form, we then consider a few specific examples of physical interest. In future work we plan to also relax the coordinate space spherical symmetry requirement.

II. SETUP AND PREPARATION

To fix notation and set the stage for our analysis we first review the setup of the spherically symmetric phase space axion cluster model [14,16]. Axions of mass density ρ_a are produced nonthermally during the QCD phase transition with a cosmological density parameter today [7–9] of

$$\Omega_a \equiv \frac{\rho_a}{\rho_c} \cong \left(\frac{10^{-5} \text{ eV}}{m_a} \right)^{7/6},$$

where ρ_c is the critical density. Hence axions are a dominant dark matter (DM) candidate if $m_a \sim 10^{-5} \text{ eV}$.

Our purpose here is to give detailed calculations of the stimulated emission rate for axion clusters. There are many possible sources of initial photons of frequency in the $\sim 10^{-5} \text{ eV}$ range; therefore spontaneous decays are not required to start the lasing process, although spontaneous decay of axions is a lower bound on axion cluster luminosity. We neglect all normal matter in the cluster; hence we neglect the attenuation of photons by nonaxionic matter, as well as any other effects due to the local environment.

For any species of particles with occupation number $f(p, r, t)$, the particle number density is

$$n(r, t) = \int \frac{d^3 p}{8\pi^3} f(p, r, t),$$

and the total particle number of these particles in volume V is

$$N = \int_V d^3 r n(r, t).$$

Angular momentum conservation requires that a pair of photons emitted by the decay of a spin zero particle (scalar or pseudoscalar) have the same helicity. The rate of change of the photon number density of helicity $\lambda = \pm 1$ within an axion cluster, due to the process $a \leftrightarrow \gamma\gamma$, can be written as a Lorentz invariant phase space integral

$$\begin{aligned} \frac{dn_\lambda}{dt} = & \int dX_{\text{LIPS}}^{(3)} [f_a(1 + f_{1\lambda})(1 + f_{2\lambda}) - f_{1\lambda}f_{2\lambda}(1 + f_a)] \\ & \times |M(a \rightarrow \gamma\gamma)|^2, \end{aligned}$$

where $f_a = f_a(\vec{p})$, $f_{i\lambda} = f_{i\lambda}(\vec{k}_i)$, and $M = M(a \rightarrow \gamma(+)\gamma(+)) = M(a \rightarrow \gamma(-)\gamma(-))$. In detail, the phase space integral is

$$\begin{aligned} \int dX_{\text{LIPS}}^{(3)} = & \int \frac{d^3 p}{(2\pi)^3 2p^0} \int \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \\ & \times \int \frac{d^3 k_2}{(2\pi)^3 2k_2^0} \delta^4(p - k_1 - k_2), \end{aligned}$$

where p is the axion momentum and k_1 and k_2 are the momenta of the photons.

Upon defining $k = k_2^0$ and using the above results, we arrive at the rate equation for the photon occupation number

$$\begin{aligned} 2k \frac{df_\lambda(\vec{k})}{dt} = & \frac{4m_a \Gamma_a}{\pi} \int \frac{d^3 k_1}{2k_1^0} \frac{d^3 p}{2p^0} \delta^4(p - k_1 - k) \\ & \times f_a(\vec{p}) [1 + f_\lambda(\vec{k}) + f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k}) f_\lambda(\vec{k}_1). \end{aligned}$$

In the case of spherical symmetry we set $f(\vec{q}) = f(|\vec{q}|) = f(q)$ and the integral simplifies to give

$$\begin{aligned} \frac{df_\lambda(k)}{dt} = & \frac{4m_a \Gamma_a}{\pi} \int_{k+\frac{m_a^2}{4k}}^\infty dp^0 \delta^4(p - k_1 - k) \\ & \times f_a(p^0) [1 + f_\lambda(k) + f_{p^0-k}] - f_\lambda(k) f_\lambda(p^0 - k). \end{aligned}$$

In [14,16] a specific model with spherical symmetry was chosen where the initial axions were contained in a ball of radius R , with a maximum momentum value of $p_{\text{max}} \approx m_a \beta$ in the nonrelativistic case. That is, the initial axion occupation number was chosen to be

$$f_a(p, r) = C_a \Theta(p_{\text{max}} - p) \Theta(R - r),$$

where C_a is a constant that can be written in terms of axion mass and initial number density. Details of this model were

worked out in [14,16] and will be recovered below when we take the spherical symmetric limit of the general case.

The direction of axion momentum can be characterized by the infinitesimal solid momentum angle $\Omega_p = (\theta_p, \phi_p)$. $f_a(p, r, \Omega_p, t)$ and $n_a(r, \Omega_p, t)$ are the axion occupation number and axion number density, respectively. An integral of $f_a(p, r, \Omega_p, t)$ over p or (p, Ω_p) gives number density $n_a(r, \Omega_p, t)$ or $n_a(r, t)$.

The photons produced in the decay of axions are initially contained in the ball of radius R , and in a momentum spherical shell of inner and outer radius $k_- = \frac{m_a \gamma}{2}(1 - \beta)$ and $k_+ = \frac{m_a \gamma}{2}(1 + \beta)$, respectively. $f_\lambda(k, r, \theta, t)$ and $n_\lambda(r, \theta, t)$ are the photon occupation number and photon number density, respectively, of helicity $\lambda = \pm 1$. We assume that the number density of each helicity state is the same, so the total photon number density n_γ can be written as $n_\gamma(r, \theta, t) = n_+(r, \theta, t) + n_-(r, \theta, t) = 2n_\lambda(r, \theta, t)$.

Since the momentum distribution of axions is assumed to be spherically symmetric, there is an equal chance for an axion to be moving in any direction. Likewise the lasing process is equally likely to commence in any direction. In [14,16] this model was used to give lasing bounds and find the stimulated decay rate equations for axions and photons. Numerical results were also given for the evolution of such an axion cluster.

Here we want to investigate how the momenta of photons are distributed if a general nonspherically symmetric distribution of the axion momenta is specified. We do this by generalizing the model of [14,16] and proceeding without making this assumption of spherical symmetry in momentum space. We proceed by expanding in spherical harmonics and then integrating over all angles of photon momentum.

III. EVOLUTION EQUATIONS FOR INDIVIDUAL COMPONENTS OF PHOTON OCCUPATION NUMBER

Once we relax spherical symmetry, we must take angular dependencies into account. There are several sets of angles to consider, those for both axion and photon position and momentum, hence a potential total of eight angles on which our results may depend. Let us begin with the axions.

No spherical symmetry in momentum space means $f_a(\vec{p}) = f_a(p, \theta_p, \phi_p)$ is a general function of angles θ_p and ϕ_p . Since $f_a(\vec{p})$ is the occupation number of axions, it has to be integrable.

A. Harmonic expansion

We can write $f_a(\vec{p})$ as the square of a square integrable function formed in a complex spherical harmonics expansion,

$$\begin{aligned} f_a(\vec{p}) &= \left[\sum_{l'm'} a_{l'm'}^{m'}(p, t) Y_{l'm'}^{m'}(\Omega_p) \right]^* \left[\sum_{lm} a_{lm}^m(p, t) Y_{lm}^m(\Omega_p) \right] \\ &= \sum_{l'l'm'm} \sum_L (-1)^{m'} a_{l'm'}^{-m'*} a_{lm}^m \sqrt{\frac{(2l'+1)(2l+1)}{4\pi(2L+1)}} \\ &\quad \times C(l', l, L|0, 0, 0) C(l', l, L|m', m, m' + m) \\ &\quad \times Y_L^{m'+m}(\Omega_p), \end{aligned}$$

where the Y_l^m 's are complex spherical harmonics and the $C(l, l', l''|m, m', m'')$'s are Clebsch-Gordan coefficients.

By regrouping and renaming coefficients, $f_a(\vec{p})$ can be written directly as a real spherical harmonic expansion,

$$f_a(\vec{p}) = \sum_{lm} a_{lm}(p, t) Y_{lm}(\Omega_p). \quad (1)$$

Similarly, photon occupation numbers can also be written as

$$f_\lambda(\vec{k}) = \sum_{lm} b_{lm}(k, t) Y_{lm}(\Omega_k), \quad (2)$$

$$f_\lambda(\vec{k}_1) = \sum_{lm} b_{lm}(k_1, t) Y_{lm}(\Omega_{k_1}). \quad (3)$$

The following calculations do not put any restrictions on the coefficients a_{lm} 's or b_{lm} 's, and neither do the calculations require these coefficients to be positive or negative. But in the real physical world, only positive occupation numbers are allowed. In short, these coefficients can be of any value, but we should scrutinize the resulting occupation numbers so that they are positive and have real world meaning. For example, occupation number $f_a(\vec{p}) = a_{10}(p, t) Y_{10}(\Omega_p)$ cannot describe any real scenario since it is negative in half of the phase space, but occupation number $f_a(\vec{p}) = a_{00}(p, t) Y_{00}(\Omega_p) + a_{10}(p, t) Y_{10}(\Omega_p)$ may be allowed since it can be positive everywhere by adjusting the coefficients a_{00} and a_{10} . Nevertheless, the following calculations are applicable to both of these occupation numbers, whether they have real physical meaning or not.

B. Integration over k_1, Ω_{k_1}

We start our analysis from the general rate equation given in [16], which describes the evolution process of spontaneous and stimulated decay of axions to photons, and the backreaction of photons to axions,

$$\begin{aligned} 2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{4m_a \Gamma_a}{\pi} \int \frac{d^3 k_1}{2k_1^0} \frac{d^3 p}{2p^0} \delta^4(p - k - k_1) \\ &\quad \times \{f_a(\vec{p})[1 + f_\lambda(\vec{k}) + f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k})f_\lambda(\vec{k}_1)\}, \end{aligned} \quad (4)$$

where $\Gamma_a = 1/\tau_a$ is the axion decay width. Writing out the differential $d^3 k_1$ and the δ function explicitly, we have

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{(k_1)^2 dk_1 \sin \theta_{k_1} d\theta_{k_1} d\phi_{k_1}}{k_1^0 p^0} \\
&\times \frac{1}{(k_1)^2 \sin \theta_{k_1}} \delta(p^0 - k^0 - k_1^0) \delta(|\vec{p} - \vec{k}| - k_1) \\
&\times \delta(\theta_{\vec{p}-\vec{k}} - \theta_{\vec{k}_1}) \delta(\phi_{\vec{p}-\vec{k}} - \phi_{\vec{k}_1}) \\
&\times \{f_a(\vec{p})[1 + f_\lambda(\vec{k}) + f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k})f_\lambda(\vec{k}_1)\},
\end{aligned}$$

canceling common factors and substituting from the harmonic expansions, and we find

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{dk_1}{k_1^0} \frac{d^3 p}{p^0} \delta(p^0 - k^0 - k_1^0) \delta(|\vec{p} - \vec{k}| - k_1) \\
&\times \delta(\theta_{\vec{p}-\vec{k}} - \theta_{\vec{k}_1}) \delta(\phi_{\vec{p}-\vec{k}} - \phi_{\vec{k}_1}) \\
&\times \{f_a(\vec{p})[1 + f_\lambda(\vec{k}) + \sum_{lm} b_{lm}(k_1, t) Y_{lm}(\Omega_{k_1})] \\
&- f_\lambda(\vec{k}) \sum_{lm} b_{lm}(k_1, t) Y_{lm}(\Omega_{k_1})\} d\theta_{k_1} d\phi_{k_1}.
\end{aligned}$$

We can do the integration over θ_{k_1} and ϕ_{k_1} most efficiently by changing from Ω_{k_1} to $\Omega_{\vec{p}-\vec{k}}$. Upon using the fact that for photons $k_1^0 = k_1$ and $k^0 = k$ along with the identity

$$\int \delta(x - y) \delta(x - z) dx = \delta(y - z),$$

we find

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{d^3 p}{p^0(p^0 - k)} \delta[|\vec{p} - \vec{k}| - (p^0 - k)] \\
&\times \{f_a(\vec{p})[1 + f_\lambda(\vec{k}) \\
&+ \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}})] \\
&- f_\lambda(\vec{k}) \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}})\}. \quad (5)
\end{aligned}$$

C. Integration over ϕ_p : Preparation

Substituting $d^3 p = p^2 dp d\Omega_p = \sqrt{(p^0)^2 - m_a^2} p^0 dp^0 d\Omega_p$ and various relations from Appendix A into Eq. (5), we find

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{\sqrt{(p^0)^2 - m_a^2} p^0 dp^0 d\Omega_p}{p^0(p^0 - k)} \\
&\times \frac{p^0 - k}{k \sqrt{(p^0)^2 - m_a^2} \sin \theta_{p0}} \delta(\theta_p - \theta_{p0}) \\
&\times \{f_a(\vec{p})[1 + f_\lambda(\vec{k}) \\
&+ \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}})] \\
&- f_\lambda(\vec{k}) \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}})\}.
\end{aligned}$$

Canceling common factors and writing the equation in a form that is suitable for doing the two angular integrations, we have

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{dp^0 \delta(\theta_p - \theta_{p0})}{k \sin \theta_{p0}} \sin \theta_p d\theta_p \\
&\times \{[1 + f_\lambda(\vec{k})] \int f_a(\vec{p}) d\phi_p \\
&+ \int f_a(\vec{p}) \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&- f_\lambda(\vec{k}) \int \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p\}. \quad (6)
\end{aligned}$$

The momentum space angular integration requires some algebra that can be found in Appendix C. Adopting θ_{p1} notation from Eq. (C1) in Appendix C, Eq. (6) becomes

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= 2m_a \Gamma_a \int \frac{dp^0}{k} \{[1 + f_\lambda(\vec{k})] \sum_l a_{l0}(p, t) Y_{l0}(\theta_{p0}) \\
&+ \sum_{l'l'm} a_{l'l'm}(p, t) b_{lm}(p^0 - k, t) Y_{l'l'}^m(\theta_{p0}, 0) Y_l^m(\theta_{p1}, 0) \\
&- f_\lambda(\vec{k}) \times \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{p1})\}. \quad (7)
\end{aligned}$$

D. Evolution of individual components

The δ function in (4) imposes conservation of 4-momentum, since the integration has been carried out, and k and p^0 in (C2) of Appendix C are independent. Hence, the k in the denominator of (C2) can be taken outside of the integral,

$$\begin{aligned}
\frac{df_\lambda(\vec{k})}{dt} &= \sum_{lm} \frac{db_{lm}(k, t)}{dt} Y_{lm}(\Omega_k) \\
&= \frac{m_a \Gamma_a}{k^2} \int dp^0 \left\{ \sum_l a_{l0}(p, t) Y_{l0}(\theta_{p0}) \right. \\
&+ \sum_{l'l'm} a_{l'l'm}(p, t) b_{lm}(p^0 - k, t) Y_{l'l'}^m(\theta_{p0}, 0) Y_l^m(\theta_{p1}, 0) \\
&+ f_\lambda(\vec{k}) \left[\sum_l a_{l0}(p, t) Y_{l0}(\theta_{p0}) \right. \\
&\left. \left. - \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{p1}) \right] \right\}.
\end{aligned}$$

Upon writing out the Ω_k dependence of the right-hand side (RHS) of the equation above explicitly, and identifying the coefficients of $Y_{lm}(\Omega_k)$, we obtain differential equations for components $b_{lm}(k, t)$,

$$\begin{aligned} \frac{db_{lm}(k, t)}{dt} = & \frac{m_a \Gamma_a}{k^2} \int dp^0 \left\{ \delta_{l0} \delta_{m0} 2\sqrt{\pi} \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) + \sum_{l'l''m'} a_{l'l''m'}(p, t) b_{l'l''m'}(p^0 - k, t) Y_{l''m'}^m(\theta_{p0}, 0) Y_{l''m'}^m(\theta_{p1}, 0) \right] \right. \\ & \left. + b_{lm}(k, t) \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) - \sum_{l'} b_{l'0}(p^0 - k, t) Y_{l'0}(\theta_{p1}) \right] \right\}. \end{aligned}$$

Replacing p^0 with $k_1 + k$, also using the following substitution,

$$\int_{k+\frac{m_a^2}{4k}} dp^0 \Leftrightarrow \int_{\frac{m_a^2}{4k}} dk_1,$$

we find our final form of the evolution equations for individual components of occupation number,

$$\begin{aligned} \frac{db_{lm}(k, t)}{dt} = & \frac{m_a \Gamma_a}{k^2} \int_{\frac{m_a^2}{4k}} dk_1 \left\{ \delta_{l0} \delta_{m0} 2\sqrt{\pi} \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) + \sum_{l'l''m'} a_{l'l''m'}(p, t) b_{l'l''m'}(k_1, t) Y_{l''m'}^m(\theta_{p0}, 0) Y_{l''m'}^m(\theta_{p1}, 0) \right] \right. \\ & \left. + b_{lm}(k, t) \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) - \sum_{l'} b_{l'0}(k_1, t) Y_{l'0}(\theta_{p1}) \right] \right\}. \end{aligned} \quad (8)$$

The terms inside the curly brackets correspond to spontaneous decay, half of stimulated decay, the other half of stimulated decay, and photon annihilation, respectively. Spontaneous decay and half of the stimulated decay of axions only contribute to the order of $l=0, m=0$. In Eq. (8), we see that only axions of $m=0$ momentum configurations take part in spontaneous decay. $l \neq 0, m \neq 0$ photon momentum distribution can be produced from only half of the stimulated decay and backreaction.

IV. EVOLUTION EQUATIONS FOR INDIVIDUAL COMPONENTS OF AXION AND PHOTON NUMBER DENSITIES

Axion and photon occupation numbers depend on position. This fact was not included in previous discussion, since it is not involved in the dynamical process. Here we will assume that spherical symmetry is present in position space and axions sit inside a ball of radius R ,

$$f_a(\vec{p}) = f_a(t, p, \Omega_p, \vec{r}) = \sum_{lm} a_{lm}(t, p, \vec{r}) Y_{lm}(\Omega_p),$$

where the occupation number components can be written

$$a_{lm}(t, p, \vec{r}) = a_{lm}(t) \Theta(p_{\max} - p) \Theta(R - r). \quad (9)$$

Hence, the axion number density is

$$n_a(t, \Omega_p, \vec{r}) = \sum_{lm} n_{lm}^a(t) \Theta(R - r) Y_{lm}(\Omega_p),$$

where the components $n_{lm}^a(t)$ and $a_{lm}(t)$ are related through or

$$n_{lm}^a(t) = \int \frac{p^2 dp}{(2\pi)^3} a_{lm}(t) \Theta(p_{\max} - p) = \frac{(m_a \beta)^3}{24\pi^3} a_{lm}(t)$$

or

$$a_{lm}(t) = \frac{24\pi^3 n_{lm}^a(t)}{m_a^3 \beta^3}. \quad (10)$$

Photo occupation number and number density can be treated in a similar fashion. We write

$$f_\lambda(\vec{k}) = f_\lambda(t, k, \Omega_k, \vec{r}) = \sum_{lm} b_{lm}(t, k, \vec{r}) Y_{lm}(\Omega_k),$$

where

$$b_{lm}(t, k, \vec{r}) = b_{lm}(t) \Theta(k_+ - k) \Theta(k - k_-) \Theta(R - r) \quad (11)$$

and

$$n_\lambda(t, \vec{r}, \Omega_k) = \sum_{lm} n_{lm}^\lambda(t) \Theta(R - r) Y_{lm}(\Omega_k).$$

The components $n_{lm}^\lambda(t)$ and $b_{lm}(t)$ are related through

$$n_{lm}^\lambda(t) = \int \frac{k^2 dk}{(2\pi)^3} b_{lm}(t) \Theta(k_+ - k) \Theta(k - k_-)$$

$$b_{lm}(t) = \frac{32\pi^3 n_{lm}^{\gamma}(t)}{m_a^3 \beta}. \quad (12)$$

In Appendix C, we replaced all the occupation number components in (8) with the corresponding number density components and integrated over the momentum space, which gives us the evolution equations for the individual components of number density of photon, normal axion, and sterile axion,

$$\begin{aligned} \frac{dn_{lm}^{\gamma}(t)}{dt} = & \frac{2\Gamma_a}{m_a^2 \beta^2} \left\{ \delta_{l0} \delta_{m0} \sqrt{\pi} \left[\sum_{l'} \frac{6}{\beta} K_{l'l'}^0 n_{l'0}^a(t) \right. \right. \\ & + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l'l'm'} n_{l'l'm'}^a(t) n_{l'l'm'}^{\gamma}(t) K_{l'l'm'}^{01} \left. \right] \\ & + \frac{16\pi^3 n_{lm}^{\gamma}(t)}{m_a^3} \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'l'}^0 - 2 \sum_{l'} n_{l'0}^{\gamma}(t) B_{l'l'}^1 \right] \left. \right\} \\ & - \frac{3c}{2R} n_{lm}^{\gamma}(t), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{dn_{lm}^a(t)}{dt} = & -\frac{\Gamma_a}{m_a^2 \beta^2} \left\{ \delta_{l0} \delta_{m0} \sqrt{\pi} \left[\sum_{l'} \frac{6}{\beta} K_{l'l'}^0 n_{l'0}^a(t) \right. \right. \\ & + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l'l'm'} n_{l'l'm'}^a(t) n_{l'l'm'}^{\gamma}(t) K_{l'l'm'}^{01} \left. \right] \\ & + \frac{16\pi^3 n_{lm}^{\gamma}(t)}{m_a^3} \\ & \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'l'}^0 - 2 \sum_{l'} n_{l'0}^{\gamma}(t) N_{l'l'}^1 \right] \left. \right\}, \end{aligned} \quad (14)$$

$$\frac{dn_{lm}^{as}(t)}{dt} = \frac{\Gamma_a}{m_a^2 \beta^2} \frac{16\pi^3 n_{lm}^{\gamma}(t)}{m_a^3} \times 2 \sum_{l'} n_{l'0}^{\gamma}(t) S_{l'l'}^1. \quad (15)$$

The $K_{l'l'}^0$ are defined to be constant coefficients describing spontaneous and half of stimulated decay. $K_{l'l'm'}^{01}$ are constant coefficients describing the other half of stimulated decay. The $B_{l'l'}^1$ are constant coefficients describing the backreaction of photons, which, when necessary, can be split into $N_{l'l'}^1$ and $S_{l'l'}^1$, the part of backreactions that produce normal axions and sterile axions, respectively, and hence $B_{l'l'}^1 = N_{l'l'}^1 + S_{l'l'}^1$.

V. EXAMPLES

A. Y_{00} momentum distribution

The choice Y_{00} corresponds to the momentum distribution of axion with no preferred direction. The number density of axion for such a momentum distribution is

$$\begin{aligned} n_a(t, \vec{r}, \Omega_p) &= \sum_{lm} n_{lm}^a(t) \Theta(R-r) Y_{lm}(\Omega_p) \\ &= n_{00}^a(t) \Theta(R-r) Y_{00}(\Omega_p). \end{aligned}$$

Thus, for any $l \neq 0, m \neq 0$, the number density component is zero,

$$n_{lm}^a(t) = 0 \quad (lm \neq 00).$$

This means that the evolution equations (14) for these components reduce to

$$\begin{aligned} \frac{dn_{00}^a(t)}{dt} = & -\frac{\Gamma_a}{m_a^2 \beta^2} \times \frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3} \\ & \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'l'}^0 - 2 \sum_{l'} n_{l'0}^{\gamma}(t) N_{l'l'}^1 \right] = 0. \end{aligned}$$

There are two cases regarding the solution of the equation above.

The first case is

$$n_{lm}^{\gamma}(t) = \delta_{l0} \delta_{m0} n_{00}^{\gamma}(t),$$

where all the photon number density components n_{lm}^{γ} vanish except $l = 0$, and $m = 0$,

$$n_{\gamma}(t, \vec{r}, \Omega_k) = n_{00}^{\gamma}(t) \Theta(R-r) Y_{00}(\Omega_k).$$

Since the photon field is fixed in the Y_{00} momentum distribution, the exact form of number density of normal axion, photon, and sterile axion can be obtained by solving the evolution equations (13), (14), and (15). In this first case we find

$$\begin{aligned} \frac{dn_{00}^{\gamma}(t)}{dt} = & \frac{2\Gamma_a}{m_a^2 \beta^2} \left\{ \sqrt{\pi} \left[\frac{6}{\beta} K_{00}^0 n_{00}^a(t) + \frac{96\pi^3}{m_a^3 \beta^2} n_{00}^a(t) n_{00}^{\gamma}(t) K_{000}^{01} \right] \right. \\ & + \frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3} \times \left[\frac{3}{\beta^2} n_{00}^a(t) K_{00}^0 - 2 n_{00}^{\gamma}(t) B_{00}^1 \right] \left. \right\} \\ & - \frac{3c}{2R} n_{00}^{\gamma}(t), \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{dn_{00}^a(t)}{dt} = & -\frac{\Gamma_a}{m_a^2 \beta^2} \left\{ \sqrt{\pi} \left[\frac{6}{\beta} K_{00}^0 n_{00}^a(t) + \frac{96\pi^3}{m_a^3 \beta^2} n_{00}^a(t) n_{00}^{\gamma}(t) K_{000}^{01} \right] \right. \\ & + \frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3} \times \left[\frac{3}{\beta^2} n_{00}^a(t) K_{00}^0 - 2 n_{00}^{\gamma}(t) N_{00}^1 \right] \left. \right\}, \end{aligned} \quad (17)$$

$$\frac{dn_{00}^{as}(t)}{dt} = \frac{\Gamma_a}{m_a^2 \beta^2} \frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3} \times 2 n_{00}^{\gamma}(t) S_{00}^1. \quad (18)$$

These equations are the same as (34'), (37'), and (38') of [16], when we set

$$\begin{aligned}
K_0^0 &= \frac{m_a^2 \beta^3}{6\sqrt{\pi}} = N_0^1, & K_{000}^{01} &= \frac{m_a^2 \beta^3}{12\pi}, \\
B_0^1 &= \frac{m_a^2 \beta^2}{4\sqrt{\pi}} \left(1 + \frac{2\beta}{3}\right), & N_0^1 &= \frac{m_a^2 \beta^3}{6\sqrt{\pi}}, & S_0^1 &= \frac{m_a^2 \beta^2}{4\sqrt{\pi}}, \\
n_{00}^a(t) &= \frac{n_a(t)}{2\sqrt{\pi}}, & n_{00}^\gamma(t) &= \frac{n_\gamma(t)}{2\sqrt{\pi}}.
\end{aligned}$$

The second case is

$$\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_l^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 = 0.$$

This solution to Eqs. (16), (17), and (18) means that the effect of half of the stimulated decay exactly cancels the effect of photon annihilation back into a normal axion. This also expresses the $l=0, m=0$ component of axion number density as a sum of components of photon number density,

$$n_{00}^a(t) = \frac{2\beta^2}{3K_0^0} \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1.$$

Evolution equation (14) for the $l=0, m=0$ component then becomes

$$\begin{aligned}
\frac{dn_{00}^a(t)}{dt} &= -\frac{6\Gamma_a \sqrt{\pi}}{m_a^2 \beta^3} \left[K_0^0 n_{00}^a(t) + \frac{16\pi^3}{m_a^3 \beta} \sum_{l'} n_{00}^a(t) n_{l'0}^\gamma(t) K_{0l'0}^{01} \right].
\end{aligned}$$

Evolution equation (13) for any $l \neq 0, m \neq 0$ components of photon number density in this second case is

$$\frac{dn_{lm}^\gamma(t)}{dt} = -\frac{64\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{lm}^\gamma(t) \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 - \frac{3c}{2R} n_{lm}^\gamma(t).$$

Photon annihilation back into sterile axion and surface loss are the only mechanisms that contribute to $n_{lm}^\gamma(t) (l \neq 0, m \neq 0)$ modes. There is no source of decay providing photons to $l \neq 0, m \neq 0$ number density components. Therefore $n_{lm}^\gamma(t) (l \neq 0, m \neq 0)$ modes are expected to vanish quickly. The majority of photons would be in the $n_{00}^\gamma(t)$ component.

In both cases, either $n_{lm}^\gamma(t) = 0$ or $\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_l^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 = 0$, if the axions are locked in Y_{00} momentum state, then so would be the photons (or at least predominantly in the second case).

B. Y_{20} momentum distribution

In this configuration the direction of the momentum of axions is preferentially parallel to the polar axis. It describes the movement of axions between northern and southern hemispheres

$$\begin{aligned}
n_a(t, \vec{r}, \Omega_p) &= \sum_{lm} n_{lm}^a(t) \Theta(R-r) Y_{lm}(\Omega_p) \\
&= n_{20}^a(t) \Theta(R-r) Y_{20}(\Omega_p).
\end{aligned}$$

$n_{20}^a(t)$ is the only nonzero axion number density component,

$$n_{lm}^a(t) = 0 \quad (l \neq 2, m \neq 0).$$

Evolution equation (14) for $n_{20}^a(t)$ becomes

$$\frac{dn_{20}^a(t)}{dt} = -\frac{16\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{20}^\gamma(t) \left[\frac{3}{\beta^2} n_{20}^a(t) K_2^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \neq 0.$$

At the same time, except for $n_{00}^a(t)$ and $n_{20}^a(t)$, evolution equation (14) becomes

$$\begin{aligned}
\frac{dn_{lm}^a(t)}{dt} &= -\frac{16\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{lm}^\gamma(t) \left[\frac{3}{\beta^2} n_{20}^a(t) K_2^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
&= 0 \quad (lm \neq 00, 20).
\end{aligned}$$

Comparing these two equations, we have

$$n_{lm}^\gamma(t) = 0 \quad (lm \neq 00, 20).$$

This shows that there are only two nonzero components for photon number density, $n_{00}^\gamma(t)$ and $n_{20}^\gamma(t)$.

Since the $n_{00}^a(t)$ component of axion number density is 0, evolution equation (14) reduces to

$$\begin{aligned}
\frac{dn_{00}^a(t)}{dt} &= 0 \\
&= -\frac{\Gamma_a}{m_a^2 \beta^2} \left\{ \sqrt{\pi} \left[\frac{6}{\beta} K_2^0 n_{20}^a(t) \right. \right. \\
&\quad \left. \left. + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l'} n_{20}^a(t) n_{l'0}^\gamma(t) K_{2l'0}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \left[\frac{3}{\beta^2} n_{20}^a(t) K_2^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \right\}.
\end{aligned}$$

This can be used to simplify the photon evolution equation (13) for component $n_{00}^\gamma(t)$,

$$\begin{aligned}
\frac{dn_{00}^\gamma(t)}{dt} &= \frac{2\Gamma_a}{m_a^2\beta^2} \left\{ \sqrt{\pi} \left[\frac{6}{\beta} K_2^0 n_{20}^a(t) + \frac{96\pi^3}{m_a^3\beta^2} \sum_{l''} n_{20}^a(t) n_{l''0}^\gamma(t) K_{2l''0}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \left[\frac{3}{\beta^2} n_{20}^a(t) K_2^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \right\} - \frac{3c}{2R} n_{00}^\gamma(t) \\
&= -\frac{64\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{00}^\gamma(t) \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 - \frac{3c}{2R} n_{00}^\gamma(t).
\end{aligned}$$

The $n_{00}^\gamma(t)$ component is expected to vanish quickly since the only contributions to this mode are due to backreaction to sterile axions and surface losses. Meanwhile, the photon component $n_{20}^\gamma(t)$ evolves according to Eq. (13),

$$\begin{aligned}
\frac{dn_{20}^\gamma(t)}{dt} &= \frac{32\pi^3 \Gamma_a}{m_a^2 \beta^5} n_{20}^\gamma(t) \left[\frac{3}{\beta^2} n_{20}^a(t) K_2^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \\
&\quad - \frac{3c}{2R} n_{20}^\gamma(t).
\end{aligned}$$

If axions were locked in a Y_{20} momentum state, the majority of photons would be expected to be in this state, which means more photons would travel in the direction that is to some extent parallel to the polar axis. However, this is not an acceptable physical number density for particles because Y_{20} is negative in some regions, but in the appropriate combinations with Y_{00} , the total number density can be positive.

C. $Y_1^{\pm 1*} Y_1^{\pm 1} \sim \sin^2 \theta$ momentum distribution

This configuration has the direction of momentum of axions preferentially parallel to the equatorial plane and is a good approximation to the case of superradiant axions near a Kerr black hole [19]. It describes axions rotating around a polar axis. The axion number density has two nonzero components, n_{00}^a and n_{20}^a ,

$$\begin{aligned}
n_a(t, \vec{r}, \Omega_p) &= \sum_{lm} n_{lm}^a(t) Y_{lm}(\Omega_p) \Theta(R-r) \\
&= n^a(t) \sin^2 \theta_p \Theta(R-r) \\
&= n^a(t) \frac{4\sqrt{\pi}}{3} \left[Y_{00}(\Omega_p) - \frac{1}{\sqrt{5}} Y_{20}(\Omega_p) \right] \Theta(R-r).
\end{aligned}$$

To maintain the $\sin^2 \theta$ distribution, there is a relation between these two nonzero components,

$$n_{20}^a(t) = -\frac{1}{\sqrt{5}} n_{00}^a(t). \quad (19)$$

Evolution equation (14) for $n_{20}^a(t)$ becomes

$$\begin{aligned}
\frac{dn_{20}^a(t)}{dt} &= -\frac{16\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{20}^a(t) \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
&\neq 0,
\end{aligned} \quad (20)$$

while for components other than $n_{00}^a(t)$ and $n_{20}^a(t)$, evolution equation (14) becomes

$$\begin{aligned}
\frac{dn_{lm}^a(t)}{dt} &= -\frac{16\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{lm}^a(t) \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
&= 0 \quad (lm \neq 00, 20).
\end{aligned}$$

Comparing these two equations, we have

$$n_{lm}^\gamma(t) = 0 \quad (lm \neq 00, 20).$$

This shows that there are only two nonzero components for photon number density, $n_{00}^\gamma(t)$ and $n_{20}^\gamma(t)$.

In Appendix D, we find that the quantity $n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)$ evolves based on the following equation:

$$\begin{aligned}
\frac{d}{dt} [n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)] \\
= - \left[\frac{64\pi^3 \Gamma_a}{m_a^5 \beta^2} \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 + \frac{3c}{2R} \right] [n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)].
\end{aligned}$$

Therefore $n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t) = 0$ is a possible solution to this differential equation although it is not the unique solution, which indicates that axions of $\sin^2 \theta$ momentum distribution can still generate photons of $\sin^2 \theta$ momentum distribution. However, since only photon annihilation back into sterile axion and surface loss contribute to the change of the quantity $n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)$, we expect

$$n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t) \approx 0.$$

So approximately, decay from $\sin^2 \theta$ momentum distribution of axions results in a similar directional profile of photons.

VI. SUMMARY AND CONCLUSIONS

Equations (13)–(15) contain the main result of this work, where we have modeled clusters of axions with spherically symmetric spatial but arbitrary momentum distributions and have studied the directional profile of photons produced in their evolution through spontaneous and stimulated axion decay via the process $a \rightarrow \gamma\gamma$. These results can be used in situations where astrophysical axions cluster [20–22]. Axions are a prime candidate for dark matter, and clustering can be due to any number of reasons ranging from primordial density perturbations to superradiance

around black holes. Three specific examples were presented, one with spherical symmetry to make contact with previous work, and two others of typical but simple examples without spherical symmetry. It is straightforward to use our result to model any cluster of axions with spherically symmetric spatial but arbitrary momentum distributions so we believe our results will have wide application.

Here we have only considered homogeneous distributions in spatial coordinates; i.e., our distributions are homogeneous within a spherically symmetric region set by the Θ function that fixes the radial extent of the distributions. The clump will in general evolve from this initial homogeneous spatial distributions via partial differential equations in (t, x_i) . We plan to relax the requirement of spherical symmetry of the spatial distribution and discuss the evolution of distributions in future work where we also consider both special and general relativistic corrections for dense clusters in curved space.

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APPENDIX A: KINEMATICS

Here we collect some useful kinematic relations needed in the text.

1. $\delta[|\vec{p} - \vec{k}| - (p^0 - k)]$ conversion

Choose the z axis to align with photon momentum $\vec{k}$, and then

$$\begin{aligned}\cos\theta_p &= \frac{\vec{p} \cdot \vec{k}}{pk}, \\ (\vec{p} - \vec{k})^2 &= p^2 + k^2 - 2pk\cos\theta_p \\ &= (p^0)^2 - m_a^2 + k^2 - 2k\sqrt{(p^0)^2 - m_a^2}\cos\theta_p.\end{aligned}$$

Now let $g(\cos\theta_p) = |\vec{p} - \vec{k}|$, to find

$$\begin{aligned}\delta[|\vec{p} - \vec{k}| - (p^0 - k)] &= \delta[g(\cos\theta_p) - (p^0 - k)] \\ &= \delta\left[\sqrt{(p^0)^2 - m_a^2 + k^2 - 2k\sqrt{(p^0)^2 - m_a^2}\cos\theta_p} - (p^0 - k)\right] \\ &= \delta(\cos\theta_p - \cos\theta_{p0}) \times \left|\frac{d\cos\theta_p}{dg(\cos\theta_p)}\right|_{\cos\theta_p=\cos\theta_{p0}},\end{aligned}$$

where $\cos\theta_{p0}$ is the root of the equation $g(\cos\theta_p) - (p^0 - k) = 0$. Clearly

$$\begin{aligned}g(\cos\theta_{p0}) - (p^0 - k) &= 0 \Leftrightarrow (\vec{p} - \vec{k})^2 = (p^0 - k)^2 \\ p^2 + k^2 - 2pk\cos\theta_{p0} &= (p^0)^2 + k^2 - 2p^0k \\ \Rightarrow \cos\theta_{p0} &= \frac{2p^0k - m_a^2}{2k\sqrt{(p^0)^2 - m_a^2}}.\end{aligned}\quad (A1)$$

The derivative at $\cos\theta_{p0}$ can also be calculated,

$$\frac{dg(\cos\theta_p)}{d\cos\theta_p} = \frac{-k\sqrt{(p^0)^2 - m_a^2}}{g(\cos\theta_p) + p^0 - k},$$

which can be evaluated to give

$$\left|\frac{d\cos\theta_p}{dg(\cos\theta_p)}\right|_{\cos\theta_p=\cos\theta_{p0}} = \frac{p^0 - k}{k\sqrt{(p^0)^2 - m_a^2}}.$$

Substituting the root and the derivative, $\delta[|\vec{p} - \vec{k}| - (p^0 - k)]$ is converted to a δ function with respect to θ_p ,

$$\begin{aligned}\delta[|\vec{p} - \vec{k}| - (p^0 - k)] &= \frac{p^0 - k}{k\sqrt{(p^0)^2 - m_a^2}} \delta\left[\cos\theta_p - \frac{2p^0k - m_a^2}{2k\sqrt{(p^0)^2 - m_a^2}}\right]\end{aligned}$$

or

$$= \frac{p^0 - k}{k\sqrt{(p^0)^2 - m_a^2} \sin\theta_{p0}} \delta(\theta_p - \theta_{p0}).$$

Then the common way to obtain various trigonometric quantities about $\vec{p}$ is through relations for the angles given below,

$$\begin{aligned}\cos\theta_p &= \frac{\vec{p} \cdot \vec{k}}{pk}, \\ \sin\theta_p \cos\phi_p &= \frac{\vec{p} \cdot \vec{e}_x}{p}, \quad \sin\theta_p \sin\phi_p = \frac{\vec{p} \cdot \vec{e}_y}{p}.\end{aligned}$$

Similarly, trigonometric quantities about $\vec{p} - \vec{k}$ can be obtained via the following relations:

$$\begin{aligned}\cos\theta_{\vec{p}-\vec{k}} &= \frac{(\vec{p} - \vec{k}) \cdot \vec{k}}{|\vec{p} - \vec{k}|k} = \frac{p\cos\theta_p - k}{\sqrt{p^2 + k^2 - 2pk\cos\theta_p}}, \\ \sin\theta_{\vec{p}-\vec{k}} &= \sqrt{1 - \cos^2\theta_{\vec{p}-\vec{k}}} = \frac{p\sin\theta_p}{\sqrt{p^2 + k^2 - 2pk\cos\theta_p}},\end{aligned}$$

$$\begin{aligned}
\sin \theta_{\vec{p}-\vec{k}} \cos \phi_{\vec{p}-\vec{k}} &= \frac{(\vec{p}-\vec{k}) \cdot \vec{e}_x}{|\vec{p}-\vec{k}|} = \frac{\vec{p} \cdot \vec{e}_x}{|\vec{p}-\vec{k}|} \\
&= \frac{p \sin \theta_p \cos \phi_p}{|\vec{p}-\vec{k}|} = \sin \theta_{\vec{p}-\vec{k}} \cos \phi_p, \\
\sin \theta_{\vec{p}-\vec{k}} \sin \phi_{\vec{p}-\vec{k}} &= \frac{(\vec{p}-\vec{k}) \cdot \vec{e}_y}{|\vec{p}-\vec{k}|} = \frac{\vec{p} \cdot \vec{e}_y}{|\vec{p}-\vec{k}|} \\
&= \frac{p \sin \theta_p \sin \phi_p}{|\vec{p}-\vec{k}|} = \sin \theta_{\vec{p}-\vec{k}} \sin \phi_p.
\end{aligned}$$

Thus $\phi_{\vec{p}-\vec{k}}$ and ϕ_p are equal,

$$\cos \phi_{\vec{p}-\vec{k}} = \cos \phi_p \quad \sin \phi_{\vec{p}-\vec{k}} = \sin \phi_p \quad \Rightarrow \quad \phi_{\vec{p}-\vec{k}} = \phi_p.$$

Then the third ϕ_p integration from Eq. (6) can be carried out,

$$\begin{aligned}
&\int \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&= \sum_{lm} b_{lm}(p^0 - k, t) \int Y_{lm}(\theta_{\vec{p}-\vec{k}}, \phi_p) d\phi_p \\
&= \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\theta_{\vec{p}-\vec{k}}, \phi_p) \times 2\pi \delta_{m0} \\
&= 2\pi \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{\vec{p}-\vec{k}}).
\end{aligned}$$

APPENDIX B: RELATIONS BETWEEN REAL AND COMPLEX SPHERICAL HARMONICS

If $m', m < 0$, then we can write

$$\begin{aligned}
&\int Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&= \int \frac{i}{\sqrt{2}} [Y_{l'}^{m'}(\Omega_p) - (-1)^{m'} Y_{l'}^{-m'}(\Omega_p)] \\
&\quad \times \frac{i}{\sqrt{2}} [Y_l^m(\Omega_{\vec{p}-\vec{k}}) - (-1)^m Y_l^{-m}(\Omega_{\vec{p}-\vec{k}})] d\phi_p.
\end{aligned}$$

Expanding and using the property $Y_l^{m*} = (-1)^m Y_l^{-m}$ we find

$$\begin{aligned}
&\int Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&= \int \frac{1}{2} [Y_{l'}^{m'}(\Omega_p) Y_l^m(\Omega_{\vec{p}-\vec{k}})^* + Y_l^m(\Omega_{\vec{p}-\vec{k}}) Y_{l'}^{m'}(\Omega_p)^* \\
&\quad - Y_{l'}^{m'}(\Omega_p) Y_l^m(\Omega_{\vec{p}-\vec{k}}) - Y_l^m(\Omega_{\vec{p}-\vec{k}})^* Y_{l'}^{m'}(\Omega_p)^*] d\phi_p.
\end{aligned}$$

We are integrating over ϕ_p , but since $\phi_{\vec{p}-\vec{k}} = \phi_p$, it does not matter that $\theta_{\vec{p}-\vec{k}} \neq \theta_p$ because Y_l^m is of the form $P_l^m(\cos \theta) e^{im\phi}$. $Y_l^m(\theta, 0) = Y_l^m(\theta, \phi = 0)$, which is a real

function, and is used to denote the result from integration of complex spherical harmonics over azimuthal angle,

$$\begin{aligned}
&\int Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&= \frac{1}{2} [2\pi \delta_{m'm} Y_{l'}^{m'}(\theta_p, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}, 0) \\
&\quad + 2\pi \delta_{m'm} Y_l^m(\theta_{\vec{p}-\vec{k}}, 0) Y_{l'}^{m'}(\theta_p, 0)] \\
&= 2\pi \delta_{m'm} Y_{l'}^m(\theta_p, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}, 0) (m', m < 0).
\end{aligned}$$

This result actually is applicable to all cases of signs of m' and m , so we can drop the restriction and write simply

$$\int Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p = 2\pi \delta_{m'm} Y_{l'}^m(\theta_p, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}, 0).$$

APPENDIX C: MOMENTUM SPACE ANGULAR INTEGRATION

1. Integration over ϕ_p : $Y_{lm}(\Omega_p)$ and $Y_{lm}(\Omega_{\vec{p}-\vec{k}})$

The ϕ_p integration of the first term in braces in Eq. (6) can be calculated directly using Eq. (1). Because of the factor $e^{im\phi}$ contained in Y_{lm} , the integration is 0 unless $m = 0$,

$$\int f_a(\vec{p}) d\phi_p = 2\pi \sum_l a_{l0}(p, t) Y_{l0}(\theta_p).$$

As stated before, $\vec{k}$ is chosen as the z axis.

2. Integration over ϕ_p : $Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}})$

Again substituting the form of f_a from Eq. (1), we see the ϕ_p integration in the second term of Eq. (6) has to be performed on $Y_{l'm'}(\Omega_p) Y_{lm}(\Omega_{\vec{p}-\vec{k}})$. The calculation is easier if we convert from real back to the complex spherical harmonics. But the relation between real and complex spherical harmonics depends on the signs of m' and m . Using the relations among spherical harmonics found in Appendix B facilitates the integration, and we find

$$\begin{aligned}
&\int f_a(\vec{p}) \sum_{lm} b_{lm}(p^0 - k, t) Y_{lm}(\Omega_{\vec{p}-\vec{k}}) d\phi_p \\
&= 2\pi \sum_{l'm} a_{l'm}(p, t) b_{lm}(p^0 - k, t) Y_{l'}^m(\theta_p, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}, 0).
\end{aligned}$$

The third integral is similar to the first.

Collecting terms after all ϕ_p integrations, Eq. (6) becomes

$$\begin{aligned}
2k \frac{df_\lambda(\vec{k})}{dt} &= \frac{m_a \Gamma_a}{\pi} \int \frac{dp^0 \delta(\theta_p - \theta_{p0})}{k \sin \theta_{p0}} \sin \theta_p d\theta_p \\
&\times \left\{ [1 + f_\lambda(\vec{k})] \times 2\pi \sum_l a_{l0}(p, t) Y_{l0}(\theta_p) \right. \\
&+ 2\pi \sum_{l'lm} a_{l'm}(p, t) b_{lm}(p^0 - k, t) \\
&\times Y_{l'}^m(\theta_p, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}, 0) \\
&\left. - f_\lambda(\vec{k}) \times 2\pi \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{\vec{p}-\vec{k}}) \right\}.
\end{aligned}$$

Integration over θ_p is straightforward and leads to

$$\begin{aligned}
k \frac{df_\lambda(\vec{k})}{dt} &= m_a \Gamma_a \int \frac{dp^0}{k} \left\{ [1 + f_\lambda(\vec{k})] \sum_l a_{l0}(p, t) Y_{l0}(\theta_{p0}) \right. \\
&+ \sum_{l'lm} a_{l'm}(p, t) b_{lm}(p^0 - k, t) \\
&\times Y_{l'}^m(\theta_{p0}, 0) Y_l^m(\theta_{\vec{p}-\vec{k}}|_{\theta_p=\theta_{p0}}, 0) \\
&\left. - f_\lambda(\vec{k}) \times \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{\vec{p}-\vec{k}}|_{\theta_p=\theta_{p0}}) \right\},
\end{aligned}$$

but $Y_l^m(\theta_{\vec{p}-\vec{k}}, 0)$ and $Y_{l0}(\theta_{\vec{p}-\vec{k}})$ need to be evaluated at $\theta_p = \theta_{p0}$ since $\theta_{\vec{p}-\vec{k}}$ is an implicit function of θ_p . $\cos \theta_p$ has been evaluated at θ_{p0} by (A1),

$$\cos \theta_p|_{\theta_p=\theta_{p0}} = \cos \theta_{p0} = \frac{2p^0 k - m_a^2}{2k \sqrt{(p^0)^2 - m_a^2}}.$$

Note that $|\vec{p} - \vec{k}| = (p^0 - k)$ is equivalent to $\theta_p = \theta_{p0}$ because that is how (A1) was derived. This gives us a relation between $\cos \theta_{\vec{p}-\vec{k}}$ and $\cos \theta_p$,

$$\begin{aligned}
\cos \theta_{\vec{p}-\vec{k}} &= \frac{(\vec{p} - \vec{k}) \cdot \vec{k}}{|\vec{p} - \vec{k}| k} = \frac{p \cos \theta_p - k}{p^0 - k}, \\
\cos \theta_{\vec{p}-\vec{k}}|_{\theta_p=\theta_{p0}} &= \frac{p^0 - \frac{m_a^2}{2k} - k}{p^0 - k} = 1 - \frac{m_a^2}{2k(p^0 - k)} \\
&= \cos \theta_{\vec{p}-\vec{k}}|_{\theta_{\vec{p}-\vec{k}}=\theta_{p1}}.
\end{aligned}$$

The last line suggests that $\cos \theta_{\vec{p}-\vec{k}}$ can also be directly evaluated at $\theta_{\vec{p}-\vec{k}} = \theta_{p1}$ if

$$\cos \theta_{p1} = 1 - \frac{m_a^2}{2k(p^0 - k)}. \quad (C1)$$

Adopting this θ_{p1} notation, Eq. (6) becomes

$$\begin{aligned}
k \frac{df_\lambda(\vec{k})}{dt} &= m_a \Gamma_a \int \frac{dp^0}{k} \left\{ [1 + f_\lambda(\vec{k})] \sum_l a_{l0}(p, t) Y_{l0}(\theta_{p0}) \right. \\
&+ \sum_{l'lm} a_{l'm}(p, t) b_{lm}(p^0 - k, t) Y_{l'}^m(\theta_{p0}, 0) Y_l^m(\theta_{p1}, 0) \\
&\left. - f_\lambda(\vec{k}) \times \sum_l b_{l0}(p^0 - k, t) Y_{l0}(\theta_{p1}) \right\}. \quad (C2)
\end{aligned}$$

3. Integration over k_1 and k

Equation (8) needs to be integrated over k to obtain an equation for the components of number density. To do this we first rewrite Eq. (8) as

$$\begin{aligned}
\int \frac{k^2 dk}{(2\pi)^3} \frac{db_{lm}(k, t)}{dt} &= \int \frac{k^2 dk}{(2\pi)^3} \frac{m_a \Gamma_a}{k^2} \int_{\frac{m_a^2}{4k}} dk_1 \left\{ \delta_{l0} \delta_{m0} 2\sqrt{\pi} \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) \right. \right. \\
&+ \sum_{l''m'} a_{l''m'}(p, t) b_{l''m'}(k_1, t) Y_{l''}^{m'}(\theta_{p0}, 0) Y_{l'}^{m'}(\theta_{p1}, 0) \Big] \\
&\left. + b_{lm}(k, t) \left[\sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) - \sum_{l'} b_{l'0}(k_1, t) Y_{l'0}(\theta_{p1}) \right] \right\}. \quad (C3)
\end{aligned}$$

Apart from writing down step function $\Theta(R - r)$, integrations are carried over (8) term by term in the following calculation:

$$\int \frac{k^2 dk}{(2\pi)^3} \frac{db_{lm}(k, t)}{dt} = \frac{db_{lm}(t)}{dt} \int \Theta(k_+ - k) \Theta(k - k_-) \frac{k^2 dk}{(2\pi)^3} = \frac{dn_{lm}^\lambda(t)}{dt}.$$

The integration on the RHS of (C3) is over k and k_1 . The following calculations omit writing down common factor $\frac{m_a \Gamma_a}{(2\pi)^3}$. The first term on the RHS of (C3) is

$$\begin{aligned}
& \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) \\
&= \sum_{l'} a_{l'0}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \Theta(p_{\max} - p) Y_{l'0}(\theta_{p0}) \\
&= \sum_{l'} a_{l'0}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \sqrt{\frac{2l'+1}{4\pi}} P_{l'}^0(\cos \theta_{p0}) \\
&= \sum_{l'} a_{l'0}(t) K_{l'}^0,
\end{aligned}$$

where $K_{l'}^0$ are defined to be constant coefficients describing spontaneous and half of stimulated decay, and are related to associated Legendre polynomials $P_{l'}^0$.

$$K_{l'}^0 = \sqrt{\frac{2l'+1}{4\pi}} \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 P_{l'}^0 \left[\frac{2(k_1+k)k - m_a^2}{2k\sqrt{(k_1+k)^2 - m_a^2}} \right].$$

The second term on the RHS of (8) can be treated similarly,

$$\begin{aligned}
& \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \sum_{l'l'm'} a_{l'm'}(p, t) b_{l'm'}(k_1, t) Y_{l'm'}^{m'}(\theta_{p0}, 0) Y_{l'm'}^{m'}(\theta_{p1}, 0) \\
&= \sum_{l'l'm'} a_{l'm'}(t) b_{l'm'}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \Theta(p_{\max} - p) \\
&\quad \times Y_{l'm'}^{m'}(\theta_{p0}, 0) Y_{l'm'}^{m'}(\theta_{p1}, 0) \Theta(k_+ - k_1) \Theta(k_1 - k_-) \\
&= \sum_{l'l'm'} a_{l'm'}(t) b_{l'm'}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'm'}^{m'}(\theta_{p0}, 0) Y_{l'm'}^{m'}(\theta_{p1}, 0) \\
&= \sum_{l'l'm'} a_{l'm'}(t) b_{l'm'}(t) K_{l'l'm'}^{01},
\end{aligned}$$

where $K_{l'l'm'}^{01}$ are constant coefficients describing the other half of stimulated decay and are related to associated Legendre polynomials $P_{l'}^{m'}$ and $P_{l''}^{m'}$,

$$\begin{aligned}
K_{l'l'm'}^{01} &= \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'm'}^{m'}(\theta_{p0}, 0) Y_{l'm'}^{m'}(\theta_{p1}, 0) \\
&= \sqrt{\frac{(2l'+1)(l'-m')!}{4\pi(l'+m')!}} \sqrt{\frac{(2l''+1)(l''-m')!}{4\pi(l''+m')!}} \\
&\quad \times \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \\
&\quad \times P_{l'}^{m'} \left[\frac{2(k_1+k)k - m_a^2}{2k\sqrt{(k_1+k)^2 - m_a^2}} \right] P_{l''}^{m'} \left[1 - \frac{m_a^2}{2kk_1} \right].
\end{aligned}$$

The third term on the RHS of (C3) is simplified by using newly defined coefficients $K_{l'}^0$,

$$\begin{aligned}
& \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 b_{lm}(k, t) \sum_{l'} a_{l'0}(p, t) Y_{l'0}(\theta_{p0}) \\
&= b_{lm}(t) \sum_{l'} a_{l'0}(t) K_{l'}^0.
\end{aligned}$$

The last term on the RHS of (C3) is

$$\begin{aligned}
& \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 b_{lm}(k, t) \sum_{l'} b_{l'0}(k_1, t) Y_{l'0}(\theta_{p1}) \\
&= b_{lm}(t) \sum_{l'} b_{l'0}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 \\
&\quad \times \Theta(k_+ - k_1) \Theta(k_1 - k_-) Y_{l'0}(\theta_{p1}) \\
&= b_{lm}(t) \sum_{l'} b_{l'0}(t) \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'0}(\theta_{p1}) \\
&= b_{lm}(t) \sum_{l'} b_{l'0}(t) B_{l'}^1,
\end{aligned}$$

where $B_{l'}^1$ are constant coefficients describing the backreaction of photons and are related to associated Legendre polynomials $P_{l'}^0$,

$$\begin{aligned}
B_{l'}^1 &= \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'0}(\theta_{p1}) \\
&= \sqrt{\frac{2l'+1}{4\pi}} \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 P_{l'}^0 \left[1 - \frac{m_a^2}{2kk_1} \right].
\end{aligned}$$

However, backreaction can produce sterile axions with energy higher than $m_a\gamma$ which need to be excluded from axion number counting since we focus on nonrelativistic axions. Dividing the integral interval $[\frac{m_a^2}{4k}, k_+]$ into two parts gives two sets of constant coefficients $N_{l'}^1$ and $S_{l'}^1$ describing backreactions that produce normal axions and sterile axions, respectively, where $N_{l'}^1 + S_{l'}^1 = B_{l'}^1$,

$$\begin{aligned}
N_{l'}^1 &= \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'0}(\theta_{p1}) \\
&= \sqrt{\frac{2l'+1}{4\pi}} \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 P_{l'}^0 \left[1 - \frac{m_a^2}{2kk_1} \right], \\
S_{l'}^1 &= \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 Y_{l'0}(\theta_{p1}) \\
&= \sqrt{\frac{2l'+1}{4\pi}} \int dk \int_{\frac{m_a^2}{4k}}^{\frac{m_a^2}{4k}} dk_1 P_{l'}^0 \left[1 - \frac{m_a^2}{2kk_1} \right].
\end{aligned}$$

4. Evolution equation for number density components

Plugging the previously defined constant coefficients into (C3), we have differential equations about the photon number density components for helicity state λ ,

$$\begin{aligned}
\frac{dn_{lm}^\lambda(t)}{dt} &= \frac{m_a \Gamma_a}{(2\pi)^3} \left\{ \delta_{l0} \delta_{m0} 2\sqrt{\pi} \left[\sum_{l'} a_{l'0}(t) K_{l'}^0 \right. \right. \\
&\quad \left. \left. + \sum_{l''m'} a_{l''m'}(t) b_{l''m'}(t) K_{l''m'}^{01} \right] \right. \\
&\quad \left. + b_{lm}(t) \times \left[\sum_{l'} a_{l'0}(t) K_{l'}^0 - \sum_{l'} b_{l'0}(t) B_{l'}^1 \right] \right\}. \\
\frac{dn_{lm}^\gamma(t)}{dt} &= \frac{2\Gamma_a}{m_a^2 \beta^2} \left\{ \delta_{l0} \delta_{m0} \sqrt{\pi} \left[\sum_{l'} \frac{6}{\beta} K_{l'}^0 n_{l'0}^a(t) \right. \right. \\
&\quad \left. \left. + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l''m'} n_{l''m'}^a(t) n_{l''m'}^\gamma(t) K_{l''m'}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{lm}^\gamma(t)}{m_a^3} \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \right\} \\
&\quad - \frac{3c}{2R} n_{lm}^\gamma(t), \tag{C4}
\end{aligned}$$

The components of occupation number a_{lm} and b_{lm} can be replaced by the components of number density, using (10) and (12),

$$\begin{aligned}
\frac{dn_{lm}^\lambda(t)}{dt} &= \frac{m_a \Gamma_a}{(2\pi)^3} \left\{ \delta_{l0} \delta_{m0} 2\sqrt{\pi} \left[\sum_{l'} \frac{24\pi^3 n_{l'0}^a(t)}{m_a^3 \beta^3} K_{l'}^0 \right. \right. \\
&\quad \left. \left. + \sum_{l''m'} \frac{24\pi^3 n_{l''m'}^a(t)}{m_a^3 \beta^3} \frac{32\pi^3 n_{l''m'}^\lambda(t)}{m_a^3 \beta} K_{l''m'}^{01} \right] \right. \\
&\quad \left. + \frac{32\pi^3 n_{lm}^\lambda(t)}{m_a^3 \beta} \right. \\
&\quad \left. \times \left[\sum_{l'} \frac{24\pi^3 n_{l'0}^a(t)}{m_a^3 \beta^3} K_{l'}^0 - \sum_{l'} \frac{32\pi^3 n_{l'0}^\lambda(t)}{m_a^3 \beta} B_{l'}^1 \right] \right\}. \\
\frac{dn_{lm}^a(t)}{dt} &= -\frac{\Gamma_a}{m_a^2 \beta^2} \left\{ \delta_{l0} \delta_{m0} \sqrt{\pi} \left[\sum_{l'} \frac{6}{\beta} K_{l'}^0 n_{l'0}^a(t) \right. \right. \\
&\quad \left. \left. + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l''m'} n_{l''m'}^a(t) n_{l''m'}^\gamma(t) K_{l''m'}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{lm}^\gamma(t)}{m_a^3} \right. \\
&\quad \left. \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \right\}, \tag{C5} \\
\frac{dn_{lm}^{as}(t)}{dt} &= \frac{\Gamma_a}{m_a^2 \beta^2} \frac{16\pi^3 n_{lm}^\gamma(t)}{m_a^3} \times 2 \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1. \tag{C6}
\end{aligned}$$

Combining common factors simplifies this to

$$\begin{aligned}
\frac{dn_{lm}^\lambda(t)}{dt} &= \frac{2\Gamma_a}{m_a^2 \beta^2} \left\{ \delta_{l0} \delta_{m0} \sqrt{\pi} \left[\sum_{l'} \frac{3}{\beta} K_{l'}^0 n_{l'0}^a(t) \right. \right. \\
&\quad \left. \left. + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l''m'} n_{l''m'}^a(t) n_{l''m'}^\lambda(t) K_{l''m'}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{lm}^\lambda(t)}{m_a^3} \right. \\
&\quad \left. \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 4 \sum_{l'} n_{l'0}^\lambda(t) B_{l'}^1 \right] \right\}.
\end{aligned}$$

We assume that the same component of number density of the photon of each helicity state is the same, meaning that

$$n_{lm}^+(t) = n_{lm}^-(t), \quad n_{lm}^\gamma(t) = 2n_{lm}^\lambda(t).$$

This will give us the evolution equations for the individual components of number density of photon (13), normal axion (14), and sterile axion (15),

APPENDIX D: DETAILS OF THE DERIVATION OF THE EVOLUTION EQUATIONS

Equation (14) for axion number density component $n_{00}^a(t)$ gives

$$\begin{aligned}
\frac{dn_{00}^a(t)}{dt} &= -\frac{\Gamma_a}{m_a^2 \beta^2} \left\{ \sqrt{\pi} \left[\frac{6}{\beta} \sum_{l'} K_{l'}^0 n_{l'0}^a(t) \right. \right. \\
&\quad \left. \left. + \frac{96\pi^3}{m_a^3 \beta^2} \sum_{l''m'} n_{l''0}^a(t) n_{l''0}^\gamma(t) K_{l''m'}^{01} \right] \right. \\
&\quad \left. + \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \right\}.
\end{aligned}$$

But the evolution of $n_{00}^a(t)$ can also be obtained by combining Eqs. (19) and (20),

$$\begin{aligned}
\frac{dn_{00}^a(t)}{dt} &= \frac{16\pi^3 \Gamma_a}{m_a^5 \beta^2} n_{00}^\gamma(t) \times \frac{\sqrt{5} n_{20}^\gamma(t)}{n_{00}^\gamma(t)} \\
&\quad \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right].
\end{aligned}$$

An algebraic relation for $n_{00}^\gamma(t)$ can be derived from these two differential equations, which is

$$\begin{aligned}
& \frac{16\pi^3\Gamma_a}{m_a^5\beta^2} n_{00}^\gamma(t) \times \left[\frac{\sqrt{5}n_{20}^\gamma(t)}{n_{00}^\gamma(t)} + 1 \right] \\
& \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
& = -\frac{\Gamma_a}{m_a^2\beta^2} \sqrt{\pi} \left[\frac{6}{\beta} \sum_{l'} K_{l'}^0 n_{l'0}^a(t) \right. \\
& \left. + \frac{96\pi^3}{m_a^3\beta^2} \sum_{l'l''} n_{l'0}^a(t) n_{l''0}^\gamma(t) K_{l'l''}^{01} \right] \\
& + \frac{16\pi^3}{m_a^3\beta^2} \sum_{l'l''} n_{l'0}^a(t) n_{l''0}^\gamma(t) K_{l'l''}^{01} \Big]. \quad (D1)
\end{aligned}$$

The photon evolution equation (13) for component $n_{00}^\gamma(t)$ gives

$$\begin{aligned}
\frac{dn_{00}^\gamma(t)}{dt} &= \frac{2\Gamma_a}{m_a^2\beta^2} \left\{ \sqrt{\pi} \left[\sum_{l'} \frac{6}{\beta} K_{l'}^0 n_{l'0}^a(t) \right. \right. \\
&+ \frac{96\pi^3}{m_a^3\beta^2} \sum_{l'l''} n_{l'0}^a(t) n_{l''0}^\gamma(t) K_{l'l''}^{01} \Big] \\
&+ \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 \right. \\
&\left. \left. - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \right\} - \frac{3c}{2R} n_{00}^\gamma(t).
\end{aligned}$$

Substituting relation (D1) into this equation, we have

$$\begin{aligned}
\frac{dn_{00}^\gamma(t)}{dt} &= \frac{2\Gamma_a}{m_a^2\beta^2} \left\{ -\frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \times \left[\frac{\sqrt{5}n_{20}^\gamma(t)}{n_{00}^\gamma(t)} + 1 \right] \right. \\
&\times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
&+ \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 \right. \\
&\left. \left. - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \right\} - \frac{3c}{2R} n_{00}^\gamma(t).
\end{aligned}$$

Moving the surface loss term $-\frac{3c}{2R} n_{00}^\gamma(t)$ to the left-hand side (LHS) of the equation, combining common factors, and noting that $B_{l'}^1 = N_{l'}^1 + S_{l'}^1$, we have

$$\begin{aligned}
\left(\frac{d}{dt} + \frac{3c}{2R} \right) n_{00}^\gamma(t) &= \frac{2\Gamma_a}{m_a^2\beta^2} \left\{ -\frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \times \frac{\sqrt{5}n_{20}^\gamma(t)}{n_{00}^\gamma(t)} \right. \\
&\times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) N_{l'}^1 \right] \\
&+ \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \times \left[-2 \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 \right] \Big\}.
\end{aligned}$$

Using $N_{l'}^1 = B_{l'}^1 - S_{l'}^1$ inside the first square bracket leads to

$$\begin{aligned}
\left(\frac{d}{dt} + \frac{3c}{2R} \right) n_{00}^\gamma(t) &= \frac{2\Gamma_a}{m_a^2\beta^2} \left\{ \frac{16\pi^3 [-\sqrt{5}n_{20}^\gamma(t)]}{m_a^3} \right. \\
&\times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] \\
&+ \frac{16\pi^3 [-\sqrt{5}n_{20}^\gamma(t)]}{m_a^3} \times 2 \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 \\
&+ \frac{16\pi^3 n_{00}^\gamma(t)}{m_a^3} \times \left[-2 \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 \right] \Big\}. \quad (D2)
\end{aligned}$$

The photon evolution equation (13) for component $n_{20}^\gamma(t)$ gives

$$\begin{aligned}
\frac{dn_{20}^\gamma(t)}{dt} &= \frac{2\Gamma_a}{m_a^2\beta^2} \times \frac{16\pi^3 n_{20}^\gamma(t)}{m_a^3} \\
&\times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right] - \frac{3c}{2R} n_{20}^\gamma(t).
\end{aligned}$$

Move the surface loss term $-\frac{3c}{2R} n_{20}^\gamma(t)$ to the LHS of the equation,

$$\begin{aligned}
\left(\frac{d}{dt} + \frac{3c}{2R} \right) n_{20}^\gamma(t) &= \frac{2\Gamma_a}{m_a^2\beta^2} \times \frac{16\pi^3 n_{20}^\gamma(t)}{m_a^3} \\
&\times \left[\frac{3}{\beta^2} \sum_{l'} n_{l'0}^a(t) K_{l'}^0 - 2 \sum_{l'} n_{l'0}^\gamma(t) B_{l'}^1 \right]. \quad (D3)
\end{aligned}$$

Substituting Eq. (D3) into (D2) yields

$$\begin{aligned}
\left(\frac{d}{dt} + \frac{3c}{2R} \right) n_{00}^\gamma(t) &= \left(\frac{d}{dt} + \frac{3c}{2R} \right) [-\sqrt{5}n_{20}^\gamma(t)] \\
&- \frac{32\pi^3\Gamma_a}{m_a^5\beta^2} [\sqrt{5}n_{20}^\gamma(t) + n_{00}^\gamma(t)] \\
&\times 2 \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1.
\end{aligned}$$

This means that the quantity $n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)$ evolves based on the following equation:

$$\begin{aligned}
& \frac{d}{dt} [n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)] \\
&= - \left[\frac{64\pi^3\Gamma_a}{m_a^5\beta^2} \sum_{l'} n_{l'0}^\gamma(t) S_{l'}^1 + \frac{3c}{2R} \right] [n_{00}^\gamma(t) + \sqrt{5}n_{20}^\gamma(t)].
\end{aligned}$$

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