Soft contribution to the damping rate of a hard photon in a weakly magnetized hot medium

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We consider weakly magnetized hot QED plasma comprising electrons and positrons. There are three distinct dispersive (longitudinal and two transverse) modes of a photon in a thermomagnetic medium. At lowest order in the coupling constant, a photon is damped in this medium via Compton scattering and pair creation process. We evaluate the damping rate of hard photon by calculating the imaginary part of the each transverse dispersive modes in a thermomagnetic QED medium. We note that one of the fermions in the loop of one-loop photon self-energy is considered as soft and the other one is hard. Considering the resummed fermion propagator in a weakly magnetized medium for the soft fermion and the Schwinger propagator for hard fermion, we calculate the soft contribution to the damping rate of hard photon. In weak field approximation the thermal and thermomagnetic contributions to damping rate get separated out for each transverse dispersive mode. The total damping rate for each dispersive mode in presence of magnetic field is found to be reduced than that of the thermal one. This formalism can easily be extended to QCD plasma.

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I. INTRODUCTION

Astrophysical plasma is almost always immersed in magnetic field. Extreme, magnetized plasma is found in interiors of neutron star, magnetospheres of magnetars and central engines of supernovae and gamma ray bursts [1]. The propagation of photon through the hot magnetized plasma, *viz.*, electron-positron plasma (EPP), is of great interest. Because the magnetar phenomena are found by analyzing the high-energy radiation detected at earth. Thus it is very important to have a good understanding of the propagation of photon through the EPP. Furthermore, the phenomenon of Faraday rotation *i.e.*, change of polarization of photon while propagating through a medium has been studied in Ref. [2] in a field theoretical viewpoint. This has also been detected in several astrophysical objects [3]. Also high-intensity laser beams are used to create ultrarelativistic EPP of temperature around 10 MeV [4]. This EPP may play an important role in various astrophysical situations. Some properties of such plasma, *viz.*, the equation of state, dispersion relation of collective

plasma modes of photon and electron, damping rates, mean free paths, transport coefficients and particle production rates, are studied using QED at finite temperature [5,6].

On the other hand in noncentral heavy ion collisions, the magnetic field as high as $(15 - 20)m_\pi^2$ can be generated [7] at LHC energies. After a few fm/c of the collision, the magnetic field strength¹ decreases to $(1 - 2)m_\pi^2$. The effect of magnetic field on the properties of the QCD matter [*viz.* quark-gluon plasma (QGP)] and on the phase diagram of QCD is of great interest. Recently, several studies have found the effect of magnetic catalysis [13–16], *i.e.*, the enhancement of phase transition temperature of QCD matter in presence of external magnetic field, whereas, some results of inverse magnetic catalysis [17–25] have been reported. Various properties of QCD matter at weak coupling in presence of magnetic field is being studied including the equation of state [26–28], transport properties [29–31]. Modification of QCD Debye mass and the two point correlation functions of quarks [32] and gluons [33–35] *i.e.*, partons have been analyzed recently. Dilepton production rate from a hot magnetized QCD plasma [12,36–42] has been calculated. The photon is also

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¹The initial magnitude of this magnetic field can be very high at the initial time of the heavy-ion collisions and then it decreases very fast, being inversely proportional to the square of time [8,9]. However for a different point of view, see Refs. [10–12], where the time dependence of magnetic field is shown to be adiabatic due to the high conductivity of the medium.

considered as a good probe of the QGP medium as photon only interacts electromagnetically. Thus, it comes out of the hot QCD system without interacting much. The damping rate of the hard photon is associated with the mean free path of photon [43] and hard photon production rate in QGP [44].

Damping rate of photon is related to the imaginary part of photon dispersion in the medium [45] which is again related to the scattering cross section of the process that we find by cutting the photon self-energy diagram [46]. In lowest order coupling constant, photons are damped by Compton scattering and pair creation process. In case of low momentum transfer, the damping rate shows infrared singularity. Thus one should consider the effective resummed propagator instead of bare propagator for soft momentum of fermion. We will call this as the soft contribution to the damping rate of photon. The hard contribution refers to the case where all the fermions in loop have momentum order of or much greater than the system temperature T . Both soft and hard contributions to the damping rate of hard photon in thermal medium have been calculated in Ref. [45]. The dispersion relations of photon are modified for a hot magnetized medium [33]. So the damping rate of photon will also get modified in a thermomagnetic medium. In this article we intend to compute the soft contribution to the hard photon damping rate for a weakly magnetized hot medium in one loop approximation of photon self-energy. In a thermomagnetic medium this would be a good indicator as higher loop calculation contributing to higher order would be extremely involved.

We consider hard photon of momentum $P^\mu = (p_0, \mathbf{p})$ where $p = |\mathbf{p}| \gg T$ in a relativistic hot magnetized QED medium. To find the soft contribution of the damping rate we introduce a separation scale Λ where $eT \ll \Lambda \ll T$ ($gT \ll \Lambda \ll T$ in case of QCD). In the soft part of the damping rate, the contribution from soft loop momentum involving a fermion is taken into account up to the separation scale Λ . Here we assume that the magnetic field strength is weak i.e., $\sqrt{eB} < eT < T$ ($\sqrt{q_f B} < gT < T$ for QCD). We use the recently obtained effective fermion propagator [32] in presence of weak magnetic field for the soft fermion and Schwinger propagator for the hard fermion in the loop. The Braaten-Pisarki-Yuan formalism [47] has been used here to calculate the imaginary part of photon self-energy. Extension to the case of damping rate of hard photon in weakly magnetized hot QCD medium is straightforward. We need to consider the loop fermions as quark and antiquark in that case.

In Sec. II we describe the set up to calculate the photon damping rate associated with imaginary part of photon self-energy. In Sec. III the self-energy is obtained in a weak field approximation. The imaginary parts of various components of photon self-energy is obtained in Sec. IV. Results are given in Sec. V. We conclude in Sec. VI.

II. SETUP

We consider plasma of electrons and positrons at temperature T . The z -axis of the lab frame is oriented along the magnetic field. The general structure of the gauge boson self-energy and corresponding effective propagator have been evaluated in Ref. [33]. The general covariant structure of photon self-energy in a magnetized hot medium can be written as

$$\Pi^{\mu\nu} = \beta B^{\mu\nu} + \sigma R^{\mu\nu} + \delta Q^{\mu\nu} + \alpha N^{\mu\nu}, \quad (1)$$

where various form factors can be written as

$$\begin{aligned} \beta &= B^{\mu\nu} \Pi_{\mu\nu}, \\ \sigma &= R^{\mu\nu} \Pi_{\mu\nu}, \\ \delta &= Q^{\mu\nu} \Pi_{\mu\nu}, \\ \alpha &= \frac{1}{2} N^{\mu\nu} \Pi_{\mu\nu}. \end{aligned} \quad (2)$$

The general covariant structure of photon propagator can be obtained [33] as

$$\begin{aligned} D_{\mu\nu} &= \frac{\xi P_\mu P_\nu}{P^4} + \frac{(P^2 - \delta) B_{\mu\nu}}{(P^2 - \beta)(P^2 - \delta) - \alpha^2} + \frac{R_{\mu\nu}}{P^2 - \sigma} \\ &+ \frac{(P^2 - \beta) Q_{\mu\nu}}{(P^2 - \beta)(P^2 - \delta) - \alpha^2} + \frac{\alpha N_{\mu\nu}}{(P^2 - \beta)(P^2 - \delta) - \alpha^2}. \end{aligned} \quad (3)$$

We note that the thermal medium (absence of magnetic field) has two dispersive modes of photon i.e., one degenerate transverse mode and one medium induced plasmon mode due to breaking of boost invariance. Now breaking of rotational invariance in the presence of a magnetic field leads to three dispersive modes of photon by lifting the degeneracy of the transverse modes. These three dispersive modes can be seen from the pole of Eq. (3). Now, the dispersion relations can be written as

$$P^2 - \sigma = 0, \quad (4)$$

$$\begin{aligned} (P^2 - \delta)(P^2 - \beta) - \alpha^2 &= \left(P^2 - \frac{\beta + \delta + \sqrt{(\beta - \delta)^2 + 4\alpha^2}}{2} \right) \\ &\times \left(P^2 - \frac{\beta + \delta - \sqrt{(\beta - \delta)^2 + 4\alpha^2}}{2} \right) \\ &= 0. \end{aligned} \quad (5)$$

In the weak magnetic field approximation α does not contribute up to $\mathcal{O}[(eB)]^2$, one gets the simple form of the above dispersive modes [26]

$$\begin{aligned}
P^2 - \sigma &= 0, \\
P^2 - \beta &= 0, \\
P^2 - \delta &= 0.
\end{aligned} \tag{6}$$

Damping rate is defined as the imaginary part of photon dispersion relation. The medium induced longitudinal (plasmon) mode does not contribute to the damping rate² and the dispersion relations for two transverse modes of a photon are given, respectively, as

$$P^2 - \sigma = 0, \quad P^2 - \delta = 0, \tag{7}$$

Damping rates $\gamma_\delta(p)$ and $\gamma_\sigma(p)$ (for no overdamping i.e., $\gamma_i \ll p_0$ where $i = \delta, \sigma$) of hard photon are given by imaginary part of the form factors as [48]

$$\gamma_\sigma(p) = -\frac{1}{2p} \text{Im} \sigma(p_0 = p), \tag{8}$$

$$\gamma_\delta(p) = -\frac{1}{2p} \text{Im} \delta(p_0 = p). \tag{9}$$

The tensor structures of $R^{\mu\nu}$ and $Q^{\mu\nu}$ [33] are given as

$$\begin{aligned}
R^{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
Q^{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\cos^2\theta_p & 0 & \sin\theta_p \cos\theta_p \\ 0 & 0 & 0 & 0 \\ 0 & \sin\theta_p \cos\theta_p & 0 & -\sin^2\theta_p \end{pmatrix}.
\end{aligned} \tag{10}$$

Using Eq. (10) in Eq. (2) we can write the form factors σ and δ in the weak field approximation as

$$\sigma = -(\Pi_0^{22} + \Pi_2^{22}), \tag{11}$$

$$\begin{aligned}
\delta &= -\cos^2\theta_p(\Pi_0^{11} + \Pi_2^{11}) - \sin^2\theta_p(\Pi_0^{33} + \Pi_2^{33}) \\
&\quad + 2 \sin\theta_p \cos\theta_p(\Pi_0^{13} + \Pi_2^{13}).
\end{aligned} \tag{12}$$

Combining Eq. (8) with Eq. (11) and Eq. (9) with Eq. (12), the damping rates become

$$\gamma_\sigma(p) = \frac{1}{2p} (\text{Im}\Pi_0^{22} + \text{Im}\Pi_2^{22}), \tag{13}$$

²The longitudinal dispersive mode merges with the light cone at high photon momentum.

FIG. 1. Photon self-energy where the blob represents the effective electron propagator in magnetic field and double line represents bare electron propagator in magnetic field.

$$\begin{aligned}
\gamma_\delta(p) &= \frac{1}{2p} [\cos^2\theta_p (\text{Im}\Pi_0^{11} + \text{Im}\Pi_2^{11}) \\
&\quad + \sin^2\theta_p (\text{Im}\Pi_0^{33} + \text{Im}\Pi_2^{33}) \\
&\quad - 2 \sin\theta_p \cos\theta_p (\text{Im}\Pi_0^{13} + \text{Im}\Pi_2^{13})].
\end{aligned} \tag{14}$$

The damping rates in Eqs. (13) and (14) can now be written as

$$\gamma_\sigma(p) = \gamma_{\text{th}}(p) + \gamma_\sigma^B(p), \tag{15}$$

$$\gamma_\delta(p) = \gamma_{\text{th}}(p) + \gamma_\delta^B(p). \tag{16}$$

where γ_{th} is the $\mathcal{O}[(eB)^0]$ contribution or thermal contribution is given as

$$\begin{aligned}
\gamma_{\text{th}}(p) &= \frac{1}{2p} \text{Im}\Pi_0^{22} = \frac{1}{2p} [\cos^2\theta_p \text{Im}\Pi_0^{11} + \sin^2\theta_p \text{Im}\Pi_0^{33} \\
&\quad - 2 \sin\theta_p \cos\theta_p \text{Im}\Pi_0^{13}].
\end{aligned} \tag{17}$$

The thermomagnetic corrections of $\mathcal{O}[(eB)^2]$ are given as

$$\gamma_\sigma^B(p) = \frac{1}{2p} \text{Im}\Pi_2^{22}, \tag{18}$$

$$\begin{aligned}
\gamma_\delta^B(p) &= \frac{1}{2p} [\cos^2\theta_p \text{Im}\Pi_2^{11} + \sin^2\theta_p \text{Im}\Pi_2^{33} \\
&\quad - 2 \sin\theta_p \cos\theta_p \text{Im}\Pi_2^{13}].
\end{aligned} \tag{19}$$

We need to obtain the imaginary parts of 11, 22, 33, and 13 components of the photon self-energy $\Pi^{\mu\nu}$ which are computed in the following sections.

III. PHOTON SELF-ENERGY IN HOT MAGNETIZED MEDIUM

The photon self-energy as shown in Fig. 1 can be written as

$$\begin{aligned}
\Pi_{\mu\nu}(P) &= ie^2 \int \frac{d^4K}{(2\pi)^4} \{ \text{Tr}[\gamma_\mu S^*(K) \gamma_\nu S(Q)] \\
&\quad + \text{Tr}[\gamma_\nu S^*(K) \gamma_\mu S(Q)] \},
\end{aligned} \tag{20}$$

where $S^*(K)$ is effective electron propagator and $S(K)$ is Schwinger propagator for bare electron. As the external

photon is hard, we consider one bare and one effective fermion propagator in the loop. In the following we would obtain the propagators for fermion.

A. Fermion propagator in weak field approximation

In the weak magnetic field limit, i.e., $\sqrt{eB} < m_{\text{th}} \sim eT < T$ the Schwinger propagator for fermion can be written up to $\mathcal{O}[(eB)^2]$ as [49]

$$\begin{aligned} S(K) &= \frac{\not{K} + m_f}{K^2 - m_f^2} + i\gamma^1 \gamma^2 \frac{\not{K}_{\parallel} + m_f}{(K^2 - m_f^2)^2} (eB) \\ &+ 2 \left[\frac{\{(K \cdot u)\not{u} - (K \cdot n)\not{n}\} - \not{K}}{(K^2 - m_f^2)^3} - \frac{k_{\perp}^2 (\not{K} + m_f)}{(K^2 - m_f^2)^4} \right] (eB)^2 \\ &+ \mathcal{O}[(eB)^3] \\ &= S_0 + S_1 + S_2 + \mathcal{O}[(eB)^3]. \end{aligned} \quad (21)$$

The general form of fermion self-energy in a weakly magnetized medium can be written as [32]

$$\Sigma(K) = -a\not{K} - b\not{u} - b'\gamma_5\not{u} - c'\gamma_5\not{u}. \quad (22)$$

In one loop order, the form factors are given as

$$a(k_0, k) = -\frac{m_{\text{th}}^2}{k^2} Q_1\left(\frac{k_0}{k}\right), \quad (23)$$

$$b(k_0, k) = \frac{m_{\text{th}}^2}{k} \left[\frac{k_0}{k} Q_1\left(\frac{k_0}{k}\right) - Q_0\left(\frac{k_0}{k}\right) \right], \quad (24)$$

$$b'(k_0, k) = 4e^2 M^2(T, m_f, eB) \frac{k_3}{k^2} Q_1\left(\frac{k_0}{k}\right), \quad (25)$$

$$c'(k_0, k) = 4e^2 M^2(T, m_f, eB) \frac{1}{k} Q_0\left(\frac{k_0}{k}\right), \quad (26)$$

where Legendre function of second kind are given as

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad (27)$$

$$Q_1(x) = xQ_0(x) - 1, \quad (28)$$

and the thermomagnetic mass is given as

$$M^2(T, m_f, eB) = \frac{eB}{16\pi^2} \left[\ln 2 - \frac{\pi T}{2m_f} \right], \quad (29)$$

whereas thermal mass is given as

$$m_{\text{th}}^2 = \frac{1}{8} e^2 T^2. \quad (30)$$

The effective fermion propagator can be written [32] as

$$\begin{aligned} S^*(K) &= P_- \frac{\not{L}(K)}{L^2} P_+ + P_+ \frac{\not{R}(K)}{R^2} P_- \\ &= S_L^*(K) + S_R^*(K), \end{aligned} \quad (31)$$

where chirality projection operators are given by

$$P_{\pm} = \frac{1}{2} (1 \pm \gamma_5), \quad (32)$$

and L^μ and R^μ are given as

$$L^\mu = (1+a)K^\mu + (b+b')u^\mu + c'n^\mu, \quad (33)$$

$$R^\mu = (1+a)K^\mu + (b-b')u^\mu - c'n^\mu. \quad (34)$$

For simplicity of calculation we expand the effective fermion propagator in Eq. (31) in powers of eB and keep terms up to $\mathcal{O}[(eB)^2]$ as

$$S^*(K) = S_0^*(K) + S_1^*(K) + S_2^*(K) + \mathcal{O}[(eB)^3], \quad (35)$$

where $S_0^*(K)$ is $\mathcal{O}[(eB)^0]$ and given as

$$S_0^*(K) = \frac{(1+a)\not{K} + b\not{u}}{D^2} = \frac{(1+a)\not{K} + b\not{u}}{D_+ D_-}, \quad (36)$$

where $D_{\pm} = (1+a)(k_0 \mp k) + b$.

Equation (36) is the effective HTL fermion propagator [50,51] in thermal medium. The $\mathcal{O}[(eB)]$ is obtained as

$$\begin{aligned} S_1^*(K) &= \frac{1}{D^4} [2(1+a)\not{K}\gamma_5\{-(1+a)k_3c' - (1+a)k_0b' - bb'\} + \not{u}\gamma_5\{((1+a)^2K^2 - b^2)b' - 2(a+1)bc'k_3\} \\ &+ c'\not{u}\gamma_5\{(2(1+a)k_0 + b)b + (a+1)^2K^2\}], \end{aligned} \quad (37)$$

whereas $\mathcal{O}[(eB)^2]$ is obtained as

$$\begin{aligned} S_2^*(K) &= \left[\frac{(2b'\{(1+a)k_0 + b\} + 2c'k_3(1+a))^2}{D^6} - \frac{b^2 - c'^2}{D^4} \right] \{(1+a)\not{K} + b\not{u}\} - \frac{(2b'\{(1+a)k_0 + b\} + 2c'k_3(1+a))(b'\not{u} + c'\not{u})}{D^4} \\ &= \left(\frac{h^2(k_0, k_{\perp}, k_3)}{D^6} - \frac{h'}{D^4} \right) \{(1+a)\not{K} + b\not{u}\} - \frac{h(k_0, k_{\perp}, k_3)}{D^4} (b'\not{u} + c'\not{u}), \end{aligned} \quad (38)$$

where $h = 2b'\{(1+a)k_0 + b\} + 2c'k_3(1+a)$ and $h' = b'^2 - c'^2$.

B. Photon self-energy in weak magnetic field

Now the $\mathcal{O}[(eB)^0]$ contribution of $\Pi^{\mu\nu}$ given in Eq. (20) can be written as

$$\begin{aligned}\Pi_0^{\mu\nu} &= ie^2 \int \frac{d^4 K}{(2\pi)^4} \{ \text{Tr}[\gamma_\mu S_0^*(K) \gamma_\nu S_0(Q)] + \text{Tr}[\gamma_\nu S_0^*(K) \gamma_\mu S_0(Q)] \} \\ &= ie^2 \int \frac{d^4 K}{(2\pi)^4} \left\{ \text{Tr} \left[\gamma^\mu \left(\frac{\gamma_0 - \vec{\gamma} \cdot \hat{k}}{2D_+} + \frac{\gamma_0 + \vec{\gamma} \cdot \hat{k}}{2D_-} \right) \gamma^\nu (f_0^{(1)} \gamma^0 - f_0^{(0)} \vec{\gamma} \cdot \vec{q}) \right] + \text{Tr} \left[\gamma^\nu \left(\frac{\gamma_0 - \vec{\gamma} \cdot \hat{k}}{2D_+} + \frac{\gamma_0 + \vec{\gamma} \cdot \hat{k}}{2D_-} \right) \gamma^\mu (f_0^{(1)} \gamma^0 - f_0^{(0)} \vec{\gamma} \cdot \vec{q}) \right] \right\} \\ &= 8ie^2 \int \frac{d^4 K}{(2\pi)^4} \frac{(1+a) \{ (K^\mu Q^\nu + K^\nu Q^\mu) - g^{\mu\nu} K \cdot Q \} + b \{ (Q^\mu u^\nu + Q^\nu u^\mu) - g^{\mu\nu} Q \cdot u \}}{D_+ D_- Q^2},\end{aligned}\quad (39)$$

where

$$f_0^{(1)} = \frac{q_0}{Q^2}, \quad f_0^{(0)} = \frac{1}{Q^2}, \quad f_1^{(1)} = \frac{q_0}{Q^4}, \quad f_1^{(0)} = \frac{1}{Q^4}.\quad (40)$$

The $\mathcal{O}[(eB)]$ contribution of $\Pi^{\mu\nu}$ is given as

$$\Pi_1^{\mu\nu} = ie^2 \int \frac{d^4 K}{(2\pi)^4} \{ \text{Tr}[\gamma_\mu S_0^*(K) \gamma_\nu S_1(Q)] + \text{Tr}[\gamma_\nu S_0^*(K) \gamma_\mu S_1(Q)] + \text{Tr}[\gamma_\mu S_1^*(K) \gamma_\nu S_0(Q)] + \text{Tr}[\gamma_\nu S_1^*(K) \gamma_\mu S_0(Q)] \},\quad (41)$$

which becomes zero.

The $\mathcal{O}[(eB)^2]$ contribution of $\Pi^{\mu\nu}$ is given as

$$\begin{aligned}\Pi_2^{\mu\nu} &= ie^2 \int \frac{d^4 K}{(2\pi)^4} \{ \text{Tr}[\gamma_\mu S_1^*(K) \gamma_\nu S_1(Q)] + \text{Tr}[\gamma_\nu S_1^*(K) \gamma_\mu S_1(Q)] + \text{Tr}[\gamma_\mu S_0^*(K) \gamma_\nu S_2(Q)] \\ &\quad + \text{Tr}[\gamma_\nu S_0^*(K) \gamma_\mu S_2(Q)] + \text{Tr}[\gamma_\mu S_2^*(K) \gamma_\nu S_0(Q)] + \text{Tr}[\gamma_\nu S_2^*(K) \gamma_\mu S_0(Q)] \}.\end{aligned}\quad (42)$$

We calculate the above mentioned trace as follows. The trace of the first and second terms of Eq. (42) can be calculated as

$$\begin{aligned}\text{Tr}[\gamma_\mu S_1^*(K) \gamma_\nu S_1(Q)] + \text{Tr}[\gamma_\nu S_1^*(K) \gamma_\mu S_1(Q)] &= \frac{8(eB)}{D^2(Q^2 - m_f^2)^2} [b' \{ (u^\mu n^\nu + u^\nu n^\mu)(Q \cdot u) - 2u^\mu u^\nu (Q \cdot n) + g^{\mu\nu} (Q \cdot n) \} \\ &\quad + c' \{ 2n^\mu n^\nu (Q \cdot u) - (n^\mu u^\nu + n^\nu u^\mu)(Q \cdot n) + g^{\mu\nu} (Q \cdot u) \}] \\ &\quad - \frac{8(eB)}{D^4(Q^2 - m_f^2)^2} [h \{ (1+a)(g^{\mu\nu}((K \cdot u)(Q \cdot n) - (K \cdot n)(Q \cdot u)) \\ &\quad - (K^\mu u^\nu + K^\nu u^\mu)Q \cdot n + (K^\mu n^\nu + K^\nu n^\mu)Q \cdot u) + b(g^{\mu\nu}Q \cdot n \\ &\quad + (u^\mu n^\nu + u^\nu n^\mu)Q \cdot u - 2u^\mu u^\nu Q \cdot n) \}].\end{aligned}\quad (43)$$

The trace of third and fourth terms in Eq. (42) can be obtained as

$$\begin{aligned}\text{Tr}[\gamma_\mu S_0^*(K) \gamma_\nu S_2(Q)] + \text{Tr}[\gamma_\nu S_0^*(K) \gamma_\mu S_2(Q)] &= \frac{8(eB)^2}{D_+(Q^2 - m_f^2)^3} [q_0(\hat{K}^\mu g^{0\nu} + \hat{K}^\nu g^{0\mu} - g^{\mu\nu}) - q_3(\hat{K}^\mu g^{3\nu} + \hat{K}^\nu g^{3\mu} - g^{\mu\nu} \hat{k}_3) \\ &\quad - (\hat{K}^\mu Q^\nu + \hat{K}^\nu Q^\mu - g^{\mu\nu} \hat{K} \cdot Q)] - \frac{8(eB)^2 q_1^2}{D_+(Q^2 - m_f^2)^4} [\hat{K}^\mu Q^\nu + \hat{K}^\nu Q^\mu - g^{\mu\nu} \hat{K} \cdot Q] \\ &\quad + \frac{8(eB)^2}{D_-(Q^2 - m_f^2)^3} [q_0(\hat{K}'^\mu g^{0\nu} + \hat{K}'^\nu g^{0\mu} - g^{\mu\nu}) - q_3(\hat{K}'^\mu g^{3\nu} + \hat{K}'^\nu g^{3\mu} + g^{\mu\nu} \hat{k}_3) \\ &\quad - (\hat{K}'^\mu Q^\nu + \hat{K}'^\nu Q^\mu - g^{\mu\nu} \hat{K}' \cdot Q)] - \frac{8(eB)^2 q_1^2}{D_-(Q^2 - m_f^2)^4} [\hat{K}'^\mu Q^\nu + \hat{K}'^\nu Q^\mu - g^{\mu\nu} \hat{K}' \cdot Q],\end{aligned}\quad (44)$$

where $\hat{K}'^\mu = (1, -\hat{\mathbf{k}})$. The trace of fifth and sixth terms in Eq. (42) are obtained as

$$\begin{aligned} & \text{Tr}[\gamma_\mu S_2^*(K) \gamma_\nu S_0(Q)] + \text{Tr}[\gamma_\nu S_2^*(K) \gamma_\mu S_0(Q)] \\ &= \frac{8}{(Q^2 - m_f^2)} \left[\left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) (1+a)(K^\mu Q^\nu + K^\nu Q^\mu - g^{\mu\nu} K \cdot Q) + \left(b \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) - b' \frac{h}{D^4} \right) \right. \\ & \quad \left. \times (g^{\mu 0} Q^\nu + g^{\nu 0} Q^\mu - g^{\mu\nu} q_0) - c' \frac{h}{D^4} (g^{\mu 3} Q^\nu + g^{\nu 3} Q^\mu - g^{\mu\nu} q_3) \right]. \end{aligned} \quad (45)$$

The photon self-energy in weak field approximation now can be decomposed using Eqs. (39), (41), (42) as

$$\Pi^{\mu\nu}(P) = \Pi_0^{\mu\nu}(P) + \Pi_2^{\mu\nu}(P), \quad (46)$$

where the first term is a pure thermal ($\mathcal{O}[(eB)^0]$) contribution and second term is thermomagnetic correction of $\mathcal{O}[(eB)^2]$. Now the $\mathcal{O}[(eB)^0]$ expression of Π^{11} , Π^{22} , Π^{33} , and Π^{13} can be written from Eq. (39) as

$$\begin{aligned} \Pi_0^{11}(p_0, p) &= 8ie^2 \int \frac{d^4 K}{(2\pi)^4} \frac{(1+a)(k_0 q_0 + 2k_1 q_1 - \vec{k} \cdot \vec{q}) + bq_0}{\{((1+a)k_0 + b)^2 - (1+a)^2 k^2\} Q^2} \\ &= -4e^2 \oint \left[\left(\frac{f_0^{(1)}}{D_+} + \frac{f_0^{(1)}}{D_-} \right) + (2\hat{k}_1 q_1 - \hat{k} \cdot q) \left\{ \frac{f_0^{(0)}}{D_+} - \frac{f_0^{(0)}}{D_-} \right\} \right], \\ \Pi_0^{22}(p_0, p) &= 8ie^2 \int \frac{d^4 K}{(2\pi)^4} \frac{(1+a)(k_0 q_0 + 2k_2 q_2 - \vec{k} \cdot \vec{q}) + bq_0}{\{((1+a)k_0 + b)^2 - (1+a)^2 k^2\} Q^2} \\ &= -4e^2 \oint \left[\left(\frac{f_0^{(1)}}{D_+} + \frac{f_0^{(1)}}{D_-} \right) + (2\hat{k}_2 q_2 - \hat{k} \cdot q) \left\{ \frac{f_0^{(0)}}{D_+} - \frac{f_0^{(0)}}{D_-} \right\} \right], \\ \Pi_0^{33}(p_0, p) &= 8ie^2 \int \frac{d^4 K}{(2\pi)^4} \frac{(1+a)(k_0 q_0 + 2k_3 q_3 - \vec{k} \cdot \vec{q}) + bq_0}{\{((1+a)k_0 + b)^2 - (1+a)^2 k^2\} Q^2} \\ &= -4e^2 \oint \left[\left(\frac{f_0^{(1)}}{D_+} + \frac{f_0^{(1)}}{D_-} \right) - (\hat{k} \cdot q - 2\hat{k}_3 q_3) \left\{ \frac{f_0^{(0)}}{D_+} - \frac{f_0^{(0)}}{D_-} \right\} \right], \\ \Pi_0^{13}(p_0, p) &= 8ie^2 \int \frac{d^4 K}{(2\pi)^4} \frac{(1+a)(k_1 q_3 + q_1 k_3)}{\{((1+a)k_0 + b)^2 - (1+a)^2 k^2\} Q^2} = -4e^2 \oint \left[(\hat{k}_1 q_3 + q_1 \hat{k}_3) \left\{ \frac{f_0^{(0)}}{D_+} - \frac{f_0^{(0)}}{D_-} \right\} \right]. \end{aligned} \quad (47)$$

Using Eqs. (42), (43), (44), and (45), one can write the $\mathcal{O}[(eB)^2]$ expression of Π^{11} , Π^{22} , Π^{33} , and Π^{13} as

$$\begin{aligned} \Pi_2^{11} &= -e^2 \oint \left[-\frac{8eB}{D^2(Q^2 - m_f^2)^2} \{b'q_3 + c'q_0\} + \frac{8eB}{D^4(Q^2 - m_f^2)^2} h \{ (1+a)(k_0 q_3 - k_3 q_0) + bq_3 \} \right. \\ & \quad + \frac{8(eB)^2}{D_+(Q^2 - m_f^2)^3} (\hat{k}_2 q_2 - \hat{k}_1 q_1) - \frac{8(eB)^2 q_\perp^2}{D_+(Q^2 - m_f^2)^4} (q_0 - \hat{k} \cdot q + 2\hat{k}_1 q_1) - \frac{8(eB)^2}{D_-(Q^2 - m_f^2)^3} (\hat{k}_2 q_2 - \hat{k}_1 q_1) \\ & \quad - \frac{8(eB)^2 q_\perp^2}{D_-(Q^2 - m_f^2)^4} (q_0 + \hat{k} \cdot q - 2\hat{k}_1 q_1) \\ & \quad \left. + \frac{8}{(Q^2 - m_f^2)} \left\{ \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) (1+a)(2k_1 q_1 + K \cdot Q) + \left(b \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) - \frac{b'h}{D^4} \right) q_0 - \frac{c'h}{D^4} q_3 \right\} \right], \end{aligned} \quad (48)$$

$$\begin{aligned}
\Pi_2^{22} = & -e^2 \oint \left[-\frac{8eB}{D^2(Q^2 - m_f^2)^2} \{b'q_3 + c'q_0\} + \frac{8eB}{D^4(Q^2 - m_f^2)^2} h\{(1+a)(k_0q_3 - k_3q_0) + bq_3\} \right. \\
& + \frac{8(eB)^2}{D_+(Q^2 - m_f^2)^3} (\hat{k}_1q_1 - \hat{k}_2q_2) - \frac{8(eB)^2q_\perp^2}{D_+(Q^2 - m_f^2)^4} (q_0 - \hat{k} \cdot q + 2\hat{k}_2q_2) - \frac{8(eB)^2}{D_-(Q^2 - m_f^2)^3} (\hat{k}_1q_1 - \hat{k}_2q_2) \\
& - \frac{8(eB)^2q_\perp^2}{D_-(Q^2 - m_f^2)^4} (q_0 + \hat{k} \cdot q - 2\hat{k}_2q_2) + \frac{8}{(Q^2 - m_f^2)} \left\{ \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) (1+a)(2k_2q_2 + K \cdot Q) \right. \\
& \left. \left. + \left(b \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) - \frac{b'h}{D^4} \right) q_0 - \frac{c'h}{D^4} q_3 \right\} \right], \quad (49)
\end{aligned}$$

$$\begin{aligned}
\Pi_2^{33} = & -e^2 \oint \left[\frac{8eB}{D^2(Q^2 - m_f^2)^2} \{-b'q_3 + c'q_0\} + \frac{8eB}{D^4(Q^2 - m_f^2)^2} h\{(1+a)(k_0q_3 + k_3q_0) + bq_3\} \right. \\
& + \frac{8(eB)^2}{D_+(Q^2 - m_f^2)^3} (-q_3\hat{k}_3 + \hat{k} \cdot q) - \frac{8(eB)^2q_\perp^2}{D_+(Q^2 - m_f^2)^4} (q_0 - \hat{k} \cdot q + 2q_3\hat{k}_3) \\
& + \frac{8(eB)^2}{D_-(Q^2 - m_f^2)^3} (q_3\hat{k}_3 - \hat{k} \cdot q) - \frac{8(eB)^2q_\perp^2}{D_-(Q^2 - m_f^2)^4} (q_0 + \hat{k} \cdot q - 2q_3\hat{k}_3) \\
& \left. + \frac{8}{(Q^2 - m_f^2)} \left\{ \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) (1+a)(2k_3q_3 + K \cdot Q) + \left(b \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) - \frac{b'h}{D^4} \right) q_0 + \frac{c'h}{D^4} q_3 \right\} \right], \quad (50)
\end{aligned}$$

$$\begin{aligned}
\Pi_2^{13} = & -e^2 \oint \left[\frac{8eB}{D^4(Q^2 - m_f^2)^2} h(1+a)(k_1q_0) - \frac{8(eB)^2}{D_+(Q^2 - m_f^2)^3} \hat{k}_3q_1 - \frac{8(eB)^2q_\perp^2}{D_+(Q^2 - m_f^2)^4} \right. \\
& \times (\hat{k}_1q_3 + \hat{k}_3q_1) + \frac{8(eB)^2}{D_-(Q^2 - m_f^2)^3} \hat{k}_3q_1 + \frac{8(eB)^2q_\perp^2}{D_-(Q^2 - m_f^2)^4} (\hat{k}_1q_3 + \hat{k}_3q_1) + \frac{8}{(Q^2 - m_f^2)} \\
& \left. \times \left\{ \left(\frac{h^2}{D^6} - \frac{h'}{D^4} \right) (1+a)(k_1q_3 + k_3q_1) + \frac{c'h}{D^4} q_1 \right\} \right]. \quad (51)
\end{aligned}$$

IV. IMAGINARY PARTS OF THE COMPONENTS OF THE PHOTON SELF-ENERGY

Before obtaining the imaginary parts, we discuss below the various approximations used in this calculation.

- (1) We have considered the momentum of photon as hard ($p \gg T$). The momentum of soft fermion $k \ll T$. Thus we can take the following approximations:

$$n_F(\omega) \sim 1, \quad n_F(p - \omega) \sim e^{-p/T}, \quad e^{-p/T} \sim 0. \quad (52)$$

- (2) An upper cutoff $\Lambda (< T)$ of the soft fermion momentum k has been introduced in the integrations.
- (3) We consider $m_f = m_{\text{th}}$ for electron.
- (4) To perform the various integrations we choose a frame of reference as shown in Fig. 2 in which the external momentum of the photon in xz plane with $0 < \theta_p < \pi/2$. So one can write

$$\vec{p} \equiv (p \sin \theta_p, 0, p \cos \theta_p), \quad (53)$$

and then the loop momentum as

$$\vec{k} \equiv (k \sin \theta \cos \phi, k \sin \theta \sin \phi, k \cos \theta). \quad (54)$$

In the following subsection we will obtain imaginary parts of various self-energy components.

A. Imaginary parts of the magnetic field independent part, i.e., $\mathcal{O}[(eB)^0]$

We evaluate the imaginary parts of Π_0^{11} , Π_0^{22} , Π_0^{33} , and Π_0^{13} using the Braaten-Pisarski-Yuan method [45,47].

FIG. 2. Choice of reference frame for computing the various components of photon self-energy. The magnetic field is along z -direction and θ_p is the angle between momentum of photon and the external magnetic field.

$$\begin{aligned} \text{Im}\Pi_0^{11} = & -4e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \{ \rho_0^{(1)}(\omega') (\rho_{D_+}(\omega) + \rho_{D_-}(\omega)) \\ & - (\hat{k} \cdot q - 2\hat{k}_1 q_1) (\rho_{D_+}(\omega) - \rho_{D_-}(\omega)) \rho_0^{(0)}(\omega') \} \delta(\omega + \omega' - p), \end{aligned} \quad (55)$$

$$\begin{aligned} \text{Im}\Pi_0^{22} = & -4e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \{ \rho_0^{(1)}(\omega') (\rho_{D_+}(\omega) + \rho_{D_-}(\omega)) \\ & - (\hat{k} \cdot q - 2\hat{k}_2 q_2) (\rho_{D_+}(\omega) - \rho_{D_-}(\omega)) \rho_0^{(0)}(\omega') \} \delta(\omega + \omega' - p), \end{aligned} \quad (56)$$

$$\begin{aligned} \text{Im}\Pi_0^{33} = & -4e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \{ \rho_0^{(1)}(\omega') (\rho_{D_+}(\omega) + \rho_{D_-}(\omega)) \\ & - (\hat{k} \cdot q - 2\hat{k}_3 q_3) (\rho_{D_+}(\omega) - \rho_{D_-}(\omega)) \rho_0^{(0)}(\omega') \} \delta(\omega + \omega' - p), \end{aligned} \quad (57)$$

$$\begin{aligned} \text{Im}\Pi_0^{13} = & -4e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \{ (\hat{k}_1 q_3 + q_1 \hat{k}_3) (\rho_{D_+}(\omega) - \rho_{D_-}(\omega)) \rho_0^{(0)}(\omega') \} \delta(\omega + \omega' - p), \end{aligned} \quad (58)$$

where $\rho_0^{(0)}, \rho_0^{(1)}, \rho_1^{(0)}, \rho_1^{(1)}, \rho_{D_+}$, and ρ_{D_-} are spectral representation of $f_0^{(0)}, f_0^{(1)}, f_1^{(0)}, f_1^{(1)}, 1/D_+$, and $1/D_-$ respectively. These spectral functions are obtained in Appendix. We know that both ρ_{D_+} and ρ_{D_-} have pole containing the mass shell δ -function + Landau cut part in space like region whereas $\rho_0^{(0)}, \rho_0^{(1)}, \rho_1^{(0)}, \rho_1^{(1)}$ have only pole containing the mass shell δ function. Since imaginary parts of various components of the self-energy contain the product of two spectral functions, it would then have the *pole-pole* and the *pole-cut* contributions.

The pole-pole parts of $\text{Im}\Pi_{11}, \text{Im}\Pi_{22}, \text{Im}\Pi_{33}$ and $\text{Im}\Pi_{13}$ contain $\delta(p - \omega_{\pm} - q)$ where $\omega_{\pm}$ is the energy of the fermion quasiparticle, $\vec{k}$ and $\vec{q} = \vec{k} - \vec{p}$ are the momenta of soft and hard fermion, respectively. Hence $\omega_{\pm} > k$. The δ -function yields

$$p - \omega_{\pm} - q = 0$$

$$\cos \phi \approx \frac{\omega_{\pm}/k - \cos \theta \cos \theta_p}{\sin \theta \sin \theta_p}.$$

The value of $\frac{\omega_{\pm}/k - \cos \theta \cos \theta_p}{\sin \theta \sin \theta_p}$ excludes the range $[-1, 1]$ for all values of the parameters θ and θ_p . This restriction is valid for both thermal and the magnetic case. Thus pole-pole parts do not contribute in this calculation [44,45]. In $\mathcal{O}[(eB)^0]$ the contribution comes only from the *pole-cut* part.

1. Pole-cut part of $\mathcal{O}[(eB)^0]$

Now we would find the *pole-cut* part of the above self-energy components in Eqs. (55), (56), (57), and (58) as

$$\begin{aligned} \text{Im}\Pi_0^{11}|_{\text{pole-cut}} = & 2e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \left\{ \delta(\omega' - q) \Theta(k^2 - \omega^2) \times (\beta_+(\omega) + \beta_-(\omega)) \right. \\ & \left. - \frac{1}{q} (\hat{k} \cdot q - 2\hat{k}_1 q_1) \delta(\omega' - q) \Theta(k^2 - \omega^2) (\beta_+(\omega) - \beta_-(\omega)) \right\} \delta(p - \omega - \omega'), \\ = & -e^2\pi \int_0^{\Lambda} \frac{k^2 dk}{2\pi^2} \int_0^{\pi} \frac{1}{2} \sin \theta d\theta \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-k}^k d\omega \left\{ (\beta_+(\omega) + \beta_-(\omega)) - \frac{1}{q} (\hat{k} \cdot q - 2\hat{k}_1 q_1) (\beta_+(\omega) - \beta_-(\omega)) \right\} \\ & \times \delta(p - \omega - q), \end{aligned} \quad (59)$$

$$\begin{aligned} \text{Im}\Pi_0^{22}|_{\text{pole-cut}} = & 2e^2\pi(1 - e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \left\{ \delta(\omega' - q) \Theta(k^2 - \omega^2) (\beta_+(\omega) + \beta_-(\omega)) \right. \\ & \left. - \frac{1}{q} (\hat{k} \cdot q - 2\hat{k}_2 q_2) \delta(\omega' - q) \Theta(k^2 - \omega^2) (\beta_+(\omega) - \beta_-(\omega)) \right\} \delta(p - \omega - \omega') \\ = & -e^2\pi \int_0^{\Lambda} \frac{k^2 dk}{2\pi^2} \int_0^{\pi} \frac{1}{2} \sin \theta d\theta \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-k}^k d\omega \left\{ (\beta_+(\omega) + \beta_-(\omega)) - \frac{1}{q} (\hat{k} \cdot q - 2\hat{k}_2 q_2) (\beta_+(\omega) - \beta_-(\omega)) \right\} \\ & \times \delta(p - \omega - q), \end{aligned} \quad (60)$$

$$\begin{aligned}
\text{Im}\Pi_0^{33}|_{pole-cut} &= 2e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \left\{ \delta(\omega' - q) \Theta(k^2 - \omega^2) (\beta_+(\omega) + \beta_-(\omega)) \right. \\
&\quad \left. - \frac{1}{q} (\hat{k}_\perp q_\perp - \hat{k}_3 q_3) \delta(\omega' - q) \Theta(k^2 - \omega^2) (\beta_+(\omega) - \beta_-(\omega)) \right\} \delta(p - \omega - \omega') \\
&= -e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int_0^\pi \frac{1}{2} \sin\theta d\theta \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-k}^k d\omega \left\{ (\beta_+(\omega) + \beta_-(\omega)) - \frac{1}{q} (\hat{k}_\perp q_\perp - \hat{k}_3 q_3) (\beta_+(\omega) - \beta_-(\omega)) \right\} \\
&\quad \times \delta(p - \omega - q), \tag{61}
\end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_0^{13}|_{pole-cut} &= 2e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \left\{ \frac{\hat{k}_1 q_3 + q_1 \hat{k}_3}{q} \delta(\omega' - q) \right. \\
&\quad \left. \times (\beta_+(\omega) - \beta_-(\omega)) \Theta(k^2 - \omega^2) \right\} \delta(p - \omega - \omega') \\
&= -e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int_0^\pi \frac{1}{2} \sin\theta d\theta \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-k}^k d\omega \frac{1}{q} (\hat{k}_1 q_3 + q_1 \hat{k}_3) (\beta_+(\omega) - \beta_-(\omega)) \delta(p - \omega - q). \tag{62}
\end{aligned}$$

Here we note that the terms with $\delta(\omega' + q) \delta(p - \omega - \omega') \Theta(k^2 - \omega^2)$ will not contribute because $k^2 - (p + q)^2$ cannot be greater than zero. So we have excluded those terms.

B. Imaginary part of magnetic field dependent part of $\mathcal{O}[(eB)^2]$

Similar to $\mathcal{O}[(eB)^0]$ case, the imaginary part of Π_2^{11} , Π_2^{22} , Π_2^{33} and Π_2^{13} can be written as

$$\begin{aligned}
\text{Im}\Pi_2^{11} &= -8e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \\
&\quad \times [eB\{q_3\rho_1^{(0)}(\rho_9^{(1)} + \rho_{10} - \rho_7) - \rho_1^{(1)}(\rho_8 + k_3\rho_9^{(0)})\} + (eB)^2(\hat{k}_1 q_1 - \hat{k}_2 q_2)\rho_2^{(0)}(\rho_{D_-} - \rho_{D_+}) \\
&\quad - (eB)^2 q_\perp^2 \{\rho_3^{(1)}(\rho_{D_+} + \rho_{D_-}) + (2\hat{k}_1 q_1 - \hat{k} \cdot q)\rho_3^{(0)}(\rho_{D_+} - \rho_{D_-})\} \\
&\quad + \rho_0^{(1)}(\rho_{15}^{(1)} - \rho_{14}^{(1)} + \rho_{16} - \rho_{13} - \rho_{11}) + (2k_1 q_1 - \vec{k} \cdot \vec{q})\rho_0^{(0)}(\rho_{15}^{(0)} - \rho_{14}^{(0)}) - q_3\rho_0^{(0)}\rho_{12}] \delta(p - \omega - \omega'), \tag{63}
\end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_2^{22} &= -8e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \\
&\quad \times [eB\{q_3\rho_1^{(0)}(\rho_9^{(1)} + \rho_{10} - \rho_7) - \rho_1^{(1)}(\rho_8 + k_3\rho_9^{(0)})\} + (eB)^2(\hat{k}_2 q_2 - \hat{k}_1 q_1)\rho_2^{(0)}(\rho_{D_-} - \rho_{D_+}) \\
&\quad - (eB)^2 q_\perp^2 \{\rho_3^{(1)}(\rho_{D_+} + \rho_{D_-}) + (2\hat{k}_2 q_2 - \hat{k} \cdot q)\rho_3^{(0)}(\rho_{D_+} - \rho_{D_-})\} + \rho_0^{(1)}(\rho_{15}^{(1)} - \rho_{14}^{(1)} + \rho_{16} - \rho_{13} - \rho_{11}) \\
&\quad + (2k_2 q_2 - \vec{k} \cdot \vec{q})\rho_0^{(0)}(\rho_{15}^{(0)} - \rho_{14}^{(0)}) - q_3\rho_0^{(0)}\rho_{12}] \delta(p - \omega - \omega'), \tag{64}
\end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_2^{33} &= -8e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \\
&\quad \times [eB\{q_3\rho_1^{(0)}(\rho_9^{(1)} + \rho_{10} - \rho_7) + \rho_1^{(1)}(\rho_8 + k_3\rho_9^{(0)})\} + (eB)^2(q_3\hat{k}_3 - \hat{k} \cdot q)\rho_2^{(0)}(\rho_{D_-} - \rho_{D_+}) \\
&\quad - (eB)^2 q_\perp^2 \{\rho_3^{(1)}(\rho_{D_+} + \rho_{D_-}) + (2q_3\hat{k}_3 - \hat{k} \cdot q)\rho_3^{(0)}(\rho_{D_+} - \rho_{D_-})\} \\
&\quad + \rho_0^{(1)}(\rho_{15}^{(1)} - \rho_{14}^{(1)} + \rho_{16} - \rho_{13} - \rho_{11}) + (2k_3 q_3 - \vec{k} \cdot \vec{q})\rho_0^{(0)}(\rho_{15}^{(0)} - \rho_{14}^{(0)}) + q_3\rho_0^{(0)}\rho_{12}] \delta(p - \omega - \omega'), \tag{65}
\end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_2^{13} &= -8e^2\pi(1-e^{p/T}) \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' n_F(\omega) n_F(\omega') \\
&\quad \times [eB\{\rho_1^{(1)}k_3\rho_9^{(0)}\} + (eB)^2\hat{k}_3 q_1\rho_2^{(0)}\{\rho_{D_-} - \rho_{D_+}\} - (eB)^2 q_\perp^2 \{(\hat{k}_1 q_3 + \hat{k}_3 q_1)\rho_3^{(0)}(\rho_{D_+} - \rho_{D_-})\} \\
&\quad + (k_1 q_3 + k_3 q_1)\rho_0^{(0)}(\rho_{15}^{(0)} - \rho_{14}^{(0)}) + q_1\rho_0^{(0)}\rho_{12}] \delta(p - \omega - \omega'). \tag{66}
\end{aligned}$$

Various spectral functions are obtained in Appendix. As discussed before we also note that the imaginary part of various components of the self-energy contain the *pole-pole* and the *pole-cut* contributions. As explained earlier the phase space does not allow the *pole-pole* part to contribute in this order. In $\mathcal{O}[(eB)^2]$ the contribution comes only from the *pole-cut* part.

1. Pole-cut part of $\mathcal{O}[(eB)^2]$

Now the expressions of pole-cut parts of Eqs. (63), (64), (65), and (66) after using the approximations, are given below:

$$\begin{aligned}
 \text{Im}\Pi_2^{11}|_{\text{pole-cut}} = & 4e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int \frac{1}{2} \sin\theta d\theta \int \frac{d\phi}{2\pi} \int_{-k}^k d\omega \int_{-\infty}^\infty d\omega' \left[\delta'''(\omega' - \omega_q) \left\{ -\frac{(eB)^2 q_\perp^2}{96\omega_q^3} (\beta_+ + \beta_-) \right. \right. \\
 & \left. \left. - \frac{(eB)^2 q_\perp^2 (2\hat{k}_1 q_1 - \hat{k} \cdot q)}{96\omega_q^4} (\beta_+ - \beta_-) \right\} \right. \\
 & - \delta''(\omega' - \omega_q) \left\{ \frac{3(eB)^2}{64\omega_q^3} \left(\frac{q_\perp^2 (\hat{k}_1 q_1 - \hat{k} \cdot q)}{\omega_q^2} - (\hat{k}_1 q_1 - \hat{k}_2 q_2) \right) (\beta_+ - \beta_-) + \frac{3(eB)^2 q_\perp^2}{128\omega_q^4} (\beta_+ + \beta_-) \right\} \\
 & + \delta'(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^2} (\beta_9^{(1)} + \beta_{10} - \beta_7) - \frac{eB}{4\omega_q} (\beta_8 + k_3 \beta_9^{(0)}) - \frac{(eB)^2}{16\omega_q^4} \left(\hat{k}_2 q_2 - \hat{k}_1 q_1 + \frac{5q_\perp^2 (\hat{k}_1 q_1 - \hat{k} \cdot q)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
 & \left. \left. - \frac{(eB)^2 q_\perp^2}{32\omega_q^5} (\beta_+ + \beta_-) \right\} \right. \\
 & + \delta(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^3} (\beta_9^{(1)} + \beta_{10} - \beta_7) - \frac{(eB)^2}{16\omega_q^5} \left(3\hat{k}_2 q_2 - 3\hat{k}_1 q_1 + \frac{5q_\perp^2 (\hat{k}_1 q_1 - \hat{k} \cdot q)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
 & \left. \left. - \frac{1}{2} (\beta_{15}^{(1)} + \beta_{16} - (\beta_{14}^{(1)} + \beta_{11} + \beta_{13})) - \frac{(2k_1 q_1 - \vec{k} \cdot \vec{q})}{2\omega_q} (\beta_{15}^{(0)} - \beta_{14}^{(0)}) + \frac{q_3}{2\omega_q} \beta_{12} \right\} \right] \delta(p - \omega - \omega'), \quad (67)
 \end{aligned}$$

$$\begin{aligned}
 \text{Im}\Pi_2^{22}|_{\text{pole-cut}} = & 4e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int \frac{1}{2} \sin\theta d\theta \int \frac{d\phi}{2\pi} \int_{-k}^k d\omega \int_{-\infty}^\infty d\omega' \left[\delta'''(\omega' - \omega_q) \left\{ -\frac{(eB)^2 q_\perp^2}{96\omega_q^3} (\beta_+ + \beta_-) \right. \right. \\
 & \left. \left. - \frac{(eB)^2 q_\perp^2 (2\hat{k}_2 q_2 - \hat{k} \cdot q)}{96\omega_q^4} (\beta_+ - \beta_-) \right\} \right. \\
 & - \delta''(\omega' - \omega_q) \left\{ \frac{3(eB)^2}{64\omega_q^3} \left(\frac{q_\perp^2 (\hat{k}_2 q_2 - \hat{k} \cdot q)}{\omega_q^2} - (\hat{k}_2 q_2 - \hat{k}_1 q_1) \right) \times (\beta_+ - \beta_-) + \frac{3(eB)^2 q_\perp^2}{128\omega_q^4} (\beta_+ + \beta_-) \right\} \\
 & + \delta'(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^2} (\beta_9^{(1)} + \beta_{10} - \beta_7) - \frac{eB}{4\omega_q} (\beta_8 + k_3 \beta_9^{(0)}) - \frac{(eB)^2}{16\omega_q^4} \left(\hat{k}_1 q_1 - \hat{k}_2 q_2 + \frac{5q_\perp^2 (\hat{k}_2 q_2 - \hat{k} \cdot q)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
 & \left. \left. - \frac{(eB)^2 q_\perp^2}{32\omega_q^5} (\beta_+ + \beta_-) \right\} \right. \\
 & + \delta(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^3} (\beta_9^{(1)} + \beta_{10} - \beta_7) - \frac{(eB)^2}{16\omega_q^5} \left(3\hat{k}_1 q_1 - 3\hat{k}_2 q_2 + \frac{5q_\perp^2 (\hat{k}_2 q_2 - \hat{k} \cdot q)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
 & \left. \left. - \frac{1}{2} (\beta_{15}^{(1)} + \beta_{16} - (\beta_{14}^{(1)} + \beta_{11} + \beta_{13})) - \frac{(2k_2 q_2 - \vec{k} \cdot \vec{q})}{2\omega_q} (\beta_{15}^{(0)} - \beta_{14}^{(0)}) + \frac{q_3}{2\omega_q} \beta_{12} \right\} \right] \delta(p - \omega - \omega'), \quad (68)
 \end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_2^{33}|_{pole-cut} = & 4e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int \frac{1}{2} \sin\theta d\theta \int \frac{d\phi}{2\pi} \int_{-k}^k d\omega \int_{-\infty}^\infty d\omega' \left[\delta'''(\omega' - \omega_q) \left\{ -\frac{(eB)^2 q_\perp^2}{96\omega_q^3} (\beta_+ + \beta_-) \right. \right. \\
& - \frac{(eB)^2 q_\perp^2 (\hat{k}_3 q_3 - \hat{k}_\perp q_\perp)}{96\omega_q^4} (\beta_+ - \beta_-) \left. \right\} \\
& - \delta''(\omega' - \omega_q) \left\{ \frac{3(eB)^2}{64\omega_q^3} \left(\frac{q_\perp^2 (\hat{k}_3 q_3 - \hat{k}_\perp q_\perp)}{\omega_q^2} + \hat{k}_\perp q_\perp \right) \times (\beta_+ - \beta_-) + \frac{3(eB)^2 q_\perp^2}{128\omega_q^4} (\beta_+ + \beta_-) \right\} \\
& + \delta'(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^2} ((\beta_9^{(1)} + \beta_{10} - \beta_7) + \frac{eB}{4\omega_q} (\beta_8 + k_3 \rho_9^{(0)})) \right. \\
& - \frac{(eB)^2}{16\omega_q^4} \left(\hat{k}_\perp q_\perp + \frac{5q_\perp^2 (\hat{k}_3 q_3 - \hat{k}_\perp q_\perp)}{2\omega_q^2} \right) (\beta_+ - \beta_-) - \frac{(eB)^2 q_\perp^2}{32\omega_q^5} (\beta_+ + \beta_-) \left. \right\} \\
& + \delta(\omega' - \omega_q) \left\{ \frac{eB q_3}{4\omega_q^3} (\beta_9^{(1)} + \beta_{10} - \beta_7) - \frac{(eB)^2}{16\omega_q^5} \left(3\hat{k}_\perp q_\perp + \frac{5q_\perp^2 (\hat{k}_3 q_3 - \hat{k}_\perp q_\perp)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
& - \frac{1}{2} (\beta_{15}^{(1)} + \beta_{16} - (\beta_{14}^{(1)} + \beta_{13} + \beta_{11})) - \frac{(2k_3 q_3 - \vec{k} \cdot \vec{q})}{2\omega_q} (\beta_{15}^{(0)} - \beta_{14}^{(0)}) - \frac{q_3}{2\omega_q} \beta_{12} \left. \right\} \delta(p - \omega - \omega'), \quad (69)
\end{aligned}$$

$$\begin{aligned}
\text{Im}\Pi_2^{13}|_{pole-cut} = & 4e^2\pi \int_0^\Lambda \frac{k^2 dk}{2\pi^2} \int \frac{1}{2} \sin\theta d\theta \int \frac{d\phi}{2\pi} \int_{-k}^k d\omega \int_{-\infty}^\infty d\omega' \left[\delta'''(\omega' - \omega_q) \right. \\
& \times \left\{ -\frac{(eB)^2 q_\perp^2 (\hat{k}_1 q_3 + \hat{k}_3 q_1)}{96\omega_q^4} (\beta_+ - \beta_-) \right\} - \delta''(\omega' - \omega_q) \left\{ \frac{3(eB)^2}{64\omega_q^3} \left(\frac{q_\perp^2 (\hat{k}_1 q_3 + \hat{k}_3 q_1)}{\omega_q^2} - \hat{k}_3 q_1 \right) (\beta_+ - \beta_-) \right\} \\
& + \delta'(\omega' - \omega_q) \left\{ \frac{eB}{4\omega_q} k_1 \beta_9^{(0)} - \frac{(eB)^2}{16\omega_q^4} \left(-\hat{k}_3 q_1 + \frac{5q_\perp^2 (\hat{k}_1 q_3 + \hat{k}_3 q_1)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right\} \\
& + \delta(\omega' - \omega_q) \left\{ -\frac{(eB)^2}{16\omega_q^5} \left(-3\hat{k}_3 q_1 + \frac{5q_\perp^2 (\hat{k}_1 q_3 + \hat{k}_3 q_1)}{2\omega_q^2} \right) (\beta_+ - \beta_-) \right. \\
& \left. - \frac{(k_1 q_3 + k_3 q_1)}{2\omega_q} (\beta_{15}^{(0)} - \beta_{14}^{(0)}) - \frac{q_1}{2\omega_q} \beta_{12} \right\} \delta(p - \omega - \omega'). \quad (70)
\end{aligned}$$

V. RESULTS

We perform the integrations in Eqs. (59), (60), (61), (62), (67), (68), (69), and (70) numerically. In this calculation we have taken $m_\pi = 0.14$ GeV. The results are shown for $\Lambda = 0.25$ GeV which satisfies $eT \ll \Lambda \ll T$.

The damping rate of photon in presence of magnetic field depends on the angle, θ_p , between the momentum of photon and the magnetic field. Figure 3 shows the variation of the damping rate of a hard photon with the propagation angle. It increases with the increasing propagation angle. One can see that the two transverse modes of a hard photon are damped in a similar fashion. Since the magnetic field strength is very weak, this difference appears to be very small. We note that the magnetic correction is $\sim \mathcal{O}[(eB)^2]$ and switching the magnetic field from z to $-z$ direction would not affect the result. These two orientations of the magnetic field correspond to the propagation angle of photon θ_p and $\pi - \theta_p$. These two situations are identical which correspond to the same damping rates of photon at θ_p and $\pi - \theta_p$.

FIG. 3. Plot of damping rate of photon with the propagation angle θ_p for $p = 3$ GeV, $T = 0.5$ GeV and $eB = m_\pi^2/4$.

FIG. 4. Plot of damping rate of photon with the energy for $T = 0.5$ GeV and $eB = m_\pi^2/4$ at propagation angles $\theta_p = \pi/10$ and $\pi/2$.

In Fig. 4 we display the damping rate as a function of photon momentum for two propagation angles $\pi/10$ and $\pi/2$. The soft contribution of the damping rate in a thermal medium agrees well with that obtained in Ref. [45]. In presence of a thermomagnetic medium, the soft contribution to the damping rate is found to be reduced than that of the thermal one. For small propagation angle, the reduction of the damping rate is more compared to that of thermal medium. For higher momentum the damping rate approaches the thermal value as the temperature becomes the dominant scale as compared to the strength of the magnetic field considered.

Figure 5 displays the variation of damping rate with temperature for a specific value of momentum and magnetic

FIG. 5. Plot of damping rate of the hard photon with temperature at $p = 3$ GeV and $eB = m_\pi^2/4$ for two propagation angles $\pi/10$ and $\pi/2$.

FIG. 6. Plot of damping rate of the hard photon with the magnetic field strength at $T = 0.5$ GeV and $p = 3$ GeV for two propagation angles $\pi/10$ and $\pi/2$.

field for two propagation angles $\pi/10$ and $\pi/2$. It is found that the soft contribution to the damping rate increases with the increase in temperature both in thermal and thermomagnetic medium. For small propagation angle the damping rate is more reduced compared to that of large propagation angle. This observation is consistent with Fig. 4.

Figure 6 shows the variation of the damping rate with the magnetic field strength for specific values of photon momentum and temperature for two propagation angles. The thermal damping rate ($\mathcal{O}[(eB)^0]$) is represented by the black dashed horizontal line. The thermomagnetic damping rate decreases with the increasing magnetic field. At smaller propagation angles the photons are less damped

FIG. 7. Plot of damping rate of photon with Λ for $\theta_p = \pi/4$, $p = 3$ GeV, $T = 0.5$ GeV and $eB = m_\pi^2/4$.

than that of higher propagation angles which are consistent with Fig. 4.

Figure 7 shows the variation of the photon damping rate with the separation scale Λ keeping the scale hierarchy $eT \ll \Lambda \ll T$. As the allowed phase space increases with the increase of Λ , the damping rate is also found to increase with it.³ The magnetic correction to the thermal damping rate is negative. So, the difference between the thermal and thermomagnetic damping rate increases with Λ .

VI. CONCLUSION

We have calculated the soft contribution to the damping rate of a hard photon in a weakly magnetized QED medium where momentum of one of the fermion in the loop is considered as soft. The two degenerate transverse modes of photon in thermal medium are damped in a similar fashion in presence of weak magnetic field as shown in Fig. 3. The difference between two transverse modes is very marginal due to weak field approximation. The soft contribution to the damping rate in the thermomagnetic medium is reduced compared to that of the thermal medium. When the magnetic field is switched off, the thermomagnetic damping modes reduce to its thermal value. The effect of magnetic field is found to be dominant at low temperature and low photon momentum.

The soft contribution to the hard photon damping rate is $\sim 10^{-6}$ GeV. Thus, a photon of a few GeV energy traversing in the QED medium of temperature ~ 0.5 GeV and background magnetic field ~ 0.005 GeV² has a mean free path ($\lambda = \gamma^{-1}/2$) of a few Å. When the present calculation is extended to the case of relativistic heavy ion collisions, the mean free path of photon is found to be of a few hundred fm. This confirms that the mean free path of photon is larger than the size of the fireball and photon can be treated as a direct probe.

The damping rate is found to be dependent on the separation scale Λ . One needs to add the hard contribution with the soft contribution to cancel the Λ dependence of the result. The hard contribution to the photon damping rate comes from two-loop order with hard particles in the loop having momentum of the order of or higher than the temperature. This itself is a huge calculation which is in progress.

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APPENDIX: SPECTRAL REPRESENTATION OF THE PROPAGATORS

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{\epsilon}{x^2 + \epsilon^2} f(x) &\approx f(0) \int_{-\infty}^{\infty} dx \frac{\epsilon}{x^2 + \epsilon^2} \left\{ \begin{array}{l} \text{significant contribution comes from} \\ \text{integration, where } x \simeq 0 \end{array} \right. \\ &= f(0) \epsilon \int_{-\infty}^{\infty} dx \frac{1}{x^2 + \epsilon^2} = \pi f(0), \end{aligned} \quad (\text{A1})$$

where $f(x)$ is a test function.

From the above equations we can write,

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x), \quad (\text{A2})$$

$$\lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{x + i\epsilon} = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right] = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \frac{-2i\epsilon}{x^2 + \epsilon^2} = -\pi \delta(x). \quad (\text{A3})$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{2\epsilon x}{(x^2 + \epsilon^2)^2} f(x) &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \epsilon f(x) \frac{d}{dx} \left[-\frac{1}{(x^2 + \epsilon^2)} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx f'(x) \frac{\epsilon}{x^2 + \epsilon^2} = \pi f'(0) = \pi \int dx f'(x) \delta(x) = -\pi \int dx f(x) \delta'(x). \end{aligned} \quad (\text{A4})$$

From the above equation we find,

$$\lim_{\epsilon \rightarrow 0} \frac{2\epsilon x}{(x^2 + \epsilon^2)^2} = -\pi \delta'(x). \quad (\text{A5})$$

³Nevertheless, the damping rate is expected to be Λ independent when hard contribution is added.

Now using Eq. (A5) one can calculate,

$$\lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{(x + i\epsilon)^2} = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{(x + i\epsilon)^2} - \frac{1}{(x - i\epsilon)^2} \right] = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \frac{-i4\epsilon x}{(x^2 + \epsilon^2)^2} = \pi \delta'(x). \quad (\text{A6})$$

Similarly,

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{\epsilon^3 - 3x^2\epsilon}{(x^2 + \epsilon^2)^3} f(x) = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{\epsilon^3}{(x^2 + \epsilon^2)^3} f(x) - \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{3x^2\epsilon}{(x^2 + \epsilon^2)^3} f(x) = I_1 + I_2, \quad (\text{A7})$$

where

$$\begin{aligned} I_1 &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{\epsilon^3 f(x)}{(x^2 + \epsilon^2)^3} \approx \lim_{\epsilon \rightarrow 0} \epsilon^3 f(0) \int_{-\infty}^{\infty} dx \frac{1}{(x^2 + \epsilon^2)^3} = \lim_{\epsilon \rightarrow 0} \epsilon^3 f(0) \frac{3}{8} \frac{1}{\epsilon^5} = \lim_{\epsilon \rightarrow 0} \frac{3}{8\epsilon^2} f(0), \\ I_2 &= -3 \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \frac{x^2 \epsilon f(x)}{(x^2 + \epsilon^2)^3} = -3 \lim_{\epsilon \rightarrow 0} \epsilon \int_{-\infty}^{\infty} dx f(x) \left[\frac{1}{8} \frac{d^2}{dx^2} \left(\frac{1}{x^2 + \epsilon^2} \right) + \frac{1}{8} \frac{2}{(x^2 + \epsilon^2)^2} \right] \\ &= -\lim_{\epsilon \rightarrow 0} \frac{3}{8\epsilon^2} f(0) - \frac{3}{8} \lim_{\epsilon \rightarrow 0} \epsilon \int_{-\infty}^{\infty} dx f(x) \frac{d^2}{dx^2} \left(\frac{1}{x^2 + \epsilon^2} \right). \end{aligned} \quad (\text{A8})$$

So,

$$\begin{aligned} I_1 + I_2 &= -\frac{3}{8} \lim_{\epsilon \rightarrow 0} \epsilon \int_{-\infty}^{\infty} dx f(x) \frac{d^2}{dx^2} \left(\frac{1}{x^2 + \epsilon^2} \right) = \frac{3}{8} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \epsilon f'(x) \frac{d}{dx} \left(\frac{1}{x^2 + \epsilon^2} \right) \\ &= -\frac{3}{8} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \epsilon f''(x) \frac{1}{x^2 + \epsilon^2} = -\frac{3}{8} \pi \int_{-\infty}^{\infty} dx f''(x) \delta(x) = -\frac{3}{8} \pi \int_{-\infty}^{\infty} dx f(x) \delta''(x). \end{aligned} \quad (\text{A9})$$

We can conclude from the last few steps that,

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon^3 - 3x^2\epsilon}{(x^2 + \epsilon^2)^3} = -\frac{3}{8} \pi \delta''(x). \quad (\text{A10})$$

Using Eq. (A10) we can find

$$\lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{(x + i\epsilon)^3} = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{(x + i\epsilon)^3} - \frac{1}{(x - i\epsilon)^3} \right] = \lim_{\epsilon \rightarrow 0} \frac{\epsilon^3 - 6\epsilon x^2}{(x^2 + \epsilon^2)^3} = -\frac{3}{8} \pi \delta''(x) \quad (\text{A11})$$

Now,

$$\lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{(x + i\epsilon)^4} = \frac{1}{2i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{(x + i\epsilon)^4} - \frac{1}{(x - i\epsilon)^4} \right] = \lim_{\epsilon \rightarrow 0} \frac{4x\epsilon^3 - 4x^3\epsilon}{(x^2 + \epsilon^2)^4}, \quad (\text{A12})$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx f(x) \frac{4x\epsilon^3 - 4x^3\epsilon}{(x^2 + \epsilon^2)^4} &= \lim_{\epsilon \rightarrow 0} 4\epsilon \int_{-\infty}^{\infty} dx f(x) \frac{1}{24} \frac{d^3}{dx^3} \left(\frac{1}{x^2 + \epsilon^2} \right) \\ &= -\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{6} \int_{-\infty}^{\infty} dx f'''(x) \frac{1}{x^2 + \epsilon^2} = -\frac{\pi f'''(0)}{6} = \frac{\pi}{6} \int_{-\infty}^{\infty} dx f(x) \delta'''(x). \end{aligned} \quad (\text{A13})$$

$$\text{Thus, } \lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{(x + i\epsilon)^4} = \frac{\pi}{6} \delta'''(x). \quad (\text{A14})$$

Now we write the spectral representations for the free propagators as

$$\rho_0^{(1)}(\omega', q) = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{\omega' + i\epsilon + \omega_q} + \frac{1}{\omega' + i\epsilon - \omega_q} \right] = -\frac{1}{2} [\delta(\omega' + \omega_q) + \delta(\omega' - \omega_q)], \quad (\text{A15})$$

$$\rho_0^{(0)}(\omega', q) = \frac{1}{2\pi\omega_q} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{\omega' + i\epsilon - \omega_q} - \frac{1}{\omega' + i\epsilon + \omega_q} \right] = \frac{1}{2\omega_q} [\delta(\omega' + \omega_q) - \delta(\omega' - \omega_q)], \quad (\text{A16})$$

$$\rho_1^{(1)}(\omega', q) = \frac{1}{4\pi\omega_q} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon - \omega_q)^2} - \frac{1}{(\omega' + i\epsilon + \omega_q)^2} \right] = -\frac{1}{4\omega_q} [\delta'(\omega' + \omega_q) - \delta'(\omega' - \omega_q)], \quad (\text{A17})$$

$$\begin{aligned} \rho_1^{(0)}(\omega', q) &= \frac{1}{4\pi\omega_q^2} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon + \omega_q)^2} + \frac{1}{(\omega' + i\epsilon - \omega_q)^2} - \frac{1}{\omega_q} \left(\frac{1}{\omega' + i\epsilon - \omega_q} - \frac{1}{\omega' + i\epsilon + \omega_q} \right) \right] \\ &= \frac{1}{4\omega_q^2} \left[\delta'(\omega' + \omega_q) + \delta'(\omega' - \omega_q) - \frac{1}{\omega_q} (\delta(\omega' + \omega_q) - \delta(\omega' - \omega_q)) \right], \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \rho_2^{(1)}(\omega', q) &= \frac{1}{8\pi\omega_q^2} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon + \omega_q)^3} + \frac{1}{(\omega' + i\epsilon - \omega_q)^3} + \frac{1}{2\omega_q} \left\{ \frac{1}{(\omega' + i\epsilon + \omega_q)^2} - \frac{1}{(\omega' + i\epsilon - \omega_q)^2} \right\} \right] \\ &= \frac{1}{8\omega_q^2} \left[-\frac{3}{8} (\delta''(\omega' + \omega_q) + \delta''(\omega' - \omega_q)) + \frac{1}{2\omega_q} (\delta'(\omega' + \omega_q) - \delta'(\omega' - \omega_q)) \right], \end{aligned} \quad (\text{A19})$$

$$\begin{aligned} \rho_2^{(0)}(\omega', q) &= -\frac{1}{8\pi\omega_q^3} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon + \omega_q)^3} - \frac{1}{(\omega' + i\epsilon - \omega_q)^3} \right. \\ &\quad \left. + \frac{1}{2\omega_q} \left\{ \frac{1}{(\omega' + i\epsilon + \omega_q)^2} + \frac{1}{(\omega' + i\epsilon - \omega_q)^2} + \frac{3}{\omega_q} \left(\frac{1}{\omega' + i\epsilon + \omega_q} - \frac{1}{\omega' + i\epsilon - \omega_q} \right) \right\} \right] \\ &= -\frac{1}{8\omega_q^3} \left[-\frac{3}{8} (\delta''(\omega' + \omega_q) - \delta''(\omega' - \omega_q)) + \frac{1}{2\omega_q} \left\{ \delta'(\omega' + \omega_q) + \delta'(\omega' - \omega_q) - \frac{3}{\omega_q} (\delta(\omega' + \omega_q) - \delta(\omega' - \omega_q)) \right\} \right], \end{aligned} \quad (\text{A20})$$

$$\begin{aligned} \rho_3^{(1)}(\omega', q) &= -\frac{1}{16\pi\omega_q^3} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon + \omega_q)^4} - \frac{1}{(\omega' + i\epsilon - \omega_q)^4} \right. \\ &\quad \left. + \frac{1}{\omega_q} \left\{ \frac{1}{(\omega' + i\epsilon + \omega_q)^3} + \frac{1}{(\omega' + i\epsilon - \omega_q)^3} + \frac{1}{2\omega_q} \left(\frac{1}{(\omega' + i\epsilon + \omega_q)^2} - \frac{1}{(\omega' + i\epsilon - \omega_q)^2} \right) \right\} \right] \end{aligned} \quad (\text{A21})$$

$$= -\frac{1}{16\omega_q^3} \left[\frac{1}{6} (\delta'''(\omega' + \omega_q) - \delta'''(\omega' - \omega_q)) + \frac{1}{\omega_q} \left\{ -\frac{3}{8} (\delta''(\omega' + \omega_q) + \delta''(\omega' - \omega_q)) + \frac{1}{2\omega_q} (\delta'(\omega' + \omega_q) - \delta'(\omega' - \omega_q)) \right\} \right], \quad (\text{A22})$$

$$\begin{aligned} \rho_3^{(0)}(\omega', q) &= \frac{1}{16\pi\omega_q^4} \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{1}{(\omega' + i\epsilon + \omega_q)^4} + \frac{1}{(\omega' + i\epsilon - \omega_q)^4} + \frac{1}{2\omega_q} \left\{ \frac{4}{(\omega' + i\epsilon + \omega_q)^3} - \frac{4}{(\omega' + i\epsilon - \omega_q)^3} \right. \right. \\ &\quad \left. \left. + \frac{1}{2\omega_q} \left(\frac{10}{(\omega' + i\epsilon + \omega_q)^2} + \frac{10}{(\omega' + i\epsilon - \omega_q)^2} + \frac{1}{2\omega_q} \left(\frac{20}{\omega' + i\epsilon + \omega_q} - \frac{20}{\omega' + i\epsilon - \omega_q} \right) \right) \right\} \right] \end{aligned} \quad (\text{A23})$$

$$\begin{aligned} &= \frac{1}{16\omega_q^4} \left[\frac{1}{6} (\delta'''(\omega' + \omega_q) + \delta'''(\omega' - \omega_q)) + \frac{1}{2\omega_q} \left\{ -\frac{3}{2} (\delta''(\omega' + \omega_q) - \delta''(\omega' - \omega_q)) + \frac{5}{\omega_q} \left(\delta'(\omega' + \omega_q) + \delta'(\omega' - \omega_q) \right. \right. \right. \\ &\quad \left. \left. - \frac{1}{\omega_q} (\delta(\omega' + \omega_q) - \delta(\omega' - \omega_q)) \right) \right\} \right]. \end{aligned} \quad (\text{A24})$$

The effective propagators are given as,

$$\frac{1}{D^2} = \frac{1}{D_+ D_-} = \sum_i \left(\frac{\partial D^2}{\partial \omega} \right)^{-1} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)}, \quad (\text{A25})$$

$$\begin{aligned} \frac{1}{D^4} &= \frac{1}{(D_+ D_-)^2} = \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \right)^{-2} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^2} + \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)} \right] \\ &= \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \right)^{-2} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^2} - \frac{\partial^2 D^2}{\partial \omega^2} \left(\frac{1}{3!} \frac{\partial^3 D^6}{\partial \omega^3} \right)^{-1} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)} \right], \end{aligned} \quad (\text{A26})$$

$$\begin{aligned} \frac{1}{D^6} &= \frac{1}{(D_+ D_-)^3} = \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \right)^{-3} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^3} + \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^2} + \frac{1}{2} \frac{\partial^2}{\partial \omega^2} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)} \right] \\ &= \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \right)^{-3} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^3} - \frac{3}{2} \frac{\partial^2 D^2}{\partial \omega^2} \left(\frac{1}{4!} \frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)^2} \right. \\ &\quad \left. - \frac{3}{5} \left(\frac{\partial^3 D^2}{\partial \omega^3} \left\{ 6 \left(\frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} + \frac{7}{12} \left(\frac{\partial D^2}{\partial \omega} \right)^{-4} \right\} + 6 \frac{\partial^2 D^2}{\partial \omega^2} \frac{\partial}{\partial \omega} \left(\frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} \right) \Big|_{\omega=\omega_i} \frac{1}{(\omega - \omega_i)} \right], \end{aligned} \quad (\text{A27})$$

where $\omega_i = \pm \omega_{\pm}$ are the poles of D_+ and D_- .

The spectral functions of the dressed propagators are given as

$$\rho_{D_{\pm}} = -\frac{(\omega^2 - k^2)}{2m_{\text{th}}^2} [\delta(\omega - \omega_{\pm}) + \delta(\omega + \omega_{\mp})] + \beta_{\pm} \Theta(k^2 - \omega^2), \quad (\text{A28})$$

where

$$\beta_{\pm} = \frac{-\frac{1}{2}(k \mp \omega)m_{\text{th}}^2}{[k(\omega \mp k) - m_{\text{th}}^2 \{Q_0(\frac{\omega}{k}) \mp Q_1(\frac{\omega}{k})\}]^2 + [\frac{1}{2}(1 \mp \frac{\omega}{k})m_{\text{th}}^2 \pi]^2}, \quad (\text{A29})$$

where we use the Legendre function of second kind

$$Q_0\left(\frac{\omega}{k}\right) = \frac{1}{2} \ln \left| \frac{\omega + k}{\omega - k} \right| - i \frac{\pi}{2} \Theta(k^2 - \omega^2). \quad (\text{A30})$$

$$\begin{aligned} \rho_4(\omega, k) &= \frac{1}{\pi} \text{Im} \left(\frac{1}{D^2} \right) = -\sum_i \left(\frac{\partial D^2}{\partial \omega} \right)^{-1} \Big|_{\omega=\omega_i} \delta(\omega - \omega_i) + \beta_4 \Theta(k^2 - \omega^2) \\ &= \frac{\omega^2 - k^2}{4m_{\text{th}}^2(k^2 - \omega^2 + m_{\text{th}}^2)} [(\omega - k)(\delta(\omega - \omega_+) + \delta(\omega + \omega_-)) + (\omega + k)(\delta(\omega - \omega_-) + \delta(\omega + \omega_+))] + \beta_4 \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A31})$$

$$\rho_5(\omega, k) = \frac{1}{\pi} \text{Im} \left(\frac{1}{D^4} \right) = \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \right)^{-2} \Big|_{\omega=\omega_i} \delta'(\omega - \omega_i) + \frac{\partial^2 D^2}{\partial \omega^2} \left(\frac{1}{3!} \frac{\partial^3 D^6}{\partial \omega^3} \right)^{-1} \Big|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_5 \Theta(k^2 - \omega^2), \quad (\text{A32})$$

$$\begin{aligned} \rho_6(\omega, k) &= \frac{1}{\pi} \text{Im} \left(\frac{1}{D^6} \right) = \sum_i \left[-\frac{3}{8} \left(\frac{\partial D^2}{\partial \omega} \right)^{-3} \Big|_{\omega=\omega_i} \delta''(\omega - \omega_i) - \frac{3}{2} \frac{\partial^2 D^2}{\partial \omega^2} \left(\frac{1}{4!} \frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} \Big|_{\omega=\omega_i} \delta'(\omega - \omega_i) \right. \\ &\quad \left. + \frac{3}{5} \left(\frac{\partial^3 D^2}{\partial \omega^3} \left\{ 6 \left(\frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} + \frac{7}{12} \left(\frac{\partial D^2}{\partial \omega} \right)^{-4} \right\} + 6 \frac{\partial^2 D^2}{\partial \omega^2} \frac{\partial}{\partial \omega} \left(\frac{\partial^4 D^8}{\partial \omega^4} \right)^{-1} \right) \Big|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_6 \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A33})$$

$$\rho_7(\omega, k) = \frac{1}{\pi} \text{Im} \left(\frac{b'}{D^2} \right) = -b' \sum_i \left(\frac{\partial D^2}{\partial \omega} \Big|_{\omega=\omega_i} \right)^{-1} \delta(\omega - \omega_i) + \beta_7 \Theta(k^2 - \omega^2), \quad (\text{A34})$$

$$\rho_8(\omega, k) = \frac{1}{\pi} \text{Im} \left(\frac{c'}{D^2} \right) = -c' \sum_i \left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-1} \delta(\omega - \omega_i) + \beta_8 \Theta(k^2 - \omega^2), \quad (\text{A35})$$

$$\begin{aligned} \rho_9^{(0)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left(\frac{h(1+a)}{D^4} \right) = h(1+a) \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] \\ &\quad + \beta_9^{(0)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A36})$$

$$\begin{aligned} \rho_9^{(1)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left(\frac{h(1+a)\omega}{D^4} \right) = \omega h(1+a) \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] \\ &\quad + \beta_9^{(1)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A37})$$

$$\rho_{10}(\omega, k) = \frac{1}{\pi} \text{Im} \left(\frac{hb}{D^4} \right) = hb \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{10} \Theta(k^2 - \omega^2), \quad (\text{A38})$$

$$\rho_{11}(\omega, k) = \frac{1}{\pi} \text{Im} \left[\frac{hb'}{D^4} \right] = hb' \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{11} \Theta(k^2 - \omega^2), \quad (\text{A39})$$

$$\rho_{12}(\omega, k) = \frac{1}{\pi} \text{Im} \left[\frac{hc'}{D^4} \right] = hc' \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{12} \Theta(k^2 - \omega^2), \quad (\text{A40})$$

$$\rho_{13}(\omega, k) = \frac{1}{\pi} \text{Im} \left[\frac{h'b}{D^4} \right] = h'b \times \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{13} \Theta(k^2 - \omega^2), \quad (\text{A41})$$

$$\begin{aligned} \rho_{14}^{(0)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left[\frac{h'(1+a)}{D^4} \right] = h'(1+a) \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] \\ &\quad + \beta_{14}^{(0)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A42})$$

$$\begin{aligned} \rho_{14}^{(1)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left[\frac{k_0 h'(1+a)}{D^4} \right] = \omega h'(1+a) \sum_i \left[\left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-2} \delta'(\omega - \omega_i) - \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^2}{D^4} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] \\ &\quad + \beta_{14}^{(1)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A43})$$

$$\begin{aligned} \rho_{15}^{(0)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left[\frac{h^2(1+a)}{D^6} \right] = h^2(1+a) \times \sum_i \left[-\frac{3}{8} \left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-3} \delta''(\omega - \omega_i) + \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta'(\omega - \omega_i) \right. \\ &\quad \left. - \frac{\partial^2}{\partial \omega^2} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{15}^{(0)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A44})$$

$$\begin{aligned} \rho_{15}^{(1)}(\omega, k) &= \frac{1}{\pi} \text{Im} \left[\frac{h^2(1+a)k_0}{D^6} \right] = \omega h^2(1+a) \times \sum_i \left[-\frac{3}{8} \left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-3} \delta''(\omega - \omega_i) + \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta'(\omega - \omega_i) \right. \\ &\quad \left. - \frac{\partial^2}{\partial \omega^2} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{15}^{(1)} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A45})$$

$$\begin{aligned} \rho_{16}(\omega, k) &= \frac{1}{\pi} \text{Im} \left[\frac{h^2 b}{D^6} \right] = h^2 b \times \sum_i \left[-\frac{3}{8} \left(\frac{\partial D^2}{\partial \omega} \bigg|_{\omega=\omega_i} \right)^{-3} \delta''(\omega - \omega_i) + \frac{\partial}{\partial \omega} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta'(\omega - \omega_i) \right. \\ &\quad \left. - \frac{\partial^2}{\partial \omega^2} \left\{ \frac{(\omega - \omega_i)^3}{D^6} \right\} \bigg|_{\omega=\omega_i} \delta(\omega - \omega_i) \right] + \beta_{16} \Theta(k^2 - \omega^2), \end{aligned} \quad (\text{A46})$$

where cut parts of the spectral functions are given as

$$\beta_4 = \frac{1}{\pi} \text{Im} \left(\frac{1}{D^2} \right) = -\frac{1}{\pi} \frac{\text{Im } D^2}{(\text{Re } D^2)^2 + (\text{Im } D^2)^2}, \quad (\text{A47})$$

$$\beta_5 = \frac{1}{\pi} \text{Im} \left(\frac{1}{D^4} \right) = -\frac{1}{\pi} \frac{2 \text{Re } D^2 \text{Im } D^2}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A48})$$

$$\beta_6 = \frac{1}{\pi} \text{Im} \left(\frac{1}{D^6} \right) = \frac{1}{\pi} \frac{(\text{Im } D^2)^3 - 3(\text{Re } D^2)^2 \text{Im } D^2}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^3}, \quad (\text{A49})$$

$$\beta_7 = \frac{1}{\pi} \text{Im} \left(\frac{b'}{D^2} \right) = \frac{1}{\pi} \frac{-\text{Re } b' \text{Im } D^2 + \text{Im } b' \text{Re } D^2}{(\text{Re } D^2)^2 + (\text{Im } D^2)^2}, \quad (\text{A50})$$

$$\beta_8 = \frac{1}{\pi} \text{Im} \left(\frac{c'}{D^2} \right) = \frac{1}{\pi} \frac{-\text{Re } c' \text{Im } D^2 + \text{Im } c' \text{Re } D^2}{(\text{Re } D^2)^2 + (\text{Im } D^2)^2}, \quad (\text{A51})$$

$$\beta_9^{(0)} = \frac{1}{2} \text{Im} \left(\frac{h(1+a)}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(h(1+a))[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(h(1+a))}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A52})$$

$$\beta_9^{(1)} = \frac{1}{2} \text{Im} \left(\frac{h(1+a)k_0}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(hk_0(1+a))[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(hk_0(1+a))}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A53})$$

$$\beta_{10} = \frac{1}{2} \text{Im} \left(\frac{hb}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(hb)[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(hb)}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A54})$$

$$\beta_{11} = \frac{1}{2} \text{Im} \left(\frac{hb'}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(hb')[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(hb')}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A55})$$

$$\beta_{12} = \frac{1}{2} \text{Im} \left(\frac{hc'}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(hc')[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(hc')}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A56})$$

$$\beta_{13} = \frac{1}{2} \text{Im} \left(\frac{h'b}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(h'b)[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(h'b)}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A57})$$

$$\beta_{14}^{(0)} = \frac{1}{2} \text{Im} \left(\frac{h'(1+a)}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(h'(1+a))[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(h'(1+a))}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A58})$$

$$\beta_{14}^{(1)} = \frac{1}{2} \text{Im} \left(\frac{h'(1+a)k_0}{D^4} \right) = \frac{1}{\pi} \frac{\text{Im}(h'(1+a)k_0)[(\text{Re } D^2)^2 - (\text{Im } D^2)^2] - 2 \text{Re } D^2 \text{Im } D^2 \text{Re}(h'(1+a)k_0)}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^2}, \quad (\text{A59})$$

$$\beta_{15}^{(0)} = \frac{1}{\pi} \text{Im} \left(\frac{h^2(1+a)}{D^6} \right) = \frac{1}{\pi} \frac{\text{Im}(h^2(1+a))[(\text{Re } D^2)^3 - 3\text{Re } D^2 (\text{Im } D^2)^2] + \text{Re}(h^2(1+a))[(\text{Im } D^2)^3 - 3\text{Im } D^2 (\text{Re } D^2)^2]}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^3}, \quad (\text{A60})$$

$$\beta_{15}^{(1)} = \frac{1}{\pi} \text{Im} \left(\frac{h^2(1+a)k_0}{D^6} \right) = \frac{1}{\pi} \frac{\text{Im}(h^2(1+a)k_0)[(\text{Re } D^2)^3 - 3\text{Re } D^2 (\text{Im } D^2)^2] + \text{Re}(h^2(1+a)k_0)[(\text{Im } D^2)^3 - 3\text{Im } D^2 (\text{Re } D^2)^2]}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^3}, \quad (\text{A61})$$

$$\beta_{16} = \frac{1}{\pi} \text{Im} \left(\frac{h^2 b}{D^6} \right) = \frac{1}{\pi} \frac{\text{Im}(h^2 b)[(\text{Re } D^2)^3 - 3\text{Re } D^2 (\text{Im } D^2)^2] + \text{Re}(h^2 b)[(\text{Im } D^2)^3 - 3\text{Im } D^2 (\text{Re } D^2)^2]}{[(\text{Re } D^2)^2 + (\text{Im } D^2)^2]^3}, \quad (\text{A62})$$

where

$$\text{Im}(D^2) = -\frac{\pi m_{\text{th}}^4}{k^2} \left[\frac{\omega}{k} + \left(1 - \frac{\omega^2}{k^2} \right) Q_0 \left(\frac{\omega}{k} \right) \right], \quad (\text{A63})$$

$$\text{Re}(D^2) = \omega^2 - k^2 - 2m_{\text{th}}^2 + \frac{m_{\text{th}}^4}{k^2} \left(\frac{2\omega}{k} Q_0 \left(\frac{\omega}{k} \right) - 1 \right) + \frac{m_{\text{th}}^4}{k^2} \left(1 - \frac{\omega^2}{k^2} \right) \left(Q_0^2 \left(\frac{\omega}{k} \right) - \frac{\pi^2}{4} \right), \quad (\text{A64})$$

$$\text{Im}(b') = -4e^2 M^2 \frac{\pi k_3 \omega}{2k^3}, \quad (\text{A65})$$

$$\text{Re}(b') = 4e^2 M^2 \frac{k_3}{k^2} \left[\frac{\omega}{k} Q_0 \left(\frac{\omega}{k} \right) - 1 \right], \quad (\text{A66})$$

$$\text{Im}(c') = -4e^2 M^2 \frac{\pi}{2k}, \quad (\text{A67})$$

$$\text{Re}(c') = 4e^2 M^2 \frac{1}{k} Q_0 \left(\frac{\omega}{k} \right), \quad (\text{A68})$$

$$\begin{aligned} \text{Im}(h(1+a)) &= \frac{4\pi e^2 M^2 k_3}{2k^7} (-2k^6 - 2k^4(k_0^2 + 3m_{\text{th}}^2) + 12k^3 k_0 m_{\text{th}}^2 Q_0 - 4k^2 m_{\text{th}}^2 (k_0^2 + m_{\text{th}}^2) \\ &\quad + 4kk_0 m_{\text{th}}^2 Q_0 (k_0^2 + 4m_{\text{th}}^2) + k_0^2 m_{\text{th}}^4 (\pi^2 - 12Q_0^2)), \\ \text{Re}(h(1+a)) &= \frac{4e^2 M^2 k_3}{2k^7} (4k^6 Q_0 - 4k^5 k_0 + 4k^4 Q_0 (k_0^2 + 3m_{\text{th}}^2) + k^3 k_0 m_{\text{th}}^2 (-12Q_0^2 + 3\pi^2 - 4) \\ &\quad + 8k^2 m_{\text{th}}^2 Q_0 (k_0^2 + m_{\text{th}}^2) + kk_0 m_{\text{th}}^2 (\pi^2 - 4Q_0^2) (k_0^2 + 4m_{\text{th}}^2) + 2k_0^2 m_{\text{th}}^4 Q_0 (4Q_0^2 - 3\pi^2)), \end{aligned} \quad (\text{A69})$$

$$\begin{aligned} \text{Im}(hb) &= \frac{4\pi e^2 M^2 k_3 m_{\text{th}}^2}{2k^7} (4k^2 k_0 (k_0^2 + m_{\text{th}}^2) + 4k Q_0 (k^4 + 2m_{\text{th}}^2 (k^2 - 2k_0^2) - k_0^4) \\ &\quad + 12k_0 m_{\text{th}}^2 Q_0^2 (k_0 - k) (k + k_0) + \pi^2 k_0 m_{\text{th}}^2 (k - k_0) (k + k_0)), \end{aligned} \quad (\text{A70})$$

$$\begin{aligned} \text{Re}(hb) &= \frac{4e^2 M^2 k_3 m_{\text{th}}^2}{2k^7} (k^5 (\pi^2 - 4Q_0^2) + 2k^3 (2k_0^2 + m_{\text{th}}^2 (\pi^2 - 4Q_0^2)) - 2k^2 k_0 Q_0 (4k_0^2 + m_{\text{th}}^2 (-4Q_0^2 + 3\pi^2 + 4)) \\ &\quad - kk_0^2 (\pi^2 - 4Q_0^2) (k_0^2 + 4m_{\text{th}}^2) + 2k_0^3 m_{\text{th}}^2 Q_0 (3\pi^2 - 4Q_0^2)), \end{aligned} \quad (\text{A71})$$

$$\text{Im}(hb') = \frac{16\pi e^4 M^4 k_3^2}{2k^7} (2k^4 - 4k^3 k_0 Q_0 + 4k^2 (k_0^2 + m_{\text{th}}^2) - 4kk_0 Q_0 (k_0^2 + 4m_{\text{th}}^2) - k_0^2 m_{\text{th}}^2 (\pi^2 - 12Q_0^2)), \quad (\text{A72})$$

$$\begin{aligned} \text{Re}(hb') &= -\frac{16e^4 M^4 k_3^2}{2k^7} (4k^4 Q_0 + k^3 k_0 (\pi^2 - 4(Q_0^2 + 1)) + 8k^2 Q_0 (k_0^2 + m_{\text{th}}^2) + kk_0 (\pi^2 - 4Q_0^2) (k_0^2 + 4m_{\text{th}}^2) \\ &\quad + 2k_0^2 m_{\text{th}}^2 Q_0 (4Q_0^2 - 3\pi^2)), \end{aligned} \quad (\text{A73})$$

$$\text{Im}(hc') = -\frac{16\pi e^4 M^4 k_3}{2k^5} (4k^3 Q_0 - 2k^2 k_0 + 4k Q_0 (k_0^2 + 2m_{\text{th}}^2) + k_0 m_{\text{th}}^2 (\pi^2 - 12Q_0^2)), \quad (\text{A74})$$

$$\text{Re}(hc') = -\frac{16e^4 M^4 k_3}{2k^5} (k^3 (\pi^2 - 4Q_0^2) + 4k^2 k_0 Q_0 + k (\pi^2 - 4Q_0^2) (k_0^2 + 2m_{\text{th}}^2) + 2k_0 m_{\text{th}}^2 Q_0 (4Q_0^2 - 3\pi^2)), \quad (\text{A75})$$

$$\begin{aligned} \text{Im}(h'b) &= \frac{8\pi e^4 M^4 m_{\text{th}}^2}{k^9} \left(k^6 \left(\frac{\pi^2}{4} - 3Q_0^2 \right) - 2k^5 \omega Q_0 + k^4 \left(k_3^2 - \omega^2 \left(\frac{\pi^2}{4} - 3Q_0^2 \right) \right) \right. \\ &\quad \left. - 4k^3 \omega k_3^2 Q_0 - k^2 \omega^2 k_3^2 \left(-3Q_0^2 + \frac{\pi^2}{4} + 3 \right) + 6k\omega^3 k_3^2 Q_0 + \omega^4 k_3^2 \left(\frac{\pi^2}{4} - 3Q_0^2 \right) \right), \end{aligned} \quad (\text{A76})$$

$$\begin{aligned} \text{Re}(h' b) = & \frac{16e^4 M^4 m_{\text{th}}^2}{k^9} \left(k^6 \left(Q_0^3 - \frac{3\pi^2}{4} Q_0 \right) + k^5 \omega \left(Q_0^2 - \frac{\pi^2}{4} \right) - k^4 Q_0 \left(\omega^2 \left(Q_0^2 - \frac{3\pi^2}{4} \right) + k_3^2 \right) + k^3 \omega k_3^2 \left(2Q_0^2 - \frac{\pi^2}{2} - 1 \right) \right. \\ & \left. + k^2 \omega^2 k_3^2 Q_0 \left(-Q_0^2 + \frac{3\pi^2}{4} + 3 \right) + 3k\omega^3 k_3^2 \left(\frac{\pi^2}{4} - Q_0^2 \right) + \omega^4 k_3^2 Q_0 \left(Q_0^2 - \frac{3\pi^2}{4} \right) \right), \end{aligned} \quad (\text{A77})$$

$$\begin{aligned} \text{Im}(h'(1+a)) = & \frac{8\pi e^4 M^4}{k^9} \left(2k^7 Q_0 + 2k^5 m_{\text{th}}^2 Q_0 + k^4 \omega \left(2k_3^2 + m_{\text{th}}^2 \left(\frac{\pi^2}{4} - 3Q_0^2 \right) \right) - 2k^3 \omega^2 k_3^2 Q_0 \right. \\ & \left. + 3k^2 \omega k_3^2 m_{\text{th}}^2 - 6k\omega^2 k_3^2 m_{\text{th}}^2 Q_0 - \omega^3 k_3^2 m_{\text{th}}^2 \left(\frac{\pi^2}{4} - 3Q_0^2 \right) \right), \end{aligned} \quad (\text{A78})$$

$$\begin{aligned} \text{Re}(h'(1+a)) = & \frac{16e^4 M^4}{k^9} \left(k^7 \left(\frac{\pi^2}{4} - Q_0^2 \right) + k^5 \left(k_3^2 + m_{\text{th}}^2 \left(\frac{\pi^2}{4} - Q_0^2 \right) \right) + k^4 \omega Q_0 \left(m_{\text{th}}^2 \left(Q_0^2 - \frac{3\pi^2}{4} \right) - 2k_3^2 \right) \right. \\ & \left. + k^3 k_3^2 \left(\omega^2 \left(Q_0^2 - \frac{\pi^2}{4} \right) + m_{\text{th}}^2 \right) - 3k^2 \omega k_3^2 m_{\text{th}}^2 Q_0 + 3k\omega^2 k_3^2 m_{\text{th}}^2 \left(Q_0^2 - \frac{\pi^2}{4} \right) - \omega^3 k_3^2 m_{\text{th}}^2 Q_0 \left(Q_0^2 - \frac{3\pi^2}{4} \right) \right), \end{aligned} \quad (\text{A79})$$

$$\begin{aligned} \text{Im}(h^2(1+a)) = & \frac{16\pi e^4 M^4 k_3^2}{2k^{11}} (-8k^9 Q_0 + 8k^8 k_0 - 8k^7 Q_0(2k_0^2 + 5m_{\text{th}}^2) + k^6 k_0(8k_0^2 + m_{\text{th}}^2(60Q_0^2 - 5\pi^2 + 24)) \\ & - 8k^5 Q_0(k_0^4 + 12k_0^2 m_{\text{th}}^2 + 8m_{\text{th}}^4) + 2k^4 k_0 m_{\text{th}}^2(k_0^2(36Q_0^2 - 3\pi^2 + 6) - 8m_{\text{th}}^2(-12Q_0^2 + \pi^2 - 1)) \\ & - 8k^3 m_{\text{th}}^2 Q_0(3k_0^4 - 4k_0^2 m_{\text{th}}^2(-4Q_0^2 + \pi^2 - 3) + 4m_{\text{th}}^4) - k^2 k_0 m_{\text{th}}^2(\pi^2 - 12Q_0^2)(k_0^4 + 12k_0^2 m_{\text{th}}^2 \\ & + 12m_{\text{th}}^4) + 16kk_0^2 m_{\text{th}}^4 Q_0(\pi^2 - 4Q_0^2)(k_0^2 + 3m_{\text{th}}^2) + k_0^3 m_{\text{th}}^6(80Q_0^4 - 40\pi^2 Q_0^2 + \pi^4)), \end{aligned} \quad (\text{A80})$$

$$\begin{aligned} \text{Re}(h^2(1+a)) = & -\frac{16e^4 M^4 k_3^2}{k^{11}} (k^9(\pi^2 - 4Q_0^2) + 8k^8 k_0 Q_0 + k^7(2k_0^2(-4Q_0^2 + \pi^2 - 2) + 5m_{\text{th}}^2(\pi^2 - 4Q_0^2)) \\ & + k^6 k_0 Q_0(8k_0^2 + m_{\text{th}}^2(20Q_0^2 - 15\pi^2 + 24)) + k^5(k_0^4(\pi^2 - 4Q_0^2) + 4k_0^2 m_{\text{th}}^2(-12Q_0^2 + 3\pi^2 - 1) + 8m_{\text{th}}^4(\pi^2 - 4Q_0^2)) \\ & + 2k^4 k_0 m_{\text{th}}^2 Q_0(k_0^2(12Q_0^2 - 9\pi^2 + 6) + 8m_{\text{th}}^2(4Q_0^2 - 3\pi^2 + 1)) \\ & + k^3 m_{\text{th}}^2(3k_0^4(\pi^2 - 4Q_0^2) - 2k_0^2 m_{\text{th}}^2(8(2Q_0^2 + 3)Q_0^2 - 6\pi^2(4Q_0^2 + 1) + \pi^4) \\ & + 4m_{\text{th}}^4(\pi^2 - 4Q_0^2)) + k^2 k_0 m_{\text{th}}^2 Q_0(4Q_0^2 - 3\pi^2)(k_0^4 + 12k_0^2 m_{\text{th}}^2 + 12m_{\text{th}}^4) \\ & - kk_0^2 m_{\text{th}}^4(16Q_0^4 - 24\pi^2 Q_0^2 + \pi^4)(k_0^2 + 3m_{\text{th}}^2) + k_0^3 m_{\text{th}}^6 Q_0(16Q_0^4 - 40\pi^2 Q_0^2 + 5\pi^4)), \end{aligned} \quad (\text{A81})$$

$$\begin{aligned} \text{Im}(h^2 b) = & \frac{16\pi e^4 M^4 k_3^2 m_{\text{th}}^2}{2k^{11}} (k^8(-(\pi^2 - 12Q_0^2)) - 8k^7 k_0 Q_0 - k^6(k_0^2(-12Q_0^2 + \pi^2 + 4) + 4m_{\text{th}}^2(\pi^2 - 12Q_0^2)) \\ & + 16k^5 k_0 Q_0(k_0^2 + m_{\text{th}}^2(\pi^2 - 4Q_0^2)) + k^4(k_0^4(\pi^2 - 12(Q_0^2 + 1)) - 16k_0^2 m_{\text{th}}^2 - 4m_{\text{th}}^4(\pi^2 - 12Q_0^2)) \\ & + 8k^3 k_0 Q_0(3k_0^4 + 12k_0^2 m_{\text{th}}^2 + 4m_{\text{th}}^4(-4Q_0^2 + \pi^2 + 1)) + k^2 k_0^2(k_0^4(\pi^2 - 12Q_0^2) + 12k_0^2 m_{\text{th}}^2(\pi^2 - 12Q_0^2) \\ & + m_{\text{th}}^4(80Q_0^4 - 8(18 + 5\pi^2)Q_0^2 + \pi^2(12 + \pi^2))) - 16kk_0^3 m_{\text{th}}^2 Q_0(\pi^2 - 4Q_0^2)(k_0^2 + 3m_{\text{th}}^2) \\ & - k_0^4 m_{\text{th}}^4(80Q_0^4 - 40\pi^2 Q_0^2 + \pi^4)), \end{aligned} \quad (\text{A82})$$

$$\begin{aligned}
\text{Re}(h^2 b) = & \frac{16e^4 M^4 k_3^2 m_{\text{th}}^2}{k^{11}} (k^8 (3\pi^2 Q_0 - 4Q_0^3) - k^7 k_0 (\pi^2 - 4Q_0^2) + k^6 Q_0 (k_0^2 (-4Q_0^2 + 3\pi^2 + 4) \\
& + 4m_{\text{th}}^2 (3\pi^2 - 4Q_0^2)) + k^5 k_0 (2k_0^2 (-4Q_0^2 + \pi^2 - 2) + m_{\text{th}}^2 (16Q_0^4 - 24\pi^2 Q_0^2 + \pi^4)) \\
& + k^4 Q_0 (k_0^4 (4Q_0^2 - 3\pi^2 + 12) + 16k_0^2 m_{\text{th}}^2 + 4m_{\text{th}}^4 (3\pi^2 - 4Q_0^2)) + k^3 k_0 (3k_0^4 (\pi^2 - 4Q_0^2) \\
& + 12k_0^2 m_{\text{th}}^2 (\pi^2 - 4Q_0^2) + 2m_{\text{th}}^4 (16Q_0^4 - 8(1 + 3\pi^2)Q_0^2 + \pi^2(2 + \pi^2))) + k^2 k_0^2 Q_0 \\
& \times (4Q_0^2 (k_0^4 + 12k_0^2 m_{\text{th}}^2 + 2(6 + 5\pi^2)m_{\text{th}}^4) - \pi^2(3k_0^4 + 36k_0^2 m_{\text{th}}^2 + (36 + 5\pi^2)m_{\text{th}}^4) - 16m_{\text{th}}^4 Q_0^4) \\
& - kk_0^3 m_{\text{th}}^2 (16Q_0^4 - 24\pi^2 Q_0^2 + \pi^4)(k_0^2 + 3m_{\text{th}}^2) + k_0^4 m_{\text{th}}^4 Q_0 (16Q_0^4 - 40\pi^2 Q_0^2 + 5\pi^4)), \tag{A83}
\end{aligned}$$

where M^2 is defined in Eq. (29).

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