Heavy baryon spectrum on the lattice with NRQCD bottom and HISQ lighter quarks

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We determine the mass spectra of heavy baryons containing one or more bottom quarks along with their hyperfine splittings and various mass differences on MILC $2 + 1$ Asqtad lattices at three different lattice spacings. NRQCD action is used for bottom quarks whereas relativistic HISQ action for the lighter up/down, strange, and charm quarks. We consider all possible combinations of bottom and lighter quarks to construct the bottom baryon operators for the states $J^P = 1/2^+$ and $3/2^+$.

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I. INTRODUCTION

Lattice QCD has been extensively employed to study B physics phenomenology, especially the decay constants and mixing parameters needed for CKM matrix elements and the mass differences in the meson sector [1]. The B mesons spectroscopy and mass splittings have undergone thorough investigations on lattice, see [1] and references therein, with an increasing impact on heavy flavor phenomenology; see, for instance, [2]. However, studying heavy baryons with bottom quark(s) on lattice is relatively a recent pursuit. Some of the early studies of heavy baryons on lattice can be found in [3–7]. Of late, a slew of low lying $J^P = 1/2^+$ bottom baryons, such as Λ_b , Σ_b , Ξ'_b , and Ω_b , have made entries in Particle Data Group (PDG) [8]. The possibilities of the discoveries of $J^P = 3/2^+$ are rather high, whereas doubly and triply bottom baryons are right now beyond the reaches of present experiments. In this state, lattice QCD can provide an insight into the masses, mass splittings, and other properties of such bottom baryons from the first principle. To this end, quite a few lattice investigations of heavy baryons containing one, two, or three bottom quarks have been undertaken using a range of light quark actions [9–11]. For an extensive list on contemporary lattice literature on heavy baryons, see [11].

These studies on heavy hadrons with bottom quark(s) are largely made possible by the use of nonrelativistic QCD action, proposed and formulated in [12,13], because of the well-known fact that the current lattice spacings, even for as low as 0.045 fm, render $am_b \gtrsim 1$. Although almost all of

the above studies employed different heavy quark actions for charm quark, HISQ action [14] is becoming an increasingly popular choice for the charm quark. This approach of simulating bottom quark with NRQCD and the rest of the quarks i.e., charm, strange, and up/down with HISQ for calculation of bottom baryon spectra has been adopted in this work.

In this paper, we present our lattice QCD results of heavy baryons involving one and two bottom quarks. We consider all possible combinations of bottom quark(s) with charm, strange, and up/down *lighter* quarks of the form (lbb) , (llb) , and (l_1l_2b) , where b is the bottom quark and l are the lighter charm, strange, and up/down quarks. We are addressing the charm quark as “light” quark in the sense that we have used relativistic action for it. The action for the lighter quarks is HISQ [14] and NRQCD [12,13] for the bottom. We discuss these actions in Sec. II. The propagators generated using nonrelativistic and relativistic actions are required to be combined to construct baryon states of appropriate quantum numbers. A discussion to achieve this combination is spread over both Secs. II and III. The bottom baryon operators are described in details in Sec. III. In the following Sec. IV, we present the simulation details including the lattice ensembles used, various parameters, and tuning of different quark masses. The lattice calculations are carried out at three different lattice spacings with fixed $m_{u/d}/m_s$ value and several quark masses. We assemble our bottom baryon spectrum results along with hyperfine and various other mass splittings in the Sec. V. Finally, we conclude and summarize in Sec. VI, which also includes a comparison of our results to the existing ones.

II. QUARK ACTIONS

As of now, the bottom (b) quark masses are not small, i.e., $am_b \not\ll 1$ in units of the lattice spacings available.

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The use of improved NRQCD is the action of choice for the b quarks. We have used $\mathcal{O}(v^6)$ NRQCD action in this paper. The charm (c) quark is also similarly heavy enough for existing lattices, but the Fermilab proposal [15] made it possible to work with relativistic actions for the c quark, provided we trade the pole mass with the kinetic mass. Subsequently, HISQ action [14] became available for relativistic c quark. In this paper, we choose HISQ action for the c quark along with s and u/d quarks. In this work, as because we use the same relativistic HISQ action for all quarks except the bottom, we use the word *light* quarks to refer to c , s , and u/d quarks. This is similar to what has been done in [16] for the B meson states calculation. Besides, one of the big advantages of this choice of action is the ability to use the MILC code [17] for this bit.

A. NRQCD action and b quark

In order to perform a lattice QCD computation of hadrons containing bottom quarks in publicly available relatively coarse lattices, NRQCD [12,13] is perhaps the most suitable and widely used quark action for the bottom. As is understood, the typical velocity of a b quark inside a hadron is nonrelativistic. Comparison of masses of bottomonium states to the mass of b quark supports the fact that the velocity of b quark inside hadron ($v^2 \sim 0.1$) is much smaller than the bottom mass. For example $M_\Upsilon = 9460$ MeV whereas $2 \times m_b = 8360$ MeV in the $\overline{\text{MS}}$ scheme. For bottom hadrons containing lighter valence quarks, the velocity of the bottom quark is even smaller. This allows us to study the b quark with a nonrelativistic effective field theory. NRQCD will remain the action of choice for the b quark until finer lattices with $am_b < 1$ become widely available.

In NRQCD, the upper and lower components of the Dirac spinor decouple, and the b quark is described by a two component spinor field, denoted by ψ_h . The NRQCD Lagrangian has the following form:

$$\mathcal{L} = \psi_h^\dagger(\vec{x}, t) [U_4(x) \psi_h(\vec{x}, t+1) - \psi_h(\vec{x}, t) + aH\psi_h(\vec{x}, t)], \quad (1)$$

where a is the lattice spacing and $U_4(x)$ is the temporal gauge link operator. $H = H_0 + \delta H$ is the NRQCD Hamiltonian, where

$$H_0 = -\frac{\vec{\Delta}^2}{2m_b} - \frac{a}{4n} \frac{(\Delta^2)^2}{4m_b^2} \quad \text{and} \quad \delta H = \sum_i \delta H^{(i)}. \quad (2)$$

The H_0 is the leading $\mathcal{O}(v^2)$ term; the $\mathcal{O}(v^4)$ and $\mathcal{O}(v^6)$ terms are in δH with coefficients c_1 through c_7 ,

$$\begin{aligned} \delta H^{(1)} &= -c_1 \frac{(\Delta^2)^2}{8m_b^3} \\ \delta H^{(2)} &= c_2 \frac{ig}{8m_b^2} (\vec{\Delta}^\pm \cdot \vec{E} - \vec{E} \cdot \vec{\Delta}^\pm) \\ \delta H^{(3)} &= -c_3 \frac{g}{8m_b^2} \vec{\sigma} \cdot (\vec{\Delta}^\pm \times \vec{E} - \vec{E} \times \vec{\Delta}^\pm) \\ \delta H^{(4)} &= -c_4 \frac{g}{2m_b} \vec{\sigma} \cdot \vec{B} \\ \delta H^{(5)} &= -c_5 \frac{g}{8m_b^3} \{\Delta^2, \vec{\sigma} \cdot \vec{B}\} \\ \delta H^{(6)} &= -c_6 \frac{3g}{64m_b^4} \{\Delta^2, \vec{\sigma} \cdot (\vec{\Delta}^\pm \times \vec{E} - \vec{E} \times \vec{\Delta}^\pm)\} \\ \delta H^{(7)} &= -c_7 \frac{ig^2}{8m_b^3} \vec{\sigma} \cdot \vec{E} \times \vec{E}. \end{aligned} \quad (3)$$

The b quark propagator is generated by the time evolution of this Hamiltonian,

$$\begin{aligned} G(\vec{x}, t+1; 0, 0) &= \left(1 - \frac{aH_0}{2n}\right)^n \left(1 - \frac{a\delta H}{2}\right) U_4(\vec{x}, t)^\dagger \\ &\quad \times \left(1 - \frac{a\delta H}{2}\right) \left(1 - \frac{aH_0}{2n}\right)^n G(\vec{x}, t; 0, 0) \end{aligned} \quad (4)$$

with

$$G(\vec{x}, t; 0, 0) = \begin{cases} 0 & \text{for } t < 0 \\ \delta_{\vec{x}, 0} & \text{for } t = 0 \end{cases}.$$

The tree level value of all the coefficients $c_1, c_2, c_3, c_4, c_5, c_6$, and c_7 is 1. Here, n is the factor introduced to ensure numerical stability at small am_b [13], where $n > 3/2m_b$. The symmetric derivative $\Delta^\pm$ and Laplacian Δ^2 in terms of forward and backward covariant derivatives are

$$\begin{aligned} a\Delta_\mu^+ \psi(x) &= U_\mu(x) \psi(x + a\hat{\mu}) - \psi(x) \\ a\Delta_\mu^- \psi(x) &= \psi(x) - U_\mu^\dagger(x - a\hat{\mu}) \psi(x - a\hat{\mu}) \\ \Delta^\pm &= \frac{1}{2} (\Delta^+ + \Delta^-) \\ \Delta^2 &= \sum_i \Delta_i^+ \Delta_i^- = \sum_i \Delta_i^- \Delta_i^+. \end{aligned} \quad (5)$$

By Taylor expanding the symmetric derivative and the Laplacian operator, we can find their forms corrected up to $\mathcal{O}(a^4)$ [12] that are used in the above Eq. (3),

$$\begin{aligned}\tilde{\Delta}_i^\pm &= \Delta_i^\pm - \frac{a^2}{6} \Delta_i^+ \Delta_i^\pm \Delta_i^- \\ \tilde{\Delta}^2 &= \Delta^2 - \frac{a^2}{12} \sum_i [\Delta_i^+ \Delta_i^-]^2.\end{aligned}\quad (6)$$

In the same way, the gauge fields are improved to $\mathcal{O}(a^4)$ using cloverleaf plaquette,

$$\begin{aligned}a\Delta_\rho^+ F_{\mu\nu}(x) &= U_\rho(x) F_{\mu\nu}(x + a\hat{\rho}) U_\rho^\dagger(x) - F_{\mu\nu}(x) \\ a\Delta_\rho^- F_{\mu\nu}(x) &= F_{\mu\nu}(x) - U_\rho^\dagger(x - a\hat{\rho}) F_{\mu\nu}(x) U_\rho(x - a\hat{\rho}) \\ g\tilde{F}_{\mu\nu}(x) &= gF_{\mu\nu}(x) - \frac{a^4}{6} [\Delta_\mu^+ \Delta_\nu^- + \Delta_\nu^+ \Delta_\mu^-] gF_{\mu\nu}(x).\end{aligned}\quad (7)$$

The chromoelectric $\tilde{E}$ and chromomagnetic $\tilde{B}$ fields in $\delta H^{(3)}$ and $\delta H^{(4)}$ of Eq. (3) are thus $\mathcal{O}(a^4)$ improved.

B. HISQ charm and lighter quarks

For the lighter quarks—charm, strange, and up/down—relativistic HISQ action [14] is used. Apart from anything else, from a practical point of coding the bottom-light operators (lbb , llb , $l_1 l_2 b$), using the same relativistic action for all lighter quarks offers a great degree of simplification. The HISQ action is given by

$$\mathcal{S} = \sum_x \bar{l}(x) (\gamma^\mu D_\mu^{\text{HISQ}} + m) l(x), \quad (8)$$

where

$$D_\mu^{\text{HISQ}} = \Delta_\mu(W) - \frac{a^2}{6} (1 + \epsilon) \Delta_\mu^3(X) \quad (9)$$

with $W_\mu(x) = F_\mu^{\text{HISQ}} U_\mu(x)$ and $X_\mu(x) = \mathcal{U} F_\mu U_\mu(x)$. The F_μ^{HISQ} has the form,

$$F_\mu^{\text{HISQ}} = \left(F_\mu - \sum_{\rho \neq \mu} \frac{a^2 (\delta_\rho)^2}{2} \right) \mathcal{U} F_\mu. \quad (10)$$

Here, the $\mathcal{U}$ is the unitarizing operator, it unitarizes whatever it acts on, and the smearing operator F_μ is given by

$$F_\mu = \prod_{\rho \neq \mu} \left(1 + \frac{a^2 \delta_\rho^{(2)}}{4} \right). \quad (11)$$

The δ_ρ and $\delta_\rho^{(2)}$ in the Eqs. (10) and (11) are covariant first and second order derivatives. Because HISQ action reduces $\mathcal{O}(\alpha_s a^2)$ discretization errors found in Asqtad action, it is well suited for s and u/d quarks. The parameter ϵ in the coefficient of the Naik term can be appropriately tuned to use the action for c quarks. For s and u/d quarks, the $\epsilon = 0$. Later, in the Table IV, we listed the parameters used for HISQ quarks. We have taken the values of ϵ from [16] and used MILC subroutines for generating HISQ propagators.

Since HISQ action is diagonal in spin space, propagators obtained do not have any spin structure. The full 4×4 spin structure can be regained by multiplying the propagators by the Kawamoto-Smit multiplicative phase factor [18],

$$\Omega(x) = \prod_{\mu=1}^4 (\gamma_\mu)^{x_\mu} = \gamma_1^{x_1} \gamma_2^{x_2} \gamma_3^{x_3} \gamma_4^{x_4}. \quad (12)$$

MILC library uses a different representation of γ matrices than the ones used in NRQCD. However, γ matrices of these two representations are related by the unitary transformation of the form,

$$S \gamma_\mu^{\text{MILC}} S^\dagger = \gamma_\mu^{\text{NR}} \quad \text{where, } S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_y & \sigma_y \\ -\sigma_y & \sigma_y \end{pmatrix}. \quad (13)$$

III. TWO-POINT FUNCTIONS

In this section, we discuss the construction of the bottom baryons by combining spin and color indices of the appropriate quark fields to form necessary baryon operators and two-point functions. The b quark field is universally represented with Q throughout the paper. It is defined later in the Eq. (17).

A. Bottom-bottom meson two-point function

After the b quark propagators are generated according to Eq. (4), we calculate the masses of bottomonium states from the exponential falloff of two-point functions, i.e., correlators of the state with quantum numbers of interest. The meson creation operators are constructed from two component quark and antiquark creation operators $\psi_h^\dagger$ and $\chi_h^\dagger$ [19,20]. As antiquarks transform as $\bar{3}$ under color rotation, we rename the antiquark spinor as $\chi_h \equiv \chi_h^*$ [13]. The meson creation operator is, thus,

$$\mathcal{O}_{hh}(x) = \psi_h^\dagger(x) \Gamma \chi_h(x). \quad (14)$$

Heavy-heavy, i.e., bottom-bottom meson two-point function is then given by [13,21]

$$\begin{aligned}C_{hh}(\vec{p}, t) &= \sum_{\vec{x}} \langle \mathcal{O}_{hh}^\dagger(x) \mathcal{O}_{hh}(0) \rangle \\ &= \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \text{Tr}[G^\dagger(x, 0) \Gamma_{\text{sink}}^\dagger G(x, 0) \Gamma_{\text{src}}],\end{aligned}\quad (15)$$

$\Gamma_{\text{sink}} = \Gamma_{\text{src}} = I$ and σ_i for the pseudoscalar and vector mesons, respectively. The heavy-heavy propagator $G(x, 0)$ is a 2×2 matrix in spin space. If we think of $G(x, 0)$ as a 4×4 matrix with vanishing lower components then we can rewrite the above Eq. (15) as [9]

$$C_{hh}(\vec{p}, t) = \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \text{Tr}[\gamma_5 G^\dagger(x, 0) \gamma_5 \Gamma_{\text{sink}}^\dagger G(x, 0) \Gamma_{\text{src}}], \quad (16)$$

where Γ matrices now changed to $\Gamma_{\text{sink}} = \Gamma_{\text{src}} = \gamma_5$ and γ_i for pseudoscalar and vector mesons, respectively. In Eq. (16), we have used the nonrelativistic Dirac representation of γ matrices. In the Eqs. (15), (16), and (19), the trace is taken over both the spin and color indices.

B. Heavy-light meson two-point function

As discussed above, b quark field ψ_b has only two spin components. We convert it to a four-component spinor having vanishing lower components,

$$Q = \begin{pmatrix} \psi_b \\ 0 \end{pmatrix}. \quad (17)$$

This helps us to combine the b and light quark fields in the usual way,

$$\mathcal{O}_{hl}(x) = \bar{Q}(x)\Gamma l(x), \quad (18)$$

where $l(x)$ stands for the light quark fields, $\bar{Q} = Q^\dagger \gamma_4$, and depending on pseudoscalar and vector mesons $\Gamma = \gamma_5$ and γ_i , respectively. Note that in the Dirac, i.e., the NR representation of γ matrices $\gamma_4 Q = Q$. The zero momentum bottom-light two-point function becomes [10,22]

$$\begin{aligned} C_{hl}(t) &= \sum_{\vec{x}} \langle \mathcal{O}_{hl}^\dagger(x) \mathcal{O}_{hl}(0) \rangle \\ &= \sum_{\vec{x}} \text{Tr}[\gamma_5 M^\dagger(x, 0) \gamma_5 \Gamma_{\text{sink}}^\dagger G(x, 0) \Gamma_{\text{src}}], \end{aligned} \quad (19)$$

where $M(x, 0)$ is the light quark propagator. It has the usual full 4×4 spin structure. As before, $G(x, 0)$ is the b quark propagator having vanishing lower components. However, before implementing Eq. (19), $G(x, 0)$ has to be rotated to the MILC basis.

C. Bottom baryon two-point functions

The bottom quark field Q has vanishing lower components and hence, can be projected to positive parity states only. Besides, the use of $\Gamma = C\gamma_5$ in a diquark operator made from same flavor, i.e., $l^T C\gamma_5 l$ is not allowed by the Pauli exclusion principle. In other words, the insertion of $C\gamma_5$ between two quark fields of the same flavor creates a combination which is antisymmetric in spin indices, while the presence of ϵ_{abc} makes the combination antisymmetric in color indices. This makes the overall operator become symmetric under the interchange of the same flavored quark fields. Keeping this in mind, the constructions of various bottom baryon two-point functions are described below.

1. Triply bottom baryon

A triply bottom baryon operator is defined by

$$(\mathcal{O}_k^{hhh})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_k Q^b) Q_\alpha^c, \quad (20)$$

where $C = \gamma_4 \gamma_2$. Here, a, b, c are the color indices, α is the spinor index, and k is the Lorentz index, which runs from 1 to 3. The zero momentum two-point function reads [9]

$$\begin{aligned} C_{jk;\alpha\delta}^{hhh}(t) &= \sum_{\vec{x}} \langle [\mathcal{O}_j^{hhh}(x)]_\alpha [\mathcal{O}_k^{hhh}(0)]_\delta^\dagger \rangle \\ &= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} G_{\alpha\delta}^{ch}(x, 0) \\ &\quad \times \text{Tr}[C \gamma_j G^{bg}(x, 0) \gamma_k \gamma_2 G^{afT}(x, 0)]. \end{aligned} \quad (21)$$

In the above Eq. (21) and the subsequent ones, the transpose and traces are taken over spin indices. Baryon operators having $C\gamma_k$ in the diquark component have overlap with both spin- $\frac{3}{2}$ and $\frac{1}{2}$ states. For example, the correlator defined in Eq. (21) can be written explicitly as an overlap with both spin- $\frac{3}{2}$ and $\frac{1}{2}$ states [23],

$$C_{ij}^{hhh}(t) = Z_{3/2}^2 e^{-E_{3/2}t} \Pi P_{ij}^{3/2} + Z_{1/2}^2 e^{-E_{1/2}t} \Pi P_{ij}^{1/2}, \quad (22)$$

where $\Pi = (1 + \gamma_4)/2$ and the spin projection operators $P_{ij}^{3/2} = \delta_{ij} - \gamma_i \gamma_j / 3$ and $P_{ij}^{1/2} = \gamma_i \gamma_j / 3$. The individual contribution to the respective spin states can be obtained by taking appropriate projections,

$$\begin{aligned} P_{ij}^{3/2} C_{jk}^{hhh} &= Z_{3/2}^2 \Pi e^{-E_{3/2}t} P_{ik}^{3/2} \\ P_{ij}^{1/2} C_{jk}^{hhh} &= Z_{1/2}^2 \Pi e^{-E_{1/2}t} P_{ik}^{1/2}. \end{aligned} \quad (23)$$

In this paper, we use these projections to separate the different spin states. We would like to point out that the spin- $\frac{1}{2}$ state of triply bottom baryon is not a physical state as it violates Pauli exclusion principle even though we can take the projection in practice.

2. Bottom-bottom-light baryon

Interpolating operator for baryons, having two b quarks and a light quark, can be constructed in two ways based on how the diquark component is formed [24],

$$(\mathcal{O}_k^{hhl})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_k Q^b) l_\alpha^c \quad (24)$$

$$(\mathcal{O}_k^{hlh})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_k l^b) Q_\alpha^c. \quad (25)$$

The corresponding baryon correlators are

$$\begin{aligned} C_{jk;\alpha\delta}^{hhl}(t) &= \sum_{\vec{x}} \langle [\mathcal{O}_j^{hhl}(x)]_\alpha [\mathcal{O}_k^{hhl}(0)]_\delta^\dagger \rangle \\ &= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} [M^{ch}(x, 0) \gamma_4]_{\alpha\delta} \\ &\quad \times \text{Tr}[\gamma_4 \gamma_2 \gamma_j G^{bg}(x, 0) \gamma_k \gamma_2 G^{afT}(x, 0)] \end{aligned} \quad (26)$$

$$\begin{aligned}
C_{jk;a\delta}^{hlh}(t) &= \sum_{\vec{x}} \langle [\mathcal{O}_j^{hlh}(x)]_\alpha [\mathcal{O}_k^{hlh}(0)]_\delta^\dagger \rangle \\
&= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} G_{a\delta}^{ch}(x, 0) \\
&\quad \times \text{Tr}[\gamma_4 \gamma_2 \gamma_j M^{bg}(x, 0) \gamma_k \gamma_2 G^{afT}(x, 0)]. \quad (27)
\end{aligned}$$

The propagator $G(x, 0)$ is required to be converted to the MILC basis using the unitary matrix S defined in Eq. (13).

An additional spin- $\frac{1}{2}$ operator can be defined for the $\mathcal{O}^{hlh}$ type operator as

$$(\mathcal{O}_5^{hlh})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_5 l^b) Q_\alpha^c. \quad (28)$$

The two-point function for this operator is obtained by replacing γ_j and γ_k by γ_5 in Eq. (27). We cannot have a $C\gamma_5$ between two Q in diquark and hence, no $\mathcal{O}_5^{hlh}$.

In Table I, we tabulated the full list of triple and double bottom baryon operators that are used in this work. We have broadly followed the nomenclature adopted in [11] but with certain modifications as needed for this work. The baryons having the same quark content and J^P are obtained in two different ways, as mentioned above. The operators with “tilde”, for instance, $\tilde{\Omega}_{bb}(1/2^+)$, are obtained by projecting the relevant $(QC\gamma_k l)Q$ operator with $P_{ij}^{1/2}$. The operators with “prime”, such as $\Omega'_{bb}(1/2^+)$, are obtained from $(QC\gamma_5 l)Q$ diquark construction. The “prime” states so constructed on lattice correspond to the “prime” continuum states, such as Ξ'_b or till unobserved Ω'_{cb} etc. It is obvious that baryon states calculated by projecting out definite spin states from a two-point function share the same interpolating operator. The starred baryons are for $J^P = 3/2^+$ states.

3. Bottom-light baryon

The natural choice for an interpolating operator, as motivated by the Heavy Quark Effective Theory (HQET) [25], in llh -baryon kind is

TABLE I. Operators for triple and double bottom baryons. Q is used for b field and l for any of the $c, s, u/d$ lighter quarks.

Baryon	Quark content	J^P	Operator
Ω_{bbb}	bbb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k Q^b) Q^c$
$\Omega_{cbb}^*, \Omega_{cbb}$	cbb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k Q^b) l^c$
$\tilde{\Omega}_{cbb}^*, \tilde{\Omega}_{cbb}$	cbb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k l^b) Q^c$
Ω'_{cbb}	cbb	$\frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_5 l^b) Q^c$
$\Omega_{bb}^*, \Omega_{bb}$	sbb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k Q^b) l^c$
$\tilde{\Omega}_{bb}^*, \tilde{\Omega}_{bb}$	sbb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k l^b) Q^c$
Ω'_{bb}	sbb	$\frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_5 l^b) Q^c$
Ξ_{bb}^*, Ξ_{bb}	ubb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k Q^b) l^c$
$\tilde{\Xi}_{bb}^*, \tilde{\Xi}_{bb}$	ubb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_k l^b) Q^c$
Ξ'_{bb}	ubb	$\frac{1}{2}^+$	$\epsilon_{abc} (Q^{aT} C \gamma_5 l^b) Q^c$

$$(\mathcal{O}_k^{hl_1 l_2})_\alpha = \epsilon_{abc} (l_1^{aT} C \gamma_k l_2^b) Q_\alpha^c, \quad (29)$$

and the corresponding two-point function is

$$\begin{aligned}
C_{jk;a\delta}^{hl_1 l_2}(t) &= \sum_{\vec{x}} \langle [\mathcal{O}_j^{hl_1 l_2}(x)]_\alpha [\mathcal{O}_k^{hl_1 l_2}(0)]_\delta^\dagger \rangle \\
&= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} G_{a\delta}^{ch}(x, 0) \\
&\quad \times \text{Tr}[\gamma_4 \gamma_2 \gamma_j M_2^{bg}(x, 0) \gamma_k \gamma_2 \gamma_4 M_1^{afT}(x, 0)]. \quad (30)
\end{aligned}$$

In HISQ formalism for light quarks, the corresponding propagators $M_1(x, 0)$ and $M_2(x, 0)$ have the same Kawamoto-Smit multiplicative factor $\Omega(x)$. Hence, they have the same spin structure irrespective of color indices. As a result, the trace over spin indices in Eq. (30) vanishes if $\gamma_j \neq \gamma_k$. Therefore, we can not separate the two spin- $\frac{3}{2}$ and $\frac{1}{2}$ states. If we want to use the same diquark structure as in Eq. (29), for baryons having different light quark flavors, we can define the spin- $\frac{1}{2}$ operator by

$$(\mathcal{O}_5^{hl_1 l_2})_\alpha = \epsilon_{abc} (l_1^{aT} C \gamma_5 l_2^b) Q_\alpha^c, \quad (31)$$

and the corresponding two-point function is

$$\begin{aligned}
C_{55;a\delta}^{hl_1 l_2}(t) &= \sum_{\vec{x}} \langle [\mathcal{O}_5^{hl_1 l_2}(x)]_\alpha [\mathcal{O}_5^{hl_1 l_2}(0)]_\delta^\dagger \rangle \\
&= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} G_{a\delta}^{ch}(x, 0) \\
&\quad \times \text{Tr}[\gamma_4 \gamma_2 \gamma_5 M_2^{bg}(x, 0) \gamma_5 \gamma_2 \gamma_4 M_1^{afT}(x, 0)]. \quad (32)
\end{aligned}$$

Besides, instead of Eq. (29), we can choose our $(hl_1 l_2)$ operator as

$$(\mathcal{O}_k^{hl_2 l_1})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_k l_2^b) l_{1\alpha}^c. \quad (33)$$

The two-point function is now

$$\begin{aligned}
C_{jk;a\delta}^{hl_2 l_1}(t) &= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} [M_1^{ch}(x, 0) \gamma_4]_{a\delta} \\
&\quad \times \text{Tr}[\gamma_4 \gamma_2 \gamma_j M_2^{bg}(x, 0) \gamma_k \gamma_2 G^{afT}(x, 0)]. \quad (34)
\end{aligned}$$

Because the light quark propagators $M_1(x, 0)$ and $M_2(x, 0)$ are proportional to each other, the relative positions of the quark fields l_1 and l_2 in Eq. (33) is irrelevant. The interpolating operator defined in Eq. (33) has overlap with both spin- $\frac{3}{2}$ and $\frac{1}{2}$ states and can be projected out by appropriate projection operators $P_{ij}^{1/2, 3/2}$.

As before, we can also define an additional spin- $\frac{1}{2}$ operator here too,

$$(\mathcal{O}_5^{hl_2 l_1})_\alpha = \epsilon_{abc} (Q^{aT} C \gamma_5 l_2^b) l_{1\alpha}^c. \quad (35)$$

The two-point function for this operator has the same form as in Eq. (34) with γ_j and γ_k replaced by γ_5 .

TABLE II. Operators for single bottom baryons. Q is used for b field as before. Interchange in the position of two lighter quarks keeps the operator unchanged.

Baryon	Quark content	J^P	Operator
$\tilde{\Omega}_{ccb}^*, \tilde{\Omega}_{ccb}$	ccb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k c^b)c^c$
Ω'_{ccb}	ccb	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 c^b)c^c$
Ω_{cb}	scb	$\frac{1}{2}^+$	$\epsilon_{abc}(s^{aT}C\gamma_5 c^b)Q^c$
$\tilde{\Omega}_{cb}^*, \tilde{\Omega}_{cb}$	scb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k c^b)s^c$
Ω'_{cb}	scb	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 c^b)s^c$
Ξ_{cb}	ucb	$\frac{1}{2}^+$	$\epsilon_{abc}(u^{aT}C\gamma_5 c^b)Q^c$
$\tilde{\Xi}_{cb}^*, \tilde{\Xi}_{cb}$	ucb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k c^b)u^c$
Ξ'_{cb}	ucb	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 c^b)u^c$
$\tilde{\Omega}_b^*, \tilde{\Omega}_b$	ssb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k s^b)s^c$
Ω'_b	ssb	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 s^b)s^c$
Ξ_b	usb	$\frac{1}{2}^+$	$\epsilon_{abc}(u^{aT}C\gamma_5 s^b)Q^c$
$\tilde{\Xi}_b^*, \tilde{\Xi}_b$	usb	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k s^b)u^c$
Ξ'_b	usb	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 s^b)u^c$
$\tilde{\Sigma}_b^*, \tilde{\Sigma}_b$	uub	$\frac{3}{2}^+, \frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_k u^b)u^c$
Σ'_b	uub	$\frac{1}{2}^+$	$\epsilon_{abc}(Q^{aT}C\gamma_5 u^b)u^c$
Λ_b	udb	$\frac{1}{2}^+$	$\epsilon_{abc}(u^{aT}C\gamma_5 d^b)Q^c$

In Table II, we tabulate our full list of single bottom baryon operators that we made use of in this work. The “tilde” and “prime” states that appear in the table have been explained before in the context of multi bottom baryon operators.

4. Light baryon

We occasionally need charmed baryon states like qcc or qqc , where q is any of s or u/d quarks or both; hence, we include a discussion on charmed baryon operators. The c quark in the present case is relativistic. For the reason discussed above, with HISQ action for lighter quarks we can define only the spin- $\frac{1}{2}$ operators. Consider a $(l_1 l_2 l_3)$ -baryon where at least two quarks are differently flavored, say $l_1 \neq l_2$. The spin- $\frac{1}{2}$ operator and the corresponding two-point function in such case is

$$(\mathcal{O}_5^{l_1 l_2 l_3})_\alpha = \epsilon_{abc}(l_1^{aT}C\gamma_5 l_2^b)l_3^c_{3\alpha} \quad (36)$$

$$C_{55,a\delta}^{l_1 l_2 l_3}(t) = \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} [M_3^{ch}(x, 0) \gamma_4]_{a\delta} \times \text{Tr}[\gamma_4 \gamma_2 \gamma_5 M_2^{bg}(x, 0) \gamma_5 \gamma_2 \gamma_4 M^{afT}_1(x, 0)]. \quad (37)$$

The two light baryon states ($J^P = 1/2^+$) that we are interested in this work are

$$\begin{aligned} \Sigma_c(uuc): & \epsilon_{abc}(c^{aT}C\gamma_5 u^b)u^c \\ \Xi_{cc}(ucc): & \epsilon_{abc}(c^{aT}C\gamma_5 u^b)c^c. \end{aligned} \quad (38)$$

IV. SIMULATION DETAILS

We calculated the bottom baryon spectra using the publicly available $N_f = 2 + 1$ Asqtad gauge configurations generated by the MILC Collaboration. Details about these lattices can be found in [26]. It uses Symanzik-improved Lüscher-Weisz action for the gluons and Asqtad action [27,28] for the sea quarks. The lattices we choose have a fixed ratio of $am_l/am_s = 1/5$ and lattice spacings ranging from 0.15 to 0.09 fm corresponding to the same physical volume. We have not determined the lattice spacings independently but use those given in [26]. In Table III, we listed the ensembles used in this work.

In NRQCD, the rest mass term does not appear in Eq. (3), and therefore, we cannot determine hadron masses from their energies at zero momentum directly from the exponential falloff of the correlation functions. Instead, we calculate the kinetic mass M_k of heavy-heavy mesons from its energy-momentum relation, which to $\mathcal{O}(p^2)$ is [15],

$$\begin{aligned} E(p) &= E(0) + \sqrt{p^2 + M_k^2} - M_k \\ \Rightarrow E(p)^2 &= E(0)^2 + \frac{E(0)}{M_k} p^2. \end{aligned} \quad (39)$$

We calculate the $E(p)$ at different values of lattice momenta $p = 2n\pi/L$, where, $n = (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0), (2, 1, 0)$, and $(2, 1, 1)$.

A. m_b tuning

The b quark mass is tuned from the spin average Υ and η_b masses,

$$M_{b\bar{b}} = \frac{3M_\Upsilon + M_{\eta_b}}{4}, \quad (40)$$

using kinetic mass for both Υ and η_b . The experimental value to which $M_{b\bar{b}}$ is tuned to is not 9443 MeV, as obtained from spin averaging Υ (9460 MeV) and η_b (9391 MeV) experimental masses, but to an appropriately adjusted value of 9450 MeV [16], which we denote as $M_{\text{phys}}^{\text{mod}}$ later in the Eq. (41). The reasons being, firstly, electromagnetic interaction among the quarks are not considered here. Secondly, the disconnected diagrams while computing a two-point function are also not considered thus, not allowing $b, \bar{b}$ quarks to annihilate to gluons. And

 TABLE III. MILC configurations used in this work. The gauge coupling is β , lattice spacing a , u/d and s sea quark masses are m_l and m_s , respectively, and lattice size is $L^3 \times T$. The N_{cfg} is the number of configurations used in this work.

$\beta = 10/g^2$	a (fm)	am_l	am_s	$L^3 \times T$	N_{cfg}
6.572	0.15	0.0097	0.0484	$16^3 \times 48$	400
6.76	0.12	0.01	0.05	$20^3 \times 64$	400
7.09	0.09	0.0062	0.031	$28^3 \times 96$	300

TABLE IV. Tuned b , c , and s quark bare masses for lattices used in this work. For the s -quark mass, we mentioned the particle states to which it is tuned to. The values of ϵ parameter used for the c quark are given in the last column.

a (fm)	am_b	am_c	$am_s (\eta_s)$	$am_s (B_s)$	ϵ [16]
0.15	2.76	0.850	0.065	0.215	-0.34
0.12	2.08	0.632	0.049	0.155	-0.21
0.09	1.20	0.452	0.0385	0.114	-0.115

finally, we do not have sea c quarks in our simulation. For a more detailed discussion on this, see [16].

The b quark mass m_b and the coefficient c_4 in Eq. (3) are then tuned to obtain a modified spin average mass and the hyperfine splitting of Υ and η_b , which is $\sim 60 - 65$ MeV [29]. In order to achieve the desired hyperfine splittings, we tuned only c_4 since at $\mathcal{O}(1/m_b)$, this is the only term that contains Pauli spin matrices. Therefore, it allows the mixing of spin components of ψ_h . This term contributes maximally to the hyperfine splitting compared to the others that contain Pauli spin matrices. The one-loop radiative correction to c_4 [30] has been used to tune hyperfine splitting in [31], where it was found to change, but only mildly, over lattice spacings $\sim 0.15 - 0.09$ fm for $N_f = 2 + 1 + 1$ HISQ gauge configurations. In our present mixed action study, the changes in the tuned c_4 on various lattice ensembles are small enough. Taking an average of those values, we choose $c_4 = 1.9$ for which the hyperfine splittings obtained on three different lattices 0.15, 0.12, and 0.09 fm are 60.6, 61.1, and 61.8 MeV, respectively.

All other coefficients c_i in Eq. (3) are set to 1.0. We set stability factor $n = 4$ throughout our simulation. The Table IV lists the values of m_b used in this work.

B. m_c tuning

The c -quark mass is tuned pretty much in the same way as m_b , except that $M_{c\bar{c}}$ is tuned to the spin average of J/ψ and η_c experimental masses. In this case, however, the adjustment to the spin averaged value due to the absence of electromagnetic interaction, c quarks in sea, and disconnected diagrams are very small and hence, neglected. The bare c -quark masses used in this work are given in Table IV.

C. m_s tuning

The s -quark mass is tuned to two different values. In the first case, we tune to the mass of fictitious $s\bar{s}$ pseudoscalar meson η_s while in the second case, to the mass of B_s . The η_s is a fictitious meson that is not allowed to decay through $s\bar{s}$ annihilation. Hence, no disconnected diagrams arise in the two-point function calculation. From chiral perturbation theory, its mass is estimated to be $m_{\eta_s} = \sqrt{2m_K^2 - m_\pi^2} = 689$ MeV [32,33]. The s -quark mass thus tuned is checked against D_s meson, making use of the c -quark mass obtained

TABLE V. D and B meson masses in MeV with the tuned am_b , am_c , and am_s .

$L^3 \times T$	B_c	D_s	
		η_s	B_s
$16^3 \times 48$	6260(8)	1994(3)	2197(2)
$20^3 \times 64$	6263(12)	1977(4)	2172(2)
$28^3 \times 96$	6255(10)	1971(3)	2167(2)
PDG [8]	6275	1968	

above and found to agree with the experimental D_s (1968 MeV).

Next, we explore, tuning m_s when a s quark is in a bound state with a heavy b quark. Here, we are assuming that the potential experienced by the s quark in the field of a b quark in a B_s meson remains the same in other strange bottom baryons, and there is no spin-spin interactions taking place between the quarks. In the infinite mass limit, the HQET Lagrangian becomes invariant under arbitrary spin rotations [25]. Thereby, we can argue that for sbb and scb systems the spin-spin interactions do not contribute significantly in spectrum calculation. However, this argument is perhaps not valid in systems like bss or bsd but still with s quark thus tuned; we possibly can obtain their masses close to their physical masses without resorting to any extrapolation.

In this paper, we will present our results obtained at these two different values of m_s . In Table IV, we listed these values of s -quark masses. In the Table V, we calculate B_c and D_s mesons using tuned b , c , and s masses. As is seen, when m_s is tuned with η_s the D_s mass obtained is fairly close to the PDG value, whereas when tuned to B_s , we see an upward shift by an average 200 MeV. We have observed similar differences when s quark appears together with c in (scb)-baryon masses.

D. u/d quarks

For the valence u/d quark mass, we used a range of bare quark masses varying from the lightest sea quark masses all the way to a little above, where the s mass is tuned to B_s . Whenever the mass of a bottom baryon containing u/d quark(s) is quoted, it will correspond to the u/d quark mass tuned at the B mass. Since we are not including either

TABLE VI. Values of $am_{u/d}$ used in this work.

$L^3 \times T$	$am_{u/d}$
$16^3 \times 48$	0.065, 0.10, 0.13, 0.14, 0.155,
(0.15 fm)	0.165, 0.185, 0.215, 0.225
$20^3 \times 64$	0.05, 0.075, 0.090, 0.10,
(0.12 fm)	0.115, 0.13, 0.155, 0.165
$28^3 \times 96$	0.04, 0.07, 0.085, 0.09,
(0.09 fm)	0.10, 0.114, 0.12, 0.13

FIG. 1. Tuning of s and u/d quark masses in various lattices. The experimental values of B^0 and B_s^0 are shown by bands whose thickness are to enhance visibility and have nothing to do with experimental errors.

electromagnetic or isospin breaking in our calculation, we do not distinguish between u and d quarks, and it is always $am_u = am_d$. This tuning of u/d mass to B works well in capturing the b -baryon states containing a single u/d , such as (usb) or (ucb) baryons, when compared to either PDG or other works.

In Table VI, we listed u/d quark masses (am_q) against the lattice spacing. We show, in the Fig. 1, our strategy used to tune m_s and $m_{u/d}$.

The tuned $am_{u/d}$ for different lattices to use with b quarks are

$$16^3 \times 48: 0.165 \quad 20^3 \times 64: 0.115 \quad 28^3 \times 96: 0.085.$$

The s and u/d masses so tuned, $m_s/m_{u/d}$ turns out to be ~ 1.3 compared to ~ 6 that we get when m_s is obtained from η_s and $m_{u/d}$ is the bare sea quark mass.

For states containing two u/d quarks, such as $\Sigma_b(uub)$ and $\Lambda_b(udb)$, we have mixed success with the above approach. When b and u form diquarks ($QC\gamma_{\{k,5\}}u$) for Σ_b state, the masses obtained are consistent with other lattice studies. However, this tuning scheme involving B fails for Λ_b where the diquark part is ($uC\gamma_5 d$) (see the Table II). Hence, for Λ_b containing both u and d quarks, we have to resort to different tuning to account for 190 MeV of mass difference with Σ_b . The mass of Λ_b is obtained from this specially tuned $m'_{u/d}$ (to be used only for Λ_b). Thus, tuned differently, the mass differences $\Xi_b - \Lambda_b$ and $\Sigma_b^* - \Lambda_b$ are well within 2σ of PDG values while $\Lambda_b - B$ is about 60 MeV higher; see the Table XIV.

V. RESULTS AND DISCUSSION

In order to extract the masses of the bottom baryons, we perform a two-exponential uncorrelated fit to the two-point functions. We then cross-checked it with fitting the

FIG. 2. Σ_b , $\tilde{\Omega}_b$ and $\tilde{\Omega}_{ccb}$ correlators in $28^3 \times 96$ lattice.

FIG. 3. Effective mass plots for the states in Fig. 2. The bands are placed over what we consider plateau.

FIG. 4. Variation of single b baryon masses in MeV against the same light quark masses as in Fig. 1. $m_q = 0.085$ and 0.114 are the tuned u/d and s quark masses, respectively, indicated by dashed vertical lines. The bands correspond to the PDG values, except for Ξ_{cb} , which is taken from [11].

effective masses over the same range of time slices. However, this zero momentum energy does not directly give us the mass of the bottom baryons because of unphysical shift in zero of energy. To account for it, the mass is obtained considering energy splittings,

$$M_{\text{latt}} = E_{\text{latt}} + \frac{n_b}{2} (M_{\text{phys}}^{\text{mod}} - E_{\text{latt}}^{n_b}), \quad (41)$$

where E_{latt} is the lattice zero momentum energy in MeV and n_b is the number of b quarks in the bottom baryon. For bottom mesons ($b\bar{l}$), n_b is obviously always 1. As discussed before, $M_{\text{phys}}^{\text{mod}}$ is the modified spin average mass of

Υ and η_b and is equal to 9450 MeV, and M_{latt} is the lattice bottom baryon mass in MeV.

In the calculation of mass splittings, this shift in energies is canceled by subtraction among energies of hadrons having equal number of b quarks (n_b) in them. For this calculation, we use a jackknifed ratio of the correlation functions for fitting [34],

$$C^{Y-X}(t) = \frac{C^Y(t)}{C^X(t)} \sim e^{-(M_Y - M_X)t}. \quad (42)$$

Below in the Fig. 2, we show a few correlators for single b baryons containing exclusively either two c or s or u/d .

TABLE VII. Masses, in lattice unit, of baryons involving single b quark and no s quark. The bare u/d -quark masses are 0.165 for $16^3 \times 48$, 0.115 for $20^3 \times 64$, and 0.085 for $28^3 \times 96$.

Baryons	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average (MeV)
$\tilde{\Omega}_{ccb}^*$	2.954(5)	2.497(4)	2.088(3)	7807(11)
$\tilde{\Omega}_{ccb}$	2.933(5)	2.482(3)	2.078(3)	7780(9)
Ω'_{ccb}	2.952(4)	2.497(3)	2.078(3)	7797(11)
$\tilde{\Xi}_{cb}^*$	2.222(6)	1.899(4)	1.648(6)	6835(20)
Ξ_{cb}	2.177(11)	1.881(4)	1.623(5)	6787(12)
$\tilde{\Xi}_{cb}$	2.199(8)	1.886(4)	1.631(4)	6805(16)
Ξ'_{cb}	2.227(6)	1.904(4)	1.653(6)	6843(19)
$\tilde{\Sigma}_b^*$	1.468(8)	1.292(5)	1.189(5)	5836(22)
$\tilde{\Sigma}_b$	1.460(7)	1.290(3)	1.174(6)	5820(21)
Σ'_b	1.470(7)	1.305(3)	1.194(9)	5848(18)
Λ_b	1.322(7)	1.208(6)	1.109(9)	5667(14)

TABLE VIII. Masses, in lattice unit, of baryons containing single b quark and s quark(s).

Baryons	Tuning	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average (MeV)
$\tilde{\Omega}_{cb}^*$	η_s	2.035(5)	1.782(5)	1.542(3)	6611(9)
	B_s	2.292(7)	1.957(6)	1.693(4)	6930(19)
Ω_{cb}	η_s	2.010(8)	1.754(5)	1.532(3)	6578(9)
	B_s	2.248(11)	1.937(7)	1.684(2)	6893(16)
$\tilde{\Omega}_{cb}$	η_s	2.012(7)	1.765(5)	1.536(3)	6587(10)
	B_s	2.267(8)	1.943(7)	1.686(2)	6906(17)
Ω'_{cb}	η_s	2.052(5)	1.785(5)	1.548(3)	6625(8)
	B_s	2.297(6)	1.966(6)	1.705(2)	6946(17)
$\tilde{\Xi}_b^*$	η_s	0.987(4)	0.945(2)	0.918(3)	5237(8)
	B_s	1.541(8)	1.352(6)	1.235(6)	5935(22)
Ξ_b	η_s	0.986(5)	0.947(2)	0.909(4)	5231(11)
	B_s	1.520(9)	1.345(3)	1.207(6)	5901(20)
$\tilde{\Xi}_b$	η_s	0.978(5)	0.944(2)	0.904(5)	5222(13)
	B_s	1.532(11)	1.350(4)	1.224(4)	5921(19)
Ξ'_b	η_s	0.987(4)	0.948(3)	0.913(5)	5235(11)
	B_s	1.544(10)	1.366(4)	1.238(6)	5946(16)
$\tilde{\Omega}_b^*$	η_s	1.129(5)	1.058(3)	1.012(4)	5430(11)
	B_s	1.611(8)	1.412(6)	1.264(3)	6019(20)
$\tilde{\Omega}_b$	η_s	1.118(7)	1.050(3)	0.997(4)	5410(10)
	B_s	1.600(11)	1.411(7)	1.264(3)	6014(17)
Ω'_b	η_s	1.131(9)	1.057(3)	1.007(2)	5427(9)
	B_s	1.615(8)	1.425(7)	1.295(2)	6051(15)

The fitting range is typically chosen (i) looking at positions of what we consider plateau in the effective mass plots and (ii) exponential fits of the correlators. Both fittings return same masses over suitably chosen range. In the effective mass plot Fig. 3, the zero momentum energies and the errors of the same three states as in the Fig. 2 are represented as bands.

In these figures, we choose to present the data from $28^3 \times 96$ lattices but the data from $16^3 \times 48$ and $20^3 \times 64$ are similar. Just to remind, in order to obtain the masses in MeV from these, we need the Eq. (41).

A. Single bottom baryons

A couple of baryon states containing one b quark have been listed in the PDG [8], such as $\Lambda_b(udb)$, $\Omega_b(ssb)$, $\Xi_b'(usb)$ etc., and they provide a good matching opportunity. In the Fig. 4, we show the agreement of some of these baryon masses with PDG at the tuned $m_{u/d}$ and m_s .

In the following Tables VII, VIII, X, and XI, we present our results of the single and multi b baryon states corresponding to the operators given in the Tables I and II. In the columns corresponding to various lattice ensembles, we show the masses in lattice unit, aE_{latt} of the Eq. (41). In the last column of each table, we provide the average M_{latt} and the statistical errors, calculated assuming the lattice configurations of different lattice spacings are statistically uncorrelated. Additionally, we include Tables XV and XVI for M_{latt} for the bottom baryons for each lattice spacing in the Appendix.

We collect our results for single bottom baryons, not containing s quark(s), in the Table VII and those with s quark(s) in Table VIII. For the $m_{u/d}$, we state the results when the valence m_l gives physical B meson mass.

Since s quark has been tuned in two different ways, we quote both the b -baryon masses at η_s , B_s points.

As is evident from our results, the numbers coming from the s quark tuned to η_s are about 300 MeV smaller from those tuned to B_s (600 MeV in baryons with two s). If we take $\Omega_b(ssb)$ and compare with the PDG value 6046 MeV, it becomes obvious.

FIG. 5. The ratio of correlators for the calculation of the two splittings shown in Table IX. The bands overlaid on the data points represent single exponential fits.

Next, we determine mass differences in single bottom sector, including the hyperfine splittings.

The mass splittings are calculated using a ratio of correlators as given in the Eq. (42). As an example, in Fig. 5, we provide the plots for the ratio of correlators, $\Omega'_b - \Lambda_b$ and $\Xi_b - \Lambda_b$, for comfortable viewing because of their relatively large mass differences; i.e., slopes are prominent and well separated. In the case of smaller differences, for instance, $\tilde{\Omega}_b^* - \tilde{\Omega}_b$ or $\tilde{\Xi}_{cb}^* - \tilde{\Xi}_{cb}$, the slopes of the ratio of correlators are rather small and not quite visible. In the Table IX above, we collect the results of single b baryons mass splittings.

A heavy quark basically acts as a static color source, and therefore, we expect that the hyperfine splittings between states containing single or multiple s and u/d quark(s) to depend only weakly on the tuning of m_s and $m_{u/d}$. For $m_{u/d} \leq 0.085$ and two values of m_s , we show this pattern for a few hyperfine splittings in Fig. 6.

B. Double bottom baryons

For the baryons containing more than one b quark, the data are relatively less noisy than those containing a single

TABLE IX. Single bottom baryons mass splittings in MeV.

Baryon splittings	$16^3 \times 48$ (MeV)	$20^3 \times 64$ (MeV)	$28^3 \times 96$ (MeV)	Average (MeV)
$\tilde{\Omega}_{ccb}^* - \tilde{\Omega}_{ccb}$	28(3)	23(2)	...	26(3)
$\tilde{\Omega}_{cb}^* - \tilde{\Omega}_{cb}$	59(8)	62(13)	61(22)	61(15)
$\tilde{\Xi}_{cb}^* - \tilde{\Xi}_{cb}$	37(6)	44(5)	44(9)	42(7)
$\tilde{\Omega}_b^* - \tilde{\Omega}_b$	29(5)	28(11)	29(4)	29(7)
$\Omega_b' - \Lambda_b$	396(4)	391(9)	406(10)	398(9)
$\Xi_b^* - \Xi_b$	138(20)	122(38)	138(46)	133(36)
$\Xi_b - \Lambda_b$	170(9)	166(11)	163(6)	166(9)
$\Lambda_b - B$	391(20)	431(20)	397(22)	406(21)
$\tilde{\Sigma}_b^* - \tilde{\Sigma}_b$	30(9)	30(8)	29(8)	30(8)
$\tilde{\Sigma}_b^* - \Lambda_b$	224(13)	203(12)	175(13)	201(13)

FIG. 6. Hyperfine splittings at various m_s and $m_{u/d}$ for a selected few bottom baryons on $28^3 \times 96$ lattice. Horizontal bands are the average values of the splittings and are used to guide the eye.

FIG. 7. Ω_{bbb}^* , $\tilde{\Omega}_{cbb}^*$ and $\tilde{\Omega}_{ccb}^*$ effective masses.

b. The effective mass plots in Fig. 7, shown for only $16^3 \times 48$ lattices but similar for two other lattices, are an evidence for this. The fitting ranges are chosen the same way as is done for Fig. 3.

The plot for Ω_{bbb}^* appears counterintuitive since being the heaviest, it is showing a lower mass compared to the other two. However, it receives a large correction because of shift in the rest mass of three b quarks.

We tabulate our results for double bottom nonstrange baryons in the Table X, while those containing a s quark in Table XI.

It is to note that Ω_{bbb} is a spin-3/2 state having no spin-1/2 counterpart. But in practice, we can take a spin-1/2 projection to get such a fictitious state. Therefore, we label the physical (bbb) spin-3/2 state with Ω_{bbb}^* to keep consistency with our remaining notation. In this case, none of the states have PDG entries.

We would like to point out that the variation of the $\Xi_{bb}(ubb)$ masses with $m_{u/d}$ is almost absent as the major contribution to these baryons are coming from the two b quarks. Similarly, from the Table XI, we see that the different tuning of the s quark has significantly less influence on the double bottom baryon masses, a situation unsurprisingly similar to double bottom baryons with a u/d quark.

The splittings in double bottom sector are tabulated in the Table XII.

In the double bottom sector, the splittings between the spin-3/2 and 1/2 states are particularly interesting because HQET relates this mass differences with hyperfine splittings of heavy-light mesons, which in the heavy-quark limit [35],

$$\frac{\Delta M_{bb}^{\text{baryon}}}{\Delta m_b^{\text{meson}}} \rightarrow \frac{3}{4}. \quad (43)$$

This behavior is consistent with our results within errors as can be seen in Table XIII.

A few GMO mass relations involving b quarks are provided in Ref. [36], which we try to verify in this work,

$$M_{\Omega_{ccb}^*} - M_{\Omega_{ccb}} \approx M_{\Omega_{cbb}^*} - M_{\Omega_{cbb}} \quad (44)$$

TABLE X. Triple and double bottom nonstrange baryon masses.

Baryon	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average (MeV)
Ω_{bbb}^*	1.983(4)	2.031(3)	2.154(4)	14403(7)
Ω_{bbb}	1.974(4)	2.023(5)	2.148(4)	14390(8)
Ω_{cbb}^*	2.429(16)	2.259(4)	2.117(3)	11081(21)
Ω_{cbb}	2.409(16)	2.246(5)	2.110(3)	11060(23)
$\tilde{\Omega}_{cbb}^*$	2.431(8)	2.255(4)	2.113(3)	11077(14)
$\tilde{\Omega}_{cbb}$	2.432(10)	2.251(4)	2.113(3)	11075(13)
Ω_{ccb}^*	2.434(8)	2.250(4)	2.114(4)	11076(12)
Ξ_{bb}^*	1.721(12)	1.643(10)	1.666(5)	10103(24)
Ξ_{bb}	1.700(12)	1.640(7)	1.664(5)	10091(17)
$\tilde{\Xi}_{bb}^*$	1.720(12)	1.635(8)	1.668(3)	10100(27)
$\tilde{\Xi}_{bb}$	1.703(16)	1.634(8)	1.661(4)	10087(22)
Ξ'_{bb}	1.704(16)	1.635(10)	1.672(3)	10096(24)

TABLE XI. Double bottom strange baryon spectra.

Baryon	Tuning	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average (MeV)
Ω_{bb}^*	η_s	1.545(11)	1.536(6)	1.576(4)	9902(12)
	B_s	1.791(12)	1.703(11)	1.716(3)	10203(22)
Ω_{bb}	η_s	1.553(9)	1.527(7)	1.570(4)	9896(13)
	B_s	1.768(12)	1.699(8)	1.715(3)	10190(17)
$\tilde{\Omega}_{bb}^*$	η_s	1.542(9)	1.529(7)	1.575(4)	9896(12)
	B_s	1.791(12)	1.693(10)	1.717(3)	10199(28)
$\tilde{\Omega}_{bb}$	η_s	1.541(12)	1.527(7)	1.571(4)	9891(13)
	B_s	1.789(9)	1.695(7)	1.714(3)	10197(24)
Ω'_{bb}	η_s	1.552(9)	1.539(7)	1.578(3)	9908(11)
	B_s	1.782(9)	1.699(7)	1.720(3)	10200(20)

TABLE XII. Double bottom baryon mass splittings in MeV. None of the splittings have PDG entries.

Baryon splittings	$16^3 \times 48$ (MeV)	$20^3 \times 64$ (MeV)	$28^3 \times 96$ (MeV)	Average (MeV)
$\tilde{\Omega}_{cbb}^* - \tilde{\Omega}_{cbb}$	...	25(5)	35(2)	30(5)
$\Omega_{bb}^* - \Omega_{bb}$	34(5)	25(8)	37(9)	32(7)
$\Xi_{bb}^* - \Xi_{bb}$	...	25(4)	39(7)	32(5)

 TABLE XIII. Ratio of hyperfine splittings of doubly heavy baryons to heavy mesons in the heavy quark limit in $28^3 \times 96$ lattice.

Baryon splittings	Our results (MeV)	Meson splittings	Our results (MeV)	Ratio
$\tilde{\Omega}_{bbc}^* - \tilde{\Omega}_{bbc}$	35(2)	$B_c^* - B_c$	46(4)	0.76(4)
$\Omega_{bb}^* - \Omega_{bb}$	37(9)	$B_s^* - B_s$	45(9)	0.82(9)
$\Xi_{bb}^* - \Xi_{bb}$	39(7)	$B^* - B$	47(7)	0.83(8)

$$M_{\Sigma_b^*} - M_{\Sigma_b} \approx M_{\Xi_{bb}^*} - M_{\Xi_{bb}}. \quad (45)$$

For the GMO relation (44), both sides are expected to be approximately 31 MeV. In our case, for $20^3 \times 64$ lattice, for which we have data for both the sides, they are approximately equal but are around 24 MeV as against 31 MeV given in [36]. Our lattice data are also consistent with the approximate GMO relation (45), where each side is about 30 MeV against 20 MeV calculated in [36].

VI. SUMMARY

In this paper, we presented the lattice QCD determination of masses of the baryons containing one or more b quark(s) using NRQCD action for the b quark and HISQ action for the c , s , and u/d quarks. This combination of NRQCD and HISQ has previously been employed in [16] for bottom mesons; however, the exact implementation was rather different. In this work, we converted the one component HISQ propagators to 4×4 matrices by the Kawamoto-Smit transformation and the two component NRQCD propagators to 4×4 matrices using the prescription suggested in [11].

We have discussed the construction of one and two bottom baryon operators in details and pointed out the difficulty for constructing operators motivated by HQET for single bottom baryons. Consequently, we modified the operators accordingly. For some of the baryons, we have multiple operators for the same state, i.e., baryons having the same quantum numbers. It would be natural in such cases to construct correlation matrices and obtain the lowest lying, i.e., ground states by solving the generalized eigenvalue method.

Single bottom baryons can have isodoublets with the same overall quantum numbers J^P . For instance, there exist three isodoublets of Ξ_b , which are not radially or orbitally excited states [37]. These states have been categorized by the spin of the us or ds diquark denoted by j and the spin-parity of the baryon. These baryons are referred to as $\Xi_b(j=0, J^P=\frac{1}{2}^+)$, $\Xi_b'(j=1, J^P=\frac{1}{2}^+)$, and $\Xi_b^*(j=1, J^P=\frac{3}{2}^+)$. The same pattern is observed in Ξ_c states [8]. The mass difference between Ξ_b' and Ξ_b is about 150 MeV. So depending upon the choice of the wave function having the same overall quantum numbers, we can have different baryon states. If we choose $(s^T C \gamma_5 d)Q$ as our $j=0$ baryon operator, then we will be simulating Ξ_b state, and if we

FIG. 8. Comparison of our single bottom baryon spectra with Brown *et al.* [11], Burch [10], Mathur *et al.* [6], and PDG [8], where available.

project out the spin-1/2 state of $j = 1$ operator $(s^T C \gamma_k d)Q$, then we will get the Ξ'_b . For the reason discussed before, we can not define $j = 1$ light-light diquark state. In our case, the wave function corresponds to Ξ'_b is $(Q^T C \gamma_5 s)d$. Constructing an operator in this way allows the s and d quarks to have parallel spin configurations. By simple physical reasoning, we can argue that the explicit construction of $j = 0$ diquark for Ξ_b is more likely to have a significant overlap with physical Ξ_b compared to Ξ'_b ($j = 1$) upon gauge averaging. However, the operator $(Q^T C \gamma_5 s)d$ is expected to have a good overlap with the Ξ'_b state, and this is also supported by our result. For an antiparallel s and d spin configuration, Ξ'_b can also have an overlap with the Ξ_b state. On lattice, operators for states having the same quantum numbers can mix, and therefore, a detailed generalized eigenvalue problem analysis can only resolve the issue of mutual overlap of Ξ_b and Ξ'_b states, which we did not include in this work. This is perhaps the reason we see discrepancies in their values with PDG and others in Fig. 8.

The b mass has been tuned to modified $\Upsilon - \eta_b$ spin average mass, while c quark to $J/\psi - \eta_c$ spin average mass. The s quark is required to be tuned to both the fictitious η_s and B_s mass since we expect the bottom-strange bound state to be more appropriate than the $s - \bar{s}$ bound state in bottom baryons. For the light u/d quarks, we have considered a wide range of bare masses and tune it using B meson. However, this scheme of tuning u/d quarks has not worked for Λ_b . There, u/d are tuned to capture the 190 MeV mass difference $\Sigma_b - \Lambda_b$. This specially tuned $m'_{u/d}$, which is used only for Λ_b , gives it a mass of 5667 MeV. The PDG value for Λ_b mass is 5620 MeV. We demonstrated the variation of bottom baryons as well as hyperfine splittings against varying m_s and $m_{u/d}$. We showed that the hyperfine splittings are almost independent of s and u/d quark masses.

We compare our bottom baryon results with other works, mostly with [6,10,11], in Figs. 8 and 9. NRQCD has been

FIG. 9. Comparison of our triple and double bottom baryon spectra with Brown *et al.* [11] and Burch [10].

the standard action of choice for the b quark in these three cited studies, but the actions used for c are all different—NRQCD, Clover-Wilson, and relativistic heavy quark action [38]. Whatever differences we see in the results for single b baryons with c quark(s), particularly in the cases with two c , possibly have stemmed from the differences in actions. However, in this work, we do not address the systematics involved, which could be significant, because of these differences. (The study of such systematics needed to arrive at phenomenologically relevant numbers will be reported elsewhere.) But otherwise, the results of bottom baryon spectra in the present study with NRQCD b quark and HISQ $c, s, u/d$ quarks appear to agree with each other. We would like to emphasize again that the errors we reported here are only statistical.

The comparison of the hyperfine splittings is shown in Fig. 10.

Apart from the hyperfine splittings, a few other mass splittings calculated in this work are assembled in Table XIV. The bottom baryon spectra and various mass splittings reported in this paper and those appearing in

FIG. 10. Hyperfine splittings of bottom baryons calculated in this work and compared with Brown *et al.* [11], Burch [10], Mathur *et al.* [6], and PDG [8], where available.

TABLE XIV. Bottom baryon mass differences in MeV. PDG values without error are simply the differences of the two states.

Mass splittings	This work	Brown <i>et al.</i> [11]	PDG [8]
$\Omega'_b - \Lambda_b$	398(9)	...	426.4(2.2)
$\Xi_b^* - \Xi_b$	133(36)	189(29)	155.5
$\Xi_b - \Lambda_b$	166(9)	...	172.5(0.4)
$\Lambda_b - B$	406(21)	...	339.2(1.4)
$\Sigma_b^* - \Lambda_b$	201(13)	251(46)	213.5

[8,11] are well comparable given the wide choice of actions and tuning employed in achieving them.

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APPENDIX: BOTTOM BARYON MASSES IN MEV

For the reason of completeness, we provide the bottom baryon masses in MeV for different lattices used in this work. In these tables, we provide only B_s tuned strange bottom baryon masses and not η_s tuned values.

TABLE XV. Single bottom baryon masses in MeV, corresponding to the Tables VII and VIII.

Baryons	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average	Baryons	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average
$\tilde{\Omega}_{ccb}^*$	7816(6)	7794(6)	7805(6)	7807(11)	$\tilde{\Omega}_{cb}^*$	6945(9)	6906(10)	6938(9)	6930(19)
$\tilde{\Omega}_{ccb}$	7788(6)	7769(5)	7782(6)	7780(9)	Ω_{cb}	6887(14)	6873(11)	6919(4)	6893(16)
Ω'_{ccb}	7813(5)	7794(5)	7782(6)	7797(11)	$\tilde{\Omega}'_{cb}$	6912(10)	6883(11)	6923(4)	6906(17)
$\tilde{\Xi}_{cb}^*$	6853(8)	6810(6)	6840(13)	6835(20)	Ω'_{cb}	6951(8)	6921(10)	6965(4)	6946(17)
Ξ_{cb}	6794(14)	6781(6)	6785(11)	6787(12)	$\tilde{\Xi}_b^*$	5957(10)	5911(10)	5934(13)	5935(22)
$\tilde{\Xi}_{cb}$	6822(10)	6789(6)	6802(9)	6805(16)	Ξ_b	5929(12)	5900(5)	5873(13)	5901(20)
Ξ'_{cb}	6859(8)	6819(6)	6851(13)	6843(19)	$\tilde{\Xi}_b$	5945(14)	5908(7)	5910(9)	5921(19)
$\tilde{\Sigma}_b^*$	5861(10)	5812(8)	5833(11)	5836(22)	Ξ'_b	5961(13)	5934(7)	5941(13)	5946(16)
$\tilde{\Sigma}_b$	5850(9)	5809(5)	5800(13)	5820(21)	$\tilde{\Omega}_b^*$	6049(10)	6010(10)	5998(7)	6019(20)
Σ'_b	5864(9)	5834(5)	5844(20)	5848(18)	$\tilde{\Omega}_b$	6035(14)	6008(11)	5998(7)	6014(17)
Λ_b	5669(9)	5674(10)	5658(20)	5667(14)	Ω'_b	6054(10)	6031(11)	6066(4)	6051(15)

TABLE XVI. Triple and double bottom baryon masses in MeV, corresponding to the Tables X and XI.

Baryons	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average	Baryons	$16^3 \times 48$ (0.15 fm)	$20^3 \times 64$ (0.12 fm)	$28^3 \times 96$ (0.09 fm)	Average
Ω_{bbb}^*	14399(5)	14405(5)	14403(9)	14403(7)	Ω_{bb}^*	10217(16)	10177(18)	10216(7)	10203(22)
Ω_{bbb}	14388(5)	14392(8)	14390(9)	14390(8)	$\tilde{\Omega}_{bb}$	10187(16)	10171(13)	10214(7)	10190(17)
Ω_{cbb}^*	11056(21)	11091(6)	11095(7)	11081(21)	$\tilde{\Omega}'_{bb}$	10217(16)	10161(16)	10218(7)	10199(28)
Ω_{cbb}	11029(21)	11070(8)	11080(7)	11060(23)	$\tilde{\Omega}_{bb}$	10214(12)	10164(11)	10212(7)	10197(24)
$\tilde{\Omega}_{cbb}^*$	11058(10)	11085(7)	11086(7)	11077(14)	Ω'_{bb}	10205(12)	10171(11)	10225(7)	10200(20)
$\tilde{\Omega}_{cbb}$	11060(13)	11078(7)	11086(7)	11075(13)					
Ω_{cbb}	11062(10)	11076(7)	11088(9)	11076(12)					
Ξ_{bb}^*	10124(16)	10078(16)	10106(11)	10103(24)					
Ξ_{bb}	10097(16)	10073(11)	10102(11)	10091(17)					
$\tilde{\Xi}_{bb}^*$	10123(16)	10065(13)	10111(7)	10100(27)					
$\tilde{\Xi}_{bb}$	10101(21)	10063(13)	10095(9)	10087(22)					
Ξ'_{bb}	10102(21)	10065(16)	10119(7)	10096(24)					

- [1] C. T. H. Davies, Proc. Sci., LATTICE2011 (**2011**) 019.
- [2] E. Gamiz, C. T. H. Davies, G. P. Lepage, J. Shigemitsu, and M. Wingate, *Phys. Rev. D* **80**, 014503 (2009).
- [3] A. A. Khan, T. Bhattacharya, S. Collins, C. T. H. Davies, R. Gupta, C. Morningstar, J. Shigemitsu, and J. Sloan, *Phys. Rev. D* **62**, 054505 (2000).
- [4] R. Woloshyn, *Phys. Lett. B* **476**, 309 (2000).
- [5] R. Lewis, N. Mathur, and R. M. Woloshyn, *Phys. Rev. D* **64**, 094509 (2001).
- [6] N. Mathur, R. Lewis, and R. M. Woloshyn, *Phys. Rev. D* **66**, 014502 (2002).
- [7] H. Na and S. A. Gottlieb, Proc. Sci., LATTICE2006 (**2006**) 191; LATTICE2007 (**2007**) 124; LATTICE2008 (**2008**) 119.
- [8] M. Tanabashi *et al.* (Particle Data Group), *Phys. Rev. D* **98**, 030001 (2018) and 2019 update.
- [9] S. Meinel, *Phys. Rev. D* **82**, 114514 (2010).
- [10] T. Burch, [arXiv:1502.00675](https://arxiv.org/abs/1502.00675).
- [11] Z. S. Brown, W. Detmold, S. Meinel, and K. Orginos, *Phys. Rev. D* **90**, 094507 (2014).
- [12] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, *Phys. Rev. D* **46**, 4052 (1992).
- [13] B. A. Thacker and G. P. Lepage, *Phys. Rev. D* **43**, 196 (1991).
- [14] E. Follana, Q. Mason, C. Davies, K. Hornbostel, G. P. Lepage, J. Shigemitsu, H. Trotter, and K. Wong, *Phys. Rev. D* **75**, 054502 (2007).
- [15] A. X. El-Khadra, A. S. Kronfeld, and P. B. Mackenzie, *Phys. Rev. D* **55**, 3933 (1997).
- [16] E. B. Gregory, C. T. H. Davies, I. D. Kendall, J. Koponen, K. Wong, E. Follana, E. Gámiz, G. P. Lepage, E. H. Müller, H. Na, and J. Shigemitsu (HPQCD Collaboration), *Phys. Rev. D* **83**, 014506 (2011).
- [17] www.physics.utah.edu/~detar/milc/.
- [18] N. Kawamoto and J. Smit, *Nucl. Phys. B* **192**, 100 (1981).
- [19] C. T. H. Davies, K. Hornbostel, A. Langnau, G. P. Lepage, A. Lidsey, J. Shigemitsu, and J. Sloan, *Phys. Rev. D* **50**, 6963 (1994).
- [20] C. T. H. Davies, K. Hornbostel, G. P. Lepage, A. J. Lidsey, J. Shigemitsu, and J. Sloan, *Phys. Rev. D* **52**, 6519 (1995).
- [21] H. D. Trotter, *Phys. Rev. D* **55**, 6844 (1997).
- [22] M. Wingate, J. Shigemitsu, C. T. H. Davies, G. P. Lepage, and H. D. Trotter, *Phys. Rev. D* **67**, 054505 (2003).
- [23] K. C. Bowler, R. D. Kenway, O. Oliveira, D. G. Richards, P. Ueberholz, L. Lellouch, J. Nieves, C. T. Sachrajda, N. Stella, and H. Wittig (UKQCD Collaboration), *Phys. Rev. D* **54**, 3619 (1996).
- [24] J. M. Flynn, F. Mescia, and A. S. B. Tariq, *J. High Energy Phys.* **07** (2003) 066.
- [25] J. G. Korner, M. Kramer, and D. Pirjol, *Prog. Part. Nucl. Phys.* **33**, 787 (1994).
- [26] A. Bazavov *et al.*, *Rev. Mod. Phys.* **82**, 1349 (2010).
- [27] K. Orginos and D. Toussaint, *Phys. Rev. D* **59**, 014501 (1998).
- [28] K. Orginos, D. Toussaint, and R. L. Sugar, *Phys. Rev. D* **60**, 054503 (1999).
- [29] N. Zerf, Proc. Sci., RADCOR2015 (**2015**) 089.
- [30] T. C. Hammant, A. G. Hart, G. M. von Hippel, R. R. Horgan, and C. J. Monahan, *Phys. Rev. Lett.* **107**, 112002 (2011).
- [31] R. J. Dowdall, C. T. H. Davies, T. C. Hammant, and R. R. Horgan, *Phys. Rev. D* **86**, 094510 (2012).
- [32] C. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, C. DeTar, S. Datta, S. Gottlieb, U. M. Heller, and R. Sugar, *Phys. Rev. D* **64**, 054506 (2001).
- [33] C. T. H. Davies, E. Follana, I. D. Kendall, G. P. Lepage, and C. McNeile, *Phys. Rev. D* **81**, 034506 (2010).
- [34] S. R. Beane, K. Orginos, and M. J. Savage, *Phys. Lett. B* **654**, 20 (2007).
- [35] N. Brambilla, A. Vairo, and T. Röscher, *Phys. Rev. D* **72**, 034021 (2005).
- [36] X.-Z. Weng, X.-L. Chen, and W.-Z. Deng, *Phys. Rev. D* **97**, 054008 (2018).
- [37] R. Aaij *et al.* (LHCb Collaboration), *Phys. Rev. Lett.* **114**, 062004 (2015).
- [38] S. Aoki, Y. Kuramashi, and S. Tominaga, *Prog. Theor. Phys.* **109**, 383 (2003).