Two-loop octagons, algebraic letters and $\bar{Q}$ equationsSong He,^{*} Zhenjie Li[†], and Chi Zhang[‡]

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We compute the symbol of the first two-loop amplitudes in planar $\mathcal{N} = 4$ SYM with algebraic letters, the eight-point Next to Maximally Helicity Violating (NMHV) amplitude (or the dual octagon Wilson loops). We show how to apply $\bar{Q}$ equations of [S. Caron-Huot and S. He, *J. High Energy Phys.* **07** (2012) 174] for computing the differential of two-loop n -point NMHV amplitudes and present the result for $n = 8$ explicitly. The symbol alphabet for octagon consists of 180 independent rational letters and 18 algebraic ones involving Gram-determinant square roots. We comment on all-loop predictions for final entries and aspects of the result valid for all multiplicities.

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I. INTRODUCTION

Scattering amplitudes are central objects in fundamental physics: not only do they connect to high energy experiments such as Large Hadron Collider, but also provide new insights into quantum field theory itself. Tremendous progress has been made in unravelling hidden mathematical structures of scattering amplitudes, especially in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory (c.f. [1–3]). It has become an extremely fruitful playground for new methods of evaluating Feynman loop integrals, which is a subject of enormous interests (for recent works, c.f. [4–6] and references there in). Relatedly, formidable progress has been made in computing multiloop scattering amplitudes (or equivalently lightlike Wilsons loops) of the theory, most notably by the hexagon and heptagon bootstrap program [7,8]. The first nontrivial amplitude in planar $\mathcal{N} = 4$ SYM, the six-point amplitude (or hexagon) has been determined through seven and six loops for Maximally Helicity Violating (MHV) and Next to Maximally Helicity Violating (NMHV) cases, respectively [9], and similarly the seven-point amplitude (or heptagon) has been determined through four loops for these cases, respectively [10]. After subtracting infrared divergences which are known to all loops [11], loop amplitudes are

transcendental functions of cross ratios, and for MHV and NMHV cases they are expected to contain only generalized polylogarithms [2] which are characterized by their symbol [12]. A crucial assumption for the hexagon and heptagon bootstrap is that the collection of letters entering the symbol, or the *alphabet*, consists of only 9 and 42 variables known as cluster coordinates [13]. After imposing conditions on physical discontinuities [8,14,15], the spaces of such functions are remarkably small and certain information from physical limits and symmetries suffices to fix the result.

It is natural to wonder about scattering amplitudes for $n \geq 8$ (octagons etc.), which are expected to be much more intricate than $n = 6, 7$ cases. In general, we need N^k MHV sectors with $k \geq 2$, which are expected to contain elliptic polylogarithms [16] and beyond [17]; even for MHV and NMHV, the main challenge is the lack of control for the symbol alphabet. Starting at $n = 8$, the cluster algebra becomes infinite type and it is unclear which letters can appear for L -loop N^k MHV amplitudes (the only data so far being the symbol of two-loop n -point MHV amplitudes [18]). A new feature for $n \geq 8$ is the appearance of *algebraic letters* which can no longer be written as rational functions of momentum twistors [19] parametrizing the kinematics. It is of great interests already at two loops to understand what letters with square roots appear for amplitudes. In this paper, we compute the symbol of two-loop NMHV octagon using $\bar{Q}$ equations [20], which provides the first example of this phenomenon, and initiates the study for higher-point and even higher-loop amplitudes.

It is well known that scattering amplitudes in planar $\mathcal{N} = 4$ SYM enjoy both superconformal and dual superconformal symmetries [21–23], which close into the infinite-dimensional Yangian symmetry [24]. In [20], *exact* all-loop equations

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obeyed by the S-matrix were derived by determining the quantum corrections to the generators of Yangian symmetry acting on the Bern-Dixon-Smirnov (BDS)-renormalized S-matrix [25]. The equations consist of $\bar{Q}$ equations for dual superconformal symmetry, and its parity-conjugate [20]. In practice, MHV and NMHV amplitudes are particularly simple since they can be solved using $\bar{Q}$ equations alone. Since the equations can be essentially derived from the dual Wilson loop picture, these are computations from first principles, which we exploit to produce higher-point amplitudes currently not accessible from loop integrals or bootstrap [26].

As a first step, we compute the complete differential of two-loop NMHV octagon. By $\bar{Q}$ equations, it is given in terms of one-dimensional integrals over one-loop N^2 MHV amplitudes in the collinear limit. As we will review shortly, the differential of L -loop MHV and NMHV amplitudes is given by a combination of generalized polylogarithmic functions of weight $(2L - 1)$, dressed with certain *universal* objects that are independent of loop order; the latter are a collection of Yangian invariants times $d \log$ of *final entries* of the symbol, known for MHV to all n and NMHV for $n = 6, 7$ [18,20]. We compute these universal NMHV final entries for all n [28], though here we only need the result for $n = 8$. The generalized polylogarithm functions accompanying them have to be computed order by order, and we will present the weight-three functions for two-loop octagons which in turn give the complete symbol (there is no qualitative difference for higher n , which will be reported elsewhere [28]).

II. REVIEW OF $\bar{Q}$ EQUATIONS

The infrared divergences of planar $\mathcal{N} = 4$ SYM are captured by the so-called BDS ansatz [25], and we are interested in the infrared-finite object, the *BDS-subtracted amplitude* $R_{n,k} = A_{n,k}/A_n^{\text{BDS}}$, for n -point, N^k MHV amplitude $A_{n,k}$. As shown in [20], $R_{n,k}$ is dual conformal invariant and enjoys a chiral half of the dual superconformal symmetries; however, it is not invariant under the action of the other half of dual superconformal generators,

$$\bar{Q}_a^A = \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a}, \quad (1)$$

where Z_i are momentum twistors and χ_i denote their Grassmann parts. We further define the basic $\text{SL}(4)$ -invariant $\langle ijkl \rangle := \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$ (or the Plücker coordinates of $G(4, n)$), and the basic R invariant [22,29],

$$[ijklm] := \frac{\delta^{0|4}(\chi_i^A \langle jklm \rangle + \text{cyclic})}{\langle ijkl \rangle \langle jklm \rangle \langle klmi \rangle \langle lmi j \rangle \langle mi jk \rangle}. \quad (2)$$

Nevertheless, this anomaly can be restored by an integral over collinear limits of higher-point amplitudes,

$$\bar{Q}_a^A R_{n,k} = \frac{1}{4} \Gamma_{\text{cusp}} \text{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3} \mathcal{Z}_{n+1})_a^A \times [R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}] + \text{cyclic}, \quad (3)$$

where Γ_{cusp} is the cusp anomalous dimension [30], and we have explicitly shown the term with particle $n+1$ added in collinear limit with n , and its (super-) momentum twistor $\mathcal{Z}_{n+1} = (Z_{n+1}, \chi_{n+1})$ parametrized by ϵ, τ ,

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + C \epsilon \tau \mathcal{Z}_1 + C' \epsilon^2 \mathcal{Z}_2, \quad (4)$$

with $C = \frac{\langle n-1 \ n \ 2 \ 3 \rangle}{\langle n \ 1 \ 2 \ 3 \rangle}$ and $C' = \frac{\langle n-2 \ n-1 \ n \ 1 \rangle}{\langle n-2 \ n-1 \ 2 \ 1 \rangle}$. The integral measure $\int (d^{2|3} \mathcal{Z}_{n+1})_a^A$ consists of the bosonic part $(d^2 \mathcal{Z}_{n+1})_a := \epsilon_{abcd} Z_{n+1}^b dZ_{n+1}^c dZ_{n+1}^d$ and the fermionic part $(d^3 \chi_{n+1})_a^A$; using (4), the bosonic measure reads

$$C(\bar{n})_a \text{Res}_{\epsilon=0} \int d\epsilon d\tau \int_0^\infty d\tau, \quad (5)$$

with $(\bar{n})_a := (n-1 \ n \ 1)_a$. The notation $\text{Res}_{\epsilon=0}$ means to extract the coefficient of $d\epsilon/\epsilon$ under the collinear limit of $\epsilon \rightarrow 0$, and we integrate over “momentum fraction” τ . The perturbative expansion of (3) relates $R_{n,k}^{(L)}$ to $R_{n+1,k+1}^{(L-1)}$ etc., and precisely the difference of the two terms in the bracket ensures that the RHS of (3) is finite.

For $k = 0, 1$, the $\bar{Q}$ equations, (3), allow us to compute the differential of $R_{n,k}^{(L)}$ which can be expressed as [31]

$$dR_{n,k}^{(L)} = \sum_{\alpha} Y_{n,k}^{\alpha} d \log(a_{\alpha}) \mathcal{I}_{\alpha}^{(2L-1)}. \quad (6)$$

Each term consists of (loop-independent) Yangian invariants $Y_{n,k}$ times final entries $d \log a$, labeled collectively by α , multiplied by (pure) transcendental functions of weight $2L - 1$, $\mathcal{I}_{\alpha}$. The symbol of $R_{n,k}$ is obtained iteratively as $\mathcal{S}[R_{n,k}^{(L)}] = \sum_{\alpha} Y_{n,k}^{\alpha} \mathcal{S}[\mathcal{I}_{\alpha}^{(2L-1)}] \otimes (a_{\alpha})$.

To derive (6), first note that all Yangian invariants $Y_{n,k}$ can be computed from positive Grassmannian [2,32,33] (recall $Y_{n,0} = 1$ and all NMHV Yangian invariants are of the form (2), and recursively the RHS of (3) consists of terms of the form $Y_{n+1,k+1} F_{n+1}^{(2L-2)}$). A remarkable feature of the integral $\text{Res}_{\epsilon=0} \int d^{2|3} \mathcal{Z}_{n+1}$ is that the fermionic integral $\int d^3 \chi_{n+1}$ and residue $\text{Res}_{\epsilon=0}$ part can be performed on Yangian invariants $Y_{n+1,k+1}$ independent of transcendental functions F 's. One may worry about the $\log^{L-1} \epsilon$ divergences arising from the collinear limit of but the divergences are always canceled after integrating over τ , as shown in [20]. For MHV ($k = 0$), the effect of $\text{Res}_{\epsilon=0} \int d\epsilon d\tau \int d^3 \chi_{n+1}$ yields terms of the form

$$\bar{Q} \log \frac{\langle \bar{n} i \rangle}{\langle \bar{n} j \rangle} \times \int d \log f_{i,j}(\tau) F_{n+1}(\tau, \epsilon \rightarrow 0), \quad (7)$$

with some rational function $f_{i,j}(\tau)$ for each term. The last step is trivial for MHV: we can simply replace χ_i^A in $\bar{Q}_a^A$ with dZ_i^A then take the trace to obtain the external derivative $d := \sum_{i,a} dZ_i^a \partial / \partial Z_i^a$, which reproduces the well-known MHV final entries $d \log \langle i-1 i i+1 j \rangle$ after collecting all cyclic terms. The one-dimensional integrals for $F_{n+1}^{(2L-2)}$ gives weight $(2L-1)$ functions in (6).

For NMHV ($k=1$), the effect of $\text{Res}_{\epsilon=0} \int \epsilon d\epsilon \int d^3 \chi_{n+1}$ on $N^2\text{MHV}$ Yangian invariants, $Y_{n+1,2}$ on the RHS of (3) gives a list of possible $Y_{n,1}$ in (2) times final entries as

$$Y_{n,1}^a \bar{Q} \log(a_\alpha) \in \left\{ [i j k l m] \bar{Q} \log \frac{\langle \bar{n} I \rangle}{\langle \bar{n} J \rangle} \right\}, \quad (8)$$

where I, J can generally be intersections of momentum twistors of the form, e.g., $(ij) \cap (klm)$ (see [1,34]). To obtain the differential of $R_{n,1}$ in this case, the naive replacement above has an ambiguity due to the existence of nontrivial kernel of $\bar{Q}$, which always take the form

$$\bar{Q}_a^A \left([12345] \log \frac{\langle 1234 \rangle}{\langle 1235 \rangle} \right) = 0. \quad (9)$$

Nevertheless, since the kernel of $\bar{Q}$ cannot contain nontrivial functions of *dual conformal invariants* (DCIs) in this case, the replacement $\chi_i \rightarrow dZ_i$ has no ambiguity once we convert the arguments of $\bar{Q} \log$ to DCI by adding “0” of the form (9). It is a straightforward but tedious algorithm to arrive at such a manifestly DCI form, which gives the final answer for $dR_{n,1}$ [35].

III. THE COMPUTATION

We start with the chiral box expansion for $N^2\text{MHV}$, BDS-subtracted amplitudes $R_{n+1,2}^{(1)}$ [36],

$$R_{n+1,2}^{(1)} = \sum_{a < b < c < d} (f_{a,b,c,d} - R_{n+1,2}^{\text{tree}} f_{a,b,c,d}^{\text{MHV}}) \mathcal{I}_{a,b,c,d}^{\text{fin}}, \quad (10)$$

where $\mathcal{I}_{a,b,c,d}^{\text{fin}}$ denote the finite part of DCI-regulated box functions which are dilogarithms; the box coefficients $f_{a,b,c,d}$ are one-loop $N^2\text{MHV}$ leading singularities which are linear combinations of $Y_{n+1,2}$'s; $f_{a,b,c,d}^{\text{MHV}} = 0, 1$ are MHV one-loop box coefficients. It is straightforward to perform $\text{Res}_{\epsilon=0} \int \epsilon d\epsilon d^3 \chi_{n+1}$ on $f_{a,b,c,d}$ which gives a list of the form (8), except that a special prescription is needed for four-mass box coefficients $f_{a,b,c,d}$. The latter involve square root of Gram determinant, and they read

$$\frac{1-u-v \pm \Delta}{2\Delta} [\alpha_\pm, b-1, b, c-1, c] [\gamma_\pm, d-1, d, a-1, a],$$

where u, v , and Δ are shown in Fig. 1, and $\alpha_\pm$ and $\gamma_\pm$ are two solutions of the Schubert problem $\alpha = (a-1a) \cap (dd-1\gamma)$, $\gamma = (c-1c) \cap (bb-1\alpha)$. It is important to

FIG. 1. A general four-mass box (left) and an octagon one (right).

note that in the result produced by such leading singularities, (8) is accompanied by $d \log f(\tau)$ where $f(\tau)$ may no longer be a rational function. After τ integral, they produce algebraic letters in two-loop NMHV amplitudes.

Let us focus on the octagon ($n=8$), where we need to consider nine 9-point four-mass boxes, and it is clear that only four can potentially contribute square root after taking the collinear limit of $\Delta(\tau)$. They are $f_{2,4,7,9}, f_{2,5,7,9}, f_{2,4,6,9}, f_{3,5,7,9}$, and it is clear that in the limit only two square roots can appear, $\Delta_{1,3,5,7}$ (see Fig. 1) and its cyclic image $\Delta_{2,4,6,8}$. The algebraic letters in the answer must involve $z, \bar{z}$ satisfying

$$\begin{aligned} z\bar{z} = u &= \frac{\langle 1278 \rangle \langle 3456 \rangle}{\langle 1256 \rangle \langle 3478 \rangle}, \\ (1-z)(1-\bar{z}) = v &= \frac{\langle 1234 \rangle \langle 5678 \rangle}{\langle 1256 \rangle \langle 3478 \rangle} \end{aligned} \quad (11)$$

and $z', \bar{z}'$ for $\Delta_{2,4,6,8}$ by $i \rightarrow i+1$. Now, let us discuss how to perform τ integrals involving square roots.

The τ integrals.—To perform these τ integrals, it is convenient to rationalize square roots by a change of variable based on rational points of the quadratic. For example, the rational parametrization $(x, y) = (2t/(1-t^2), (1+t^2)/(1-t^2))$ of the quadratic $y^2 = x^2 + 1$ is obtained by inserting $y-1 = (x-0)t$. Let us consider the collinear limit of the four-mass box coefficient $f_{2,4,6,9}$ as an example, which has $\tau = \infty$ as its rational point, and it becomes a rational function of t under the substitution $\tau = \frac{\rho(t-z)(t-\bar{z})}{(t-\sigma)}$ where

$$\begin{aligned} \rho &= \frac{\langle 1238 \rangle \langle 1256 \rangle \langle 3478 \rangle}{\langle 2378 \rangle \langle 1(34)(56)(82) \rangle}, \\ \sigma &= \frac{\langle 1348 \rangle \langle 3456 \rangle \langle 1278 \rangle}{\langle 3478 \rangle \langle 1(34)(56)(82) \rangle}, \end{aligned}$$

with the notation $\langle a(bc)(de)(fg) \rangle = \langle abde \rangle \langle acfg \rangle - \langle acde \rangle \langle abfg \rangle$. Now, the square root in z enters the final result via the limit of integration $\int_0^\infty d\tau \rightarrow \int_z^\infty dt \frac{dt}{dt}$. Similarly, one can find change of variables for the other 3 four-mass boxes, e.g., the square root in z' comes from the box $f_{3,5,7,9}$. Interestingly, the other two boxes $f_{2,4,7,9}$ and $f_{2,5,7,9}$, which have both $\tau = 0$ and $\tau = \infty$ as rational points, do not contain algebraic letters after the substitution.

This argument explains why there is no square roots in NMHV heptagons from N^2 MHV octagons.

A crucial check at this stage is the convergence of τ integrals. First, we collect all terms proportional to $\log \epsilon$ and numerically verify that they integrate to zero. Next, each term of τ -integrand naively has poles at $\tau = 0$ or ∞ ; we numerically verified that they are spurious poles and can be removed by a simple subtraction. Once this is done, we can easily perform all the subtracted τ integrals, e.g., using POLYLOGTOOLS [37].

Last entry conditions and DCI.—Even before performing the τ integral, we have found 147 linear independent objects of the form $Y_{8,1} \bar{Q} \log \frac{\langle \bar{n}i \rangle}{\langle \bar{n}j \rangle}$, from performing $\text{Res}_{\epsilon=0} \int \epsilon d\epsilon \int d^3\chi_9$ on all $Y_{9,2}$'s; out of them, only 17 elements are of the form $[i j k l m] \bar{Q} \log \frac{\langle \bar{n}(ij) \cap (klm) \rangle}{\langle \bar{n}i \rangle \langle jklm \rangle}$, while others do not involve intersections of momentum twistors. After taking cyclic permutations, we have a basis of $8 \times 147 = 1176$ such objects, which is an all-loop prediction for NMHV octagon. The last step is converting these final entries to DCI combinations to remove ambiguities from the kernel of $\bar{Q}$. To do this, we transform the above basis to a basis with minimal number of elements that break DCI by adding 0 of the form (9). Asking their coefficients to vanish gives constraints on our weight-three functions from τ integrals. There are 21 constraints, each being lengthy combination of weight-3 Goncharov polylogarithms, and we have numerically checked that they all vanish. This leaves us with 1155 DCI combinations as last entry conditions.

IV. RESULTS

The differential of two-loop NMHV octagon can be written as $dR_{8,1}^{(2)} = \sum_{\alpha} ([i j k l m] d \log x)_{\alpha} F_{\alpha}$, where for each combination of R invariant times $d \log$ of DCI x , we have a function F_{α} ; these weight-three Goncharov polylogarithms are recorded in funcdata.m [38]. From here, it is trivial to obtain the complete symbol, $\mathcal{S}[R_{8,1}^{(2)}]$; one can expand the symbol in a basis of $\binom{7}{4} = 35$ R invariants, e.g., $[i j k l 8]$ for $1 \leq i < j < k < l < 8$, and we record symbols of the 35 coefficients, $\mathcal{S}_{i,j,k,l}$ in symdata.m [38]. On average, each symbol has around 10^5 terms.

The symbol alphabet of $R_{8,1}^{(2)}$ consists of 180 multiplicative-independent rational letters, which can form $180 - 8 = 172$ DCI cross ratios, and 18 independent algebraic letters which we choose to write in a DCI form directly. The 180 rational letters are contained in the prediction of rational symbol alphabet from Landau equations [39], but we find that the last class of them is missing.

The 18 algebraic letters represent the first appearance of square roots in a two-loop amplitudes [40] and let us spell them out. The collinear integrals over nine-point

four-mass boxes result in 22×2 algebraic letters of the form

$$\left\{ \frac{x_* - z}{x_* - \bar{z}}, \frac{x'_* - z'}{x'_* - \bar{z}'} \right\}, \quad (12)$$

where z, z' , etc. are defined in (11); four of these are known: $\frac{z}{\bar{z}}, \frac{1-z}{1-\bar{z}}$ in $f_{1,3,5,7}$ and their cyclic images in $f_{2,4,6,8}$ (with z'), which correspond to $x, x' = 0, 1$, respectively. The 40 new algebraic letters take the form of (12) with x_* and x'_* generated by cyclic shifts of the following five seeds [41]:

$$\begin{aligned} x_a &= \frac{\langle 1(52)(34)(78) \rangle \langle 3456 \rangle}{\langle 1345 \rangle \langle 1256 \rangle \langle 3478 \rangle}, & x_b &= x_a|_{5 \leftrightarrow 6}, \\ x_c &= \frac{\langle 1378 \rangle \langle 3456 \rangle}{\langle 1356 \rangle \langle 3478 \rangle}, & x_d &= x_c|_{3 \leftrightarrow 4}, \\ x_e &= \frac{\langle 187(34) \cap (256) \rangle}{\langle 1256 \rangle \langle 3478 \rangle}, \end{aligned} \quad (13)$$

where $i \leftrightarrow j$ denotes the exchange of particle labels. These algebraic letters are not independent since they satisfy multiplicative relations and we end up with only 18 independent letters [42].

We find that all algebraic letters always enter the symbols of $\{F_{\alpha}\}$ in the following combinations:

$$\left(u \otimes \frac{1-z}{1-\bar{z}} + v \otimes \frac{\bar{z}}{z} \right) \otimes \frac{x_* - z}{x_* - \bar{z}} \quad (14)$$

and similarly for x', z' . In other words, any algebraic letter with nontrivial x appears only in the third entry, which is accompanied by the corresponding (symbol of) four-mass box in the first two entries.

It is interesting to rewrite the algebraic letters in the more familiar form $\frac{1}{2}(a \pm \sqrt{a^2 - 4b})$, where we have 44 pairs of (a, b) that are polynomials of Plücker coordinates. We remark that the discriminant $a^2 - 4b$ is always proportional to that for one of the two four-mass boxes, $\Delta_{1,3,5,7}^2$ or $\Delta_{2,4,6,8}^2$, which are square-root branch points from Landau equations [39,43]. However, Landau equations cannot predict what a and b can appear, except that branch points $b = 0$ correspond to zero locus of some rational letters. We have checked that this is indeed the case for all b of the 44 letters, e.g.,

$$\begin{aligned} (x_a - z)(x_a - \bar{z}) &\propto \langle 1(34)(56)(78) \rangle \langle 5(12)(34)(78) \rangle, \\ (x_c - z)(x_c - \bar{z}) &\propto \langle 1(34)(56)(78) \rangle \langle 3(12)(56)(78) \rangle, \\ (x_e - z)(x_e - \bar{z}) &\propto \langle 1(34)(56)(78) \rangle \langle 2(34)(56)(78) \rangle, \end{aligned}$$

where we dropped some Plücker coordinates. Like rational ones, algebraic letters are positive in positive kinematics. In demo_octagon.nb [38], we list all $172 + 18$ DCI letters

which are written by using a nine-dimensional positive parametrization for the momentum twistors [44,45].

We have performed various consistency checks on our results, for example, cyclicity, dual conformal invariance, and the condition of physical first entries; let us present details for more nontrivial checks, including integrability, collinear limits, and absence of spurious poles.

Integrability.—To integrate $dR_{8,1}^{(2)}$, it is necessary that 35 coefficients of the form $\sum_i F_i d \log x_i$ satisfy $\sum_i dF_i \wedge d \log x_i = 0$. We have numerically checked this at the symbol level by using the positive parametrization mentioned above.

Collinear limits.—The NMHV octagon smoothly reduces to NMHV and MHV heptagon under the k -preserving and the k -decreasing limit, respectively. We have verified these by taking Eq. (4) with $n = 7$ for fixed τ and then taking the limit $\epsilon \rightarrow 0$. In addition, R invariants behave as $[abc78] \rightarrow 0$ and $[abcd8] \rightarrow [abcd7]$ under the k -preserving limit, while only $[12a78]$ survive as one under the k -decreasing limit.

Cancellation of spurious poles.—Finally, we check that the residue on any spurious pole of R invariants vanishes. By cyclicity, we only need to check the following poles:

$$\langle 1235 \rangle, \langle 1236 \rangle, \langle 1237 \rangle, \langle 1246 \rangle, \langle 1247 \rangle, \langle 1257 \rangle, \langle 1357 \rangle,$$

each of which is of the form $\langle 1abc \rangle$ and merely belongs to $[1abc8]$ in our basis. We have verified numerically the coefficient of $[1abc8]$ vanishes as $\langle 1abc \rangle \rightarrow 0$ by sending $Z_1 \rightarrow \alpha Z_a + \beta Z_b + \gamma Z_c + \delta Z_8$ for fixed α, β, γ , then taking the limit $\delta \rightarrow 0$.

V. CONCLUSION AND DISCUSSIONS

Using $\bar{Q}$ equations, we computed the differential of two-loop NMHV octagon, whose symbol contains interesting algebraic letters. There is no qualitative difference for $n > 8$: we have computed the universal final entries, $Y_{n,1} \times d \log(a)$ to all n , and weight-three functions involving algebraic letters are computed from four-mass boxes in the same way as we did for $n = 8$. We are particularly interested in the explicit result for $n = 9$, which via $\bar{Q}$ equations can produce the three-loop MHV octagon [28]. We expect it to be the first three-loop amplitude with algebraic letters, and it is highly desirable to see how the

symbol alphabet changes comparing to two-loop octagons. It is plausible that with sufficient knowledge of symbol alphabet one can initiate the bootstrap for *octagon functions*, which can push the frontier to higher loops and reveal more interesting structures.

Relatedly, it is of great interests to further explore mathematical structures of our results. The connections to cluster algebra, tropical Grassmannian, etc. [46] are certainly worth exploring, and it is conceivable that our algebraic letters can be interpreted from tropical $G_+(4, 8)$ and the corresponding cluster algebra. The *cluster adjacency* [47] and *extended Steinmann relations* [48] have been observed for hexagons and heptagons and conjectured for higher n [49]. We have verified that in the list of all-loop final entries to all n , at least those involving intersections nicely satisfy adjacency; we leave the study of the full symbol to future works.

In addition to providing new data points for amplitudes, it would be fascinating to investigate potential interplays of $\bar{Q}$ equations with Feynman integral computations (see [50] for related works). An intriguing example is provided by the component $\chi_1 \chi_3 \chi_5 \chi_7$, which has been computed from numerically evaluating double-pentagon integrals at a special kinematic point in [51]. From $\bar{Q}$ equations, it is obvious that the component is completely free of algebraic letters; in our basis it is given by the coefficient of $[13578]$, which is the simplest one with 28376 terms and 76 rational letters only (see the ancillary file). By evaluating the symbol at the special kinematic point, our prediction is verified by the computation of [51].

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