Aharonov-Bohm defects

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We discuss what happens when a field that receives an Aharonov-Bohm (AB) phase develops a vacuum expectation value (VEV), using the example of an Alice string in a $U(1) \times SU(2)$ gauge theory coupled with complex triplet scalar fields. We introduce scalar fields belonging to the doublet representation of $SU(2)$, charged or chargeless under the $U(1)$ gauge symmetry, that receives an AB phase around the Alice string. When the doublet develops a VEV, the Alice string becomes a global string in the absence of the interaction depending on the relative phase between the doublet and triplet, while in the presence of such an interaction the Alice string is confined by a soliton or domain wall, and therefore the spatial rotation around the string is spontaneously broken. We call such an object induced by an AB phase an “AB defect,” and argue that such a phenomenon is ubiquitous in various systems.

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I. INTRODUCTION

A gauge potential—as opposed to a (magnetic or electric) field strength—is not merely a mathematical object, but rather a physical quantity, as manifested by the Aharonov-Bohm (AB) effect [1], which is a quantum-mechanical effect occurring when a charged particle scatters from a solenoid with nonzero magnetic flux inside. Although both the magnetic and electric fields are zero everywhere outside the solenoid, a particle going around the solenoid is affected by the gauge potential and picks up a phase, resulting in a nontrivial differential scattering cross section. The AB effect was experimentally observed [2,3] and examined in various nanomaterials such as quantum dots. Nowadays, the AB effect is also studied in various areas of physics, from particle physics, quantum field theory, and string theory to cosmology. In cosmology, AB cosmic strings—i.e., cosmic strings (vortices) exhibiting AB effects—were proposed [4], and the friction they experience due to AB effects was discussed [5,6]. It was also suggested that AB cosmic strings may provide observational evidence of string theory [7,8]. Non-Abelian vortices in supersymmetric gauge theory [9–21] exhibit AB effects [22–24], including a non-Abelian generalization of AB effects [25], once a part of flavor

symmetry is gauged [26]. In dense QCD (which may be relevant for the cores of neutron stars) a color magnetic flux tube in the two-flavor color superconducting phase exhibits AB effects for quarks [27], while a non-Abelian vortex (color magnetic flux tube) in the color-flavor locked phase [28–33] also exhibits (electromagnetic) AB effects for charged particles [34] as well as $\mathbb{Z}_3$ (color) AB effects for quarks [35–38].

An Alice string is one type of string that exhibits nontrivial AB phases. It is a kind of topological vortex that changes the charge sign of particles encircling it [39,40]. The conventional model admitting an Alice string is given by an $SO(3)$ gauge theory with scalar fields belonging to the fiveplet representation (traceless symmetric tensor), in which the $SO(3)$ gauge symmetry is spontaneously broken down to $O(2)$. The unbroken gauge group $O(2)$ is a subgroup inside the full symmetry group and becomes space dependent around the Alice string. The $U(1)$ generator in $O(2)$ —which we identify as electromagnetism—flips the sign after encircling the Alice string once. This property generates an AB phase of charged particles. In the literature, these strings were discussed for purely field-theoretical interests such as a nonlocal charge called a Cheshire charge, non-Abelian statistics, and so on [41–49]. A global analogue of Alice strings was discussed in the context of spinor Bose-Einstein condensates [50–53]. Analogues of Alice strings in string theory were also discussed [54–56]. Recently, a Bogomol’nyi completion [57,58] for an Alice string was achieved in a $U(1) \times SU(2)$ gauge theory coupled with complex triplet scalar fields, and it was shown to be a half Bogomol’nyi-Prasad-Sommerfield (BPS) state in supersymmetric theories [59,60]. The point was that the fundamental group is

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$\pi_1(M) \simeq \mathbb{Z}$, unlike the conventional case of $\pi_1(M) \simeq \mathbb{Z}_2$ where M is a vacuum manifold. Since the energy is saturated by topological charges for BPS states, one can in principle construct multiple string configurations placed at arbitrary positions. This model was in fact a local version of global Alice strings in Refs. [50–53]. Recently, an Alice string was also studied in the context of axion cosmology [61,62], in which the $U(1)$ part in the full symmetry group $U(1) \times SU(2)$ is global, making it a global (axion) string, while the $SU(2)$ part is local.

In this paper, we discuss what happens when a field that receives a nontrivial AB phase develops a vacuum expectation value (VEV), and then find that a topological defect that attaches a string or fluxoid appears, which we call an AB defect. The AB defect can be found in the Georgi-Machacek (GM) model [63] proposed as a model beyond the standard model (BSM), with three real triplet scalar fields and one doublet scalar field. If the triplet VEVs are larger than the doublet VEV, then a $\mathbb{Z}_2$ string is attached by a domain wall, which is nothing but an AB defect [64]. However, to illustrate our mechanism we discuss a simpler example of an Alice string in a $U(1) \times SU(2)$ gauge theory coupled with complex triplet scalar fields. We then introduce doublet scalar fields charged or chargeless under the $U(1)$ group, which receive nontrivial AB phases in the presence of an Alice string. We study the behavior of the Alice string when the doublet fields develop a VEV. We show that a soliton or domain wall attached to the Alice string is inevitably created in the presence of an interaction depending on the relative phase between the doublet and triplet, and therefore the spatial rotation around the string is spontaneously broken. We also show that in the absence of such an interaction, no soliton appears and the Alice string turns into a global string. We also find that the backreaction due to the existence of two coexisting condensates makes the magnetic flux inside the Alice string fractional.

We should note that axion domain walls attached to an axion string (see Ref. [65] for a review) are not AB defects, although configurations of string-wall composites [66,67] themselves look similar to AB defects. Another example of an AB defect can be found in the Georgi-Machacek model [63,64], as mentioned above. We will discuss this in Sec. VI.

This paper is organized as follows. In Sec. II we describe our model of an $SU(2) \times U(1)$ gauge theory with one complex triplet and one doublet scalar field. We consider two different interaction potentials corresponding to the two models for which the doublet field is charged or chargeless under the $U(1)$ gauge symmetry. In Sec. III we briefly review Alice strings in the $SU(2) \times U(1)$ gauge theory coupled with only complex triplet scalar fields. Although BPS-ness is not necessary, we consider a BPS Alice string just for simplicity. We also present AB phases of the charged or chargeless doublet fields. In Sec. IV we discuss the vacuum structure, a global Alice string with a

fractional flux, and an Alice string attached by a soliton in the first model where the doublet field is charged. In Sec. V we discuss the same in the second model where the doublet field is chargeless. Section VI is devoted to a summary and discussion. We argue that the appearance of AB defects is ubiquitous in various field-theoretical and condensed matter models. In the Appendix we give a detailed explanation of our numerical method.

II. THE MODELS

We start with an $SU(2) \times U(1)$ gauge theory coupled with one complex triplet scalar field Φ and one doublet scalar field Ψ . The matter contents are the same as for the BSM triplet Higgs model, but we consider a different phase and parameter region. However, we use the terminology “hypercharge” for the $U(1)$ part and label it as $U(1)_Y$. The hypercharge of the triplet is fixed as $Y = 1$, and we discuss two cases where the doublet fields carry either hypercharge $Y = \frac{1}{2}$ (as the same as the triplet Higgs model) or $Y = 0$.

The Lagrangian is given by

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}\text{Tr}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \text{Tr}|D_\mu\Phi|^2 + |D_\mu\Psi|^2 - V(\Phi, \Psi), \quad (2.1)$$

where $D_\mu\Phi^\alpha = \partial_\mu\Phi^\alpha - ie a_\mu\Phi^\alpha + g\epsilon^{\alpha\beta\gamma}A_\mu^\beta\Phi^\gamma$, $D_\mu\Psi = \partial_\mu\Psi - i\frac{e_Y}{2}a_\mu\Psi - iA_\mu\Psi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Here e and g are the coupling constants of the $U(1)_Y$ and $SU(2)$ gauge fields, respectively, where e_Y takes values $\{e, 0\}$ for the two different cases of the doublet field. The potential term is given by

$$V(\Phi, \Psi) = V_\Phi(\Phi) + V_\Psi(\Psi) + V_{\text{int}}(\Phi, \Psi). \quad (2.2)$$

The potentials for the triplet and doublet fields, given by

$$V_\Phi(\Phi) = \frac{\lambda_g}{4}\text{Tr}[\Phi, \Phi^\dagger]^2 + \frac{\lambda_e}{2}(\text{Tr}\Phi\Phi^\dagger - 2\xi^2)^2, \quad (2.3)$$

$$V_\Psi(\Psi) = M^2\Psi^\dagger\Psi + \lambda_\Psi(\Psi^\dagger\Psi)^2, \quad (2.4)$$

respectively, are common for the two models, where λ_g and λ_e are two couplings of the triplet, ξ is a parameter that gives the triplet a VEV, M is the bare mass of the doublet, and λ_Ψ is the quartic coupling of the doublet field. As for the interaction term V_{int} between the doublet and triplet fields, we consider

$$V_{\text{int}}^{(1)}(\Phi, \Psi) = \mu^{(1)}(\Psi_c^\dagger\Phi^*\Psi + \Psi^\dagger\Phi\Psi_c) + \lambda_1\Psi^\dagger\Psi\text{Tr}(\Phi^\dagger\Phi) + \lambda_2\Psi^\dagger\Phi^\dagger\Phi\Psi, \quad (2.5)$$

$$V_{\text{int}}^{(2)}(\Phi, \Psi) = \mu^{(2)}|\Psi_c^\dagger\Phi^*\Psi|^2 + \lambda_1\Psi^\dagger\Psi\text{Tr}(\Phi^\dagger\Phi) + \lambda_2\Psi^\dagger\Phi^\dagger\Phi\Psi \quad (2.6)$$

for the two models, where $\mu^{(1)}$ and $\mu^{(2)}$ are parameters of terms depending on the relative phase between the doublet and triplet fields (which are different for the two models), and λ_1 and λ_2 are other couplings between these fields (which are the same for the two models). The charge conjugation of the doublet field is defined as $\Psi_c = i\sigma^2\Psi^*$.

III. ALICE STRING AND AHARONOV-BOHM PHASES AROUND IT

A. Alice string solution: A review

In this section, we give a brief review of BPS Alice strings [59,60] in the absence of the doublet field. So in this case, we set all of the scalar couplings to zero except for λ_g and λ_e . To construct a vortex solution we write the static Hamiltonian as

$$H = \int d^3x \left[\frac{1}{2} \text{Tr} F_{ij} F^{ij} + \frac{1}{4} f_{ij} f^{ij} + \text{Tr} |D_i \Phi|^2 + \frac{\lambda_g}{4} \text{Tr} [\Phi, \Phi^\dagger]^2 + \frac{\lambda_e}{2} (\text{Tr} \Phi \Phi^\dagger - 2\xi^2)^2 \right]. \quad (3.1)$$

We choose the vacuum expectation value of the field Φ as

$$\langle \Phi \rangle_v = 2\xi \tau^1, \quad (3.2)$$

with $\tau^1 = \frac{1}{2}\sigma^1$. This vacuum is different than that generally used in the BSM triplet Higgs model. This triplet vacuum breaks the gauge symmetry group spontaneously as

$$G = U(1)_Y \times SU(2) \rightarrow H = \mathbb{Z}_2 \ltimes U(1)_1 \simeq O(2), \quad (3.3)$$

where $\ltimes$ stands for a semidirect product. Here, in $U(1)_1$ the suffix “1” stands for rotation around τ^1 . Following Eq. (3.2), we notice that any rotation around τ^1 keeps $\langle \Phi \rangle_v$ invariant. Simultaneous π rotations in the $U(1)_Y$ group and around an axis along any linear combination of τ^3 and τ^2 keep $\langle \Phi \rangle_v$ invariant, since both of the π rotations generate sign changes separately. This defines the unbroken discrete group $\mathbb{Z}_2$, and the elements of the unbroken group are defined as

$$H = \{(1, e^{i\frac{g}{2}\sigma^1}), (-1, i(c_2\sigma^2 + c_3\sigma^3)e^{i\frac{g}{2}\sigma^1})\}, \quad (3.4)$$

where c_2 and c_3 are arbitrary real constants normalized to unity as $c_2^2 + c_3^2 = 1$. The semidirect product implies that the $U(1)_1$ acts on the $\mathbb{Z}_2$. The vacuum manifold is

$$\frac{G}{H} = \frac{U(1) \times SO(3)}{O(2)} \simeq \frac{S^1 \times S^2}{\mathbb{Z}_2}. \quad (3.5)$$

The fundamental group for this symmetry-breaking process can be calculated as

$$\pi_1\left(\frac{G}{H}\right) \simeq \mathbb{Z}. \quad (3.6)$$

This nontrivial fundamental group indicates the existence of stable strings. The generator of the unbroken $U(1)$ changes sign as it encircles the string once, identifying this vortex as an Alice string.

The scalar and gauge field configurations at a large distance from an Alice string can be expressed as

$$\Phi(R, \theta) \sim \xi e^{i\frac{\theta}{2}} \begin{pmatrix} 0 & e^{i\frac{\theta}{2}} \\ e^{-i\frac{\theta}{2}} & 0 \end{pmatrix} = \xi e^{i\frac{\theta}{2}} e^{i\frac{\theta}{4}\sigma^3} \sigma^1 e^{-i\frac{\theta}{4}\sigma^3},$$

$$A_i \sim -\frac{1}{4g} \frac{\epsilon_{ij} x^j}{r^2} \sigma^3, \quad a_i \sim -\frac{1}{2e} \frac{\epsilon_{ij} x^j}{r^2}, \quad (3.7)$$

with angular coordinate θ and system size R . At $\theta = 0$ (along the x^1 axis) the scalar field takes its vacuum value $\Phi(R, \theta = 0) = \xi \sigma^1$, and the order parameter at any arbitrary θ is given by a holonomy action as

$$\Phi(R, \theta) \sim e^{ie \int \mathbf{a} \cdot \mathbf{dl}} P e^{ig \int \mathbf{A} \cdot \mathbf{dl}} \Phi(R, 0) P e^{-ig \int \mathbf{A} \cdot \mathbf{dl}} \sim U_0(\theta) U_3(\theta) \Phi(R, 0) U_3^{-1}(\theta), \quad (3.8)$$

where the holonomies are defined as

$$U_0(\theta) = e^{ie \int_0^\theta \mathbf{a} \cdot \mathbf{dl}} = e^{i\frac{\theta}{2}} \in U(1),$$

$$U_3(\theta) = P e^{ig \int_0^\theta \mathbf{A} \cdot \mathbf{dl}} = e^{i\frac{\theta}{4}\sigma^3} \in SU(2). \quad (3.9)$$

These can be understood by using the topological vortex condition, where the order parameter Φ is covariantly constant at large distances ($\mathcal{D}_i \Phi \rightarrow 0$ as $R \rightarrow \infty$). Since the scalar field configuration of the vortex is space dependent, the unbroken group generators also change around the vortex. In the case of the Alice string, as we encircle the string by an angle θ , the unbroken $U(1)$ generator Q changes as

$$Q_\theta = U_3(\theta) Q_0 U_3(\theta)^{-1}, \quad U_3(\theta) = e^{i\frac{\theta}{4}\sigma^3}, \quad (3.10)$$

where the value of the generator at $\theta = 0$ is $Q_0 = \tau^1$. This is true because Q_θ keeps Eq. (3.7) invariant. Now it is interesting to notice that the generator changes its sign after one complete encirclement,

$$Q_{2\pi} = -Q_0. \quad (3.11)$$

This is the most remarkable feature of the Alice string.

To find the minimum tension, the Bogomol'nyi completion can be performed by considering the critical couplings $\lambda_e = e^2$ and $\lambda_g = g^2$. The Bogomol'nyi completion of the tension—that is, the static energy per a unit length—is found to be

$$\begin{aligned}
\mathcal{T} &= \int d^2x \left[\text{Tr} \left[F_{12} \pm \frac{g}{2} [\Phi, \Phi^\dagger] \right]^2 + \text{Tr} |\mathcal{D}_\pm \Phi|^2 \right. \\
&\quad \left. + \frac{1}{2} [f_{12} \pm e(\text{Tr} \Phi \Phi^\dagger - 2\xi^2)]^2 \pm 2ef_{12}\xi^2 \right] \\
&\geq 2e\xi^2 \left| \int d^2x f_{12} \right| = 2\pi\xi^2,
\end{aligned} \tag{3.12}$$

with $\mathcal{D}_\pm \equiv \frac{D_1 \pm iD_2}{2}$. Here, to obtain the second line we have used the most severe bound for either choice of the sign so that the bound is inside the absolute value. Also, there are total derivative terms before the integration that disappear upon integration and do not contribute to the energy. The BPS equations are

$$\begin{aligned}
F_{12} \pm \frac{g}{2} [\Phi, \Phi^\dagger] &= 0, \\
\mathcal{D}_\pm \Phi &= 0, \\
f_{12} \pm e(\text{Tr} \Phi \Phi^\dagger - 2\xi^2) &= 0.
\end{aligned} \tag{3.13}$$

Now the vortex solutions can be constructed by starting with a vortex ansatz,

$$\begin{aligned}
\Phi(r, \theta) &= \xi \begin{pmatrix} 0 & f_1(r)e^{i\theta} \\ f_2(r) & 0 \end{pmatrix}, \\
A_i(r, \varphi) &= -\frac{1}{4g} \frac{\epsilon_{ij}x_j}{r^2} \sigma^3 A(r), \quad a_i(r, \theta) = -\frac{1}{2e} \frac{\epsilon_{ij}x_j}{r^2} a(r),
\end{aligned} \tag{3.14}$$

where $\{r, \theta\}$ are radial and angular coordinates of the two-dimensional space, respectively. The profile functions $f_1(r), f_2(r), A(r)$, and $a(r)$ depend only on the radial coordinate, and they can be solved numerically with the boundary conditions $f_1(0) = f_2'(0) = 0, f_1(\infty) = f_2(\infty) = 1, A(0) = a(0) = 0$, and $A(\infty) = a(\infty) = 1$.¹ The numerical solution for the BPS equation (3.13) is displayed in Fig. 1.

As can be seen from Eq. (3.7), the vortex solution is generated by the σ_3 generator. This could be σ_2 or in general $c_2\sigma_2 + c_3\sigma_3$, with $c_2^2 + c_3^2 = 1$. Therefore, the Alice string solution is parametrized by a $U(1)$ modulus, corresponding to the flux that it carries inside its core.

The BPS completion helps us to embed our Alice string into an $\mathcal{N} = 1$ supersymmetric theory [59], in which BPS Alice strings are shown to be 1/2 BPS, preserving half of the supersymmetries.² In this case, BPS solitons belong to short multiplets of supersymmetry and are

¹These profile functions eventually satisfy the same equations as those for a non-Abelian vortex in $U(N)$ gauge theory coupled with N Higgs scalar fields in the fundamental representation [9–21].

²While supersymmetric theories with the same matter content were studied in Ref. [68], Alice properties were not recognized there.

FIG. 1. The numerical solutions of the profile functions $f_1(r)$, $f_2(r)$, $1-a(r)$, and $1-A(r)$ for a vortex configuration of winding number one as a function of r [59].

quantum-mechanically stable [69]. In this paper, however, the BPS-ness is not necessary.

B. The Aharonov-Bohm phase around an Alice string

1. The first model: Charged doublet

Since we have an unbroken $U(1)_1$ symmetry in the bulk, there may exist massless particles in the bulk that are charged under the unbroken $U(1)_1$. These particles may interact with Alice strings, which may generate AB phases. To realize this, let us first insert scalar particles with a doublet representation without a potential. In this case, the doublet scalar field interacts at low energy with only the unbroken $U(1)$ gauge field in the bulk since the other $SU(2)$ gauge fields are massive due to the large VEV of Φ . However, the Alice string's properties generate a nontrivial AB phase due to the unbroken $\mathbb{Z}_2$ defined in Eq. (3.3). To confirm this, let us put the fields Φ, A_i , and a_i in their string configurations found in the Alice string analysis; then, the doublet field changes around the vortex as

$$\begin{aligned}
\Psi_\theta &= e^{i\frac{\theta}{2} \int_0^\theta a \cdot dl} \left(P e^{ig \int_0^\theta A \cdot dl} \right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&= e^{i\frac{\theta}{2} \int_0^\theta a \cdot dl} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2} \int_0^\theta a \cdot dl} \psi_1 \\ \psi_2 \end{pmatrix}.
\end{aligned} \tag{3.15}$$

As can be noticed, that after a complete encirclement the doublet field gets an AB phase as

$$\Psi_{2\pi} = e^{i\frac{\theta}{2} \oint a \cdot dl} \left(P e^{ig \oint A \cdot dl} \right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.16}$$

This has interesting physical consequences. Using the eigenvalues of the $U(1)_1$ generator, we may define charges as $Q = \sigma^1$, since it is the only gauge symmetry group that

lives in the bulk or far away from the vortex core. So we describe the system using eigenstates of charge operator as

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (3.17)$$

According to Eq. (3.16), when the charged state $|+\rangle$ encircles the vortex it becomes $|-\rangle$. Therefore, the charge-conjugation symmetry cannot be globally defined.

2. The second model: Chargeless doublet

Since we have chargeless doublet scalar particles in the bulk, these particles exhibit an AB phase only from the non-Abelian $SU(2)$ gauge field configuration of the Alice string. The doublet field changes around the vortex as

$$\Psi_\theta = (Pe^{ig \int_0^\theta A \cdot dl}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{i\frac{\theta}{4}\sigma^3} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{4}}\psi_1 \\ e^{-i\frac{\theta}{4}}\psi_2 \end{pmatrix}. \quad (3.18)$$

As can be noticed, after a complete encirclement the doublet field receives an AB phase as

$$\Psi_{2\pi} = Pe^{ig \oint A \cdot dl} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = i \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}. \quad (3.19)$$

According to Eq. (3.19), when the charge state $|+\rangle$ encircles the vortex, it becomes $-i|-\rangle$. In this case, the positive charge transforms into a negative one and acquires an overall π phase.

IV. AHARONOV-BOHM DEFECTS IN THE FIRST MODEL WITH A CHARGED DOUBLET

So far we have discussed the AB phase of the doublet field around an Alice string. Here and in the next section, we answer the question of what happens when such a field with a nontrivial AB phase acquires a VEV. In this section, we discuss this situation in the first model in the presence of a charged doublet scalar field with $e_\psi = \frac{1}{2}$.

A. The charged doublet potential and symmetry breaking

We now switch on the full potential in Eq. (2.2), including the interaction $V_{\text{int}}^{(1)}(\Phi, \Psi)$. The potential generates a nonzero VEV for the doublet field, $\langle \Psi \rangle \neq 0$, if we choose a negative bare mass term, $M^2 < 0$. As we will see below, the doublet field breaks the unbroken symmetry group $O(2)$ either completely or into a $\mathbb{Z}_2$ subgroup, depending on the parameter choice. To understand the vacuum in the presence of the large hierarchy $\langle |\Phi| \rangle \gg \langle |\Psi| \rangle$, we set the triplet field in its vacuum value as $\Phi = \xi\sigma^1$ and substitute it into the potential in Eq. (2.2):

$$V(\Phi, \Psi) = [M^2 + \xi^2(2\lambda_1 + \lambda_2)]\Psi^\dagger\Psi + \lambda_\psi(\Psi^\dagger\Psi)^2 + \xi\mu^{(1)}(\Psi_c^\dagger\sigma^1\Psi + \Psi^\dagger\sigma^1\Psi_c). \quad (4.1)$$

We assume the condition $-M^2 > 2\xi^2(\lambda_1 + \lambda_2)$ to trigger the symmetry breaking.

In the case of $\mu^{(1)} = 0$, the potential has an enlarged $SO(4)$ symmetry and the vacuum manifold is S^3 , defined by

$$|\psi_1|^2 + |\psi_2|^2 = -\frac{M^2 + \xi^2(2\lambda_1 + \lambda_2)}{2\lambda_\psi} = v_2^2, \quad (4.2)$$

where we have inserted $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

In the case of $\mu^{(1)} \neq 0$, we insert $\Psi_c = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}$ (where $\psi_1 = \psi_{11} + i\psi_{12}$ and $\psi_2 = \psi_{21} + i\psi_{22}$) into the potential to find

$$V(\Phi, \Psi) = -(2\lambda_\psi v_2^2 + 2\xi\mu^{(1)})(\psi_{11}^2 + \psi_{22}^2) + 2(\xi\mu^{(1)} - \lambda_\psi v_2^2)(\psi_{12}^2 + \psi_{21}^2) + \lambda_\psi(\psi_{11}^2 + \psi_{22}^2 + \psi_{12}^2 + \psi_{21}^2)^2. \quad (4.3)$$

The symmetry of this potential is reduced from $SO(4)$ to $U(1) \times U(1)$. We now discuss two cases separately, depending on the sign of the second term: case I ($\lambda_\psi v_2^2 < \xi\mu^{(1)}$) and case II ($\lambda_\psi v_2^2 > \xi\mu^{(1)}$).

In case I, the vacuum manifold is S^1 described by

$$\psi_{11}^2 + \psi_{22}^2 = \frac{(\lambda_\psi v_2^2 + \xi\mu^{(1)})}{\lambda_\psi} = \xi_{\psi_1}^2, \quad \psi_{12} = \psi_{21} = 0. \quad (4.4)$$

The fact that the potential is invariant under $S^1 \times S^1$ can be understood as follows. The actual $SO(4)$ symmetry in the $\mu^{(1)} = 0$ case is realized by taking into account the full symmetry group as $SU(2)_L \times SU(2)_R$. In the case of $\mu^{(1)} \neq 0$, the symmetry transformations $U_L = e^{i\frac{\alpha}{2}\sigma^1} \in SO(2) \subset SU(2)_L$, $e^{i\frac{\beta}{2}\sigma^1} \in SU(2)_R$ act on (Ψ, Ψ_c) as

$$\begin{aligned} (\Psi, \Psi_c) &\rightarrow U_L(\alpha)(\Psi, \Psi_c)U_R(\beta) \\ &= \begin{pmatrix} \cos \frac{\alpha}{2} & i \sin \frac{\alpha}{2} \\ i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \psi_1 & \psi_2^* \\ \psi_2 & -\psi_1^* \end{pmatrix} \\ &\quad \times \begin{pmatrix} \cos \frac{\beta}{2} & i \sin \frac{\beta}{2} \\ i \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix}. \end{aligned} \quad (4.5)$$

The components are

$$\begin{aligned}
\psi'_{11} &= \cos \frac{\alpha - \beta}{2} \psi_{11} + \sin \frac{\alpha - \beta}{2} \psi_{22}, \\
\psi'_{12} &= \cos \frac{\alpha + \beta}{2} \psi_{12} + \sin \frac{\alpha + \beta}{2} \psi_{21}, \\
\psi'_{22} &= \cos \frac{\alpha - \beta}{2} \psi_{22} - \sin \frac{\alpha - \beta}{2} \psi_{11}, \\
\psi'_{21} &= \cos \frac{\alpha + \beta}{2} \psi_{21} - \sin \frac{\alpha + \beta}{2} \psi_{12}.
\end{aligned} \quad (4.6)$$

Here $U_L(\alpha)$ is a gauge transformation, whereas $U_R(\beta)$ is a global symmetry transformation. We have two circles parametrized by the two parameters $\frac{\alpha - \beta}{2}$ and $\frac{\alpha + \beta}{2}$. In the vacuum where $\psi_{12} = \psi_{21} = 0$, we have an unbroken global $U(1)$ if we choose a gauge where $\alpha = \beta$. We denote the unbroken $U(1)$ group as the color-flavor locked $U(1)_{L+R}$ parametrized by α .

In case II, the potential in Eq. (4.3) with $\lambda_\psi v_2^2 > \xi \mu^{(1)}$ yields the vacua³

$$\psi_{11}^2 + \psi_{22}^2 = \frac{(\lambda_\psi v_2^2 + \xi \mu^{(1)})}{\lambda_\psi} = \xi_\psi^2, \quad \psi_{12} = \psi_{21} = 0. \quad (4.7)$$

These vacua S^1 are the same as those in Eq. (4.4) for case I.

For both cases, the vacua S^1 are parametrized by $\frac{\alpha - \beta}{2}$. If we set $\alpha = -\beta = \gamma$ as a gauge choice, the generic vacua are found to be

$$\Psi_\gamma = \xi_\psi \begin{pmatrix} \cos \gamma \\ -i \sin \gamma \end{pmatrix}. \quad (4.8)$$

We set our vacuum at $\gamma = 0$ in the following for simplicity.

To calculate the VEVs of the triplet and doublet in the case of $\mu^{(1)} = 0$, we may write the potential in terms of both VEVs as

$$\begin{aligned}
V(|\Phi|_v, |\Psi|_v) &= |\Phi|_v^4 \lambda_e \left(1 - \frac{\xi^2}{|\Phi|_v^2} \right)^2 - m^2 |\Psi|_v^2 + \lambda_\psi |\Psi|_v^4 \\
&\quad + (2\lambda_1 + \lambda_2) |\Psi|_v^2 |\Phi|_v^2.
\end{aligned} \quad (4.9)$$

Here we have defined $M^2 = -m^2$. By minimizing this potential, we find that the solution is

$$\begin{aligned}
|\Phi|_v &= \sqrt{\frac{4\lambda_e \lambda_\psi \xi^2 - (2\lambda_1 + \lambda_2) m^2}{4\lambda_e \lambda_\psi - (2\lambda_1 + \lambda_2)^2}}, \\
|\Psi|_v &= \sqrt{\frac{2\lambda_e [m^2 - \xi^2 (2\lambda_1 + \lambda_2)]}{4\lambda_e \lambda_\psi - (2\lambda_1 + \lambda_2)^2}}.
\end{aligned} \quad (4.10)$$

³In addition to the true vacua, the Hamiltonian has two more critical points: $\psi_{12} = \psi_{21} = 0, \psi_{11} = \psi_{22} = 0$ and $\psi_{12}^2 + \psi_{21}^2 = (\lambda_\psi v_2^2 - \xi \mu^{(1)})/\lambda_\psi, \psi_{11} = \psi_{22} = 0$. It can be checked by computing the determinant of a Hessian matrix that the first is a local maximum and the second gives saddle points.

B. The noninteractive case: A global vortex with a fractional flux

In the last subsection we discussed the symmetries of the potential in the case in which the triplet is set to its vacuum value $\Phi_v = \xi \sigma^1$ and studied the vacuum of the doublet in this case. Now we are going to ask what happens when we set the triplet field configuration to an Alice string. To understand the situation, we first consider $\xi \gg \xi_\psi$ to remove backreaction. In this case, the Alice string behaves as a background and we study a nonzero VEV Ψ_v of the doublet field in this background. We thus substitute the large-distance Alice string configurations in Eq. (3.7) into the action.

However, as we have discussed, the doublet field gets an AB phase in the presence of an Alice string. Then, the VEV Ψ_v of the doublet would be non-single-valued after one encirclement of the Alice string, which is inconsistent. To overcome this problem, when $\mu^{(1)} = 0$ the system assumes an energetically favorable configuration, in which the AB phase of the doublet is nullified by a global transformation of the doublet. So the doublet does not get any winding even if it is involved in the formation of a vortex. More precisely, the first component of the doublet field ($|\psi_1|_v$) receives the winding that arises due to the AB phase, but it should be canceled by a simultaneous global rotation. Let us imagine the following two steps (although they occur simultaneously in practice). In the first step, we take the doublet at $\theta = 0$ as $\Psi|_{\theta=0} = \frac{|\Psi|_v}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If we go along a path that encircles the Alice string, it would acquire an AB phase to become $\Psi_v(\theta) = \frac{|\Psi|_v}{\sqrt{2}} \begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix}$, which is not single valued. In the second stage, a global rotation is created to make the doublet single valued (constant).

In order to construct a full numerical vortex solution, we may take into account the backreaction due to the interaction between the doublet and triplet fields. In this case, we use the VEVs of the scalar fields in Eq. (4.10). We consider the vortex ansatz

$$\begin{aligned}
\Phi(r, \theta) &= |\Phi|_v \begin{pmatrix} 0 & f_1(r) e^{i\theta} \\ f_2(r) & 0 \end{pmatrix}, \quad \Psi_v = |\Psi|_v \begin{pmatrix} \psi(r) \\ 0 \end{pmatrix}, \\
A_i(r, \theta) &= -\frac{\mathcal{C}}{4g} \frac{\epsilon_{ij} x_j}{r^2} \sigma^3 A(r), \quad a_i(r, \theta) = -\frac{\mathcal{C}}{2e} \frac{\epsilon_{ij} x_j}{r^2} a(r),
\end{aligned} \quad (4.11)$$

where $\{r, \theta\}$ are the radial and angular coordinates of the two-dimensional space, respectively. The profile functions $f_1(r), f_2(r), A(r), a(r)$, and $\psi(r)$ depend only on the radial coordinate and they can be solved numerically with the boundary conditions $f_1(0) = f_2'(0) = \psi(0) = 0, f_1(\infty) = f_2(\infty) = \psi(\infty) = 1, A(0) = a(0) = 0$, and $A(\infty) = a(\infty) = 1$. $\mathcal{C}$ is the fractional flux arising due to backreaction, which becomes $\mathcal{C} \simeq 1$ when $\xi \gg v_2$. To understand the effect of backreaction on fluxes, we insert the

FIG. 2. Profile functions of a global vortex with a fractional flux. Panel (a): The parameter choices are $\lambda_g = 1, \lambda_1 = 0.03, \lambda_2 = 0.03, \lambda_e = 1, \lambda_\psi = 5, m = 1, \xi = 2, e = 0.05$, and $g = 0.1$. Panel (b): $\lambda_g = 0.2$, and the remaining parameters are the same as in panel (a). Panel (c): $\lambda_g = 1, \lambda_1 = \lambda_2 = 0$, and the remaining parameters are the same as in panels (a) and (b). One may notice that $\psi(r)$ has a trivial shape if $\lambda_1 = \lambda_2 = 0$.

large-distance configurations of the scalar fields into the Hamiltonian, without assuming any fixed form for the gauge fields. We insert $A_i = A_i^3 \tau^3$ and the gradient terms of the scalar fields to give

$$\mathcal{H}_{\text{grad}} \sim |\Phi|_v^2 (\partial_i \theta - (ea_i + gA_i^3))^2 + |\Phi|_v^2 (ea_i - gA_i^3)^2 + \frac{|\Psi|_v^2}{4} (ea_i + gA_i^3)^2. \quad (4.12)$$

After minimizing, we find the solution

$$ea_i + gA_i^3 = \frac{|\Phi|_v^2 \partial_i \theta}{|\Phi|_v^2 + \frac{|\Psi|_v^2}{4}}, \quad ea_i = gA_i^3. \quad (4.13)$$

We thus find

$$\mathcal{C} = \frac{|\Phi|_v^2}{|\Phi|_v^2 + \frac{|\Psi|_v^2}{4}}. \quad (4.14)$$

This makes the Abelian and non-Abelian fluxes fractional,

$$\oint a \cdot dl = \frac{\pi |\Phi|_v^2}{e(|\Phi|_v^2 + \frac{|\Psi|_v^2}{4})}, \quad \oint A^3 \cdot dl = \frac{\pi |\Phi|_v^2}{g(|\Phi|_v^2 + \frac{|\Psi|_v^2}{4})}. \quad (4.15)$$

We thus have found that the fractional nature of fluxes is due to the existence of the doublet VEV.

Now we write the equations of motion of all profile functions as

$$\begin{aligned} & -r \frac{\partial}{\partial r} \left(\frac{A'}{r} \right) + 4g^2 |\Phi|_v^2 \left[\left(\frac{a+A}{2} - \frac{1}{\mathcal{C}} \right) f_1^2 + \frac{(A-a)}{2} f_2^2 \right] + g^2 |\Psi|_v^2 (a+A) \psi^2 = 0, \\ & -r \frac{\partial}{\partial r} \left(\frac{a'}{r} \right) + 4e^2 |\Phi|_v^2 \left[\left(\frac{a+A}{2} - \frac{1}{\mathcal{C}} \right) f_1^2 + \frac{(a-A)}{2} f_2^2 \right] + e^2 |\Psi|_v^2 (a+A) \psi^2 = 0, \\ & -\frac{1}{r} \frac{\partial}{\partial r} (r f_1') + \left(1 - \mathcal{C}^2 \frac{a+A}{2} \right)^2 \frac{f_1}{r^2} + |\Phi|_v^2 \left[\lambda_g (f_1^2 - f_2^2) f_1 + \frac{\lambda_e}{2} \left(f_1^2 + f_2^2 - 2 \frac{\xi^2}{|\Phi|_v^2} \right) f_1 \right] \\ & + (\lambda_1 + \lambda_2) |\Psi|_v^2 \psi^2 f_1 = 0, \\ & -\frac{1}{r} \frac{\partial}{\partial r} (r f_2') + \mathcal{C}^2 \frac{(a-A)^2}{4r^2} f_2 + |\Phi|_v^2 \left[\lambda_g (f_2^2 - f_1^2) f_2 + \frac{\lambda_e}{2} \left(f_1^2 + f_2^2 - 2 \frac{\xi^2}{|\Phi|_v^2} \right) f_2 \right] + \lambda_1 |\Psi|_v^2 \psi^2 f_2 = 0, \\ & -\frac{1}{r} \frac{\partial}{\partial r} (r \psi') + \frac{\mathcal{C}^2}{8r^2} (a+A)^2 \psi + [m^2 + 2\lambda_\psi |\Psi|_v^2 \psi^2 + \lambda_1 (f_1^2 + f_2^2) |\Phi|_v^2 + \lambda_2 f_1^2 |\Phi|_v^2] \psi = 0. \end{aligned} \quad (4.16)$$

Numerical solutions of these equations are portrayed in Fig. 2.

C. The interactive case: An Alice string confined by a soliton

So far we discussed AB phases, symmetry-breaking patterns, and a global vortex due to the nonzero VEV of the

doublet in the case of $\mu^{(1)} = 0$ (in the absence of the interaction depending on the relative phase between the triplet and doublet). In this subsection, we switch on $\mu^{(1)}$ and discuss what happens for the global vortex discussed in the last subsection.

Let us first discuss the case where $\xi \gg \xi_\psi$ to remove the backreaction of the doublet VEV to the triplet; we will

discuss the general case later. In this case, the Alice string is heavy and behaves as a background. So we set the large-distance configurations defined in Eq. (3.7) as our background configuration. In the presence of an Alice string, the doublet field receives an AB phase, which makes the doublet VEV non-single-valued after one encirclement of the Alice string, as discussed in the previous subsection. In the case of $\mu^{(1)} \neq 0$, here we propose that the system assumes an energetically favorable configuration that generates a domain wall or soliton to preserve the single-valuedness of the doublet VEV. This can be understood by introducing an additional phase $\phi(x)$ of the doublet to ensure the single-valuedness of the doublet field. $\phi(x)$ changes as the doublet encircles the Alice string together with the AB phase. We thus have the following ansatz for the doublet:

$$\Psi_{\text{DW}} \sim \xi_{\psi} \begin{pmatrix} e^{i(\frac{\theta+\phi(\theta)}{2})} \\ 0 \end{pmatrix}, \quad \phi(0) = 0, \quad \phi(2\pi) = -2\pi, \quad (4.17)$$

where $\phi(\theta)$ is a decreasing function and the boundary condition keeps the doublet single valued (constant). We substitute this ansatz into the Hamiltonian density in the presence of the Alice string configuration in Eq. (3.7) at large distances, where the potentials are given in Eqs. (2.4) and (2.5). We thus obtain the effective Hamiltonian of the doublet,

$$\mathcal{H}_{\text{DW}}/\xi_{\psi}^2 \sim \frac{1}{4} [(\partial_i \phi)^2 + 8\mu_{\psi}^2 (1 - \cos \phi)], \quad \mu_{\psi}^2 = \xi \mu, \quad (4.18)$$

which is nothing but the sine-Gordon model. With the boundary condition for ϕ in Eq. (4.17), we inevitably encounter a single kink $\phi(x) = 4 \tan^{-1} e^{\pm 2\mu_m x}$, with energy density per unit area $T_{\text{DW}} = 4\mu_{\psi} \xi_{\psi}^2$.

To confirm our claim, we solve the full two-dimensional equations of motion numerically by the relaxation method, as shown in Fig. 3. To do this computation, we have used a 500×500 square lattice with a lattice spacing of 0.2. The details of the numerical method can be found in the Appendix. Since the wall pulls the Alice string, this

FIG. 3. A full two-dimensional numerical simulation for an Alice string confined by a soliton in the first model with a charged doublet. Each panel is for a different gauge invariant. (a) The gauge invariant $4\text{Tr}\Phi^2\text{Tr}\Phi^{\dagger 2}$, which is zero at the center of the Alice string; if we insert our Alice string ansatz into it, it is $\sim \xi^4 f(r)^2 g(r)^2$. (b) The same plot as in panel (a) in the smaller range 256.4–255 to clearly show the existence of a domain wall. (The white circular region at the center lies outside the range.) (c) The interaction potential $V_{\text{int}}^{(1)} + \frac{2\xi m^2}{\lambda_{\psi}}$, in which the domain wall is clearly visible. (d) The energy density $\mathcal{H}(x, y)$, in which the domain wall is not clearly visible because of the very low intensity. (e) The same plot as in panel (d) in the smaller range 0.05–0.5 to show the domain wall. (f) The flux squared $\text{Tr}F_{12}^2 = \sum_{a=1}^3 F_{12}^a{}^2$, in which the confined magnetic field is clearly visible. The parameter choices are $\xi = 0.1, e = 0.5, g = 1, \mu^{(1)} = 0.1, \lambda_{\psi} = 1, m = 0.1, \lambda_e = 2$, and $\lambda_g = 1$.

configuration continues to move. However, the shape is converged after some relaxation time. Figure 3 shows such a configuration.

It may be interesting to emphasize that the spatial rotation around the string is spontaneously broken once the interaction term proportional to $\mu^{(1)}$ is introduced.⁴

V. AHARONOV-BOHM DEFECTS IN THE SECOND MODEL WITH A CHARGELESS DOUBLET

In this section we consider the second model with a chargeless doublet with the potential term in Eq. (2.6).

A. The chargeless doublet potential and symmetry breaking

Here we study the vacuum of the doublet field while keeping the triplet in its original vacuum, when the triplet VEV is much larger than the doublet VEV. Following Eq. (4.2) and inserting the vacuum configuration of the triplet ($\Phi = \xi\sigma^1$) into the potential, we find that the full potential is

$$V(\Phi, \Psi) = -v_2^2 2\lambda_\psi (|\psi_1|^2 + |\psi_2|^2) + \lambda_\psi (|\psi_1|^2 + |\psi_2|^2)^2 + \xi^2 \mu^{(2)} |\psi_1^2 - \psi_2^2|^2, \quad (5.1)$$

with $v_2^2 = \frac{-M^2 + \xi^2(2\lambda_1 + \lambda_2)}{2\lambda_\psi}$. The vacuum manifold is found to be

$$|\psi_1| = |\psi_2| = \pm \frac{v_2}{\sqrt{2}}, \quad \text{or} \quad \Psi_v = \frac{v_2 e^{i\beta}}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \quad (5.2)$$

At the vacua the fields ψ_1 and ψ_2 are in the same phase or have a phase difference of π . The doublet VEV breaks $O(2)$ completely, including $\mathbb{Z}_2 \in O(2)$.

We thus can expect a domain wall configuration interpolating between $|+\rangle$ and $|-\rangle$, and we call this an ‘‘Alice wall.’’ Differently from the first model, this model admits a topologically stable domain wall.

B. The noninteractive case: A global vortex with a fractional flux

Here we discuss the case where $\mu^{(2)} = 0$ and follow the same procedure we discussed before. Similarly to the analysis discussed in the previous section, there exists a global rotation of the doublet field that cancels the AB phase of the doublet. We add a single profile function for it,

$$\begin{aligned} \Phi(r, \theta) &= |\Phi|_v \begin{pmatrix} 0 & f_1(r)e^{i\theta} \\ f_2(r) & 0 \end{pmatrix}, & \Psi_v &= \frac{|\Psi|_v}{\sqrt{2}} \psi(r) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ A_i(r, \varphi) &= -\frac{\mathcal{C}}{4g} \frac{\epsilon_{ij} x_j}{r^2} \sigma^3 A(r), & a_i(r, \theta) &= -\frac{\mathcal{C}}{2e} \frac{\epsilon_{ij} x_j}{r^2} a(r). \end{aligned} \quad (5.3)$$

The profile functions $f_1(r)$, $f_2(r)$, $A(r)$, $a(r)$, and $\psi(r)$ depend only on the radial coordinate r with the boundary conditions $f_1(0) = f_2'(0) = \psi(0) = 0$, $f_1(\infty) = f_2(\infty) = \psi(\infty) = 1$, $A(0) = a(0) = 0$, and $A(\infty) = a(\infty) = 1$. The constants $|\Phi|_v$, $|\Psi|_v$, and $\mathcal{C}$ were defined before, and they remain the same in this case. The equations of motion can be written as

$$\begin{aligned} -r \frac{\partial}{\partial r} \left(\frac{A'}{r} \right) + 4g^2 |\Phi|_v^2 \left[\left(\frac{a+A}{2} - \frac{1}{\mathcal{C}} \right) f_1^2 + \frac{(A-a)}{2} f_2^2 \right] + g^2 |\Psi|_v^2 A \psi^2 &= 0, \\ -r \frac{\partial}{\partial r} \left(\frac{a'}{r} \right) + 4e^2 |\Phi|_v^2 \left[\left(\frac{a+A}{2} - \frac{1}{\mathcal{C}} \right) f_1^2 + \frac{(a-A)}{2} f_2^2 \right] &= 0, \\ -\frac{1}{r} \frac{\partial}{\partial r} (r f') + \left(1 - \mathcal{C}^2 \frac{a+A}{2} \right)^2 \frac{f_1}{r^2} + |\Phi|_v^2 \left[\lambda_g (f_1^2 - f_2^2) f_1 + \frac{\lambda_e}{2} \left(f_1^2 + f_2^2 - 2 \frac{\xi^2}{|\Phi|_v^2} \right) f_1 \right] \\ + (\lambda_1 + \lambda_2) |\Psi|_v^2 \psi^2 f_1 &= 0, \\ -\frac{1}{r} \frac{\partial}{\partial r} (r f_2') + \mathcal{C}^2 \frac{(a-A)^2}{4r^2} f_2 + |\Phi|_v^2 \left[\lambda_g (f_2^2 - f_1^2) f_2 + \frac{\lambda_e}{2} \left(f_1^2 + f_2^2 - 2 \frac{\xi^2}{|\Phi|_v^2} \right) f_2 \right] \\ + \lambda_1 |\Psi|_v^2 \psi^2 f_2 &= 0, \\ -\frac{1}{r} \frac{\partial}{\partial r} (r \psi') + \frac{\mathcal{C}^2}{8r^2} A^2 \psi + [m^2 + 2\lambda_\psi |\Psi|_v^2 \psi^2 + \lambda_1 (f_1^2 + f_2^2) |\Phi|_v^2 + \lambda_2 f_1^2 |\Phi|_v^2] \psi &= 0. \end{aligned} \quad (5.4)$$

⁴The terminology ‘‘spontaneous symmetry breaking’’ is usually used for the vacuum or ground states, which are usually uniform. However, the notation is also used for nontrivial backgrounds such as topological solitons. In the latter case, the Nambu-Goldstone modes are localized around the solitons and attain moduli of the solitons.

FIG. 4. The numerical solution for the profile functions of the global vortex in the second model. The parameter choices are $\lambda_g = 1, \lambda_1 = 0.01, \lambda_2 = 0.01, \lambda_e = 1, \lambda_\psi = 5, m = 1, \xi = 2, e = 0.05$, and $g = 0.1$.

We solve these equations numerically, and the solution is shown in Fig. 4.

C. The interactive case: An Alice string confined by a domain wall

Since the hypercharge of the doublet is zero in this case, the AB phase of the doublet receives a contribution only from the non-Abelian gauge field around the Alice string

while encircling around the vortex, as in Eq. (3.19). The AB phase is a phase difference of π between ψ_1 and ψ_2 , and so in this case the doublet also loses its single-valuedness. Therefore, this system should create a domain wall around the vortex to recover the single-valuedness of the doublet. To understand the existence of the domain wall, we add another phase ϕ to regain its single-valuedness, similarly to the previous case. Here, we keep the Alice string away from the backreaction for simplicity. We write the domain wall ansatz as

$$\Psi_{\text{DW}} \sim \frac{|\Psi|_v}{\sqrt{2}} e^{i\frac{\theta+\phi(\theta)}{4}\sigma^3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \phi(0) = 0, \quad \phi(2\pi) = -2\pi, \quad (5.5)$$

with a decreasing function ϕ . Since the interaction term (the $\mu^{(2)}$ term) is quadratic in the triplet field, we should fix its magnitude so that it remains much lower than the bare mass term of the triplet. We write the full potential (we set $\lambda_1 = \lambda_2 = 0$ for simplicity) and insert the doublet ansatz along with the Alice string configuration in Eq. (3.7) into the potential to find

$$V(\Phi, \Psi) = 2\lambda_e (|\Phi|_v^2 - \xi^2)^2 + \frac{\lambda_4}{4} [|\Psi|_v^2 - v_2^2]^2 + \frac{\mu^{(2)}}{2} |\Phi|_v^2 |\Psi|_v^4 (1 - \cos \phi). \quad (5.6)$$

FIG. 5. An Alice string confined by a domain wall in the second model with a chargeless doublet. The plots are the same as in Fig. 3. The parameter choices are $\xi = 0.1, e = 0.5, g = 1, \mu^{(2)} = 100, \lambda_\psi = 1, m = 0.1, \lambda_e = 2$, and $\lambda_g = 0.5$. Panels (a)–(f) are for the same quantities as in Fig. 3.

From the above expression we fix the coefficients with the relation $4\lambda_e \xi^2 > \mu |\Psi|_v^4$ to ensure that the quadratic term of the triplet is always negative. Now we set $|\Psi|_v = v_2$ and $|\Phi|_v = \xi$, and the effective Hamiltonian is found to be

$$16\mathcal{H}/v_2^2 = (\partial_i \phi)^2 + 2m_0^2(1 - \cos \phi), \quad m_0^2 \equiv 4\mu^{(2)}\xi^2 v_2^2, \quad (5.7)$$

which is the sine-Gordon model, and a kink is inevitable from the boundary condition of ϕ in Eq. (5.5). The solution is the well-known kink soliton $\phi(x) = 4 \tan^{-1} e^{\pm m_0 x}$ with energy per unit area $T_{\text{DW}} = m_0 v_2^2 = 2\sqrt{\mu^{(2)}}\xi v_2^3$. In this case, the nontrivial element of $\mathbb{Z}_2$ appears as the AB phase and it is unavoidable since $\psi_1 = \pm \psi_2 \neq 0$ in the vacua.

To confirm our claim, we solve the full two-dimensional equations of motion numerically by the relaxation method, as shown in Fig. 5. The numerical settings are the same as in the first model.

It is again interesting to note that the spatial rotation around the string is spontaneously broken due to the interaction term proportional to $\mu^{(2)}$.

VI. SUMMARY AND DISCUSSION

In this paper we have examined the question of what happens when fields receiving an AB phase develop a VEV. To concretely study this problem, we considered an Alice string in the $SU(2) \times U(1)$ gauge theory with complex triplet scalar fields, and introduced doublet scalar fields that are either charged or chargeless for the $U(1)$ gauge group. The doublet scalar fields receive nontrivial AB phases when encircling around the Alice string in both models. We found that when the doublet develops a VEV the Alice string becomes a global string with fractional flux for both models in the absence of an interaction, depending on the relative phase between the doublet and triplet fields, while the Alice string is attached by a soliton and domain wall in the first and second model, respectively, in the presence of an interaction. The interaction terms also spontaneously break the spatial rotation around the Alice string. For both models, we examined this first by showing that the relative phase between the doublet and triplet fields is reduced to the sine-Gordon model at large distances from the Alice string, and we further confirmed this using full two-dimensional numerical simulations.

Although a sine-Gordon kink is attached to the Alice string in both models, the interpretation of the kink seems to be different for the two models. It is a nontopological soliton in the first model, while it is a topological domain wall connecting two disconnected vacua in the second model. This difference may be responsible for the stability of the soliton or domain wall.

The first model is related to the so-called triplet Higgs model proposed as a BSM model, for which one adds a triplet Higgs scalar field in addition to the doublet Higgs

scalar field of the SM. However, in the realistic case, the VEV of the triplet Higgs scalar field should be much smaller than the VEV of the doublet Higgs scalar field, in order to be consistent with the so-called ρ parameter, while we have considered an inverse hierarchy in this paper. In addition to this, the phases of the triplet fields are different for the two cases: a triplet VEV spontaneously breaks the $SU(2)_W \times U(1)_Y$ down to $U(1)_{W+Y}$ in the ferromagnetic phase, and to $O(2)$ for the polar phase. The former is relevant for the SM, while the latter corresponds to our model admitting an Alice string. Both cases assume the same model but in different parameter regions. Although our parameter region is not realistic in the current Universe, it could be relevant in the early Universe.

Although we have considered particular models admitting an Alice string, our conclusion seems to hold in more general cases. Namely, when fields receiving AB phases develop VEVs, a domain wall or soliton will be created in order for the fields to be single valued, or a string will become a global string with a fractional flux. We call such a soliton or domain wall induced by AB phases an ‘‘AB defect.’’

One example is the Georgi-Machacek model [63] (which was also proposed as a BSM model) which has three real triplet scalar fields in addition to the Higgs doublet. In this model, the hierarchy of the VEVs of the triplet and doublet Higgs fields can be interchanged to be consistent with the ρ parameter. If only the triplet fields develop a VEV at high energy, it admits a $\mathbb{Z}_2$ string, and if the doublet also develops a VEV at low energy, a domain wall is attached to it [64], similarly to our model.⁵ The difference with the current study is that the $\mathbb{Z}_2$ string in that case is not an Alice string, but rather a non-Abelian string carrying $\mathbb{CP}^1$ moduli in the limit of exact custodial symmetry. The $\mathbb{CP}^1$ moduli of the domain wall and string match at the junction line.

Another example is given by two-gap (or two-component) superconductors, which can be described by a $U(1)$ gauge theory with two charged complex scalars (gaps) Φ_1 and Φ_2 coupled to each other by a Josephson term $\Phi_1^* \Phi_2 + \text{c.c.}$ In fact, fractional fluxes were first found in this case [74, 75]. Let us first assume a VEV $v_1 = \langle \Phi_1 \rangle$ only for the first component, and consider a vortex in the first component Φ_1 . Then, Φ_2 receives an AB phase around the vortex. If the second component Φ_2 develops a small VEV $v_2 = \langle \Phi_2 \rangle$ at low energy with a hierarchy $v_1 \gg v_2$ between the VEVs, an AB defect arises attached to the vortex. A salient feature of this case is that the AB defect

⁵As far as we understand, the two Higgs doublet model (2HDM)—which is a more popular extension of the SM—does not exhibit AB defects, although it does admit a similar vortex–domain wall composite [70–73]. This is because one Higgs doublet does not allow a topologically stable string, and instead the relative phase of the two Higgs doublets has a topological winding of vortex strings. In this sense, the 2HDM is closer to axion models.

attached to the vortex in the first component Φ_1 can end on a vortex in the second component Φ_2 , with the total configuration being a vortex molecule. This is because in this case the second component Φ_2 also breaks the $U(1)$ gauge symmetry simultaneously, resulting in a nontrivial topology of π_1 , in contrast to our Alice model and the GM model, in which the doublet does not allow a topologically stable vortex. In the case of two-gap superconductors, the hierarchy between the VEVs is not essential for stability; both $v_1 \gg v_2$ and $v_2 \gg v_1$ admit essentially the same vortex molecule. In other words, the hierarchy of the VEVs is exchangeable. On the other hand, in our Alice model and the GM model, the hierarchy where the triplet VEV is much larger than the doublet VEV allows a string to be attached by a soliton or domain wall, as discussed in this paper. However, the inverse hierarchy allows no topological string since only the doublet breaks the $SU(2) \times U(1)$ symmetry in the same way as in the SM, allowing no topologically stable solitons; the VEV hierarchies are not exchangeable for the Alice model and GM model.

We should also address several other points.

In this paper we considered the case of only one Alice string. Our theory admits multiple BPS Alice strings at arbitrary positions when we turn off the doublet field. Since a doublet encircling two strings receives no AB phase, the two Alice strings are connected by one AB defect when the doublet develops a VEV. One natural question is how each Alice string finds a partner when there are many Alice strings.

An Alice string carries a $U(1)$ modulus, corresponding to the internal direction of the flux [60]. When a $U(1)$ modulus is twisted along a closed Alice string (called a vorton), it is nothing but a magnetic monopole [76–78] (see Ref. [51] for global analogues). It is natural to ask what is the fate of the monopole if one introduces a doublet having a VEV. In this case, a monopole as a twisted Alice ring may become a drum vorton [79,80], that is, a soliton or a wall is stretched inside the ring. The question remains of how magnetic fluxes from the monopole are confined.

In Refs. [61,62] Alice strings were applied to an axion model to solve the so-called domain-wall problem. N axion domain walls are attached to one axion string in the model with domain-wall number N . In this case, domain walls cannot decay and dominate the Universe, resulting in the domain-wall problem. Although the domain-wall number is two in the model in Refs. [61,62], one axion string attached by two domain walls decays into two Alice strings, each of which is attached by only one domain wall, thereby solving the domain-wall problem. In Ref. [61] the doublet scalar field was considered. This doublet field receives an AB phase as discussed in the present paper. Therefore, if the doublet develops a VEV, the two Alice strings resulting from the decay of one axion string are connected by another domain wall (an AB defect) suppressing the decay.

It is also important to ask whether fermions receiving nontrivial AB phases can contribute to AB defects. This depends on the form of the fermion condensation; fermion-antifermion condensation will have a trivial AB phase, while fermion-fermion condensation will possess a nontrivial AB phase. Two-gap superconductors can in fact be regarded as such an example of a gap composed of fermion-fermion condensation with a nontrivial AB phase. A non-Abelian vortex in dense QCD [28–33] provides a nontrivial AB phase for charged particles [34] as well as a $\mathbb{Z}_3$ AB phase for quarks [35–38], and so if (further) diquark condensation occurs, it may give a fermionic example of AB defects.

It would be natural to ask whether the notion of AB defects can be extended to higher-dimensional cases. The example that we discussed in this paper is a string of codimension two attached by a wall of codimension one. If we consider one higher codimension, a monopole can be attached by a string. In fact, many examples of such configurations are known [11–13,18–21,81–83], but it is unclear whether they can be understood in terms of generalized AB phases. To investigate this problem, the notion of topological obstructions of a monopole [84–86] may be useful.

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APPENDIX: NUMERICAL COMPUTATION

In this appendix we present full numerical results for the equations of motion. We use the relaxation method to find solutions. We use the static Hamiltonian with $\lambda_1 = \lambda_2 = 0$ for simplicity, given as

$$\mathcal{H} = \frac{1}{2} \text{Tr} F_{ij}^2 + \frac{1}{4} f_{ij}^2 + |\mathcal{D}_i \Phi|^2 + |\mathcal{D}_i \Psi|^2 + V(\Phi, \Psi), \quad (\text{A1})$$

$$V(\Phi, \Psi) = V_\Phi(\Phi) + V_\Psi(\Psi) + V_{\text{int}}(\Phi, \Psi), \quad (\text{A2})$$

$$V_\Phi(\Phi) = \frac{\lambda_g}{4} \text{Tr}[\Phi, \Phi^\dagger]^2 + \frac{\lambda_e}{2} (\text{Tr} \Phi \Phi^\dagger - 2\xi^2)^2, \quad (\text{A3})$$

$$V_\Psi(\Psi) = M^2 \Psi^\dagger \Psi + \lambda_\Psi (\Psi^\dagger \Psi)^2. \quad (\text{A4})$$

The interaction potential $V_{\text{int}}(\Phi, \Psi)$ is either $V_{\text{int}}^{(1)}$ or $V_{\text{int}}^{(2)}$ for the first or second model:

$$\begin{aligned} V_{\text{int}}^{(1)}(\Phi, \Psi) &= \mu^{(1)}(\Psi_c^\dagger \Phi^* \Psi + \Psi^\dagger \Phi \Psi_c), \\ V_{\text{int}}^{(2)}(\Phi, \Psi) &= \mu^{(2)}|\Psi_c^\dagger \Phi^* \Psi|^2. \end{aligned} \quad (\text{A5})$$

For the relaxation method, we use the first order in imaginary time equations of motion,

$$\frac{\partial \Phi^\alpha}{\partial \tau} = -\frac{\delta \mathcal{H}}{\delta \Phi^{*\alpha}}, \quad (\text{A6})$$

$$\frac{\partial \Psi^\alpha}{\partial \tau} = -\frac{\delta \mathcal{H}}{\delta \Psi^{*\alpha}}, \quad (\text{A7})$$

$$\frac{\partial A_i^\alpha}{\partial \tau} = -\frac{\delta \mathcal{H}}{\delta A_i^\alpha}, \quad (\text{A8})$$

$$\frac{\partial a_i}{\partial \tau} = -\frac{\delta \mathcal{H}}{\delta a_i}. \quad (\text{A9})$$

We use a 500×500 square lattice with a lattice spacing of 0.2, subject to Neumann boundary conditions.

The initial configurations used in these computations are

$$\begin{aligned} \Phi^1(x, y) &= 2\xi \tanh\left(\sqrt{(x^2 + y^2)}\right) \exp(i0.5n_v\theta) \\ &\times \cos(0.5n_v\theta), \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \Phi^2(x, y) &= -2\xi \tanh\left(\sqrt{(x^2 + y^2)}\right) \exp(i0.5n_v\theta) \\ &\times \sin(0.5n_v\theta), \end{aligned} \quad (\text{A11})$$

$$\Phi^3(x, y) = 0, \quad (\text{A12})$$

$$a_1(x, y) = -(0.5/e)n_v \frac{y}{L^2} \tanh\left(\sqrt{(x^2 + y^2)}\right), \quad (\text{A13})$$

$$a_2(x, y) = (0.5/e)n_v \frac{x}{L^2} \tanh\left(\sqrt{(x^2 + y^2)}\right), \quad (\text{A14})$$

$$A_1^1(x, y) = A_2^1(x, y) = 0, \quad (\text{A15})$$

$$A_1^2(x, y) = A_2^2(x, y) = 0, \quad (\text{A16})$$

$$A_1^3(x, y) = -(0.5/g)n_v \frac{y}{L^2} \tanh\left(\sqrt{(x^2 + y^2)}\right), \quad (\text{A17})$$

$$A_2^3(x, y) = (0.5/g)n_v \frac{x}{L^2} \tanh\left(\sqrt{(x^2 + y^2)}\right), \quad (\text{A18})$$

$$\psi_1(x, y) = \sqrt{\frac{-0.5M^2}{\lambda_\psi}}, \quad (\text{A19})$$

$$\psi_2(x, y) = 0. \quad (\text{A20})$$

Here $n_v (=1)$ is the winding number and L is the system size.

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