Threshold expansion formula of N bosons in a finite volume from a variational approach

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In present work, we show how the threshold expansion formula of N identical bosons in finite volume may be derived by iterations of Faddeev-type coupled dynamical equations. The energy shift of N -boson system near threshold is dominated by zero momenta mode of N -body amplitudes with all particles nearly static. The dominant zero momenta mode and subleading nonzero momenta mode contributions are connected through finite volume Faddeev-type coupled dynamical equations. Eliminating nonzero momenta modes by iterations ultimately yields an analytic expression that can be solved by threshold expansion.

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I. INTRODUCTION

Quantum mechanical many-body dynamics is essential for the understanding of wide range phenomena in modern physics, including Bose-Einstein condensate and superfluidity [1–3]. The many-body dynamics usually rely on approximate approaches in the past, such as Hartree-Fock method [4]. In recent years, a lot of progress has been made toward the study of few- and many-body dynamics from first principle, quantum chromodynamics (QCD) [5–18]. The calculation of lattice QCD is usually performed in Euclidean space with all particles confined in a periodic cubic box; hence, the multihadron dynamics is not directly accessible. Instead, dynamics is encoded in a discrete energy spectrum of multihadron system in finite volume. Therefore, establishing a method of mapping out infinite-volume multihadron dynamics from discrete energy spectrum in finite volume has become an important subject in past few years. Such a connection in two-body sector is established by Lüscher formula in [19] and its extensions [20–29]. Many promising developments along different approaches have been made toward few- and many-body finite volume systems recently [30–58]. One crucial thing to justify these recent developments is to perform some tests and reproduce some known results, such as the

threshold expansion formula that was originally derived by perturbation theory [59–62].

Motivated exactly by the purpose of testing our formalism on finite volume N -body dynamics based on variational approach [50,52,55], in this work, we illustrate how the well-known threshold expansion formula for N -identical-boson system [59–62] may be derived from coupled dynamical equations. The exact value of eigen energy of N -body system is given by the eigen solution of these Faddeev-type coupled dynamical equations. Faddeev-type coupled dynamical equation is a nonperturbative approach; hence, it applies in principle to both weakly and strongly coupled system. To reproduce threshold expansion formula, the perturbation expansion in terms of weak coupling is carried out by iterations of coupled dynamical equations. An energy-dependent closed form is thus obtained, and it ultimately yields the threshold expansion formula by further expansion near threshold. The threshold expansion formula up to $\mathcal{O}(\eta^4/L^6)$ for pairwise interaction and $\mathcal{O}(\eta_3/L^6)$ for three-body interaction is already known [61,62], where η and η_3 are the two-body and three-body coupling strengths respectively. The exact expression of $\mathcal{O}(\eta^4/L^6)$ expansion formula requires higher order terms by multiple iterations, which ultimately becomes a tedious task. To simplify our presentation since the result is not new, in this work, we will only show the derivation of the threshold expansion formula up to $\mathcal{O}(\eta^3/L^5)$ and $\mathcal{O}(\eta_3/L^6)$ by a single iteration, in terms of perturbation theory, they may be associated with η^2 and $\eta\eta_3$ order diagrams, respectively.

The paper is organized as follows. The formalism of finite volume N -identical-boson systems is presented in detail in Sec. II. The derivation of threshold expansion

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formula is illustrated in Sec. III. Summary is given in Sec. IV.

II. N-BOSON DYNAMICS IN FINITE VOLUME

The dynamics of N nonrelativistic identical bosons in finite volume is described by Lippmann-Schwinger-type integral equation; see Refs. [52,55],

$$\Phi_E(\{\mathbf{x}\}) = \int_{L^3} \prod_{i=1}^N d\mathbf{x}'_i G_E(\{\mathbf{x} - \mathbf{x}'_i\}) V(\{\mathbf{x}'_i\}) \Phi_E(\{\mathbf{x}'_i\}), \quad (1)$$

where the position of i th particle is denoted by $\mathbf{x}_i$ and $\{\mathbf{x}\} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$. The N -body finite volume Green's function is given by

$$G_E(\{\mathbf{x}\}) = \frac{1}{L^{3N}} \sum_{\{\mathbf{p}\}} \frac{e^{i \sum_{i=1}^N \mathbf{p}_i \cdot \mathbf{x}_i}}{E - \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m}}, \quad (2)$$

where $\{\mathbf{p}\} = \{\mathbf{p}_1, \dots, \mathbf{p}_N\}$, and $\mathbf{p}_i = \frac{2\pi}{L} \mathbf{n}_i$ with $\mathbf{n}_i \in \mathbb{Z}^3$ stands for the free momentum of i th particle. L is the size of the cubic box. The finite volume Green's function is the solution of differential equation,

$$\left(E + \sum_{i=1}^N \frac{\nabla_i^2}{2m}\right) G_E(\{\mathbf{x}\}) = \prod_{i=1}^N \sum_{\mathbf{n}_i \in \mathbb{Z}^3} \delta(\mathbf{x}_i + \mathbf{n}_i L) \quad (3)$$

and satisfies the periodic boundary condition,

$$G_E(\{\mathbf{x} + \mathbf{n}L\}) = G_E(\{\mathbf{x}\}). \quad (4)$$

Due to the periodic nature of Green's function, the periodicity of wave function, $\Phi_E(\{\mathbf{x} + \mathbf{n}L\}) = \Phi_E(\{\mathbf{x}\})$, is hence automatically warranted by Eq. (1). The interactions among particles is described by $V(\{\mathbf{x}\})$, the same form as given in [61] is used in present work, i.e., only contact pairwise and three-body interactions are considered,

$$V(\{\mathbf{x}\}) = \eta \sum_{(i<j)=1}^N \delta(\mathbf{r}_{ij}) + \eta_3 \sum_{(i<j<k)=1}^N \delta(\mathbf{r}_{ik}) \delta(\mathbf{r}_{jk}), \quad (5)$$

where $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ is relative coordinate between i th and j th particles, and η and η_3 are the coupling strengths for pairwise and three-body contact interactions, respectively.

As illustrated in Refs. [52,55], two types of finite volume Faddeev amplitudes may be introduced by

$$T_{(ij)}(\{\mathbf{k}\}) = - \int_{L^3} \left(\prod_{l=1}^N d\mathbf{x}_l e^{-i\mathbf{k}_l \cdot \mathbf{x}_l} \right) \eta \delta(\mathbf{r}_{ij}) \Phi_E(\{\mathbf{x}\}) \quad (6)$$

and

$$T_{(ijk)}(\{\mathbf{k}\}) = - \int_{L^3} \left(\prod_{l=1}^N d\mathbf{x}_l e^{-i\mathbf{k}_l \cdot \mathbf{x}_l} \right) \eta_3 \delta(\mathbf{r}_{ik}) \delta(\mathbf{r}_{jk}) \Phi_E(\{\mathbf{x}\}), \quad (7)$$

where $T_{(ij)}$'s and $T_{(ijk)}$'s are associated with pairwise and three-body contact interactions, respectively. There are totally $\frac{N(N-1)}{2} T_{(ij)}$'s and $\frac{N(N-1)(N-2)}{6} T_{(ijk)}$'s. Equation (1) is thus turned into $\frac{(N+1)N(N-1)}{6}$ coupled equations, for instance,

$$T_{(12)}(\{\mathbf{k}\}) = \frac{1}{L^3} \sum_{\mathbf{p}_2} \frac{\eta}{E - \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m}} \times \left[\sum_{i<j} T_{(ij)}(\{\mathbf{p}\}) + \sum_{i<j<k} T_{(ijk)}(\{\mathbf{p}\}) \right],$$

$$\mathbf{p}_1 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_2, \quad \mathbf{p}_l = \mathbf{k}_l, \quad l = 3, \dots, N \quad (8)$$

and

$$T_{(123)}(\{\mathbf{k}\}) = \frac{1}{L^6} \sum_{\mathbf{p}_2, \mathbf{p}_3} \frac{\eta_3}{E - \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m}} \times \left[\sum_{i<j} T_{(ij)}(\{\mathbf{p}\}) + \sum_{i<j<k} T_{(ijk)}(\{\mathbf{p}\}) \right],$$

$$\mathbf{p}_1 = \sum_{i=1}^3 \mathbf{k}_i - \mathbf{p}_2 - \mathbf{p}_3, \quad \mathbf{p}_l = \mathbf{k}_l, \quad l = 4, \dots, N. \quad (9)$$

The rest of equations for $T_{(ij)}$'s and $T_{(ijk)}$'s are thus obtained by swapping particle indices: $1 \leftrightarrow i$, $2 \leftrightarrow j$, and $3 \leftrightarrow k$.

A. Symmetry consideration

Because of exchange symmetry of N identical bosons system, only two independent amplitudes are required. Let us define

$$T(\overline{\{\mathbf{k}\}}_{(12)}) = T_{(12)}(\{\mathbf{k}\}), \quad (10)$$

where $\overline{\{\mathbf{k}\}}_{(12)} = \{\mathbf{k}_3, \dots, \mathbf{k}_N\}$ is a subset of $\{\mathbf{k}\} = \{\mathbf{k}_1, \mathbf{k}_2, \overline{\{\mathbf{k}\}}_{(12)}\}$ by removing first two elements, and

$$T_3(\overline{\{\mathbf{k}\}}_{(123)}) = T_{(123)}(\{\mathbf{k}\}), \quad (11)$$

where $\overline{\{\mathbf{k}\}}_{(123)} = \{\mathbf{k}_4, \dots, \mathbf{k}_N\}$ is a subset of $\{\mathbf{k}\} = \{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \overline{\{\mathbf{k}\}}_{(123)}\}$ by removing first three elements. According to Eq. (8), $T_{(12)}(\{\mathbf{k}\})$ in fact depends on both $\mathbf{k}_1 + \mathbf{k}_2$ and $\overline{\{\mathbf{k}\}}_{(12)}$; the $\mathbf{k}_1 + \mathbf{k}_2$ dependence has been dropped due to the fact that all momenta are constrained by

FIG. 1. Diagrammatic representation of Eqs. (13) and (14), pairwise and three-body interactions are represented by black solid circle and blue solid square.

momentum conservation $\sum_{i=1}^N \mathbf{k}_i = \mathbf{P}$, where $\mathbf{P}$ stands for total momentum of N particle. Similarly, the $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$ dependence in $T_{(123)}(\{\mathbf{k}\})$ is dropped as well because of momentum conservation constraint. The rest of amplitudes are related to T and T_3 defined in Eqs. (10) and (11), respectively, by

$$T_{(ij)}(\{\mathbf{k}\}) = T(\{\overline{\mathbf{k}}\}_{(ij)}), \quad T_{(ijk)}(\{\mathbf{k}\}) = T_3(\{\overline{\mathbf{k}}\}_{(ijk)}), \quad (12)$$

where $\{\overline{\mathbf{k}}\}_{(ij)}$ and $\{\overline{\mathbf{k}}\}_{(ijk)}$ can be obtained from sets $\{\mathbf{k}\}_{(12)}$ and $\{\mathbf{k}\}_{(123)}$ by swapping particle momenta: $\mathbf{k}_1 \leftrightarrow \mathbf{k}_i$, $\mathbf{k}_2 \leftrightarrow \mathbf{k}_j$, and $\mathbf{k}_3 \leftrightarrow \mathbf{k}_k$. Two sets of coupled equations for $T_{(ij)}$'s and $T_{(ijk)}$'s are hence reduced to two equations,

$$T(\{\overline{\mathbf{k}}\}_{(12)}) = \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{\sum_{i<j} T(\{\overline{\mathbf{p}}\}_{(ij)}) + \sum_{i<j<k} T_3(\{\overline{\mathbf{p}}\}_{(ijk)})}{E - \frac{(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_2)^2 + \mathbf{p}_2^2 + \sum_{i=3}^N \mathbf{k}_i^2}{2m}}, \quad \mathbf{p}_1 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_2, \quad \mathbf{p}_l = \mathbf{k}_l, \quad l = 3, \dots, N \quad (13)$$

and

$$T_3(\{\overline{\mathbf{k}}\}_{(123)}) = \frac{\eta_3}{L^6} \sum_{\mathbf{p}_2, \mathbf{p}_3} \frac{\sum_{i<j} T(\{\overline{\mathbf{p}}\}_{(ij)}) + \sum_{i<j<k} T_3(\{\overline{\mathbf{p}}\}_{(ijk)})}{E - \frac{(\sum_{i=1}^3 \mathbf{k}_i - \mathbf{p}_2 - \mathbf{p}_3)^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2 + \sum_{i=4}^N \mathbf{k}_i^2}{2m}}, \quad \mathbf{p}_1 = \sum_{i=1}^3 \mathbf{k}_i - \mathbf{p}_2 - \mathbf{p}_3, \quad \mathbf{p}_l = \mathbf{k}_l, \quad l = 4, \dots, N, \quad (14)$$

where $\{\mathbf{p}\} = \{\mathbf{p}_1, \mathbf{p}_2, \{\overline{\mathbf{p}}\}_{(12)}\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \{\overline{\mathbf{p}}\}_{(123)}\}$ in both Eqs. (13) and (14). The diagrammatic representation of Eqs. (13) and (14) is given in Fig. 1.

B. Three-boson dynamical equations

In the case of $N = 3$, the dynamical equations are thus given by

$$T(\mathbf{k}_3) = \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{T(\mathbf{k}_3) + 2T(\mathbf{p}_2) + T_3}{E - \frac{(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_2)^2 + \mathbf{p}_2^2 + \mathbf{k}_3^2}{2m}} \quad (15)$$

and

$$T_3 = \frac{\eta_3}{L^6} \sum_{\mathbf{p}_2, \mathbf{p}_3} \frac{3T(\mathbf{p}_2) + T_3}{E - \frac{\mathbf{p}_2^2 + \mathbf{p}_3^2 + (\mathbf{P} - \mathbf{p}_2 - \mathbf{p}_3)^2}{2m}}. \quad (16)$$

Eliminating T_3 amplitude, we find

$$T(\mathbf{k}_3) = \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{1}{E - \frac{\mathbf{p}_2^2 + (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_2)^2 + \mathbf{k}_3^2}{2m}} \left[T(\mathbf{k}_3) + 2T(\mathbf{p}_2) + \frac{\frac{\eta_3}{L^6} \sum_{\mathbf{q}_2, \mathbf{q}_3} \frac{3T(\mathbf{q}_2)}{E - \frac{\mathbf{q}_2^2 + \mathbf{q}_3^2 + (\mathbf{P} - \mathbf{q}_2 - \mathbf{q}_3)^2}{2m}}}{1 - \frac{\eta_3}{L^6} \sum_{\mathbf{q}_2, \mathbf{q}_3} \frac{1}{E - \frac{\mathbf{q}_2^2 + \mathbf{q}_3^2 + (\mathbf{P} - \mathbf{q}_2 - \mathbf{q}_3)^2}{2m}}} \right]. \quad (17)$$

III. THRESHOLD EXPANSION

In this section, we illustrate that the threshold expansion formula may be derived from Eqs. (13) and (14) by iterations. Near the ground state energy threshold, all N particles are nearly at rest for weak interactions. Hence, the dominant contribution comes from the zero momenta mode of amplitudes: $\{\mathbf{k}\} = \{\mathbf{0}\}$. Thus, we find

$$\begin{aligned} & \left[1 - \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} \right] T(\{\overline{\mathbf{0}}\}_{(12)}) \\ &= \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{\sum_{i<j}^{(ij) \neq (12)} T(\{\overline{\mathbf{p}}\}_{(ij)}) + \sum_{i<j<k} T_3(\{\overline{\mathbf{p}}\}_{(ijk)})}{E - \frac{\mathbf{p}_2^2}{m}}, \\ & \mathbf{p}_1 = -\mathbf{p}_2, \quad \mathbf{p}_l = \mathbf{0}, \quad l = 3, \dots, N \end{aligned} \quad (18)$$

and

$$\begin{aligned} & T_3(\{\overline{\mathbf{0}}\}_{(123)}) \\ &= \frac{\eta_3}{L^6} \sum_{\mathbf{p}_2, \mathbf{p}_3} \frac{\sum_{i<j} T(\{\overline{\mathbf{p}}\}_{(ij)}) + \sum_{i<j<k} T_3(\{\overline{\mathbf{p}}\}_{(ijk)})}{E - \frac{\mathbf{p}_2^2 + \mathbf{p}_3^2 + (\mathbf{p}_2 + \mathbf{p}_3)^2}{2m}}, \\ & \mathbf{p}_1 = -\mathbf{p}_2 - \mathbf{p}_3, \quad \mathbf{p}_l = \mathbf{0}, \quad l = 4, \dots, N. \end{aligned} \quad (19)$$

A. Perturbation expansion by iteration of N -body dynamical equations

As we can see from above equations, the leading order contributions of T and T_3 start at the order of $\frac{\eta}{L^3}$ and $\frac{\eta_3}{L^6}$, respectively. In present work, the aim is to just simply illustrate how the threshold expansion formula are derived from Eqs. (13) and (14). For this purpose, we will only compute up to η^3/L^5 order in threshold expansion formula by iterating Eqs. (13) and (14) only once. The contributions from three-body force are only kept at the lowest order effect, and also splitting up zero momenta mode and nonzero momenta mode contributions in Eq. (18), so we obtain

$$\left[1 - \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} - \frac{\eta}{L^3} \frac{N(N-1)}{E}\right] T(\{\mathbf{0}\}_{(12)}) = \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{\sum_{i < j}^{(ij) \neq (12)} T(\{\mathbf{p}\}_{(ij)})}{E - \frac{\mathbf{p}_2^2}{m}} + \frac{\eta}{L^3} \frac{N(N-1)(N-2)}{E} T_3(\{\mathbf{0}\}_{(123)}) + \mathcal{O}\left(\frac{\eta^2 \eta_3}{L^{12}}\right),$$

$$\mathbf{p}_1 = -\mathbf{p}_2, \quad \mathbf{p}_l = \mathbf{0}, \quad l = 3, \dots, N \quad (20)$$

and

$$T_3(\{\mathbf{0}\}_{(123)}) = \frac{\eta_3}{L^6} \frac{N(N-1)}{E} T(\{\mathbf{0}\}_{(12)}) + \mathcal{O}\left(\frac{\eta \eta_3}{L^9}\right). \quad (21)$$

Eliminating $T_3(\{\mathbf{0}\}_{(123)})$ term in Eq. (20), we find

$$\left[1 - \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} - \frac{\eta}{L^3} \frac{N(N-1)}{E}\right] T(\{\mathbf{0}\}_{(12)}) = \frac{\eta \eta_3}{L^9} \frac{N^2(N-1)^2(N-2)}{E^2} T(\{\mathbf{0}\}_{(12)}) + \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{\sum_{i < j}^{(ij) \neq (12)} T(\{\mathbf{p}\}_{(ij)})}{E - \frac{\mathbf{p}_2^2}{m}} + \mathcal{O}\left(\frac{\eta^2 \eta_3}{L^{12}}\right),$$

$$\mathbf{p}_1 = -\mathbf{p}_2, \quad \mathbf{p}_l = \mathbf{0}, \quad l = 3, \dots, N. \quad (22)$$

Now, dominant zero momenta mode and subleading non-zero momenta mode are well separated in Eq. (22). The terms that are given by nonzero momenta mode of T amplitudes in Eq. (22) can be eliminated and thus are related to zero momenta mode amplitudes by iterating Eq. (13) once.

Nonzero momenta mode of set $\{\mathbf{p}\}_{(ij)}$ in $T(\{\mathbf{p}\}_{(ij)})$ can be split into two groups: (i) $\{\mathbf{p}\}_{(1j)} = \{\mathbf{0}, \dots, \mathbf{p}_2, \dots, \mathbf{0}\}$ with only a single nonzero momentum dependence at j th position, $\mathbf{p}_j = \mathbf{p}_2$ and $j > 2$ and (ii) $\{\mathbf{p}\}_{(ij)} = \{\mathbf{0}, \dots, -\mathbf{p}_2, \dots, \mathbf{p}_2, \dots, \mathbf{0}\}$ with two nonzero momenta dependence at i th and j th positions, $\mathbf{p}_i = -\mathbf{p}_2$ and $\mathbf{p}_j = \mathbf{p}_2$, where $(i < j) = 3, \dots, N$.

- (i) For amplitudes in group one with only a single nonzero momentum dependence, using Eq. (13) again, we find that each nonzero mode $T(\{\mathbf{p}\}_{(1j)})$ is related to two amplitudes that do not depend on $\mathbf{p}_2$,

$$T(\{\mathbf{p}\}_{(1j)}) = \frac{\eta}{L^3} \left[\sum_{\mathbf{q}_2} \frac{T(\{\mathbf{q}\}_{(1j)})}{E - \frac{\mathbf{q}_1^2 + \mathbf{q}_2^2 + \mathbf{p}_2^2}{2m}} + \sum_{\mathbf{q}_1} \frac{T(\{\mathbf{q}\}_{(2j)})}{E - \frac{\mathbf{q}_1^2 + \mathbf{q}_2^2 + \mathbf{p}_2^2}{2m}} \right] + \dots$$

$$= \frac{\eta}{L^3} \sum_{\mathbf{q}_2} \frac{2T(\{\mathbf{q}\}_{(1j)})}{E - \frac{(\mathbf{p}_2 + \mathbf{q}_2)^2 + \mathbf{q}_2^2 + \mathbf{p}_2^2}{2m}} + \dots, \quad (23)$$

where $\{\mathbf{q}\}_{(1j)} = \{\mathbf{0}, \dots, \mathbf{q}_2, \dots, \mathbf{0}\}$ and $\{\mathbf{q}\}_{(2j)} = \{\mathbf{0}, \dots, \mathbf{q}_1, \dots, \mathbf{0}\}$ with $\mathbf{q}_2$ and $\mathbf{q}_1$ sitting at j th position. Splitting sum of $\mathbf{q}_2$ to zero momenta mode and nonzero momenta mode again in Eq. (23), the dominant contribution for $T(\{\mathbf{p}\}_{(1j)})$ comes from

zero momenta mode, subleading contribution from nonzero momenta mode may be eliminated by iteration again. Keeping only dominant zero momenta mode contribution, we hence find

$$T(\{\mathbf{p}\}_{(1j)}) = \frac{\eta}{L^3} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} 2T(\{\mathbf{0}\}_{(12)}) + \mathcal{O}\left(\frac{\eta \eta_3}{L^6}\right). \quad (24)$$

There are $(N-2)$ terms of $T(\{\mathbf{p}\}_{(1j)})$ in group one and another $(N-2)$ equivalent terms for $T(\{\mathbf{p}\}_{(2j)})$ amplitudes. Therefore, the total dominant contribution from group one is $\frac{\eta}{L^3} \frac{4(N-2)}{E - \frac{\mathbf{p}_2^2}{m}} T(\{\mathbf{0}\}_{(12)})$.

- (ii) For amplitudes in group two with two nonzero momentum dependence, each $T(\{\mathbf{p}\}_{(ij)})$ is related to only one amplitude that does not depend on $\mathbf{p}_i = -\mathbf{p}_2$ and $\mathbf{p}_j = \mathbf{p}_2$,

$$T(\{\mathbf{p}\}_{(ij)}) = \frac{\eta}{L^3} \sum_{\mathbf{q}_2} \frac{T(\{\mathbf{q}\}_{(ij)})}{E - \frac{\mathbf{q}_2^2 + \mathbf{p}_2^2}{m}} + \dots, \quad (25)$$

where $\{\mathbf{q}\}_{(ij)} = \{\mathbf{0}, \dots, -\mathbf{q}_2, \dots, \mathbf{q}_2, \dots, \mathbf{0}\}$ with $-\mathbf{q}_2$ and $\mathbf{q}_2$ sitting at i th and j th positions, respectively. Hence, the dominant zero momenta mode contribution from $T(\{\mathbf{p}\}_{(ij)})$ term is

$$T(\{\mathbf{p}\}_{(ij)}) = \frac{\eta}{L^3} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} T(\{\mathbf{0}\}_{(12)}) + \mathcal{O}\left(\frac{\eta \eta_3}{L^6}\right). \quad (26)$$

There are $\frac{(N-2)(N-3)}{2}$ such terms, the total number of dominant contributions from group two is thus $\frac{\eta}{L^3} \frac{(N-2)(N-3)}{E - \frac{\mathbf{p}_2^2}{m}} T(\{\mathbf{0}\}_{(12)})$.

1. Zero momenta mode N -boson dynamical equation

Combining all nonzero momenta mode terms in Eq. (22) from both group one and group two, we obtain

$$\begin{aligned} & \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{\sum_{i < j}^{(ij) \neq (12)} T(\{\overline{\mathbf{p}}\}_{(ij)})}{E - \frac{\mathbf{p}_2^2}{m}} \\ &= \frac{\eta^2}{L^6} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{4(N-2) + \frac{(N-2)(N-3)}{2}}{(E - \frac{\mathbf{p}_2^2}{m})^2} T(\{\overline{\mathbf{0}}\}_{(12)}) + \mathcal{O}\left(\frac{\eta^2 \eta_3}{L^{12}}\right). \end{aligned} \quad (27)$$

Plugging them back into Eq. (20), we thus find

$$\begin{aligned} & \left[1 - \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} - \frac{\eta}{L^3} \frac{N(N-1)}{E} \right] T(\{\overline{\mathbf{0}}\}_{(12)}) \\ &= \frac{\eta \eta_3}{L^9} \frac{N^2(N-1)^2(N-2)}{E^2} T(\{\overline{\mathbf{0}}\}_{(12)}) \\ &+ \frac{\eta^2}{L^6} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{\frac{(N-2)(N+5)}{2}}{(E - \frac{\mathbf{p}_2^2}{m})^2} T(\{\overline{\mathbf{0}}\}_{(12)}) + \mathcal{O}\left(\frac{\eta^2 \eta_3}{L^{12}}\right). \end{aligned} \quad (28)$$

Zero momenta mode amplitude $T(\{\overline{\mathbf{0}}\}_{(12)})$ is thus canceled out from both sides of the equation, and Eq. (28) yields an analytic form that depends on only E and momentum sum.

2. Three-boson example

Using three-body dynamical equation given in Eq. (17) as a specific example, setting $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{0}$, and keeping only up to $\frac{\eta \eta_3}{L^9}$ order, we obtain

$$T(\mathbf{0}) \simeq \frac{\eta}{L^3} \sum_{\mathbf{p}_2} \frac{T(\mathbf{0}) + 2T(\mathbf{p}_2)}{E - \frac{\mathbf{p}_2^2}{m}} + \frac{\eta \eta_3}{L^9} \frac{3T(\mathbf{0})}{E^2}. \quad (29)$$

Splitting up to zero momenta and nonzero momenta mode in Eq. (29) and also using Eq. (17) once to eliminate nonzero momenta mode,

$$T(\mathbf{p}_2) = \frac{\eta}{L^3} \frac{2T(\mathbf{0})}{E - \frac{\mathbf{p}_2^2}{m}} + \dots, \quad \mathbf{p}_2 \neq \mathbf{0}, \quad (30)$$

hence, we finally get

$$1 - \frac{\eta}{L^3} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{1}{E - \frac{\mathbf{p}_2^2}{m}} - \frac{\eta}{L^3} \frac{3}{E} \simeq \frac{\eta^2}{L^6} \sum_{\mathbf{p}_2 \neq \mathbf{0}} \frac{4}{(E - \frac{\mathbf{p}_2^2}{m})^2} + \frac{\eta \eta_3}{L^9} \frac{3}{E^2}. \quad (31)$$

B. Near threshold expansion and ground state energy

By assuming that energy shift near threshold is small due to weak interactions: $E \sim 0$, Eq. (28) is thus turned into a polynomial equation by near threshold expansion, keeping up to $\mathcal{O}(E^3)$, we have

$$\begin{aligned} & \mathcal{O}(E^4) + \left(\frac{1}{L^3} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\frac{\mathbf{p}^2}{m^2}} \right) E^3 \\ &+ \left(\frac{1}{\eta} + \frac{1}{L^3} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\frac{\mathbf{p}^2}{m}} - \frac{\eta}{L^6} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{\frac{(N+5)(N-2)}{2}}{\frac{\mathbf{p}^4}{m^2}} \right) E^2 - \frac{1}{L^3} \frac{N(N-1)}{2} E \\ &= \frac{\eta_3}{L^9} \frac{N^2(N-1)^2(N-2)}{12}. \end{aligned} \quad (32)$$

Introducing renormalized two-body coupling constant

$$\frac{1}{\eta} = \frac{1}{\eta_R} - \frac{m\Lambda}{\pi L}, \quad (33)$$

where Λ is a dimensionless quantity and related to the cutoff on momentum sum by $|\mathbf{p}| = \frac{2\pi|\mathbf{n}|}{L} \leq \frac{2\pi\Lambda}{L}$, and also using relations given in Ref. [61],

$$\begin{aligned} & \frac{1}{L^3} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\frac{\mathbf{p}^2}{m}} - \frac{m\Lambda}{\pi L} = \frac{m\mathcal{I}}{(2\pi)^2 L}, \quad \mathcal{I} = \sum_{\mathbf{n} \neq \mathbf{0}}^{\mathbf{n} \leq \Lambda} \frac{1}{\mathbf{n}^2} - 4\pi\Lambda, \\ & \frac{1}{L^3} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\frac{\mathbf{p}^4}{m^2}} = \frac{m^2 L \mathcal{J}}{(2\pi)^4}, \quad \mathcal{J} = \sum_{\mathbf{n} \neq \mathbf{0}} \frac{1}{\mathbf{n}^4}, \end{aligned} \quad (34)$$

we can rewrite Eq. (32) to

$$\begin{aligned} & \frac{L\mathcal{J}}{(2\pi)^4} (mE)^3 + \left(\frac{1}{m\eta_R} + \frac{\mathcal{I}}{(2\pi)^2 L} \right. \\ & \left. - \frac{(N+5)(N-2)}{2} \frac{m\eta_R \mathcal{J}}{(2\pi)^4 L^2} \right) (mE)^2 \\ & - \frac{1}{L^3} \frac{N(N-1)}{2} (mE) \simeq \frac{m\eta_3}{L^9} \frac{N^2(N-1)^2(N-2)}{12}. \end{aligned} \quad (35)$$

The cubic equation, Eq. (35), can be easily solved by perturbation theory,

$$mE = \frac{N(N-1)}{2} \frac{4\pi a_0}{L^3} \left[1 + \sum_{n=1}^3 \left(\frac{a_0}{\pi L} \right)^n c_n \right], \quad (36)$$

where a_0 is two-body scattering length and is related to coupling constant of pairwise contact interaction by

$$m\eta_R = 4\pi a_0. \quad (37)$$

The solution of cubic equation, Eq. (35) is thus given by

$$\begin{aligned} E &= \frac{N(N-1)}{2} \frac{4\pi a_0}{mL^3} \left[1 - \left(\frac{a_0}{\pi L} \right) \mathcal{I} \right. \\ &+ \left(\frac{a_0}{\pi L} \right)^2 (\mathcal{I}^2 + (2N-5)\mathcal{J}) + \mathcal{O}\left(\frac{a_0^3}{L^3}\right) \\ &+ \left. \frac{N(N-1)(N-2)}{6} \frac{\eta_3}{L^6} \right], \end{aligned} \quad (38)$$

FIG. 2. Plot of mE as the function of L : red dashed curves are exact solutions given by Eq. (39) and black solid curves are approximate solution given by threshold expansion in Eq. (42), $m\eta = 0.1$.

which is consistent with well-known results in Refs. [59–62].

C. Threshold expansion formula in 1D and comparison to exact solutions

The iteration of coupled finite volume N -body dynamical equation approach and results presented in Secs. III A and III B can be applied to N -boson interaction in one-dimensional (1D) with little changes. N identical bosons interacting with pairwise contact potentials in 1D is in fact exactly solvable; see Refs. [47,63,64]. The exact analytic solutions are given by $E = \frac{1}{2m} \sum_{i=1}^N p_i^2$, where p_i 's satisfies coupled equations,

$$\frac{p_i L}{2} = \sum_{j=1(j \neq i)}^N \cot^{-1} \left(\frac{p_i - p_j}{m\eta} \right), \quad \sum_{i=1}^N p_i = 0. \quad (39)$$

Keeping only pairwise contact interaction and expanding up to $\mathcal{O}(E^3)$, the threshold expansion equation in 1D can be obtained by replacing 3D momentum sum $\frac{1}{L^3} \sum_{\mathbf{p}}$ in Eq. (32) by 1D counterpart $\frac{1}{L} \sum_p$; hence, we obtain

$$\begin{aligned} \frac{1}{L} \frac{N(N-1)}{2} &\simeq \left(\frac{1}{L} \sum_{p \neq 0} \frac{1}{p^4} \right) (mE)^2 \\ &+ \left(\frac{1}{m\eta} + \frac{1}{L} \sum_{p \neq 0} \frac{1}{p^2} - \frac{\eta}{L^2} \sum_{p \neq 0} \frac{(N+5)(N-2)}{2p^4} \right) (mE). \end{aligned} \quad (40)$$

The infinite momentum sum in 1D can be carried out rather easily,

$$\sum_{p \neq 0} \frac{1}{p^2} = \frac{L^2}{12}, \quad \sum_{p \neq 0} \frac{1}{p^4} = \frac{L^4}{720}, \quad (41)$$

the solution of Eq. (40) is thus given by

$$mE = \frac{\sqrt{b^2 + 4c} - b}{2}, \quad (42)$$

where

$$\begin{aligned} b &= \frac{720}{L^3} \left(\frac{1}{m\eta} + \frac{L}{12} - \frac{(N+5)(N-2)}{2} \frac{m\eta L^2}{720} \right), \\ c &= \frac{N(N-1)}{2} \frac{720}{L^4}. \end{aligned} \quad (43)$$

The comparison of mE as the function of L between exact solutions given by Eq. (39) and approximate solution by threshold expansion in Eq. (42) is illustrated in Fig. 2.

IV. SUMMARY

As a sanity check and a test on the formalism of finite volume N -body system developed in [50,52,55], we illustrate how the well-known threshold expansion formula of N -identical-boson system may be derived by iterations of Faddeev-type coupled dynamical equations. The ground state energy of N -boson system near threshold is dominated by zero momenta mode of N -body amplitudes, nonzero momenta mode amplitudes are associated with subleading order contributions and are related to leading order zero momenta mode through Faddeev-type coupled dynamical equations. Eliminating nonzero momenta modes by iterations ultimately yields an analytic expression that depends on only system energy and free momentum sum; thus, it can be turned into a polynomial equation by treating energy shift near threshold as a small parameter. With only a single iteration, we are able to compute threshold expansion formula up to $\mathcal{O}(\eta^3/L^5)$ for pairwise interaction and $\mathcal{O}(\eta_3/L^6)$ for three-body interaction.

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