Vacuum polarization corrections to the $n = 2$ and $n = 3$ levels in muonic ${}^4\text{He}$

Wayne W. Repko,^{1,*} Stanley F. Radford,^{2,†} and Duane A. Dicus^{3,‡}

¹*Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA*

²*Department of Physics, The College at Brockport, State University of New York, Brockport, New York 14420, USA*

³*Department of Physics and Center for Particles and Fields, University of Texas, Austin, Texas 78712, USA*

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The $n = 2$ and $n = 3$ levels for muonic helium are calculated using a potential that includes all one-loop and recoil effects. Electronic vacuum polarization corrections are calculated using an extension of the Kinoshita and Nio method. For $n = 2$, the results are $2p_{1/2} - 2s_{1/2} = 1374.24 \pm 1.4$ meV and $2p_{3/2} - 2s_{1/2} = 1520.83 \pm 1.4$ meV, essentially in agreement with the latest summary of the current calculations. The $n = 3$ results are summarized in tabular form and give $3p_{1/2} - 3s_{1/2} = 394.48 \pm 0.43$ meV and $3d_{3/2} - 3p_{3/2} = 111.40$ meV.

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I. INTRODUCTION

The tension between the determination of the proton size measured using muonic hydrogen energy levels [1,2] and electron scattering data [3] has focused renewed attention on accurate calculations of the level splitting in muonic atoms [4–9]. This discrepancy is also being addressed by using increasingly precise measurements of the electronic energy levels of ordinary hydrogen [10,11]. Part of the problem in the muonic hydrogen extraction of the proton radius is related to the appropriate modeling of its complicated electromagnetic structure consisting of both electric and magnetic form factors. There are also many more

QED corrections since the presence of an additional spin means that there are hyperfine and tensor interactions as well as mixing between energy levels. While these corrections are well understood, they, in some sense, get in the way of figuring out the proper way to model the size parameters of the proton. As a consequence, studying a system without these additional terms could prove helpful in understanding the nuclear corrections.

The case of muonic ${}^4\text{He}$ is considerably simpler since the nucleus is spinless and there is only one nuclear form factor and a single spin [12,13]. The underlying potential that describes a muon interacting with a point nucleus, including all recoil and one-loop effects, is

$$\begin{aligned}
 V(\vec{r}) = & -Z\alpha \left\{ \frac{1}{r} - \frac{\pi}{2m_1^2} \delta(\vec{r}) - \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) \frac{\vec{L} \cdot \vec{S}}{r^3} + \frac{1}{2m_1 m_2} p_i \left(\frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r} \right) p_j \right. \\
 & - \frac{\alpha}{2\pi} \left(\frac{1}{m_1^2} + \frac{1}{m_1 m_2} \right) \frac{\vec{L} \cdot \vec{S}}{r^3} - \frac{4\alpha}{3} \ln(\mu/\lambda) \left(\frac{1}{m_1^2} + \frac{2Z}{m_1 m_2} + \frac{Z^2}{m_2^2} \right) \delta(\vec{r}) \\
 & + \frac{4\alpha}{3} \left(\frac{1}{m_1^2} \ln(\eta_2) + \frac{Z^2}{m_2^2} \ln(\eta_1) \right) \delta(\vec{r}) - \frac{2Z\alpha}{m_1 m_2} \left(\frac{m_1^2 \ln(\eta_1) - m_2^2 \ln(\eta_2)}{m_1^2 - m_2^2} \right) \delta(\vec{r}) \\
 & \left. - \frac{7Z\alpha}{6\pi m_1 m_2} \nabla^2 \left(\frac{\ln(\mu r + \gamma)}{r} \right) \right\}. \tag{1}
 \end{aligned}$$

*repko@pa.msu.edu

†sradford@brockport.edu

‡dicus@physics.utexas.edu

The first line in Eq. (1) is the fine-structure contribution to the potential derived from the one photon exchange diagram and is usually called the Breit-Pauli potential. Including the relativistic corrections to the kinetic energy completes the fine-structure contribution. The remaining terms are one-loop corrections obtained from the muonium potential of Ref. [14] by neglecting the terms containing the muon spin ($\vec{\sigma}_2/2$). Ideally, the spin-independent terms should be verified by calculating the one-loop correction to the Breit-Pauli potential. This potential will be the starting point for the calculation of corrections to both the $n = 2$ and $n = 3$ levels. The parameters are $m_1 = m_\mu = 105.658372$ MeV, $m_2 = m_{He} = 3728.400128$ MeV, and $\alpha = 7.297352566 \cdot 10^{-3}$. μ is the reduced mass, λ is the infrared cutoff, $\eta_j = m_j/(m_1 + m_2)$ and γ is the Euler constant.

Explicit calculations are given for the $n = 2$ case to establish that this approach reproduces the usual results for the $n = 2$ Lamb shift. The results for $n = 3$ follow in the same way and are summarized in tabular form. In either

case, the largest corrections to Eq. (1) are QED electron vacuum polarization corrections to the Coulomb interaction and modifications due to the nuclear form factor. These will be treated in the next two sections. The next largest corrections are those from the fine-structure portion of the potential and are calculated in Sec. IV. The one-loop corrections are treated in Sec. V and the $n = 3$ results are summarized in Sec. VI. There are also numerous smaller vacuum polarization corrections to the fine-structure terms that are summarized in the Appendixes.

II. VACUUM POLARIZATION CORRECTIONS

The primary effect of including the $e\bar{e}$ vacuum polarization corrections is the modification of the Coulomb potential. Perhaps the simplest way to treat the vacuum polarization corrections to the Coulomb potential is to use the momentum space approach of Kinoshita and Nio [15]. They show that the order by order energy corrections can be expressed as

$$\Delta E_{n,\ell} = -\frac{4\pi Z\alpha}{(2\pi)^3} \int d^3k \frac{1}{k^2} (\Pi_f^{(2)}(\vec{k}^2) + \Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(2)}(\vec{k}^2) + \Pi_f^{(4)}(\vec{k}^2) + \Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(2)}(\vec{k}^2) + 2\Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(4)}(\vec{k}^2) + \Pi_f^{(6)}(\vec{k}^2) + \dots) \rho_{n\ell}(k^2 a^2), \quad (2)$$

where a denotes the Bohr radius $(\mu\alpha)^{-1}$ and $\rho_{n\ell}(k^2 a^2)$ is the Fourier transform of the nonrelativistic probability density

$$\rho_{n\ell}(k^2 a^2) = \int d^3r \langle |\psi_{n\ell}^m(\vec{r})|^2 \rangle e^{-i\vec{k}\cdot\vec{r}}. \quad (3)$$

The angular brackets indicate an average over the degenerate m values for a given ℓ . For the $n = 2$ states [15]

$$\rho_{2s}(k^2 a^2) = \frac{Z^4(Z^4 - 3k^2 a^2 Z^2 + 2k^4 a^4)}{(Z^2 + k^2 a^2)^4}, \quad \rho_{2p}(k^2 a^2) = \frac{Z^6(Z^2 - k^2 a^2)}{(Z^2 + k^2 a^2)^4}. \quad (4)$$

Note that Eq. (2) contains both reducible e.g., $\Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(2)}(\vec{k}^2)$, and irreducible e.g., $\Pi_f^{(4)}(\vec{k}^2)$, contributions. The sum of these contributions can be obtained if one uses the dispersion representation of Källen [16] and Lehmann [17]. The electron vacuum polarization corrections to the Coulomb potential take the form

$$D(\vec{k}^2) = -\frac{Ze^2}{k^2} - Ze^2 \int_{4m_e^2}^{\infty} \frac{d\lambda}{\lambda} \frac{\Delta(\lambda)}{\lambda + k^2}, \quad (5)$$

where

$$\Delta(q^2) = -\frac{(2\pi)^3}{3q^2} \sum_n \delta^{(4)}(q - q_n) \langle 0 | j_\mu(0) | n \rangle \langle n | j^\mu(0) | 0 \rangle. \quad (6)$$

For any order in e^2 , the integral in Eq. (5) results in

$$-Ze^2 \int_{4m_e^2}^{\infty} \frac{d\lambda}{\lambda} \frac{\Delta^{(n)}(\lambda)}{\lambda + k^2} = -\frac{Ze^2}{k^2} \Pi^{(n)}(\vec{k}^2), \quad (7)$$

where $\Pi^{(n)}(\vec{k}^2)$ is the sum of all $n/2$ loop vacuum polarization corrections, both reducible and irreducible. In general, the expression for the vacuum polarization correction at order n is

$$\Delta E_{\text{VP}}^{(n)}(n, \ell, Z) = -\frac{4\pi Z\alpha}{(2\pi)^3} \int_{4m_e^2}^{\infty} \frac{d\lambda}{\lambda} \Delta^{(n)}(\lambda) \int \frac{d^3k}{k^2 + \lambda} \rho_{n\ell}(k^2 a^2). \quad (8)$$

(6) For the $2s$ and $2p$ states, the d^3k integrals are

TABLE I. Contributions to the $2p_j - 2s_{1/2}$ splitting for muonic helium. The entries are in meV. VP denotes vacuum polarization.

	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$	$2p_{1/2} - 2s_{1/2}$	$2p_{3/2} - 2s_{1/2}$
One-loop m_e VP	-2077.25	-411.46	-411.46	1665.79	1665.79
One-loop m_e relativistic VP	-0.85	-0.32	-0.04	0.53	0.81
Two-loop (Kallen-Sabry) m_e VP	-15.24	-3.67	-3.67	11.57	11.57
Three-loop (Kinoshita-Nio) m_e VP				0.023	0.023
Second-order m_e perturbation theory	-1.90	-0.19	-0.19	1.71	1.71
Third-order m_e perturbation theory				0.01	0.01
One-loop m_μ VP	-0.34	0.00	0.00	0.34	0.34
One-loop m_π VP	-0.26	0.00	0.00	0.26	0.26
Nuclear size	292.95 ± 1.4	0.00	0.00	-292.95 ± 1.4	-292.95 ± 1.4
Nuclear size VP corrections	2.41	-0.02	-0.02	-2.43	-2.43
Nuclear polarizability	-2.35 ± 0.13			2.35 ± 0.13	2.35 ± 0.13
Fine structure	-183.30	-183.00	-37.43	0.29	145.86
Fine-structure VP corrections	0.86	-0.11	0.02	-0.97	-0.84
Fine-structure VP second order	0.262	-0.17	-0.02	-0.43	-0.29
Order α^5 spin orbit	0.00	-0.31	0.16	-0.31	0.16
Order α^5 Lamb shift	11.57	0.03	0.03	-11.54	-11.54
Total	-1973.44 ± 1.4	-599.23	-452.64	1374.24 ± 1.4	1520.83 ± 1.4

$$\frac{\pi^2 Z^3 (2a^2 \lambda + Z^2)}{2a(a\sqrt{\lambda} + Z)^4} \quad \text{and} \quad \frac{\pi^2 Z^5}{2a(a\sqrt{\lambda} + Z)^4}. \quad (9)$$

With these results, the $2s$ and $2p$ states can be expressed as

$$\Delta E_{\text{VP}}^{(n)2S} = -\frac{\mu Z^4 \alpha^2}{4} \int_4^\infty \frac{dx}{x} \Delta^{(n)}(x) \frac{(2\beta^2 x + Z^2)}{(\beta\sqrt{x} + Z)^4}, \quad (10)$$

$$\Delta E_{\text{VP}}^{(n)2P} = -\frac{\mu Z^6 \alpha^2}{4} \int_4^\infty \frac{dx}{x} \Delta^{(n)}(x) \frac{1}{(\beta\sqrt{x} + Z)^4}, \quad (11)$$

where $\lambda = m_e^2 x$ and $\beta = m_e a$. In particular, for the $e\bar{e}$ intermediate state, $\Delta^{(2)}(x)$ is

$$\Delta^{(2)}(x) = \frac{\alpha}{3\pi} (1 + 2/x) \sqrt{1 - 4/x} \theta(x - 4). \quad (12)$$

For the fourth-order Kallen-Sabry [18] contribution, $\Delta^{(2)}(x)$ is replaced by $\Delta^{(4)}(\sqrt{x}/2)$, where $\Delta^{(4)}(x)$ is given

in Ref. [19]. The results are contained in Table I. All numerical integrals were evaluated using the NIntegrate routine in *Mathematica*.

From the last line in Eq. (2), there are additional α^5 corrections. The terms containing $[\Pi_f^{(2)}(\vec{k}^2)]^3 + 2\Pi_f^{(2)}(\vec{k}^2)\Pi_f^{(4)}(\vec{k}^2) = 2\Pi_f^{(2)}(\vec{k}^2)\Pi^{(4)}(\vec{k}^2) - [\Pi_f^{(2)}(\vec{k}^2)]^3$, where $\Pi^{(4)}(\vec{k}^2) = [\Pi_f^{(2)}(\vec{k}^2)]^2 + \Pi_f^{(4)}(\vec{k}^2)$, can be evaluated using previous analytic expressions. However, an analytical expression for $\Delta^{(6)}(q^2)$ is not available, which means that $\Pi_f^{(6)}(\vec{k}^2)$ must be included numerically [15].

Individually, each of the terms in Eq. (2) can, by itself, be viewed as a perturbation that can contribute corrections in higher order nonrelativistic perturbation theory. For example, there is a fourth-order electron vacuum polarization correction arising from treating the first term in Eq. (2) in second-order perturbation theory. It can be written as

$$E_{n\ell}^{(2)} = \mu^2 Z \alpha \int_0^\infty dr r^2 R_{n\ell}(Zr/a) VP(r) \int_0^\infty dr' r'^2 g_{n\ell}(Zr/a, Zr'/a) VP(r') R_{n\ell}(Zr'/a), \quad (13)$$

where $VP(r)$ is

$$VP(r) = -\frac{Z\alpha}{r} \int_4^\infty \frac{dy}{y} \Delta^{(2)}(y) e^{-m_e \sqrt{y} r}, \quad (14)$$

and $g_{n\ell}(Zr/a, Zr'/a)$ is the radial Coulomb Green's function. For $n = 2$ and $\ell = 0$ with $x = Zr/a$, $x' = Zr'/a$, $E_{20}^{(2)}$ is

$$E_{20}^{(2)} = \frac{\mu Z^2 \alpha^2}{2} \int_4^\infty \frac{dy}{y} \Delta^{(2)}(y) \int_4^\infty \frac{dy'}{y'} \Delta^{(2)}(y') \int_0^\infty dx x(1 - x/2) e^{-x/2} e^{-bx} \int_0^\infty dx' x' g_{20}(x, x') e^{-b'x'} (1 - x'/2) e^{-x'/2}. \quad (15)$$

Here $b = m_e a \sqrt{y}/Z$ and $b' = m_e a \sqrt{y'}/Z$. The integrals over x and x' can be evaluated analytically and the remaining integrals were evaluated numerically. There is a similar expression for $E_{21}^{(2)}$ with $x e^{-x/2}$ replacing $(1-x/2)e^{-x/2}$ and 24 replacing 2. The radial Green's functions $g_{20}(x, x')$ and $g_{21}(x, x')$ are [19–23]

$$g_{20}(x, x') = Z e^{-(x+x')/2} \left[(2-x)(2-x') \left(\ln(x) + \ln(x') + \frac{(x+x')}{4} + 2\gamma - \frac{15}{4} - \text{Ei}(x_{<}) \right) + 12 - 2x - 2x' - \frac{2}{x} - \frac{2}{x'} + \frac{x}{x'} + \frac{x'}{x} - xx' + (2-x_{>}) e^{x_{<}} \left(\frac{1}{x_{<}} - 1 \right) \right], \quad (16a)$$

$$g_{21}(x, x') = \frac{Z}{3} x x' e^{-(x+x')/2} \left[\ln(x) + \ln(x') + \frac{(x+x')}{4} + 2\gamma - \frac{49}{12} - \frac{3}{x} - \frac{3}{x'} - \frac{2}{x^3} - \frac{2}{x'^3} - \frac{3}{x'} - \frac{3}{x'^2} - \frac{2}{x'^3} - \text{Ei}(x_{<}) + e^{x_{<}} \left(\frac{1}{x_{<}} + \frac{1}{x_{<}^2} + \frac{2}{x_{<}^3} \right) \right], \quad (16b)$$

where $x = Zr/a$ and $x' = Zr'/a$. $g_{20}(x, x')$ differs from Ref. [23]. The numerical results are in Table I.

Here, too, there are fifth-order electron vacuum polarization corrections obtained by using $\Pi_f^{(2)}(\vec{k}^2)$ in third-order perturbation theory, and $\Pi_f^{(2)}(\vec{k}^2)$ together with $\Pi_f^{(4)}(\vec{k}^2)$ in second-order perturbation theory. The third-order correction was only partially treated in [15]. This order correction has an extra term as was pointed out in Ref. [24]. The corrected value is included in Table I.

Finally, there are relativistic corrections to the one-loop electron vacuum polarization contributions. These are usually calculated using Dirac wave functions for a particle with a mass corresponding to the reduced mass of muonic helium moving in a static Coulomb potential. While not completely rigorous, these wave functions give a reasonable estimate since two-body relativistic wave functions are not available. The corrections were calculated using the results of Ref. [25].

III. NUCLEAR SIZE CORRECTIONS

For a spinless nucleus, the simplest nuclear size connection comes from expanding the charge form factor as

$$F(\vec{k}^2) \sim 1 - \frac{1}{6} \langle r^2 \rangle \vec{k}^2 + \dots, \quad (17)$$

where $\langle r^2 \rangle \equiv r_N^2$ is the nuclear mean squared charge radius. This leads to an effective potential

$$\Delta V_N(\vec{r}) = \frac{2}{3} \pi Z \alpha r_N^2 \delta(\vec{r}), \quad (18)$$

and an energy correction in the $2s$ state of

$$\Delta E_N 2S = \frac{\mu^3 Z^4 \alpha^4 r_N^2}{12} = 105.33 r_N^2 \text{ meV fm}^{-2}, \quad (19)$$

for $Z = 2$. This result is known to overestimate the correction because any reasonable form factor will contain a term order r_N^3/a^3 with the opposite sign.

To illustrate this, consider the Gaussian form factor $F(\vec{k}^2) = e^{-k^2 r_N^2/6}$, which is often used to parametrize form factors. The corresponding form of $\Delta E_N 2S$ is

$$\begin{aligned} \Delta E_N 2S &= \frac{\mu^3 Z^4 \alpha^4 r_N^2}{12} \left(1 - \frac{8}{3} \sqrt{\frac{2}{3\pi}} \frac{Z r_N}{a} \right) \\ &= 105.33 r_N^2 (1 - 0.009335 r_N). \end{aligned} \quad (20)$$

The result for helium with $r_N = 1.681 \pm 0.004$ fm [13] is given in Table I. The majority of the error in the nuclear size correction comes from the uncertainty in r_N . The value for the nuclear polarizability is taken from Ref. [26].

IV. FINE-STRUCTURE CORRECTIONS

The contributions from the various fine-structure terms are

$$\frac{Z \alpha \pi}{2 m_1^2} \langle 2s_{1/2} | \delta(\vec{r}) | 2s_{1/2} \rangle = \frac{\mu^3 Z^4 \alpha^4}{16 m_1^2} \quad (21)$$

$$\begin{aligned} &Z \alpha \left(\frac{1}{2 m_1^2} + \frac{1}{m_1 m_2} \right) \langle 2p_j | \frac{\vec{L} \cdot \vec{S}}{r^3} | 2p_j \rangle \\ &= \frac{\mu^3 Z^4 \alpha^4}{48} \left(\frac{1}{2 m_1^2} + \frac{1}{m_1 m_2} \right) (j(j+1) - 11/4) \end{aligned} \quad (22)$$

$$-\frac{Z \alpha}{2 m_1 m_2} \langle 2s | p_i \left(\frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r} \right) p_j | 2s \rangle = -\frac{3 \mu^3 Z^4 \alpha^4}{16 m_1 m_2} \quad (23)$$

$$-\frac{Z \alpha}{2 m_1 m_2} \langle 2p | p_i \left(\frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r} \right) p_j | 2p \rangle = -\frac{\mu^3 Z^4 \alpha^4}{16 m_1 m_2} \quad (24)$$

$$-\frac{1}{8} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \langle 2s | (\vec{p}^2)^2 | 2s \rangle = -\frac{13}{128} \mu^4 Z^4 \alpha^4 \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \quad (25)$$

$$-\frac{1}{8} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \langle 2p | (\vec{p}^2)^2 | 2p \rangle = -\frac{7}{384} \mu^4 Z^4 \alpha^4 \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \quad (26)$$

Using these results, the total fine-structure contributions are

$$E_{\text{FS}}(2s_{1/2}) = -\frac{13}{128} \mu^4 Z^4 \alpha^4 \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) - \frac{3\mu^3 Z^4 \alpha^4}{16m_1 m_2} + \frac{\mu^3 Z^4 \alpha^4}{16m_1^2}, \quad (27)$$

$$E_{\text{FS}}(2p_{1/2}) = -\frac{7}{384} \mu^4 Z^4 \alpha^4 \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) - \frac{\mu^3 Z^4 \alpha^4}{24} \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) - \frac{\mu^3 Z^4 \alpha^4}{16m_1 m_2}, \quad (28)$$

$$E_{\text{FS}}(2p_{3/2}) = -\frac{7}{384} \mu^4 Z^4 \alpha^4 \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) + \frac{\mu^3 Z^4 \alpha^4}{48} \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) - \frac{\mu^3 Z^4 \alpha^4}{16m_1 m_2}. \quad (29)$$

The numerical results are given in Table I. Note that the $2p_{1/2}$ and $2s_{1/2}$ are not degenerate due to recoil effects.

V. ONE-LOOP CONTRIBUTIONS

The one-loop corrections consist of a spin-orbit contribution and a spin-independent contribution, both of order $Z^4 \alpha^5$. The spin-orbit correction is

$$\langle 2p_j | Z \alpha a_\mu \left(\frac{1}{m_1^2} + \frac{1}{m_1 m_2} \right) \frac{\vec{L} \cdot \vec{S}}{r^3} | 2p_j \rangle = \frac{\mu^3 Z^4 \alpha^4 a_\mu}{48} \left(\frac{1}{m_1^2} + \frac{1}{m_1 m_2} \right) (j(j+1) - 11/4), \quad (30)$$

where a_μ is the muon anomalous magnetic moment.

The matrix element of the remaining one-loop spin independent term V'_{SI} is [14]

$$\begin{aligned} \langle n\ell | V'_{SI} | n\ell \rangle = & \frac{\mu^3 Z^4 \alpha^5}{n^3 \pi} \left\{ \frac{4}{3} \left(\left(\log \left[\frac{1}{Z^2 \alpha^2 k_0[n, 0]} \right] + \frac{5}{6} \right) \left(\frac{1}{m_1^2} + \frac{2Z}{m_1 m_2} + \frac{Z^2}{m_2^2} \right) \right. \right. \\ & - \left(\frac{1}{m_1^2} \log(\eta_2) + \frac{Z^2}{m_2^2} \log(\eta_1) \right) \Bigg) + \frac{14Z}{3m_1 m_2} \left(\log \left[\frac{2Z\alpha}{n} \right] + \frac{n-1}{2n} + \sum_{j=0}^n \frac{1}{j} \right) \\ & + \frac{2Z}{m_1 m_2} \left(\frac{m_1^2 \log(\eta_1) - m_2^2 \log(\eta_2)}{(m_1^2 - m_2^2)} \right) \Bigg\} \delta_{\ell 0} \\ & + \frac{\mu^3 Z^4 \alpha^5}{n^3 \pi} \left\{ \frac{4}{3} \log \left[\frac{1}{k_0[n, \ell]} \right] \left(\frac{1}{m_1^2} + \frac{2Z}{m_1 m_2} + \frac{Z^2}{m_2^2} \right) - \frac{7Z}{3m_1 m_2} \frac{1}{\ell(\ell+1)(2\ell+1)} \right\} (1 - \delta_{\ell 0}). \end{aligned} \quad (31)$$

The Bethe sums $k_0[n, \ell]$ used are $\ln(k_0[1, 0]) = 2.9841285$, $\ln(k_0[2, 0]) = 2.8117699$, and $\ln(k_0[2, 1]) = -0.0300167$ [27]. Again, the numerical results are listed in Table I.

VI. RESULTS FOR $n = 3$

The corresponding results for $n = 3$ can be calculated using the same techniques that were used in the $n = 2$ case. Wave functions for the relativistic corrections can be derived from Rose [28], Eqs. (5.49a) and (5.49b). The necessary radial Green's functions are [23]

$$\begin{aligned} g_{30}(x, x') = & \frac{4Z}{9} e^{-(x+x')/2} [(x^2/2 - 3x + 3)(x'^2/2 - 3x' + 3)(\log(x) + \log(x') + (x+x')/6 + 2\gamma - 25/6 - \text{Ei}(x_-)) \\ & + ((3-x)x/3 - 5x/2 - 1/x + 7)(x'^2/2 - 3x' + 3) + (x^2/2 - 3x + 3)((3-x')x'/3 - 5x'/2 - 1/x' + 7) \\ & + (x_-^2/2 - 3x_- + 3)e^{x_-}(x_-/2 + 1/x_- - 5/2)], \end{aligned} \quad (32a)$$

$$\begin{aligned}
g_{31}(x, x') = & \frac{Z}{18} x x' e^{-(x+x')/2} [(4-x)(4-x')(\log(x) + \log(x') + (x+x')/6 + 2\gamma - 55/12 - \text{Ei}(x_{<})) \\
& + (4-x)(x'/3 - 6/x' - 4/x'^2 - 2/x'^3 + 5) + (x/3 - 6/x - 4/x^2 - 2/x^3 + 5)(4-x') \\
& + (4-x_{>}) e^{x_{<}} (2/x_{<}^3 + 2/x_{<}^2 + 3/x_{<} - 1)], \tag{32b}
\end{aligned}$$

$$\begin{aligned}
g_{32}(x, x') = & \frac{Z}{90} x^2 x'^2 e^{-(x+x')/2} [\log(x) + \log(x') + (x+x')/6 + 2\gamma - 89/20 - \text{Ei}(x_{<}) \\
& - 5/x - 10/x^2 - 20/x^3 - 30/x^4 - 24/x^5 - 5/x' - 10/x'^2 - 20/x'^3 - 30/x'^4 - 24/x'^5 \\
& + e^{x_{<}} (1/x_{<} + 1/x_{<}^2 + 2/x_{<}^3 + 6/x_{<}^4 + 24/x_{<}^5)], \tag{32c}
\end{aligned}$$

where $x = 2Zr/3a$ and $x' = 2Zr'/3a$. The $n = 3$ Bethe sums can be found in Ref. [27].

The resulting level corrections and splittings are summarized in Table II. The nuclear polarizability correction was obtained by rescaling the result in Ref. [26] using the $3s$ wave function evaluated at the origin.

VII. CONCLUSIONS

Starting from a potential describing the QED interaction between a point scalar nucleus and a muon that includes the one-loop level corrections and all recoil effects, the Lamb shifts for $n = 2$ and $n = 3$ muonic helium have been calculated. The effects of electron vacuum polarization were included by extending the Kinoshita-Nio [15] method to all operators in the potential. This significantly simplifies the calculation of the various corrections. The nuclear size correction was modeled using a simple one-parameter form factor. Given an accurate value of r_N , the resulting correction can be included in the prediction of the $2p_{1/2} - 2s_{1/2}$ and

$2p_{3/2} - 2s_{1/2}$ splittings and the corresponding results for $n = 3$, or the correction can be used to extract a value of r_N from spectroscopic data.

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APPENDIX A: DETAILS OF THE VACUUM POLARIZATION CORRECTS TO THE FINE STRUCTURE

The additional corrections to the fine-structure potential involve vacuum polarization modifications to the $1/r$ factors contained in the first line of Eq. (1) as well as traditional second-order perturbative corrections. For the former, the approach of Ref. [15] is very convenient. Using this technique, the expectation value of any potential $V(r)$ can be expressed as

TABLE II. Contributions to the $3p_{1/2} - 3s_{1/2}$ and $3d_{3/2} - 3p_{3/2}$ splittings for muonic helium. The entries are in meV. VP denotes vacuum polarization.

	$3s_{1/2}$	$3p_{1/2}$	$3p_{3/2}$	$3d_{3/2}$	$3d_{5/2}$	$3p_{1/2} - 3s_{1/2}$	$3d_{3/2} - 3p_{3/2}$
One-loop m_e VP	-603.07	-121.83	-121.83	-11.37	-11.37	481.24	110.46
One-loop m_e relativistic VP	-0.26	-0.10	-0.02	0.00	0.00	0.16	0.02
Kallen-Sabry VP	-4.41	-1.05	-1.05	-0.13	-0.13	3.36	0.92
Second-order m_e VP	-0.86	-0.10	-0.10	0.00	0.00	0.76	0.10
One-loop m_μ VP	-0.10	0.00	0.00	0.00	0.00	0.10	0.00
One-loop m_π VP	-0.08	0.00	0.00	0.00	0.00	0.08	0.00
Nuclear size	86.80 ± 0.41	0.00	0.00	0.00	0.00	-86.80 ± 0.41	0.00
Nuclear size VP corrections	1.09	-0.01	-0.01	0.00	0.00	-1.10	0.01
Nuclear polarizability	-0.70 ± 0.13	0.00	0.00	0.00	0.00	0.70 ± 0.13	
Fine structure	-65.01	-64.92	-21.79	-21.84	-7.39	0.09	-0.05
Fine-structure VP corrections	0.25	-0.06	-0.01	0.00	0.00	-0.31	0.01
Fine-structure VP second order	0.20	-0.09	-0.01	0.00	0.00	-0.29	0.01
Order α^5 spin orbit	0.00	-0.07	0.03	-0.02	0.01	-0.07	-0.05
Order α^5 Lamb shift	3.45	0.01	0.01	0.00	0.00	-3.44	-0.01
Total	-582.69 ± 0.43	-188.20	-144.76	-33.37	-18.88	394.48 ± 0.43	111.40

$$\langle n\ell | V | n\ell \rangle = \frac{1}{(2\pi)^3} \int d^3k V(k^2) \rho_{n\ell}(k^2 a^2), \quad (\text{A1})$$

where $V(k^2)$ is the Fourier transform of $V(r)$ and $a = (\mu\alpha)^{-1}$.

To see how this works, consider the fine-structure contribution from the term $Z\alpha\pi\delta(\vec{r})/(2m_1^2)$. Since the Fourier transform of $\delta(\vec{r})$ is 1, the contribution to the energy is

$$\langle n\ell | \frac{Z\alpha\pi}{2m_1^2} \delta(\vec{r}) | n\ell \rangle = \frac{\mu^3 Z\alpha^4}{4\pi m_1^2} \int_0^\infty dy y^2 \rho_{n\ell}(y), \quad (\text{A2})$$

where $y = ka$. Using Eqs. (4), this gives Eq. (21) for the $2s$ state and 0 for the $2p$ state. To include the effect of electron vacuum polarization, recall that the leading contribution to this is given by

$$\Pi_f^{(2)}(k^2) = \frac{2\alpha}{\pi} \left[\frac{1}{3} (1 - 2m_e^2/k^2) \left(\sqrt{1 + 4m_e^2/k^2} \sinh^{-1}(k/(2m_e)) - 1 \right) + \frac{1}{18} \right] = \frac{2\alpha}{\pi} \Pi^{(2)}(k^2). \quad (\text{A3})$$

To calculate the $\Pi_f^{(2)}(k^2)/k^2$ correction, it is only necessary to introduce $\Pi_f^{(2)}(k^2)$ into Eq. (A2). The result, after rescaling, is

$$EVP_\delta = \frac{\mu^3 Z\alpha^5}{2\pi^2 m_1^2} \int_0^\infty dy y^2 \Pi^{(2)}(y/\beta) \rho_{n\ell}(y), \quad (\text{A4})$$

with $\beta = m_e a$. The numerical results are given in Table III for $n = 2$.

With a slight modifications of the definition of $\rho_{n\ell}$, the other fine-structure terms can be treated in a similar way. The expectation value of the fine-structure spin-orbit term is

$$\begin{aligned} \langle n\ell j | V_{LS} | n\ell j \rangle &= \frac{Z\alpha}{2} (j(j+1) - \ell(\ell+1) - 3/4) \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) \int d^3r \frac{|\psi_{n\ell}^m(\vec{r})|^2}{r^3} \\ &= \frac{Z\alpha}{2(2\pi)^3} (j(j+1) - \ell(\ell+1) - 3/4) \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) \int d^3k \frac{4\pi}{k^2} \tilde{\rho}_{n\ell}(k^2 a^2), \end{aligned} \quad (\text{A5})$$

where

$$\tilde{\rho}_{n\ell}(k^2 a^2) = \int d^3r \frac{\langle |\psi_{n\ell}^m(\vec{r})|^2 \rangle e^{-i\vec{k}\cdot\vec{r}}}{r^2} = \int_0^\infty dr R_{n\ell}^2(r) \frac{\sin(kr)}{kr}, \quad (\text{A6})$$

and the angular brackets denote an average over the degenerate m values. For $n = 2$ and $\ell = 1$, $\tilde{\rho}_{21}(k^2 a^2)$ is

$$\tilde{\rho}_{21}(k^2 a^2) = \frac{Z^6}{12a^2(k^2 a^2 + Z^2)^2}. \quad (\text{A7})$$

Then, after setting $k = y/a$,

$$\begin{aligned} \langle 2p_j | V_{LS} | 2p_j \rangle &= \frac{\mu^3 Z^4 \alpha^4}{12\pi} \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) (j(j+1) - 11/4) \int_0^\infty dy \frac{1}{(y^2 + 1)^2} \\ &= \frac{\mu^3 Z^4 \alpha^4}{48} \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) (j(j+1) - 11/4). \end{aligned} \quad (\text{A8})$$

This agrees with Eq. (22). Because of the $1/r^3$ behavior, the vacuum polarization correction is obtained by inserting $2\alpha(\Pi^{(2)} + \bar{\Pi}^{(2)})/\pi$ into Eq. (A5), where

$$\bar{\Pi}^{(2)}(k) = \left[\frac{1}{3} - \frac{2m_e^2}{k^2} + \frac{2m_e^2}{k^2} \frac{\sinh^{-1}(k/(2m_e))}{\sqrt{(k^2/(4m_e^2))(1 + k^2/(4m_e^2))}} \right], \quad (\text{A9})$$

yielding

$$EVP_{LS} = \frac{Z^4 \mu^3 \alpha^5}{6\pi^2} \left(\frac{1}{2m_1^2} + \frac{1}{m_1 m_2} \right) (j(j+1) - 11/4) \int_0^\infty dy \frac{(\Pi^{(2)}(Zy/\beta) + \bar{\Pi}^{(2)}(Zy/\beta))}{(y^2 + 1)^2}. \quad (\text{A10})$$

The remaining fine-structure electron vacuum polarization correction can be obtained in an analogous way by defining a modified $\rho_{n\ell}(k^2 a^2)$ that involves averages of derivatives of the radial wave function. Explicitly,

$$-\frac{Z\alpha}{2m_1 m_2} \langle n\ell | p_i \left(\frac{\delta_{ij}}{r} + \hat{x}_i \frac{1}{r} \hat{x}_j \right) p_j | n\ell \rangle = -\frac{Z\alpha}{2(2\pi)^3 m_1 m_2} \int d^3 k \frac{4\pi}{k^2} \hat{\rho}_{n\ell}(k^2 a^2), \quad (\text{A11})$$

where

$$\hat{\rho}_{n\ell}(k^2 a^2) = \frac{1}{k} \int_0^\infty dr \left[2r \left(\frac{dR_{n\ell}(r)}{dr} \right)^2 + \frac{\ell(\ell+1)}{r} R_{n\ell}^2(r) \right] \sin(kr). \quad (\text{A12})$$

The vacuum polarization correction is obtained by inserting $\Pi_f^{(2)}(k^2)$ into the integral in Eq. (A11). Table III contains the numerical results.

In addition, there are second-order perturbative corrections that involve $VP(r)$ and one of the terms contributing to the fine-structure corrections. They all can be reduced to the form

$$E_{n\ell}^{(2)} = 2\mu^2 Z\alpha \int_0^\infty dr r^2 R_{n\ell}(Zr/a) VP(r) \int_0^\infty dr' r'^2 g_{n\ell}(Zr/a, Zr'/a) V_{\text{FS}}(r') R_{n\ell}(Zr'/a). \quad (\text{A13})$$

The various contributions are enumerated in Table III.

APPENDIX B: DETAILS OF THE VACUUM POLARIZATION CORRECTIONS TO THE NUCLEAR SIZE

Using the nuclear charge form factor model $F(k^2) = e^{-k^2 r_N^2/6}$, the inclusion of the vacuum polarization correction is given by

$$\Delta E_N(n\ell) = -\frac{2\mu Z\alpha^2}{\pi} \int dx \Pi_f^{(2)}(x/\beta) (F(x/a) - 1) \rho_{n\ell}(x^2). \quad (\text{B1})$$

TABLE III. Electron vacuum polarization corrections to the $n = 2$ states of muonic helium are shown. The entries are in meV. VP denotes vacuum polarization.

	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
$\delta(\vec{r})$ VP	0.872	-0.021	-0.021
Spin-orbit VP	0.000	-0.088	0.044
Spin-independent VP	-0.017	-0.003	-0.003
Total fine-structure VP	0.855	-0.112	0.020
Second-order $\delta(\vec{r})$ -VP	1.403	0.000	0.000
Second-order $(\vec{p}^2)^2$ -VP	-1.058	-0.065	-0.065
Second-order spin-independent VP	-0.083	-0.007	-0.007
Second-order spin-orbit VP	0.000	-0.096	0.048
Total second-order fine-structure VP	0.262	-0.168	-0.024

For $n = 2$, $\ell = 0, 1$ numerical integration gives the results in Table IV.

The second-order vacuum polarization correction to the nuclear size for $n = 2$, $\ell = 0$ can be written

$$\Delta E_N VP = \frac{2\mu^2 Z\alpha}{4\pi} \int d^3 r \Psi_{20}^*(Zr/a) VP(r) \times \int d^3 r' g_{20}(Zr/a, Zr'/a) \Delta V_N(\vec{r}') \Psi_{20}(Zr'/a), \quad (\text{B2})$$

where for $\Delta V_N(\vec{r}')$, the correction to $-Z\alpha/r$, we use the expression in Eq. (18). The integrals over $d^3 r$ and $d^3 r'$ can be integrated analytically and the integral over $\Delta^{(2)}(x)$ was evaluated numerically. The result is found in Table IV. The contribution from the $2p$ state is negligible.

TABLE IV. Nuclear size corrections to the $n = 2$ states of muonic helium. The entries are in meV. VP denotes vacuum polarization.

	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
Nuclear size $\langle r^2 \rangle + \langle r^2 \rangle^{3/2}$	292.95	0.000	0.000
Nuclear size $F(q^2)$ VP	0.894	-0.023	-0.023
Second-order nuclear size VP	1.515	0.000	0.000
Total nuclear size VP	2.409	-0.023	-0.023

- [1] R. Pohl *et al.*, *Nature (London)* **466**, 213 (2010).
- [2] A. Antognini *et al.*, *Science* **339**, 417 (2013).
- [3] P. J. Mohr, B. N. Taylor, and D. B. Newell, *Rev. Mod. Phys.* **84**, 1527 (2012).
- [4] E. Borie and G. A. Rinker, *Rev. Mod. Phys.* **54**, 67 (1982).
- [5] E. Borie, *Phys. Rev. A* **71**, 032508 (2005).
- [6] K. Pachucki, *Phys. Rev. A* **53**, 2092 (1996).
- [7] K. Pachucki, *Phys. Rev. A* **60**, 3593 (1999).
- [8] R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, *Annu. Rev. Nucl. Part. Sci.* **63**, 175 (2013).
- [9] C. Peset and A. Pineda, *Eur. Phys. J. A* **51**, 156 (2015).
- [10] Beyer *et al.*, *Science* **358**, 79 (2017).
- [11] H. Fleurbaey, S. Galtier, S. Thomas, M. Bonnaud, L. Julien, F. Biraben, F. Nez, M. Abgrall, and J. Guéna, *Phys. Rev. Lett.* **120**, 183001 (2018).
- [12] E. Borie, *Ann. Phys. (Amsterdam)* **327**, 733 (2012).
- [13] M. Diepold, B. Franke, J. J. Krauth, A. Antognini, F. Kottmann, and R. Pohl, *Ann. Phys. (Amsterdam)* **396**, 220 (2018).
- [14] S. N. Gupta, W. W. Repko, and C. J. Suchyta III, *Phys. Rev. D* **40**, 4100 (1989).
- [15] T. Kinoshita and M. Nio, *Phys. Rev. Lett.* **82**, 3240 (1999); **103**, 079901(E) (2009).
- [16] G. Källen, *Helv. Phys. Acta* **25**, 417 (1952).
- [17] H. Lehmann, *Nuovo Cimento* **11**, 342 (1954).
- [18] A. O. G. Kallen and A. Sabry, *Kong. Dan. Vid. Sel. Mat. Fys. Med.* **29**, 1 (1955).
- [19] W. W. Repko and D. A. Dicus, *Phys. Rev. D* **98**, 013002 (2018).
- [20] L. Hostler and R. H. Pratt, *Phys. Rev. Lett.* **10**, 469 (1963).
- [21] L. Hostler, *Phys. Rev.* **178**, 126 (1969); *J. Math. Phys. (N.Y.)* **16**, 1585 (1975).
- [22] B. R. Johnson and J. O. Hirschfelder, *J. Math. Phys. (N.Y.)* **20**, 2484 (1979).
- [23] B. J. Laurenzi and A. Flamberg, *Int. J. Quantum Chem.* **11**, 869 (1977).
- [24] V. G. Ivanov, E. Y. Korzinin, and S. G. Karshenboim, *arXiv:0905.4471*.
- [25] W. W. Repko, *Phys. Rev. D* **98**, 033006 (2018). The denominators in Eq. (3) should be $(1 + \vec{k}^2 a^2)^4$ and dx/x in Eq. (8) should be dx .
- [26] I. Sick, *Phys. Rev. C* **77**, 041302 (2008).
- [27] S. Klarsfeld and A. Maquet, *Phys. Lett. B* **43**, 201 (1973).
- [28] M. E. Rose, *Relativistic Electron Theory* (John Wiley and Sons, New York, 1961), p. 178.