Multiple modular symmetries as the origin of flavor

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We develop a general formalism for multiple moduli and their associated modular symmetries. We apply this formalism to an example based on three moduli with finite modular symmetries S_4^A , S_4^B , and S_4^C , associated with two right-handed neutrinos and the charged lepton sector, respectively. The symmetry is broken by two bitriplet scalars to the diagonal S_4 subgroup. The low energy effective theory involves the three independent moduli fields τ_A , τ_B , and τ_C , which preserve the residual modular subgroups Z_3^A , Z_2^B , and Z_3^C , in their respective sectors, leading to trimaximal TM_1 lepton mixing, consistent with current data, without flavons.

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I. INTRODUCTION

The discovery of neutrino mass and mixing implies that the Standard Model (SM) must be extended somehow. An elegant possibility remains the original type Ia seesaw mechanism [1–7] involving right-handed neutrinos, which, when integrated out, yield the Weinberg operators $HHL_i L_j$, where H is the Higgs doublet of the SM and L_i is a lepton doublet of the i th family. The minimal type Ia seesaw mechanism supplements the particle content of the SM by just two right-handed neutrinos (2RHN) [8,9], and this approach will be followed in the present paper. However, to explain the observed approximate tribimaximal lepton mixing, one must go beyond the seesaw mechanism and consider a non-Abelian discrete family symmetry [10,11]. For example, S_4 has been used to account for trimaximal TM_1 lepton mixing [12,13], enforced by a residual Z_2^{SU} symmetry in the neutrino sector, and a residual Z_3^T in the charged lepton sector.¹ However such realistic models typically involve many flavons.

The origin of such non-Abelian discrete family symmetry might be due to a continuous non-Abelian gauge

symmetry [14–21]. Alternatively, it could be due to extra dimensions [22–33]. With extra dimensions, it could either arise as an accidental symmetry of the orbifold fixed points (for recent discussion with two extra dimensions, see [30,34–36]) or as a subgroup of the symmetry of the extra dimensional lattice, known as modular symmetry [37], arising from superstring theory [38,39].² Indeed, it has been suggested that a finite subgroup of the modular symmetry group, when interpreted as a family symmetry, might help to provide a possible explanation for the neutrino mass matrices [42,43], and this will be the approach followed here.

Recently it has been suggested that finite modular symmetry might be the origin of flavor mixing with neutrino masses as modular forms [44], leading to constraints on the Yukawa couplings. This has led to a revival of the idea that modular symmetries are symmetries of the extra dimensional spacetime with Yukawa couplings determined by their modular weights [45]. The finite modular groups $\Gamma_2 \simeq S_3$ [46,47], $\Gamma_3 \simeq A_4$ [44,45,47–50], $\Gamma_4 \simeq S_4$ [51,52], and $\Gamma_5 \simeq A_5$ [53,54] have been considered, in which special Yukawa structures are consequences of the modular forms. Compared with traditional neutrino models of flavor symmetry, only a minimal set of flavon fields (or no flavons at all) need to be introduced in the new framework,³ making such an approach very attractive.

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¹We adopt the standard presentation of the S_4 generators S , T , U where $S^2=T^3=U^2=(ST)^3=(SU)^2=(TU)^2=(STU)^4=I$ [10].

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²The geometric connection between the origin of the family symmetry due to modular symmetry and the orbifolding method with two extra dimensions has recently been discussed, e.g., in [40,41]. On the other hand, massive states predicted in string theories may break the modular symmetries. This effect is naturally suppressed by the Planck scale, and thus can be safely ignored.

³Extension to the quark flavor mixing is given in [47,49,55].

Within the framework of finite modular symmetry outlined above, only a single modulus field τ is usually considered, corresponding to a single finite modular symmetry Γ_N . It has been pointed out that particular modular forms, corresponding to special values of τ , preserve a residual subgroup of the finite modular symmetry Γ_N . For example, such residual symmetries are considered in [50] as subgroups of the modular A_4 symmetry. Some of these specific values for τ have been shown to be obtained in extra dimensions through orbifolding [40]. With the help of two moduli with different residual symmetry Z_3 in the charged lepton sector and Z_2 in the neutrino sector, it was shown how trimaximal TM_2 lepton mixing may be realized [50]. Also brief discussion on residual symmetry after modular S_4 symmetry breaking is given in [52]. However, the formalism for having two or more moduli fields (as necessary for such a scheme) has not so far been developed, providing one of the main motivations for the present paper.

In the present paper, we shall extend the formalism of finite modular symmetry to the case of multiple moduli fields τ_J ($J = 1, \dots, M$) associated with the finite modular symmetry $\Gamma_{N_1}^1 \times \Gamma_{N_2}^2 \times \dots \times \Gamma_{N_M}^M$. As an example, we shall then present the first consistent example of a flavor model of leptons with multiple modular S_4 symmetries interpreted as a family symmetry. The considered model involves three finite modular symmetries S_4^A , S_4^B and S_4^C , associated with two right-handed neutrinos and the charged lepton sector, respectively, broken by two bitriplets scalars to their diagonal subgroup. The low energy effective theory consists of a single S_4 modular symmetry with three independent modular fields τ_A , τ_B and τ_C , which preserve the residual modular subgroups Z_3^A , Z_2^B and Z_3^C , in their respective sectors,⁴ leading to trimaximal TM_1 lepton mixing, consistent with current data, without requiring any flavons.

The remainder of the paper is organized as follows. In Sec. II we show how the formalism of finite modular symmetry with a single modulus field can be extended to include multiple moduli and an extended finite modular group. In Sec. III we have focused on the case of the single finite modular S_4 symmetry, and have analyzed its stabilizers and resulting remnant symmetries. In Sec. IV we have proposed a model based on three moduli fields associated with a high energy finite modular group S_4^3 , which is broken to a single diagonal S_4 with three independent moduli fields at low energies, whose stabilizers leads to different remnant symmetry in the different sectors, which may be used to enforce trimaximal TM_1

⁴Having a separate residual symmetry associated with each of the two right-handed neutrinos and the charged lepton sector was also assumed in the tridirect CP approach [56,57], although here we do not assume any (generalized) CP symmetry. An extension of modular symmetry to include general CP symmetries was given in [58].

mixing, leading to good numerical fits to the data, once right-handed neutrino mixing is taken into account. Section V concludes the paper.

II. FROM SINGLE TO MULTIPLE MODULAR SYMMETRIES

Modular invariant supersymmetric field theories have been analyzed in [38,39]. Modular invariance is involved in string compactifications and realistic Yukawa couplings arise from modular forms [59–62]. It has been invoked while addressing several aspects of the flavor problem in model building [63–69]. Direct application of modular symmetry to explain lepton flavor mixing was suggested in [44]. In the rest of this section, we will give a short review of effective modular-invariant supersymmetry and then expand the formalism to include multiple moduli fields.

A. A single modular symmetry

The modular group $\bar{\Gamma}$ acting on the complex modulus τ ($\text{Im}(\tau) > 0$) as linear fractional transformations:

$$\gamma: \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad (1)$$

where a, b, c, d are integers and satisfy $ad - bc = 1$. It is convenient to represent each element of $\bar{\Gamma}$ by a two by two matrix.⁵ Then, $\bar{\Gamma}$ is expressed as

$$\bar{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm 1), a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}. \quad (2)$$

This group is isomorphic to the projective special linear group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$. The modular group has two generators, S_τ and T_τ , which satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = 1$. They act on the modulus τ and take the following forms

$$S_\tau: \tau \rightarrow -\frac{1}{\tau}, \quad T_\tau: \tau \rightarrow \tau + 1, \quad (3)$$

respectively. Representing them by two by two matrices, we obtain

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

$\bar{\Gamma}$ is a discrete but infinite group. By requiring $a, d = 1 \pmod{N}$ and $b, c = 0 \pmod{N}$, $N = 2, 3, 4, \dots$, i.e.,

$$a = k_a N + 1, \quad d = k_d N + 1, \quad b = k_b N, \quad c = k_c N, \quad (5)$$

where k_a, k_b, k_c and k_d are integers, we obtain a subset of $\bar{\Gamma}$ which is also an infinite group and is labeled as

⁵Note that it need not be a unitary matrix.

$$\bar{\Gamma}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}), \right. \\ \left. \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \quad (6)$$

The quotient group $\bar{\Gamma}/\bar{\Gamma}(N)$, labeled as Γ_N , is a finite group, also called the finite modular group. The finite modular group Γ_N can be also obtained by imposing an additional condition for T_τ , $T_\tau^N = \mathbf{1}$, which can be achieved to identify $\tau = \tau + N$ in the upper complex plane.⁶ For N taking some small number, Γ_N is isomorphic to a permutation group, in particular, $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$ [43].

In a theory satisfying the Γ_N modular symmetry, any chiral superfield ϕ_i , as a function of τ (but does not need to be modular forms), nonlinearly transforms as [38],

$$\phi_i(\tau) \rightarrow \phi_i(\gamma\tau) = (c\tau + d)^{-2k_i} \rho_{I_i}(\gamma) \phi_i(\tau), \quad (7)$$

where $-2k_i$ with k_i an integer is the modular weight of ϕ_i , I_i is the representation of ϕ_i and $\rho_{I_i}(\gamma)$ denotes a unitary representation matrix of γ with γ an element of Γ_N .

Considering an $\mathcal{N} = 1$ supersymmetric model in the finite modular symmetry, the action in general takes the form [38,39]

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\phi_i, \bar{\phi}_i; \tau, \bar{\tau}) \\ + \left[\int d^4x d^2\theta W(\phi_i; \tau) + \text{H.c.} \right], \quad (8)$$

where h is a positive constant. The Kähler potential K can be changed at most by a Kähler transformation under Γ_N , and the superpotential W is required to be invariant, i.e.,

$$K(\phi_i, \bar{\phi}_i; \tau, \bar{\tau}) \rightarrow K(\phi_i, \bar{\phi}_i; \tau, \bar{\tau}) + f(\phi_i, \tau) + \bar{f}(\bar{\phi}_i, \bar{\tau}), \\ W(\phi_i; \tau) \rightarrow W(\phi_i; \tau). \quad (9)$$

An example of the Kähler potential satisfying the Kähler transformation takes the following form,⁷

$$K(\phi_i, \bar{\phi}_i; \tau, \bar{\tau}) = -h \log(-i\tau + i\bar{\tau}) + \sum_i \frac{\bar{\phi}_i \phi_i}{(-i\tau + i\bar{\tau})^{2k_i}}. \quad (10)$$

⁶Note that once $\tau = \tau + N$ is imposed, $\tau' = \frac{-1}{\tau} = \frac{-1}{\tau+N} = \frac{-\tau}{4\tau-1}$ is automatically satisfied.

⁷The effects of taking a different form for the Kahler potential are expected to be subdominant, analogously to the results shown by studies of Kahler corrections, e.g., [70]. Corrections to the Kahler potential may further lead to the stabilisation of the moduli vacua (see, e.g., reviews [71,72]). We, following all other papers on modular symmetries, avoid this problem by fixing moduli VEVs at typical values.

After τ gets a vacuum expectation value (VEV), the Kähler potential leaves kinetic terms for the scalar components of the supermultiplets ϕ_i and the modulus field as⁸

$$\frac{h}{\langle -i\tau + i\bar{\tau} \rangle^2} \partial_\mu \bar{\tau} \partial^\mu \tau + \sum_i \frac{\partial_\mu \bar{\phi}_i \partial^\mu \phi_i}{\langle -i\tau + i\bar{\tau} \rangle^{2k_i}}. \quad (11)$$

The superpotential $W(\phi_i; \tau)$ is in general a function of the modulus τ and superfields ϕ_i . Under the modular transformation, the superpotential should be invariant under the modular transformation [38]. Expanding the superpotential $W(\phi_i; \tau)$ in powers of ϕ_i , we obtain

$$W(\phi_i; \tau) = \sum_n \sum_{\{i_1, \dots, i_n\}} \sum_{I_Y} (Y_{I_Y} \phi_{i_1} \cdots \phi_{i_n})_{\mathbf{1}}. \quad (12)$$

Here, Y_{I_Y} represents a collection of coefficients of the relevant couplings. It transforms as a multiplet modular form of weight $2k_Y$ and representation I_Y ,

$$Y_{I_Y}(\tau) \rightarrow Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_{I_Y}(\gamma) Y_{I_Y}(\tau), \quad (13)$$

where $k_Y = k_{i_1} + \cdots + k_{i_n}$ is required to be a non-negative integral. Its representation and weight are required for the invariance of the operator under the Γ_N modular transformation.

B. Multiple modular symmetries

All lepton flavour models based on finite modular symmetries in the literature so far have been limited to the case of a single modulus field. No theoretical approach or model has so far managed to include more than one modulus fields in a self-consistent approach, although the latter case has been briefly mentioned in some references, e.g., [52]. In this subsection, we will discuss how to include multiple moduli fields consistently.

We start from the modular transformation as a series of modular groups $\bar{\Gamma}^1, \bar{\Gamma}^2, \dots, \bar{\Gamma}^M$, where the modulus field for each modular symmetry $\bar{\Gamma}^J$ for $J = 1, \dots, M$ is denoted as τ_J . Following Eq. (1), any modular transformation γ_J in $\bar{\Gamma}^J$ takes the form as

$$\gamma_J: \tau_J \rightarrow \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}. \quad (14)$$

A series of finite modular groups $\Gamma_{N_J}^J$ for $J = 1, 2, \dots, M$ can be obtained by modding out an integer N_J by following the discussion in the former section. Note that N_J does not need to be identical to $N_{J'}$ for $J \neq J'$.

For any finite modular transformations $\gamma_1, \dots, \gamma_M$ in $\Gamma_{N_1}^1 \times \Gamma_{N_2}^2 \times \cdots \times \Gamma_{N_M}^M$, the chiral superfield ϕ_i , as a function of $\tau_1, \dots, \tau_M$, now transforms as

⁸The scalar component of ϕ_i may gain a nonzero VEV, and this VEV also contributes to the kinetic term of τ . We ignore such a contribution by assuming $v_{\phi_i} \ll \sqrt{h}$.

$$\begin{aligned} \phi_i(\tau_1, \dots, \tau_M) &\rightarrow \phi_i(\gamma_1 \tau_1, \dots, \gamma_M \tau_M) \\ &= \prod_{J=1, \dots, M} (c_J \tau_J + d_J)^{-2k_{i,J}} \\ &\quad \otimes_{J=1, \dots, M} \rho_{I_{i,J}}(\gamma_J) \phi_i(\tau_1, \tau_2, \dots, \tau_M), \end{aligned} \quad (15)$$

where $k_{i,J}$ and $I_{i,J}$ are the weight and representation of ϕ_i in $\Gamma_{N_J}^J$, respectively, and $\otimes$ represents the outer product of the representation matrices for $\rho_{I_{i,1}}, \rho_{I_{i,2}}, \dots, \rho_{I_{i,M}}$.

For an $\mathcal{N} = 1$ supersymmetric model in a series of modular symmetries, the action is extended to the form

$$\begin{aligned} \mathcal{S} &= \int d^4x d^2\theta d^2\bar{\theta} K(\phi_i, \bar{\phi}_i; \tau_1, \dots, \tau_M, \bar{\tau}_1, \dots, \bar{\tau}_M) \\ &\quad + \int d^4x d^2\theta W(\phi_i; \tau_1, \dots, \tau_M) + \text{H.c.}, \end{aligned} \quad (16)$$

where h is a positive constant. The superpotential W is required to be invariant under all modular transformations and that the Kähler potential K can be changed at most by Kähler transformations.

Including multiple modulus fields, the Kähler potential can be written as,

$$\begin{aligned} K(\phi_i, \bar{\phi}_i; \tau_1, \dots, \tau_M, \bar{\tau}_1, \dots, \bar{\tau}_M) &= - \sum_{J=1, \dots, M} h_J \log(-i\tau_J + i\bar{\tau}_J) \\ &\quad + \sum_i \frac{\bar{\phi}_i \phi_i}{\prod_{J=1, \dots, M} (-i\tau_J + i\bar{\tau}_J)^{2k_{i,J}}}, \end{aligned} \quad (17)$$

where all h_J are positive constants. Since each modular symmetry is independent from each other, one modulus field getting a VEV leaves the rest of the Kähler potential still satisfying the other modular symmetries. For example, after τ_1 gets a VEV, the Kähler potential is left with

$$\begin{aligned} &- \sum_{J=2, \dots, M} h_J \log(-i\tau_J + i\bar{\tau}_J) \\ &+ \sum_i \frac{1}{\langle -i\tau_J + i\bar{\tau}_J \rangle^{2k_{i,1}}} \frac{\bar{\phi}_i \phi_i}{\prod_{J=2, \dots, M} (-i\tau_J + i\bar{\tau}_J)^{2k_{i,J}}}. \end{aligned} \quad (18)$$

Once all modulus fields get VEVs, the Kähler potential gives rise to kinetic terms for the scalar components of the supermultiplets ϕ_i and the modulus fields as

$$\begin{aligned} &\sum_{J=1, \dots, M} \frac{h_J}{\langle -i\tau_J + i\bar{\tau}_J \rangle^2} \partial_\mu \bar{\tau}_J \partial^\mu \tau_J \\ &+ \sum_i \frac{\partial_\mu \bar{\phi}_i \partial^\mu \phi_i}{\prod_{J=1, \dots, M} \langle -i\tau_J + i\bar{\tau}_J \rangle^{2k_{i,J}}}. \end{aligned} \quad (19)$$

In this example, the scalar component of each modulus field performs as a scalar field of vanishing weight in the remaining modular symmetries.

The superpotential $W(\phi_i; \tau_1, \dots, \tau_M)$ is in general a function of the modulus fields τ_1 to τ_M and superfields ϕ_i . Under the modular transformation, the superpotential should be invariant under the modular transformation [38]. Expanding the superpotential W in powers of ϕ_i , we obtain

$$W(\phi_i; \tau_1, \dots, \tau_M) = \sum_n \sum_{\{i_1, \dots, i_n\}} (Y_{(I_{Y,1}, \dots, I_{Y,M})} \phi_{i_1} \cdots \phi_{i_n})_1, \quad (20)$$

the weights of $Y_{(I_{Y,1}, \dots, I_{Y,M})}$ are given by $k_{Y,J} = k_{1,J} + \cdots + k_{n,J}$ for $J = 1, \dots, M$. And the modular form $Y_{(I_{Y,1}, \dots, I_{Y,M})}$ transforms as

$$\begin{aligned} &Y_{(I_{Y,1}, \dots, I_{Y,M})}(\tau_1, \dots, \tau_M) \\ &\rightarrow Y_{(I_{Y,1}, \dots, I_{Y,M})}(\gamma_1 \tau_1, \dots, \gamma_M \tau_M) \\ &= \prod_{J=1, \dots, M} (c_J \tau_J + d_J)^{2k_{Y,J}} \\ &\quad \otimes_{J=1, \dots, M} \rho_{I_{Y,J}}(\gamma_J) Y_{(I_{Y,1}, \dots, I_{Y,M})}(\tau_1, \dots, \tau_M). \end{aligned} \quad (21)$$

III. MODULAR S_4 SYMMETRY AND ITS REMNANT SYMMETRIES

In this section, we temporarily return to the case of a single modular symmetry, focusing on the case of a single modular S_4 symmetry and its remnant symmetries, before generalizing the results to the case of multiple S_4 symmetries in the next section.

A. Modular S_4 symmetry

S_4 is a permutation group of four objects. In the framework of modular symmetry, the S_4 modular group is obtained in the series of Γ_N by fixing $N = 4$. In other word, its generators satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^4 = I$. In previous works, it is common to use three generators S , T and U , which satisfy $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = I$ [10], to generate S_4 . These traditional generators are related to the modular generators S_τ and T_τ as

$$S = T_\tau^2, \quad T = S_\tau T_\tau, \quad U = T_\tau S_\tau T_\tau^2 S_\tau, \quad (22)$$

which provides a useful dictionary to relate the two types of generators. In the upper complex plane with the requirement $\tau = \tau + 4$, S , T , and U can be represented by two by two matrices such as

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}. \quad (23)$$

Due to the identification in Eq. (5), these representation matrices are not unique. Using Eq. (23), we write out another three elements of S_4 , namely $TS = S_\tau T_\tau^{-1}$,

$ST = T_\tau S_\tau T_\tau^{-1} S_\tau$, and $STS = T_\tau^{-1} S_\tau T_\tau S_\tau$, which are order-three elements of S_4 which will appear in our later discussion,

$$\begin{aligned} TS &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, & ST &= \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}, \\ STS &= \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}. \end{aligned} \quad (24)$$

Modular forms of even weights in a modular S_4 symmetry can be explicitly constructed in terms of the Dedekind eta function $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, with $q = e^{2\pi i \tau}$ [51]. At lowest weight $2k = 2$, there are five independent modular forms. By defining

$$\begin{aligned} Y(a_1, \dots, a_6 | \tau) &= \frac{d}{d\tau} \left[a_1 \log \eta \left(\tau + \frac{1}{2} \right) + a_2 \log \eta(4\tau) + a_3 \log \eta \left(\frac{\tau}{4} \right) \right. \\ &\quad \left. + a_4 \log \eta \left(\frac{\tau+1}{4} \right) + a_5 \log \eta \left(\frac{\tau+2}{4} \right) + a_6 \log \eta \left(\frac{\tau+3}{4} \right) \right], \end{aligned} \quad (25)$$

with $a_1 + \dots + a_6 = 0$, these five independent modular forms can be constructed to be

$$\begin{aligned} Y_1(\tau) &= Y(1, 1, \omega, \omega^2, \omega, \omega^2 | \tau), \\ Y_2(\tau) &= Y(1, 1, \omega^2, \omega, \omega^2, \omega | \tau), \\ Y_3(\tau) &= Y(1, -1, -1, -1, 1, 1 | \tau), \\ Y_4(\tau) &= Y(1, -1, -\omega^2, -\omega, \omega^2, \omega | \tau), \\ Y_5(\tau) &= Y(1, -1, -\omega, -\omega^2, \omega, \omega^2 | \tau), \end{aligned} \quad (26)$$

where $\omega = e^{2\pi i/3}$. These five independent modular forms at lowest weight $2k = 2$ form a doublet **2** and a triplet **3'** of S_4 ,

$$Y_2^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad Y_{3'}^{(2)} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}. \quad (27)$$

Modular forms with higher even weights ($2k = 4, 6, \dots$) can be constructed from these five modular forms. In general, the dimension of the linear space formed by the modular forms of weight $2k$ and level 4 is $4k + 1$ [44]. Namely, the nine independent modular forms of weight $2k = 4$, which form one **1**, one **2**, one **3** and one **3'**. Among them, the two triplet modular forms are given by

$$Y_3^{(4)} = \begin{pmatrix} Y_1 Y_4 - Y_2 Y_5 \\ Y_1 Y_5 - Y_2 Y_3 \\ Y_1 Y_3 - Y_2 Y_4 \end{pmatrix}, \quad Y_{3'}^{(4)} = \begin{pmatrix} Y_1 Y_4 + Y_2 Y_5 \\ Y_1 Y_5 + Y_2 Y_3 \\ Y_1 Y_3 + Y_2 Y_4 \end{pmatrix}. \quad (28)$$

At weight $2k = 6$, there are 13 independent forms. They form one **1**, one **1'**, one **2**, one **3** and two **3'**s of S_4 . Here we only interested in the two **3'**s of S_4 . They are given by

$$\begin{aligned} Y_3^{(6)} &= \begin{pmatrix} -Y_1^2 Y_5 + Y_2^2 Y_4 \\ -Y_1^2 Y_3 + Y_2^2 Y_5 \\ -Y_1^2 Y_4 + Y_2^2 Y_3 \end{pmatrix}, & Y_{3'_1}^{(6)} &= \begin{pmatrix} Y_1^2 Y_5 + Y_2^2 Y_4 \\ Y_1^2 Y_3 + Y_2^2 Y_5 \\ Y_1^2 Y_4 + Y_2^2 Y_3 \end{pmatrix}, \\ Y_{3'_2}^{(6)} &= Y_1 Y_2 \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}. \end{aligned} \quad (29)$$

These modular forms will be used for our model building in the next section. For modular forms with weights up to 10, a full list can be found in [52].

Extension from a single S_4 modular symmetry to a series of modular S_4 symmetries is straightforwardly achieved by following the procedure in Sec. II B with all levels fixed at $N_J = 4$. In each S_4^J , we denote their generators S , T and U by S_J , T_J and U_J , where the subscript is only used to distinguish groups. Modular forms with weights $k_{Y,1}, \dots, k_{Y,M}$ are multiplets of multiple moduli, namely of $\tau_1, \dots, \tau_M$.

B. Stabilizers and residual symmetries of modular S_4

Although a brief discussion on residual symmetry after modular S_4 symmetry breaking has been given in [52], we note that the essential correlation between the modular field and its residual symmetries has not been discussed. In this section, we will give a thorough analysis of this case, uncovering some new results along the way.

We begin by introducing and reviewing the notion of stabilizers of the symmetry which will play a crucial role in residual symmetries. Given an element γ in the modular group $S_4 \simeq \Gamma_4$, a stabilizer of γ corresponds to a fixed point τ_γ in the upper complex plane which satisfies $\gamma \tau_\gamma = \tau_\gamma$. Once the modular field τ gains a VEV at such a stabilizer, $\langle \tau \rangle = \tau_\gamma$, an Abelian residual modular symmetry generated by γ is preserved. It is obvious that acting γ on a modular form at its stabilizer leaves the modular form invariant, i.e.,

$$\gamma: Y_I(\tau_\gamma) \rightarrow Y_I(\gamma \tau_\gamma) = Y_I(\tau_\gamma). \quad (30)$$

Following the standard transformation property in Eq. (13), we obtain

$$\rho_I(\gamma) Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k} Y_I(\tau_\gamma). \quad (31)$$

This equation lead us to the following important properties for the stabilizer and the modular form:

- (i) A modular form at a stabilizer $Y_I(\tau_\gamma)$ is an eigenvector of the representation matrix $\rho_I(\gamma)$ with respective eigenvalue $(c\tau_\gamma + d)^{-2k}$.

- (ii) The stabilizer τ_γ satisfies $|c\tau_\gamma + d| = 1$ since $(c\tau_\gamma + d)^{-2k}$ is an eigenvalue of a unitary matrix.

A special case is that when $(c\tau_\gamma + d)^{-2k} = 1$ is satisfied, $\rho_I(\gamma)Y_I(\tau_\gamma) = Y_I(\tau_\gamma)$, and we recover the residual flavor symmetry generated by γ . In general, the eigenvalue does not need to be fixed at 1 in the framework of modular symmetry.

In the follow-up of this subsection, we will consider the following stabilizers,

$$\begin{aligned}\tau_S &= i\infty, & \tau_T &= \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, & \tau_U &= \frac{1}{2} + \frac{i}{2}, \\ \tau_{TS} &= -\omega^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, & \tau_{ST} &= \frac{1}{2} + \frac{i}{2\sqrt{3}}, \\ \tau_{STS} &= -\frac{1}{2} + \frac{i}{2\sqrt{3}}.\end{aligned}\quad (32)$$

Although τ_T and τ_{TS} have been discussed in [52] (identified with τ_L and τ_R therein, respectively), τ_S , τ_U , and τ_{ST} as stabilizers in the S_4 modular symmetry are discussed here for the first time. Here we apply this notation to take the advantage of modular residual symmetries generated by S , T , U , TS , ST , and STS , respectively. Following Eq. (23), it is straightforward to check that these stabilizers are invariant under the corresponding modular transformations respectively, i.e.,

$$\begin{aligned}S: \tau_S &\rightarrow S\tau_S = \tau_S + 2 = \tau_S, \\ T: \tau_T &\rightarrow T\tau_T = \frac{-1}{\tau_T + 1} = \tau_T, \\ U: \tau_U &\rightarrow U\tau_U = \frac{\tau_U - 1}{2\tau_U - 1} = \tau_U, \\ TS: \tau_{TS} &\rightarrow TS\tau_{TS} = \frac{1}{-\tau_{TS} + 1} = \tau_{TS}, \\ ST: \tau_{ST} &\rightarrow ST\tau_{ST} = \frac{2\tau_{ST} - 1}{3\tau_{ST} - 1} = \tau_{ST}, \\ STS: \tau_{STS} &\rightarrow STS\tau_{STS} = \frac{-2\tau_{STS} - 1}{3\tau_{STS} + 1} = \tau_{STS}.\end{aligned}\quad (33)$$

It is worth noting that these stabilizers are some typical examples but not the full list of stabilizers of S_4 .

At the stabilizer, the multiplets formed by the modular form may specify interesting directions. We will discuss how the triplet modular forms $Y_{\mathbf{3}}^{(2k)}$ or $Y_{\mathbf{3}'}^{(2k)}$ (for $k = 1, 2, 3$) gain these directions based on the symmetry argument in Eq. (31).

We begin our discussion from modular forms at the stabilizer γ_S . We know that $Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_S)$ is the eigenvector of $\rho_{\mathbf{3}^{(i)}}(S)$ with respective eigenvalue $1^{-2k} \equiv 1$,

$$\rho_{\mathbf{3}^{(i)}}(S)Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_S) = Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_S) \quad (34)$$

for any weight $2k$. Given the well-known representation matrix for S in $\mathbf{3}$ or $\mathbf{3}'$,

$$\rho_{\mathbf{3}^{(i)}}(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \quad (35)$$

Three eigenvalues are given by 1, -1 , and -1 . The eigenvector corresponding to the eigenvalue 1 is always fixed at $(1, 1, 1)^T$ up to an overall factor. Therefore, we conclude that $Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_S)$ always takes the form

$$Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_S) = y_{\mathbf{3}^{(i)},S}^{(2k)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (36)$$

Here, $y_{\mathbf{3}^{(i)},S}^{(2k)}$ is a overall factor determined by the weight and representation. By taking $\tau_S = i\infty$ into the exact modular form $Y_i(\tau)$ in Eq. (26), we obtain $q = 0$ and $Y_1(\tau_S) = Y_2(\tau_S) = i3\pi/8$, $Y_3(\tau_S) = Y_4(\tau_S) = Y_5(\tau_S) = i\pi/4$. For the weight $2k=2$, $y_{\mathbf{3}',S}^{(2)} = i\pi/4$ for $\mathbf{3}'$. For $2k=4$, $y_{\mathbf{3},S}^{(4)} = 0$ and $y_{\mathbf{3}',S}^{(4)} = -3\pi^2/16$. For $2k=6$, $y_{\mathbf{3},S}^{(6)} = 0$, $y_{\mathbf{3}',S}^{(6)} = 2y_{\mathbf{3}_2,S}^{(6)} = -i9\pi^3/128$. At the stabilizer τ_S , since the eigenvalue is always fixed at 1 regardless of the weight, the residual Z_2^S modular symmetry is identical to the residual Z_2^S flavor symmetry.

We perform a similar discussion for modular forms at the stabilizer τ_T . Equation (31) is simplified into

$$\rho_{\mathbf{3}^{(i)}}(T)Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_T) = (-\tau_T - 1)^{-2k}Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_T) = \omega^{2k}Y_{\mathbf{3}^{(i)}}^{(2k)}(\tau_T). \quad (37)$$

Thus, the selected eigenvector corresponds to the eigenvalue ω^{2k} , which is weight-dependent. In the T -diagonal basis we use in the paper, representation matrix for T is given by

$$\rho_{\mathbf{3}^{(i)}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \quad (38)$$

The triplet form, as an eigenvalue of T , takes a very simple form

$$\begin{aligned}Y_{\mathbf{3}'}^{(2)}(\tau_T) &= y_{\mathbf{3}',T}^{(2)} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & Y_{\mathbf{3}^{(i)}}^{(4)}(\tau_T) &= y_{\mathbf{3}^{(i)},T}^{(4)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ Y_{\mathbf{3}^{(i)}}^{(6)}(\tau_T) &= y_{\mathbf{3}^{(i)},T}^{(6)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\end{aligned}\quad (39)$$

where the overall factors are also determined by the weight and representation. These results can be checked numerically by taking τ_T into exact formulas of modular forms. It is straightforward to obtain $Y_1(\tau_T) = Y_3(\tau_T) = Y_5(\tau_T) = 0$, and we are left with only two nonzero modular forms, $Y_2(\tau_T) = 2.11219i$ and $Y_4(\tau_T) = -2.43895i$. Taking them to Eqs. (27), (28), and (29), we arrive at the same above result with $y_{3,T}^{(2)} = -2.43895i$, $y_{3,T}^{(4)} = -y_{3',T}^{(4)} = -5.15151$, and $y_{3,T}^{(6)} = y_{3',T}^{(6)} = 10.881i$, and $y_{3_2,T}^{(6)} = 0$. In this typical example, only the third direction, i.e., $(1, 0, 0)^T$, corresponding to modular forms with weights $2k = 0 \pmod{3}$, preserves the residual flavor symmetry generated by T . The other two vectors do not satisfy the residual flavor symmetry, but only the residual modular symmetry.

In the framework of flavor symmetry, the residual symmetry generated by U is usually called μ - τ symmetry. We discuss the modular form at the stabilizer of U . $Y_{3^{(i)}}^{(2k)}(\tau_U)$ is the eigenvalue of $\rho_{3^{(i)}}(U)$ with respective eigenvalue $(2\tau_U - 1)^{-2k} = (-1)^k$. Representation matrices for U are different in $\mathbf{3}$ and $\mathbf{3}'$,

$$\rho_3(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{3'}(U) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (40)$$

$\rho_3(U)$ has one eigenvalue -1 and the other two degenerate eigenvalues $+1$. The eigenvector with respective eigenvalue -1 is fixed at $(0, 1, -1)^T$ without considering an overall factor. The eigenvector with respective eigenvalue $+1$ is in principle a linear combination of two independent vectors $(2, -1, -1)^T$ and $(1, 1, 1)^T$. For odd and even k , we can express $Y_3^{(2k)}(\tau_U)$ as

$$Y_3^{(2k)}(\tau_U) = y_{3,U}^{(2k)} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ for an odd } k, \\ Y_3^{(2k)}(\tau_U) = y_{3,U}^{(2k)} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + y_{3,U}^{(2k)'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for an even } k. \quad (41)$$

The coefficients are determined by the weight. Numerically, $Y_1(\tau_U) = -Y_2(\tau_U) = 2.84287i$, $Y_3(\tau_U) = -(2\sqrt{2} + i)a$, $Y_4(\tau_U) = Y_5(\tau_U) = (\sqrt{2} - i)a$ with $a = 1.09422$. We obtain $y_{3,U}^{(4)} = \sqrt{2}Y_1(\tau_U)a$, $y_{3,U}^{(4)'} = -i2Y_1(\tau_U)a$ for $2k = 4$, and $y_{3,U}^{(6)} = 3\sqrt{2}Y_1^2(\tau_U)a$ for $2k = 6$. In $\mathbf{3}'$ representations, $\rho_{3'}(U)$ has one eigenvalue $+1$ and the other two degenerate eigenvalues -1 . For an even k the direction of $Y_{3'}^{(2k)}(\tau_U)$ is fixed along $(0, 1, -1)^T$, while for an odd k $Y_{3'}^{(2k)}(\tau_U)$ is a linear combination of $(2, -1, -1)^T$ and $(1, 1, 1)^T$,

$$Y_{3'}^{(2k)}(\tau_U) = y_{3',U}^{(2k)} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + y_{3',U}^{(2k)'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for an odd } k, \\ Y_{3'}^{(2k)}(\tau_U) = y_{3',U}^{(2k)} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ for an even } k. \quad (42)$$

Specifically, for $2k=2$, we have $y_{3',U}^{(2)} = -\sqrt{2}a$, $y_{3',U}^{(2)'} = -ia$. For $2k=6$, we have $y_{3',U}^{(6)} = \sqrt{2}Y_1^2(\tau_U)a$, $y_{3',U}^{(6)'} = -i2Y_1^2(\tau_U)a$; and $y_{3',U}^{(6)} = \sqrt{2}Y_1^2(\tau)a$, $y_{3',U}^{(6)'} = iY_1^2(\tau)a$, respectively. And for $2k = 4$, keeping the $(0, 1, -1)^T$, we have $y_{3',U}^{(4)} = 3\sqrt{2}Y_1(\tau_U)a$. We would like to mention that although the direction $(0, 1, -1)^T$ is realized in both $\mathbf{3}$ and $\mathbf{3}'$ representations, $(0, 1, -1)^T$ in $\mathbf{3}'$ preserves a μ - τ flavour symmetry, but that in $\mathbf{3}$ preserves not a μ - τ flavor symmetry, but a μ - τ modular symmetry.

In addition, we would like to consider stabilizers for the elements TS , ST , and STS . These elements are order-3 elements and stabilizer for each element preserves a Z_3 symmetry. The representation matrices of TS , ST , and STS take the forms

$$\rho_{3^{(i)}}(TS) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\omega^2 & -\omega^2 & 2\omega^2 \\ 2\omega & 2\omega & -\omega \end{pmatrix}, \\ \rho_{3^{(i)}}(ST) = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2 & -\omega^2 & 2\omega \\ 2 & 2\omega^2 & -\omega \end{pmatrix}, \\ \rho_{3^{(i)}}(STS) = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & -\omega^2 & 2 \\ 2\omega^2 & 2 & -\omega \end{pmatrix}. \quad (43)$$

They all have three eigenvalues given by 1 , ω , and ω^2 . The corresponding eigenvectors for TS are $(-1, 2\omega, 2\omega^2)^T$, $(2\omega, 2\omega^2, -1)^T$, and $(2\omega^2, -1, 2\omega)^T$, respectively; the corresponding eigenvectors for ST are $(-1, 2\omega^2, 2\omega)^T$, $(2\omega^2, 2\omega, -1)^T$, and $(2\omega, -1, 2\omega^2)^T$, respectively; and the corresponding eigenvectors for STS are $(-1, 2, 2)^T$, $(2, 2, -1)^T$, and $(2, -1, 2)^T$, respectively. $Y_{3^{(i)}}^{(2k)}(\tau_{TS})$ corresponds to the eigenvalue $(1 - \tau_{TS})^{-2k} = \omega^k$. Thus, we directly arrive at

$$Y_{3'}^{(2)}(\tau_{TS}) = y_{3',TS}^{(2)} \begin{pmatrix} 2\omega \\ 2\omega^2 \\ -1 \end{pmatrix}, \quad Y_{3'}^{(4)}(\tau_{TS}) = y_{3',TS}^{(4)} \begin{pmatrix} 2\omega^2 \\ -1 \\ 2\omega \end{pmatrix}, \\ Y_{3'}^{(6)}(\tau_{TS}) = y_{3',TS}^{(6)} \begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}. \quad (44)$$

TABLE I. Triplet (**3** and **3'**) representations of S_4 modular forms of low weights ($2k = 2, 4, 6$) at typical stabilizers $\tau_S = i\infty$, $\tau_U = \frac{1}{2} + \frac{i}{2}$, $\tau_T = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\tau_{TS} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\tau_{ST} = \frac{1}{2} + \frac{i}{2\sqrt{3}}$, and $\tau_{STS} = -\frac{1}{2} + \frac{i}{2\sqrt{3}}$. Here, we have ignored the overall factor if it is nonzero. **0** represents a vanishing modular form, namely, the one which has a zero overall factor.

τ	Weight 2	Weight 4		Weight 6		
	3'	3	3'	3	3' ₁	3' ₂
τ_S	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	0	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
τ_U	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2 - i\sqrt{2} \\ -1 - i\sqrt{2} \\ -1 - i\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2\sqrt{2} + i \\ -\sqrt{2} + i \\ -\sqrt{2} + i \end{pmatrix}$	$\begin{pmatrix} 2 - i\sqrt{2} \\ -1 - i\sqrt{2} \\ -1 - i\sqrt{2} \end{pmatrix}$
τ_T	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	0
τ_{TS}	$\begin{pmatrix} 2\omega \\ 2\omega^2 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2\omega^2 \\ -1 \\ 2\omega \end{pmatrix}$	$\begin{pmatrix} 2\omega^2 \\ -1 \\ 2\omega \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}$	0
τ_{ST}	$\begin{pmatrix} 2\omega \\ -1 \\ 2\omega^2 \end{pmatrix}$	$\begin{pmatrix} 2\omega^2 \\ 2\omega \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2\omega^2 \\ 2\omega \\ -1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2\omega^2 \\ 2\omega \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2\omega^2 \\ 2\omega \end{pmatrix}$	0
τ_{STS}	$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$	0

Taking the explicit formulas of modular forms into account, we obtain the overall factors to be $y_{\mathbf{3}',TS}^{(2)} = -Y_5(\tau_{TS}) = 0.81298i$, $y_{\mathbf{3},TS}^{(4)} = y_{\mathbf{3}',TS}^{(4)} = -Y_1(\tau_{TS})Y_5(\tau_{TS}) = -1.71717$, $y_{\mathbf{3},TS}^{(6)} = -y_{\mathbf{3}',TS}^{(6)} = -Y_1^2(\tau_{TS})Y_5(\tau_{TS}) = -3.62699i$, and $y_{\mathbf{3}',TS}^{(6)} = 0$. We turn to the modular forms at stabilizers τ_{ST} . $Y_{\mathbf{3}^{(2)}}^{(2k)}(\tau_{ST})$ are obtained by exchanging the second and the third entries of the above expressions but with care due to different weights

$$\begin{aligned}
 Y_{\mathbf{3}'}^{(2)}(\tau_{ST}) &= y_{\mathbf{3}',ST}^{(2)} \begin{pmatrix} 2\omega^2 \\ -1 \\ 2\omega \end{pmatrix}, & Y_{\mathbf{3}^{(4)}}^{(4)}(\tau_{ST}) &= y_{\mathbf{3}^{(4)},ST}^{(4)} \begin{pmatrix} 2\omega \\ 2\omega^2 \\ -1 \end{pmatrix}, \\
 Y_{\mathbf{3}}^{(6)}(\tau_{ST}) &= y_{\mathbf{3}^{(6)},ST}^{(6)} \begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}, & &
 \end{aligned} \tag{45}$$

where $y_{\mathbf{3}',ST}^{(2)} = 2.43895i$, $y_{\mathbf{3},ST}^{(4)} = -y_{\mathbf{3}',ST}^{(4)} = -15.4545$, $y_{\mathbf{3},ST}^{(6)} = y_{\mathbf{3}',ST}^{(6)} = -97.9287i$, and $y_{\mathbf{3}',ST}^{(6)} = 0$. They correspond to eigenvectors of $\rho_{\mathbf{3}^{(2)}}(ST)$ with respective eigenvalues $(3\tau_{ST} - 1)^{-2k} = \omega^{2k}$. Finally, we list modular forms at stabilizers τ_{STS} . $Y_{\mathbf{3}^{(2)}}^{(2k)}(\tau_{STS})$ are given by

$$\begin{aligned}
 Y_{\mathbf{3}'}^{(2)}(\tau_{STS}) &= y_{\mathbf{3}',ST}^{(2)} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, & Y_{\mathbf{3}^{(4)}}^{(4)}(\tau_{STS}) &= y_{\mathbf{3}^{(4)},STS}^{(4)} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \\
 Y_{\mathbf{3}}^{(6)}(\tau_{STS}) &= y_{\mathbf{3}^{(6)},STS}^{(6)} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, & &
 \end{aligned} \tag{46}$$

where $y_{\mathbf{3}',STS}^{(2)} = -2.43895i$, $y_{\mathbf{3},STS}^{(4)} = y_{\mathbf{3}',STS}^{(4)} = -15.4545$, $y_{\mathbf{3},STS}^{(6)} = -y_{\mathbf{3}',STS}^{(6)} = -97.9287i$, and $y_{\mathbf{3}',STS}^{(6)} = 0$. They correspond to eigenvectors of $\rho_{\mathbf{3}^{(2)}}(ST)$ with respective eigenvalues $(3\tau_{STS} + 1)^{-2k} = \omega^k$.

We summarize directions of triplet (**3** and **3'**) modular forms for lower weights ($2k = 2, 4, 6$) at stabilizers ($\tau = \tau_S, \tau_U, \tau_T, \tau_{TS}, \tau_{ST}, \tau_{STS}$) in Table I.

All the above discussion in this subsection is based on a single modular S_4 with a single modulus field. Extending to the case of multiple modular symmetries may allow the theory to have several different residual modular symmetries. Namely, the different moduli fields may take different values at different stabilizers. In the next section, we will apply this property to model building.

IV. A MODEL WITH THREE MODULAR S_4 SYMMETRIES

Combining the results of the previous two sections, we see that the extension from one single modular field to

TABLE II. Transformation properties of leptons, Yukawa couplings Y and right-handed neutrino masses M in $S_4^A \times S_4^B \times S_4^C$.

Field	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
L	1	1	3	0	0	0
e^c	1	1	1	0	0	-6
μ^c	1	1	1	0	0	-4
τ^c	1	1	1	0	0	-2
N_A^c	1	1	1	-6	0	0
N_B^c	1	1	1	0	-4	0
Φ_{AC}	3	1	3	0	0	0
Φ_{BC}	1	3	3	0	0	0

Yuk/Mass	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_C)$	1	1	3	0	0	6
$Y_\mu(\tau_C)$	1	1	3	0	0	4
$Y_\tau(\tau_C)$	1	1	3	0	0	2
$Y_A(\tau_A)$	3	1	1	6	0	0
$Y_B(\tau_B)$	1	3	1	0	4	0
$M_A(\tau_A)$	1	1	1	12	0	0
$M_B(\tau_B)$	1	1	1	0	8	0
$M_{AB}(\tau_A, \tau_B)$	1	1	1	6	4	0

multiple moduli fields, as discussed in Sec. II, opens a window into a new type of modular model building, in which several moduli fields can appear, with each one having a different modular form with a different residual symmetry, of the kind discussed in Sec. III.

A. A modular S_4^3 model

As a concrete example, we will show how the results of the previous sections can lead to a consistent model of trimaximal TM_1 mixing, analogous to the traditional approach [12,13]. At high energies, the model in Table II is based on three modular symmetries, S_4^A , S_4^B , and S_4^C , with moduli fields labeled by τ_A , τ_B , and τ_C , respectively. After the moduli fields gain different VEVs, different textures of mass matrices are realized in charged lepton and neutrino sectors.

The transformation properties of the leptons are given in Table II. We arrange that each lepton has no more than one nonvanishing modular weight in either S_4^A , S_4^B , or S_4^C . We note that: (1) The lepton doublets L form a triplet of S_4^C with zero weight; (2) the right-handed leptons e^c , μ^c , and τ^c are singlets of S_4^C but have different weights $2k_C = -6, -4, -2$, respectively; (3) We introduce only two right-handed neutrinos N_A^c and N_B^c , which are all singlets but have weights $2k_A = -6$ and $2k_B = -4$ in S_4^A and S_4^B , respectively. It is in principle possible to arrange one field with nonvanishing weights in more than one modular symmetry, so our choice here is just for simplicity.

In addition, we introduce two scalars Φ_{AC} and Φ_{BC} . These scalars are assumed to be bitriplets in the flavor space, arranged in $S_4^A \times S_4^B \times S_4^C$ as $\Phi_{AC} \sim (\mathbf{3}, \mathbf{1}, \mathbf{3})$ and $\Phi_{BC} \sim (\mathbf{1}, \mathbf{3}, \mathbf{3})$ with zero weights. As bitriplets, they transform as

$$\begin{aligned}\Phi_{AC} &\rightarrow \rho_3(\gamma_A) \otimes \rho_3(\gamma_C) \Phi_{AC}, \\ \Phi_{BC} &\rightarrow \rho_3(\gamma_B) \otimes \rho_3(\gamma_C) \Phi_{BC}.\end{aligned}\quad (47)$$

for any elements γ_A , γ_B and γ_C of S_4^A , S_4^B , and S_4^C , respectively. These scalars are introduced to connect three S_4 's together as shown in the superpotential below,

$$\begin{aligned}w_\ell = \frac{1}{\Lambda} &[L\Phi_{AC}Y_A(\tau_A)N_A^c + L\Phi_{BC}Y_B(\tau_B)N_B^c]H_u \\ &+ [LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c]H_d \\ &+ \frac{1}{2}M_A(\tau_A)N_A^cN_A^c + \frac{1}{2}M_B(\tau_B)N_B^cN_B^c \\ &+ M_{AB}(\tau_A, \tau_B)N_A^cN_B^c,\end{aligned}\quad (48)$$

where the leptonic superpotential includes the terms responsible for generating lepton masses.

To be invariant under the modular transformation, $Y_{e,\mu,\tau}$ are **3**-plet modular forms in the modular space S_4^C with weight $2k_C = 2, 4, 6$, respectively, Y_A and Y_B are **3**-plet modular forms in the modular space S_4^A , S_4^B with weights $2k_A = 6$ and $2k_B = 4$, respectively. A term, e.g., $L\Phi_{AC}Y_A(\tau_A)N_A^c$ is explicitly written as

$$\begin{aligned}L\Phi_{AC}Y_A(\tau_A)N_A^c &= L_1[(\Phi_{AC})_{11}(Y_A)_1 + (\Phi_{AC})_{21}(Y_A)_3 + (\Phi_{AC})_{31}(Y_A)_2]N_A^c \\ &+ L_2[(\Phi_{AC})_{13}(Y_A)_1 + (\Phi_{AC})_{23}(Y_A)_3 + (\Phi_{AC})_{33}(Y_A)_2]N_A^c \\ &+ L_3[(\Phi_{AC})_{12}(Y_A)_1 + (\Phi_{AC})_{22}(Y_A)_3 + (\Phi_{AC})_{32}(Y_A)_2]N_A^c, \\ &= (L_1, L_2, L_3)P_{23} \begin{pmatrix} (\Phi_{AC})_{11} & (\Phi_{AC})_{12} & (\Phi_{AC})_{13} \\ (\Phi_{AC})_{21} & (\Phi_{AC})_{22} & (\Phi_{AC})_{23} \\ (\Phi_{AC})_{31} & (\Phi_{AC})_{32} & (\Phi_{AC})_{33} \end{pmatrix}^T P_{23} \begin{pmatrix} (Y_A)_1 \\ (Y_A)_2 \\ (Y_A)_3 \end{pmatrix} N_A^c,\end{aligned}\quad (49)$$

where L_α , $(\Phi_{AC})_{i\alpha}$ and $(Y_A)_i$ are entries of L , Φ_{AC} and Y_A , respectively, for $i, \alpha = 1, 2, 3$, and P_{23} is the (2,3) row/column-switching transformation matrix. M_A and M_B are

singlet modular forms in the modular space S_4^A , S_4^B with weights $2k_A = 12$ and $2k_B = 8$, respectively. The cross mass term between N_A and N_B , M_{AB} , is not forbidden.

FIG. 1. Illustration of the breaking of $S_4^A \times S_4^B \times S_4^C \rightarrow S_4^D$, identified as the diagonal subgroup, via the VEVs of Φ_{AC} and Φ_{BC} .

It takes both nontrivial weights in S_4^A and S_4^B , $2k_A = 6$ and $2k_B = 4$. The general formulas for M_A , M_B , and M_{AB} are given by

$$\begin{aligned} M_A(\tau_A) &= m_A Y_1^2(\tau_A) Y_2^2(\tau_A), \\ M_B(\tau_B) &= m_{B,1} [Y_1^6(\tau_B) + Y_2^6(\tau_B)] + m_{B,2} Y_1^3(\tau_B) Y_2^3(\tau_B), \\ M_{AB}(\tau_A, \tau_B) &= m_{AB} [Y_1^3(\tau_A) + Y_2^3(\tau_A)] Y_1(\tau_B) Y_2(\tau_B), \end{aligned} \quad (50)$$

where m_A , $m_{B,1}$, $m_{B,2}$, and m_{AB} are complex free parameters with a mass dimension.

B. Symmetry breaking of S_4^3 to the diagonal S_4 subgroup

The modular symmetries are broken after the bitriplets scalars τ_A , τ_B , and τ_C gain VEVs. Unlike the flavons introduced in most flavor models in the literature, the VEVs of these scalars are not responsible for special Yukawa textures for leptons, but rather their purpose is to break three modular S_4 's to a single modular S_4 symmetry, identified as the diagonal subgroup and denoted as S_4^D ,

$$S_4^A \times S_4^B \times S_4^C \rightarrow S_4^D, \quad (51)$$

as depicted in Fig. 1.

The VEVs of Φ_{AC} and Φ_{BC} take the following forms

$$\langle \Phi_{AC} \rangle_{i\alpha} = v_{AC} (P_{23})_{i\alpha}, \quad \langle \Phi_{BC} \rangle_{m\alpha} = v_{BC} (P_{23})_{m\alpha}. \quad (52)$$

Here again, P_{23} represents the (2,3) row/column-switching transformation matrix, and $\alpha = 1, 2, 3$ corresponds the entries of the triplet of S_4^C , while $i = 1, 2, 3$ ($m = 1, 2, 3$) corresponds to those of S_4^A (S_4^B). These VEV structures are

not arbitrarily assumed, but can be simply achieved following the standard driving field method. They are essentially related to the group structure of S_4 and its explicit form is basis-dependent.⁹ For details of how to derive them without loss of generality, we refer the reader to Appendix B.

Although S_4^A , S_4^B , and S_4^C are broken by these VEVs, the diagonal subgroup S_4^D survives below the symmetry breaking scale, corresponding to the associated transformation $\gamma_A = \gamma_B = \gamma_C$. In more detail, the S_4^D survives since, given any γ_A of S_4^A , there always exists an element γ_C of S_4^C which is identical to γ_A , and the VEV of Φ_{AC} is invariant under this “contravariant” transformation. Furthermore, there also exists an element γ_B of S_4^B which is identical to γ_C , and the VEV of Φ_{BC} is also invariant under the transformation. Thus, the modular S_4^D symmetry corresponds to a universal transformation.

C. The effective low energy theory with modular S_4 symmetry

The effective low energy superpotential, below the S_4^3 breaking scale, involves only a single surviving modular S_4 symmetry, and may be written as,

$$\begin{aligned} w_{\ell}^{\text{eff}} &= \left[\frac{v_{AC}}{\Lambda} LY_A(\tau_A) N_A^c + \frac{v_{BC}}{\Lambda} LY_B(\tau_B) N_B^c \right] H_u \\ &+ [LY_e(\tau_C) e^c + LY_{\mu}(\tau_C) \mu^c + LY_{\tau}(\tau_C) \tau^c] H_d \\ &+ \frac{1}{2} M_A(\tau_A) N_A^c N_A^c + \frac{1}{2} M_B(\tau_B) N_B^c N_B^c \\ &+ M_{AB}(\tau_A, \tau_B) N_A^c N_B^c, \end{aligned} \quad (53)$$

where terms such as, e.g., $LY_A(\tau_A) N_A^c$ may be explicitly written as

$$LY_A(\tau_A) N_A^c = [L_1(Y_A)_1 + L_2(Y_A)_3 + L_3(Y_A)_2] N_A^c, \quad (54)$$

which is straightforwardly obtained from Eq. (49). This superpotential involves only the single residual S_4^D , and three modular fields τ_A , τ_B , and τ_C at the same time.

The above superpotential may be taken as a starting point for models based on a single modular S_4 symmetry, where the three moduli fields introduced in an *ad hoc* way and taken to be independent fields. However, we have shown that such a model can consistently arise from a high energy

⁹Explicit forms of scalar VEVs are dependent upon the basis of S_4 we use. As shown in Appendix A, we work in the T -diagonal basis in Table IV, where the trivial singlet contraction for two triplets is $(ab)_1 = a_1 b_1 + a_2 b_3 + a_3 b_2$. If we had worked in the real basis in Table V, where the singlet contraction can be simply given by $(\tilde{a} \tilde{b})_1 = \tilde{a}_1 \tilde{b}_1 + \tilde{a}_2 \tilde{b}_2 + \tilde{a}_3 \tilde{b}_3$, the VEVs of Φ_{AC} and Φ_{BC} would have been proportional to the identity matrix, $\langle \Phi_{AC} \rangle_{i\alpha} = v_{AC} \delta_{i\alpha}$, $\langle \Phi_{BC} \rangle_{m\alpha} = v_{BC} \delta_{m\alpha}$, following the discussion in Appendix B.

model involving three modular groups S_4^3 . The key point of such a model is that, in the low energy effective theory, the three moduli transform under the same S_4^D , i.e., for any $\gamma_D \in S_4^D$, τ_A , τ_B , and τ_C transform in the following way,

$$\gamma_D: \tau_J \rightarrow \gamma_D \tau_J = \frac{a_D \tau_J + b_D}{c_D \tau_J + d_D}, \quad (55)$$

for $J = A, B, C$. We also write out transformation properties of leptons

$$\begin{aligned} L &\rightarrow L(\gamma_D) = \rho_3(\gamma_D) L, \\ \alpha^c(\tau_C) &\rightarrow \alpha^c(\gamma_D \tau_C) = (c_D \tau_C + d_D)^{-2k_\alpha} \alpha^c(\tau_C), \\ N_A^c(\tau_A) &\rightarrow N_A^c(\gamma_D \tau_A) = (c_D \tau_A + d_D)^{-6} N_A^c(\tau_A), \\ N_B^c(\tau_B) &\rightarrow N_B^c(\gamma_D \tau_B) = (c_D \tau_B + d_D)^{-4} N_B^c(\tau_B), \end{aligned} \quad (56)$$

and those for modular forms

$$\begin{aligned} Y_\alpha(\tau_C) &\rightarrow Y_\alpha(\gamma_D \tau_C) = (c_D \tau_C + d_D)^{2k_\alpha} \rho_3(\gamma_D) Y_\alpha(\tau_C), \\ Y_A(\tau_A) &\rightarrow Y_A(\gamma_D \tau_A) = (c_D \tau_A + d_D)^6 \rho_3(\gamma_D) Y_A(\tau_A), \\ Y_B(\tau_B) &\rightarrow Y_B(\gamma_D \tau_B) = (c_D \tau_B + d_D)^4 \rho_3(\gamma_D) Y_B(\tau_B), \\ M_A(\tau_A) &\rightarrow M_A(\gamma_D \tau_A) = (c_D \tau_A + d_D)^{12} M_A(\tau_A), \\ M_B(\tau_B) &\rightarrow M_B(\gamma_D \tau_B) = (c_D \tau_B + d_D)^8 M_B(\tau_B), \\ M_{AB}(\tau_A, \tau_B) &\rightarrow M_{AB}(\gamma_D \tau_A, \gamma_D \tau_B) \\ &= (c_D \tau_A + d_D)^6 (c_D \tau_B + d_D)^4 M_{AB}(\tau_A, \tau_B), \end{aligned} \quad (57)$$

where $\alpha = e, \mu, \tau$ and $k_{e,\mu,\tau} = 3, 2, 1$.

We make a further comment on residual modular symmetries. It is well known that in classical flavor model building, the residual symmetry for Majorana neutrinos is restricted to Z_2 or $Z_2 \times Z_2$. In the framework of modular symmetry, the residual symmetry can be relaxed, e.g., Z_3 for N_A as will be applied in Sec. IV E. And the reason is that the relevant mass is not a trivial coefficient but a modular form, which can vary with residual modular transformation. This novel feature could be applied to other phenomenological model constructions. For example, the residual symmetry to stabilize a dark matter candidate is not limited to a Z_2 , while the latter is necessary in classic models of non-Abelian discrete symmetry [73].

To summarize, we have derived a low energy effective flavonless leptonic flavor model with one modular S_4 symmetry and three independent moduli fields. The importance of this for model building is that, as we shall see shortly, by making use of the different moduli fields, we can access different sets of triplet modular forms, corresponding to having different residual symmetries in different sectors of the theory. This is similar to the traditional approach to model building based on S_4 , but of course is

achieved now without having to introduce flavons with certain vacuum alignments.

D. Flavor structure in the charged lepton sector

In the charged lepton sector, only S_4^C plays a role. We assume the VEV of τ_C fixed at $\langle \tau_C \rangle = \tau_T = \omega$. Following Eq. (39), we obtain

$$\begin{aligned} Y_e(\langle \tau_C \rangle) &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & Y_\mu(\langle \tau_C \rangle) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ Y_\tau(\langle \tau_C \rangle) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \end{aligned} \quad (58)$$

for weights $2k_C = 6, 4, 2$, respectively. This is a consequence of the residual modular Z_3^T symmetry. These modular forms will lead to diagonal Yukawa couplings for the charged leptons, where all lepton mixing arises from the neutrino sector. Although the diagonal Yukawa couplings are independent, we do not gain any understanding of the charged lepton mass hierarchy in this model.

E. Flavor structure in the neutrino sector

In the neutrino sector, by selecting $\langle \tau_A \rangle = \tau_{TS} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\langle \tau_B \rangle = \tau_U = \frac{1}{2} + \frac{i}{2}$ we have residual modular symmetries Z_3^{TS} and Z_2^U , respectively. Following the discussion in Sec. III B, we obtain the modular form for the Yukawa coupling

$$Y_A(\langle \tau_A \rangle) = \begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}, \quad Y_B(\langle \tau_B \rangle) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (59)$$

by selecting the modular weights of N_A^c and N_B^c in S_4^A and S_4^B to be $2k_A = -6$ and $2k_B = -4$, respectively. $Y_A(\langle \tau_A \rangle)$ and $Y_B(\langle \tau_B \rangle)$ give rise to the 3×2 Dirac neutrino mass matrix M'_D . M_A , M_B , and M_{AB} all takes nonzero values at $\langle \tau_A \rangle = \tau_{TS}$ and $\langle \tau_B \rangle = \tau_U$. Thus, we obtain a 2×2 Majorana matrix for N_A^c and N_B^c ,

$$M_N = \begin{pmatrix} M_A & M_{AB} \\ M_{AB} & M_B \end{pmatrix}. \quad (60)$$

Here, we still use M_A , M_B , and M_{AB} to represent values of $M_A(\tau_A)$, $M_B(\tau_B)$, and $M_{AB}(\tau_A, \tau_B)$ at the relevant VEVs. M_N can be diagonalized by a unitary matrix V via $V^T M_N V = \text{diag}\{M_1, M_2\}$, with

$$V = e^{i\alpha_3} \begin{pmatrix} \hat{c}_R & \hat{s}_R^* \\ -\hat{s}_R & \hat{c}_R^* \end{pmatrix}, \quad (61)$$

where $\hat{c}_R \equiv \cos \theta_R e^{i\alpha_1}$ and $\hat{s}_R \equiv \sin \theta_R e^{i\alpha_2}$. The Dirac mass matrix M_D in the basis where charged lepton and right-handed neutrino mass matrices are diagonal is obtained through V acting on the right of M'_D , which mixes the columns:

$$M_D = e^{i\alpha_3} \begin{pmatrix} -\hat{c}_R & -\hat{s}_R^* \\ 2\omega^2 \hat{c}_R + \hat{s}_R & 2\omega^2 \hat{s}_R^* - \hat{c}_R^* \\ 2\omega \hat{c}_R - \hat{s}_R & 2\omega \hat{s}_R^* + \hat{c}_R^* \end{pmatrix}. \quad (62)$$

Applying seesaw formula, we obtain

$$M_\nu = (\mu_1 \hat{c}_R^2 + \mu_2 \hat{s}_R^{*2}) \begin{pmatrix} 1 & -2\omega^2 & -2\omega \\ -2\omega^2 & 4\omega & 4 \\ -2\omega & 4 & 4\omega^2 \end{pmatrix} \\ + (\mu_1 \hat{s}_R^2 + \mu_2 \hat{c}_R^{*2}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ + (\mu_1 \hat{c}_R \hat{s}_R - \mu_2 \hat{c}_R^* \hat{s}_R^*) \begin{pmatrix} 0 & -1 & 1 \\ -1 & 4\omega^2 & 2i\sqrt{3} \\ 1 & 2i\sqrt{3} & -4\omega \end{pmatrix}, \quad (63)$$

where μ_1 and μ_2 are real inspect of an overall phase. There are five physical parameters μ_1 , μ_2 , θ_R , α_1 , and α_2 .

The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is obtained by diagonalizing the neutrino mass matrix, $U^T M_\nu U = \text{diag}\{0, m_2, m_3\}$. Since both Y_A and Y_B are orthogonal to $(2, -1, -1)^T$, we directly arrive at the TM_1 form of lepton mixing matrix [74–77],

$$U_{\text{TM}_1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \end{pmatrix}. \quad (64)$$

TM_1 lepton mixing implies three equivalent relations:

$$\tan \theta_{12} = \frac{1}{\sqrt{2}} \sqrt{1 - 3s_{13}^2} \quad \text{or} \quad \sin \theta_{12} = \frac{1}{\sqrt{3}} \frac{\sqrt{1 - 3s_{13}^2}}{c_{13}} \quad \text{or} \\ \cos \theta_{12} = \sqrt{\frac{2}{3}} \frac{1}{c_{13}} \quad (65)$$

leading to a prediction $\theta_{12} \approx 34^\circ$, in excellent agreement with current global fits, assuming $\theta_{13} \approx 8.5^\circ$. By contrast, the corresponding TM_2 relations imply $\theta_{12} \approx 36^\circ$ [76], which is on the edge of the three sigma region, and hence disfavored by current data. TM_1 mixing also leads to an

exact sum rule relation for $\cos \delta$ in terms of the other lepton mixing angles [76],

$$\cos \delta = -\frac{\cot 2\theta_{23}(1 - 5s_{13}^2)}{2\sqrt{2}s_{13}\sqrt{1 - 3s_{13}^2}}. \quad (66)$$

F. Numerical fit

As described in previous subsections, we obtain through the use of modular symmetries a flavon-less effective theory which fulfils TM_1 lepton mixing. In this section, we make use of the above analytical sum rules for TM_1 lepton mixing as well as the diagonalization of the 2×2 symmetric matrices which result from the rotation of the neutrino mass matrix by the TB mixing matrix, following the analytic methods presented in [78]. We are thus able to express each observable (the 3 mixing angles, the squared mass ratio and the CP -violating phase δ) in terms of the model parameters $(\{x\}) = (\{\alpha_1, \alpha_2, \theta_R, \mu_1, \mu_2\})$, i.e., the phases α_1 and α_2 , the angle parametrizing the rotation originating from RH neutrino sector, and the parameters governing the contribution from Y_A and Y_B , μ_1 and μ_2 . These formulas are somewhat complicated and not particularly illustrative, but enable us to easily run a numerical minimization procedure on a χ^2 function:

$$\chi^2 = \sum \left(\frac{P_i(\{x\}) - \text{BF}_i}{\sigma_i} \right)^2, \quad (67)$$

where P_i are the model predictions, BF_i the current best-fit values, and the errors σ_i correspond here to the average of the 1σ ranges for each observable. We use the best-fit values and 1σ ranges from NuFit 4.0 [79,80]. The minimization runs over model parameters and the observables tested are the 3 PMNS mixing angles, the phase δ , and the absolute masses obtained from the square roots of the squared mass differences (taking into account that we have only 2 RH neutrinos and normal mass ordering, $m_1 = 0$).

The obtained best-fit point (BF) corresponds to a $\chi^2 = 0.74$, with the model parameters shown in Table III, together with the respective predictions for the observables, including mixing parameters, neutrino masses, and the effective neutrino mass parameter in neutrino-less double beta decay $m_{ee} = |\mu_1 \hat{c}_R^2 + \mu_2 \hat{s}_R^{*2}|$. These observables (predicted by the analytical formulas for the specific point in parameter space) completely match with the values obtained by performing an entirely numerical diagonalization for the same point in parameter space. For the best-fit point the observables are all within the 1σ range except $\delta = 290^\circ$.

For comparison we present also two other benchmark points. In Benchmark 1 (B1), the observables are all within the 1σ range except δ , which is slightly smaller (285°) than

TABLE III. Model parameters (Para.) and respective observables (Obs.) for the best-fit point (BF) and two other benchmark points (B1, B2).

BF	Parameters	χ^2	α_1	α_2	θ_R	μ_1	μ_2	
		0.74	64.53°	20.38°	43.01°	0.00633 eV	0.0114 eV	
	Observables	θ_{12}	θ_{13}	θ_{23}	δ	m_2	m_3	m_{ee}
		34.33°	8.61°	49.6°	290°	0.00860 eV	0.0502 eV	0.00206 eV
B1	Parameters	χ^2	α_1	α_2	θ_R	μ_1	μ_2	
		1.6	70.16°	16.62°	43.51°	0.00651 eV	0.0135 eV	
	Observables	θ_{12}	θ_{13}	θ_{23}	δ	m_2	m_3	m_{ee}
		34.33°	8.62°	48.6°	285°	0.00860 eV	0.0502 eV	0.00188 eV
B2	Parameters	χ^2	α_1	α_2	θ_R	μ_1	μ_2	
		55	358.73°	338.89°	24.65°	0.00533 eV	0.0114 eV	
	Observables	θ_{12}	θ_{13}	θ_{23}	δ	m_2	m_3	m_{ee}
		34.34°	8.56°	41.5°	254°	0.00860 eV	0.0502 eV	0.00319 eV

in the best fit point. Conversely, θ_{23} deviates slightly from its best-fit point. The total $\chi^2 = 1.6$ is slightly worse. In Benchmark 2 (B2), $\delta = 254^\circ$ is within the 1σ range. Conversely, θ_{13} is slightly deviated from its best fit point and $\theta_{23} = 41.5^\circ$ is strongly deviated from its best-fit point and is indeed outside the 1σ range. The total $\chi^2 = 55$ is much worse, although we note that this value is somewhat spurious, given that the expression in Eq. (67) is based on Gaussian distributions, which is not the case for θ_{23} , which for B2 contributes 0.99 of the total χ^2 . We are taking the best-fit point for $\theta_{23} = 49.6^\circ$ from NuFit 4.0 [79,80].

It is worth emphasizing that these predictions originate from the special directions Y_A and Y_B obtained from the fixed points in the respective modular symmetries. The best-fit point observables all lie within the 1σ range except δ , which nevertheless lies within its 3σ range and takes a value close to maximal (290°).

V. CONCLUSIONS AND DISCUSSION

In this paper we have considered, for the first time, leptonic flavor models based on multiple moduli fields with an extended finite modular symmetry. We reviewed the case of a single modular symmetry $\bar{\Gamma}$ with a single modulus field τ and $\mathcal{N} = 1$ supersymmetry, then extended the formalism to include a series of modular groups $\bar{\Gamma}^1, \bar{\Gamma}^2, \dots, \bar{\Gamma}^M$, where the modulus field for each modular symmetry $\bar{\Gamma}^J$ is denoted as τ_J , where $J = 1, \dots, M$, resulting in the finite modular symmetry $\Gamma_{N_1}^1 \times \Gamma_{N_2}^2 \times \dots \times \Gamma_{N_M}^M$.

We then returned to the case of a single modular symmetry, focusing on the case of modular S_4 symmetry and its remnant symmetries, exploring relations of stabilizers of modular transformations, residual symmetries, and modular forms in the framework of finite modular

symmetry. In the case of modular S_4 symmetry, several new stabilizers of residual symmetries were identified, where each stabilizer preserves a Z_2 or Z_3 residual symmetry. We discovered a strong correlation between the modular transformation and the modular form at its stabilizer, namely that *a modular form at a stabilizer of any modular transformation is an eigenvector of the representation matrix of the modular transformation*. Based on this correlation, we were able to determine some new types of modular forms without knowing exact expressions for those modular forms.

As an application of the preceding results, we constructed a flavor model of leptons involving two right-handed neutrinos and three finite modular symmetries $S_4^A \times S_4^B \times S_4^C$. Here, S_4^A and S_4^B are modular symmetries for two right-handed neutrinos, respectively, while S_4^C is the modular symmetry in charged lepton sector. They are connected by two bitriplets scalars. After they gain VEVs, three S_4 's are broken to a single S_4^D , i.e., $S_4^A \times S_4^B \times S_4^C \rightarrow S_4^D$. Independent fixed points in the extra dimensions associated with S_4^A and S_4^B specify (flavonless) special directions that preserve subgroups of the respective symmetries, whereas a scalar transforming as a triplet of both S_4^A and S_4^C and another scalar transforming one of both as a triplet of both S_4^B and S_4^C acquire vacuum expectation values that break $S_4^A \times S_4^B \times S_4^C$ to its diagonal subgroup S_4^D . We emphasize that these scalars do not carry any information about flavor.

After the three S_4 's are broken, we arrive at an effective low energy flavor mixing model with a single S_4 modular symmetry but three independent modular fields τ_A , τ_B , and τ_C . The independence of these modular fields allows us to assign different VEVs for them which determine the flavor structure. We fix the VEV of τ_C at a stabilizer which

satisfies a modular Z_3^C symmetry. A diagonal charged lepton mass matrix is obtained. VEVs of τ_A and τ_B are fixed at other two stabilizers which preserve a different Z_3^A symmetry and a Z_2^B symmetry respectively. The residual modular symmetries justify the special directions that lead to TM_1 mixing. This is similar to the traditional approach to model building based on S_4 , but of course is achieved now without having to introduce flavons with certain vacuum alignments.

Finally, we performed an analysis of the predictions of the model taking into account the existence of RH neutrino mixing (in the model-building basis). When this is taken into account, the 5 observables depend on 4 real model parameters, and we obtain an excellent fit to experiment, with all 3 mixing angles and the squared mass ratio within 1σ of their experimental values and a near-maximal value for $\delta = 290$ degrees. Having two right-handed neutrinos, the model predicts the absolute neutrino mass scale $m_1 = 0$.

In conclusion, we have developed a general formalism for multiple modular symmetries, analysed the residual symmetries of modular S_4 symmetry, and proposed a realistic model based on modular S_4^3 symmetry, which yields the successful trimaximal TM_1 lepton mixing, without requiring any flavons.

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APPENDIX A: GROUP THEORY OF S_4

S_4 is the permutation group of 4 objects, see e.g., [81]. The Kronecker products between different irreducible representations can be easily obtained:

$$\begin{aligned} 1' \otimes 1' &= 1, \quad 1' \otimes 2 = 2, \quad 1' \otimes 3 = 3', \quad 1' \otimes 3' = 3, \\ 2 \otimes 2 &= 1 \oplus 1' \oplus 2, \quad 2 \otimes 3 = 2 \otimes 3' = 3 \oplus 3', \\ 3 \otimes 3 &= 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3', \quad 3 \otimes 3' = 1' \oplus 2 \oplus 3 \oplus 3'. \end{aligned} \quad (\text{A1})$$

The generators of S_4 in different irreducible representations are listed in Table IV, in the basis we used in the main

TABLE IV. The representation matrices for the S_4 generators T , S , and U used in the main text, where ω is the cube root of unity $\omega = e^{2\pi i/3}$.

	$\rho(T)$	$\rho(S)$	$\rho(U)$
1	1	1	1
1'	1	1	-1
2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3'	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

text. This basis is widely used in the literature since the charged lepton mass matrix invariant under T is diagonal in this basis. The following basis-dependent property is satisfied, $\rho_{3^{(v)}}^T(\gamma)P_{23} = P_{23}\rho_{3^{(v)}}(\gamma^{-1})$ for any γ of S_4 . The products of two 3 dimensional irreducible representations a and b can be expressed as

$$\begin{aligned} (ab)_{1_i} &= a_1 b_1 + a_2 b_3 + a_3 b_2, \\ (ab)_{2_i} &= (a_2 b_2 + a_1 b_3 + a_3 b_1, a_3 b_3 + a_1 b_2 + a_2 b_1)^T, \\ (ab)_{3_i} &= (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 - a_2 b_1, \\ &\quad 2a_2 b_2 - a_3 b_1 - a_1 b_3)^T, \\ (ab)_{3'_i} &= (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)^T, \end{aligned} \quad (\text{A2})$$

where

$$\begin{aligned} 1_i &= 1, \quad 3_i = 3, \quad 3_j = 3' \text{ for } a \sim b \sim 3, \quad 3', \\ 1_i &= 1', \quad 3_i = 3', \quad 3_j = 3 \text{ for } a \sim 3, \quad b \sim 3'. \end{aligned} \quad (\text{A3})$$

The products of two doublets $a = (a_1, a_2)^T$ and $b = (b_1, b_2)^T$ are divided into

$$\begin{aligned} (ab)_1 &= a_1 b_2 + a_2 b_1, \quad (ab)_{1'} = a_1 b_2 - a_2 b_1, \\ (ab)_2 &= (a_2 b_2, a_1 b_1)^T. \end{aligned} \quad (\text{A4})$$

In Appendix B, we apply another basis to calculate the vacuum alignment. Both bases are widely used and

TABLE V. The representation matrices for the S_4 generators T , S , and U used for vacuum alignments.

	$\tilde{\rho}(T)$	$\tilde{\rho}(S)$	$\tilde{\rho}(U)$
1	1	1	1
1'	1	1	-1
2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3'	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

have no physical difference. However, we apply this basis because it is simpler to carry out the respective calculations. Representation matrices for generators in this basis are listed in Table V. Representation matrix for any γ of S_4 satisfies $\tilde{\rho}_{3^{(i)}}^T(\gamma) = \tilde{\rho}_{3^{(i)}}(\gamma^{-1})$. Basis transformation between the first and second basis are given by

$$\tilde{\rho}_{3^{(i)}}(\gamma) = U_{\omega} \rho_{3^{(i)}}(\gamma) U_{\omega}^{\dagger} \quad (\text{A5})$$

for any $\gamma \in S_4$, where

$$U_{\omega} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (\text{A6})$$

Irreducible products of two triplet representations $\tilde{a}$ and $\tilde{b}$ are simply written as

$$\begin{aligned}
(\tilde{a} \tilde{b})_{1_i} &= \sum_{i=1,2,3} \tilde{a}_i \tilde{b}_i, \\
((\tilde{a} \tilde{b})_2)_i &= \sum_{j=1,2,3} \omega^{i(j-1)} \tilde{a}_j \tilde{b}_i \quad \text{for } i = 1, 2, \\
((\tilde{a} \tilde{b})_{3_i})_i &= \sum_{j,k=1,2,3} |\epsilon_{ijk}| \tilde{a}_j \tilde{b}_k \quad \text{for } i = 1, 2, 3, \\
((\tilde{a} \tilde{b})_{3_j})_i &= \sum_{j,k=1,2,3} \epsilon_{ijk} \tilde{a}_j \tilde{b}_k \quad \text{for } i = 1, 2, 3,
\end{aligned} \quad (\text{A7})$$

where **1_i**, **3_i**, and **3_j** are given the same as in Eq. (A3).

APPENDIX B: VACUUM ALIGNMENTS

Vacuum alignments for the bitriplet scalars Φ_{AC} and Φ_{BC} can be realized following the general way in most supersymmetric flavor models. As seen in Table VI, we introduce four driving fields χ_{AC} , χ_A and χ_{BC} , χ_B ,

$$\begin{aligned}
\chi_{AC} &\sim (\mathbf{3}, \mathbf{1}, \mathbf{3}), & \chi_A &\sim (\mathbf{3}, \mathbf{1}, \mathbf{1}), \\
\chi_{BC} &\sim (\mathbf{1}, \mathbf{3}, \mathbf{3}), & \chi_B &\sim (\mathbf{1}, \mathbf{3}, \mathbf{1})
\end{aligned} \quad (\text{B1})$$

of $S_4^A \times S_4^B \times S_4^C$. The superpotential for vacuum alignment is given by

$$\begin{aligned}
w_d &= \Phi_{AC} \Phi_{AC} \chi_{AC} + M_A \Phi_{AC} \chi_{AC} + \Phi_{AC} \Phi_{AC} \chi_A, \\
&+ \Phi_{BC} \Phi_{BC} \chi_{BC} + M_B \Phi_{BC} \chi_{BC} + \Phi_{BC} \Phi_{BC} \chi_B,
\end{aligned} \quad (\text{B2})$$

where M_A and M_B are mass-dimensional coefficients. Minimization of the superpotential gives rise to conditions for Φ_{AC} and Φ_{BC} VEVs. In order to give a better illustration, we present the derivation of Φ_{AC} in the second basis as $(\tilde{\Phi}_{AC})_{i\alpha} = \sum_{j,\beta} (U_{\omega})_{ij} (U_{\omega})_{\alpha\beta} (\Phi_{AC})_{j\beta}$. With the help of the Clebsch-Gordan coefficients in Eq. (A7), these conditions are explicitly written as

$$\begin{aligned}
&\sum_{j,k=1,2,3} \sum_{\beta,\gamma=1,2,3} |\epsilon_{ijk}| |\epsilon_{\alpha\beta\gamma}| (\tilde{\Phi}_{AC})_{j\beta} (\tilde{\Phi}_{AC})_{k\gamma} \\
&+ M_A (\tilde{\Phi}_{AC})_{i\alpha} = 0 \quad \text{for } i = 1, 2, 3, \quad \alpha = 1, 2, 3, \\
&\sum_{j,k=1,2,3} \sum_{\alpha=1,2,3} |\epsilon_{ijk}| (\tilde{\Phi}_{AC})_{j\alpha} (\tilde{\Phi}_{AC})_{k\alpha} = 0 \quad \text{for } i = 1, 2, 3.
\end{aligned} \quad (\text{B3})$$

The full solution for the above equation is not hard to obtain. There are 24 solutions in total. It is convenient to write them as 3×3 unitary matrices,

$$\langle \tilde{\Phi}_{AC} \rangle = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \right. \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ \left. \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} v_{AC}. \quad (\text{B4})$$

The matrices in Eq. (B4) are identical to the **3**-plet representation matrices of the 24 elements of S_4 in basis in Table V. Using the basis transformation, we obtain solutions for $\langle \Phi_{AC} \rangle_{i\alpha} = \sum_{j,\beta} (U_\omega^\dagger)_{ij} (U_\omega^\dagger)_{\alpha\beta} \langle \tilde{\Phi}_{AC} \rangle_{j\beta}$ in the basis used in the main text, i.e., that in Table IV. With the help of $U_\omega^\dagger = U_\omega P_{23}$, we can express these solutions as

$$\langle \Phi_{AC} \rangle = \rho_3(\gamma) P_{23} v_{AC} \quad (\text{B5})$$

with γ any element of S_4 .

In the main text, we achieved the breaking of modular symmetry $S_4^A \times S_4^C$ to the flavor symmetry S_4^D by assuming the VEV $\langle \Phi_{AC} \rangle_{i\alpha} = (P_{23})_{i\alpha} v_{AC}$ (corresponding to the first solution in Eq. (B4), $\langle \tilde{\Phi}_{AC} \rangle_{i\alpha} \propto \delta_{i\alpha}$). We comment in the following that this VEV is not special and any VEV with the form $\rho_3(\gamma) P_{23} v_{AC}$ can lead to the breaking of two S_4 's to a single S_4 .

To see this feature more clearly, let us pick up the second solution in Eq. (B4) as an example. This solution is represented as a 3×3 matrix to be $\langle \Phi_{AC} \rangle = \rho_3(S) P_{23} v_{AC}$, i.e., corresponding to the element S . The operator $L \Phi_{AC} Y_A N_A^c H_u$, for instance, after Φ_{AC} gains the VEV $\rho_3(S) P_{23} v_{AC}$, is effectively expressed as

$$L \langle \Phi_{AC} \rangle Y_A N_A^c H_u = (L_1, L_2, L_3) P_{23} \rho_3(S) \begin{pmatrix} (Y_A)_1 \\ (Y_A)_2 \\ (Y_A)_3 \end{pmatrix} N_A^c H_u v_{AC}. \quad (\text{B6})$$

For any element γ_D of S_4^D , which leads to $(L_1, L_2, L_3) \rightarrow (L_1, L_2, L_3) \rho_3^T(\gamma_D)$, with the help of basis-dependent property $\rho_3^T(\gamma_D) P_{23} = P_{23} \rho_3(\gamma_D^{-1})$, one can always require the transformation properties for Y_A and N_A^c in the following,

$$\begin{aligned} Y_A &\rightarrow (c_D \tau_A + d_D)^6 \rho_3(S^{-1} \gamma_D S) Y_A, \\ N_A^c &\rightarrow (c_D \tau_A + d_D)^{-6} N_A^c, \end{aligned} \quad (\text{B7})$$

TABLE VI. Symmetry and field content according to sector (scalar and driving sectors), for fields responsible for vacuum alignment.

Fields	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
χ_{AC}	3	1	3	0	0	0
χ_{BC}	1	3	3	0	0	0
χ_A	3	1	1	0	0	0
χ_B	1	3	1	0	0	0

such that the effective operator $L\langle\Phi_{AC}\rangle Y_A N_A H_u$ is invariant in S_4^D . The transformation property for Y_A is equivalent to the following transformation property for $[\rho_3(S)Y_A]$,

$$[\rho_3(S)Y_A] \rightarrow (c_D \tau_A + d_D)^6 \rho_3(\gamma_D) [\rho_3(S)Y_A]. \quad (\text{B8})$$

Similarly, $S_4^B \times S_4^C$ can be broken to S_4^D by the VEV $\langle\Phi_{BC}\rangle_{ma} = (P_{23})_{ma} v_{AC}$. Eventually, we realize the breaking of three S_4 's to a single S_4 .

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- [1] P. Minkowski, *Phys. Lett.* **67B**, 421 (1977).
  - [2] T. Yanagida, in *Proceedings of the Workshop on Unified Theory and the Baryon Number of the Universe*, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95.
  - [3] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Z. Freeman (North-Holland, Amsterdam, 1979), p. 315.
  - [4] S. L. Glashow, in *Quarks and Leptons*, edited by M. Levy *et al.* (Plenum, New York, 1980), p. 707.
  - [5] R. N. Mohapatra and G. Senjanovic, *Phys. Rev. Lett.* **44**, 912 (1980).
  - [6] J. Schechter and J. W. F. Valle, *Phys. Rev. D* **22**, 2227 (1980).
  - [7] J. Schechter and J. W. F. Valle, *Phys. Rev. D* **25**, 774 (1982).
  - [8] S. F. King, *Nucl. Phys.* **B576**, 85 (2000).
  - [9] S. F. King, *J. High Energy Phys.* 09 (2002) 011.
  - [10] S. F. King and C. Luhn, *Rep. Prog. Phys.* **76**, 056201 (2013).
  - [11] S. F. King, *Prog. Part. Nucl. Phys.* **94**, 217 (2017).
  - [12] I. de Medeiros Varzielas and L. Lavoura, *J. Phys. G* **40**, 085002 (2013).
  - [13] C. Luhn, *Nucl. Phys.* **B875**, 80 (2013).
  - [14] I. de Medeiros Varzielas, S. F. King, and G. G. Ross, *Phys. Lett. B* **644**, 153 (2007).
  - [15] Y. Koide, *J. High Energy Phys.* 08 (2007) 086.
  - [16] T. Banks and N. Seiberg, *Phys. Rev. D* **83**, 084019 (2011).
  - [17] C. Luhn, *J. High Energy Phys.* 03 (2011) 108.
  - [18] A. Merle and R. Zwicky, *J. High Energy Phys.* 02 (2012) 128.
  - [19] Y. L. Wu, *Phys. Lett. B* **714**, 286 (2012).
  - [20] B. L. Rachlin and T. W. Kephart, *J. High Energy Phys.* 08 (2017) 110.
  - [21] S. F. King and Y. L. Zhou, *J. High Energy Phys.* 11 (2018) 173.
  - [22] T. Asaka, W. Buchmuller, and L. Covi, *Phys. Lett. B* **523**, 199 (2001).
  - [23] G. Altarelli, F. Feruglio, and Y. Lin, *Nucl. Phys.* **B775**, 31 (2007).
  - [24] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, *Nucl. Phys.* **B768**, 135 (2007).
  - [25] G. Altarelli, F. Feruglio, and C. Hagedorn, *J. High Energy Phys.* 03 (2008) 052.
  - [26] A. Adulpravitchai, A. Blum, and M. Lindner, *J. High Energy Phys.* 07 (2009) 053.
  - [27] T. J. Burrows and S. F. King, *Nucl. Phys.* **B835**, 174 (2010).
  - [28] A. Adulpravitchai and M. A. Schmidt, *J. High Energy Phys.* 01 (2011) 106.
  - [29] T. J. Burrows and S. F. King, *Nucl. Phys.* **B842**, 107 (2011).
  - [30] F. J. de Anda and S. F. King, *J. High Energy Phys.* 07 (2018) 057.
  - [31] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, *Phys. Rev. D* **97**, 116002 (2018).
  - [32] F. J. de Anda and S. F. King, *J. High Energy Phys.* 10 (2018) 128.
  - [33] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, *Phys. Lett. B* **795**, 7 (2019).
  - [34] T. Kobayashi, Y. Omura, and K. Yoshioka, *Phys. Rev. D* **78**, 115006 (2008).
  - [35] Y. Olguin-Trejo, R. Pérez-Martínez, and S. Ramos-Sánchez, *Phys. Rev. D* **98**, 106020 (2018).
  - [36] A. Mütter, E. Parr, and P. K. S. Vaudrevange, *Nucl. Phys.* **B940**, 113 (2019).
  - [37] A. Giveon, E. Rabinovici, and G. Veneziano, *Nucl. Phys.* **B322**, 167 (1989).
  - [38] S. Ferrara, D. Lust, A. D. Shapere, and S. Theisen, *Phys. Lett. B* **225**, 363 (1989).
  - [39] S. Ferrara, D. Lust, and S. Theisen, *Phys. Lett. B* **233**, 147 (1989).
  - [40] F. J. de Anda, S. F. King, and E. Perdomo, *Phys. Rev. D* **101**, 015028 (2020).
  - [41] T. Kobayashi and S. Tamba, *Phys. Rev. D* **99**, 046001 (2019).
  - [42] G. Altarelli and F. Feruglio, *Nucl. Phys.* **B741**, 215 (2006).
  - [43] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, *Nucl. Phys.* **B858**, 437 (2012).
  - [44] F. Feruglio, *arXiv:1706.08749*.
  - [45] J. C. Criado and F. Feruglio, *SciPost Phys.* **5**, 042 (2018).
  - [46] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, *Phys. Rev. D* **98**, 016004 (2018).
  - [47] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, *Phys. Lett. B* **794**, 114 (2019).
  - [48] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, *J. High Energy Phys.* 11 (2018) 196.
  - [49] H. Okada and M. Tanimoto, *Phys. Lett. B* **791**, 54 (2019).
  - [50] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, *Phys. Lett. B* **793**, 247 (2019).
  - [51] J. T. Penedo and S. T. Petcov, *Nucl. Phys.* **B939**, 292 (2019).
  - [52] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* 04 (2019) 005.
  - [53] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* 04 (2019) 174.
  - [54] G. J. Ding, S. F. King, and X. G. Liu, *Phys. Rev. D* **100**, 115005 (2019).
  - [55] H. Okada and M. Tanimoto, *arXiv:1905.13421* [Phys. Rev. D (to be published)].

- [56] G. J. Ding, S. F. King, and C. C. Li, *J. High Energy Phys.* **12** (2018) 003.
- [57] G. J. Ding, S. F. King, and C. C. Li, *Phys. Rev. D* **99**, 075035 (2019).
- [58] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* **07** (2019) 165.
- [59] L. E. Ibanez, *Phys. Lett. B* **181**, 269 (1986).
- [60] J. A. Casas, F. Gomez, and C. Munoz, *Int. J. Mod. Phys. A* **08**, 455 (1993).
- [61] O. Lebedev, *Phys. Lett. B* **521**, 71 (2001).
- [62] T. Kobayashi and O. Lebedev, *Phys. Lett. B* **566**, 164 (2003).
- [63] P. Brax and M. Chemtob, *Phys. Rev. D* **51**, 6550 (1995).
- [64] P. Binetruiy and E. Dudas, *Nucl. Phys.* **B451**, 31 (1995).
- [65] E. Dudas, S. Pokorski, and C. A. Savoy, *Phys. Lett. B* **369**, 255 (1996).
- [66] E. Dudas, [arXiv:hep-ph/9602231](https://arxiv.org/abs/hep-ph/9602231).
- [67] G. K. Leontaris and N. D. Tracas, *Phys. Lett. B* **419**, 206 (1998).
- [68] T. Dent, *Phys. Rev. D* **64**, 056005 (2001).
- [69] T. Dent, *J. High Energy Phys.* **12** (2001) 028.
- [70] S. F. King, I. N. R. Peddie, G. G. Ross, L. Velasco-Sevilla, and O. Vives, *J. High Energy Phys.* **07** (2005) 049.
- [71] V. Balasubramanian and P. Berglund, *J. High Energy Phys.* **11** (2004) 085.
- [72] E. Silverstein, [arXiv:hep-th/0405068](https://arxiv.org/abs/hep-th/0405068).
- [73] M. Hirsch, S. Morisi, E. Peinado, and J. W. F. Valle, *Phys. Rev. D* **82**, 116003 (2010).
- [74] Z. z. Xing and S. Zhou, *Phys. Lett. B* **653**, 278 (2007).
- [75] C. S. Lam, *Phys. Rev. D* **74**, 113004 (2006).
- [76] C. H. Albright and W. Rodejohann, *Eur. Phys. J. C* **62**, 599 (2009).
- [77] C. H. Albright, A. Dueck, and W. Rodejohann, *Eur. Phys. J. C* **70**, 1099 (2010).
- [78] S. F. King, *J. High Energy Phys.* **02** (2016) 085.
- [79] I. Esteban, M. C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni, and T. Schwetz, *J. High Energy Phys.* **01** (2019) 106.
- [80] NuFIT 4.0 (2018), www.nu-fit.org.
- [81] J. A. Escobar and C. Luhn, *J. Math. Phys. (N.Y.)* **50**, 013524 (2009).