Entropy counting from a Schwarzschild/CFT correspondence and soft hair

Artem Averin*

*Arnold–Sommerfeld–Center for Theoretical Physics, Ludwig–Maximilians–Universität,
80333 München, Germany
and Max–Planck–Institut für Physik, Werner–Heisenberg–Institut, 80805 München, Germany*

(Received 28 October 2019; accepted 22 January 2020; published 18 February 2020)

We revisit Carlip’s approach to entropy counting. This analysis reemerged in a recently obtained Schwarzschild/conformal field theory (CFT) correspondence as the Sugawara-construction of a two-dimensional (2D) stress tensor. Here, for the example of a Schwarzschild black hole, we show how to single out diffeomorphisms forming in contrast to Carlip’s analysis the full 2D local conformal algebra. We provide arguments as to why their Hamiltonian generators are expected to be the symmetry generators of a possible conformal field theory describing the part of phase space responsible for black hole microstates. Then, we can infer central charges and temperatures of this CFT by inspecting the algebra of these Hamiltonian generators. Using these data in the Cardy formula, precise agreement with the Bekenstein-Hawking entropy is found. Alternatively, we obtain the same CFT temperatures by thermodynamic considerations. Altogether, this suggests that the Hamiltonian generators need no corrections through possible noncanonical counterterms. We comment on the related recent work by Haco, Hawking, Perry and Strominger.

DOI: [10.1103/PhysRevD.101.046024](https://doi.org/10.1103/PhysRevD.101.046024)

I. INTRODUCTION

It is one of the main problems in quantum gravity to explain the microcanonical origin of the Bekenstein-Hawking entropy $S = \frac{A}{4\hbar G}$ of black holes [1–3]. In the classical $\hbar \rightarrow 0$ limit, the entropy becomes infinite. Therefore, the Hamiltonian phase space of pure Einstein gravity has to contain infinitely many points corresponding to the microstates of a black hole for a fixed mass and angular momentum parameter. On the other hand, the black hole uniqueness theorems [4] tell that asymptotically flat and stationary solutions of Einstein’s field equations are given by the Kerr family up to diffeomorphisms. This can create the impression of an arising paradox: due to the uniqueness theorems, it may naively seem that there is no place in Hamiltonian phase space that can accommodate the infinitely many microstates as required by the classically infinite entropy. Therefore, one can ask how to reconcile the black hole uniqueness theorems with the classically infinite Bekenstein-Hawking entropy.

Since the uniqueness theorems single out the Kerr solutions up to diffeomorphisms, they already suggest a possible solution. It may be that some of the diffeomorphisms are

physical, i.e., shifts in phase space rather than gauge redundancies. This phenomenon is known to happen in gauge theories typically when the gauge parameters are nonvanishing in some asymptotic region. Such asymptotic symmetries could in gravity then be responsible for microstates of a Kerr black hole.

Indeed, the study of asymptotic symmetries in three-dimensional gravity [5] brought some success in understanding the Banados-Teitelboim-Zanelli black hole. It is found that the asymptotic symmetry algebra contains the two-dimensional (2D) local conformal algebra. Conformal field theory techniques can then (under assumptions on which we briefly elaborate at the end of this section) be used to count the state degeneracy [6], and agreement with the Bekenstein-Hawking entropy is found. Carlip raised the idea [7–10] of mimicking this in the higher-dimensional case. Although it is not clear which gauge transformations are responsible for microstates, Carlip was able to single out a Witt algebra of diffeomorphisms in the presence of a black hole event horizon. The Hamiltonian generators of these diffeomorphisms are then candidates for the generators of a possible conformal symmetry that may govern the part of phase space responsible for black hole microstates. Hamiltonian methods can then be used to study the conformal algebra of the diffeomorphism generators and CFT techniques then (under assumptions) to count the state degeneracy. Indeed, agreement with the expected Bekenstein-Hawking entropy is found.

Although Carlip’s approach is universal, it tells nothing about what the possible underlying CFT describing the

* artem.averin@campus.lmu.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP³.

relevant part in phase space really is. To understand this part in phase space has always been one of the main motivations in the study of asymptotic symmetries (see, for instance, Ref. [11] and references therein). The idea recently gained new interest as the proposal of “soft black hole hair” [12,13]. To analyze the structure of Hamiltonian phase space in the vicinity of a black hole state is nevertheless still a necessary and open problem.

To improve the situation is the overall goal of our investigations, and in Ref. [20], we describe how the Hamiltonian phase space can be analyzed in a systematic way. For a Schwarzschild black hole and assuming in a sense the application of the simplest possible scenario, we proposed a concrete candidate of a dual theory describing the part of phase space responsible for microstates. This theory was given in terms of its observables and their Poisson-bracket algebra. If conformally invariant, as is expected from several directions, Carlip’s approach to entropy counting reemerges at this point as a Sugawara construction of the conformal generators of this Schwarzschild/CFT correspondence out of its observables. Accordingly, one has to find a suited choice of Witt algebra of diffeomorphisms. Carlip presented a general construction of such an algebra in the presence of a black hole.

Here, we want to revisit Carlip’s approach. In Carlip’s construction, only one copy of a Witt algebra of diffeomorphisms and associated Hamiltonian generators are found. The two-dimensional conformal algebra consists, however, of two commuting copies. Since there seems to be no reason why all black holes should be described by chiral CFTs, it is natural to seek for diffeomorphisms building two Witt-algebra copies (further motivation for this search will be provided in Secs. II A and III A). Is such a choice possible, and does it lead to something maybe even more appropriate?

That such a choice is possible follows directly from Ref. [20]. There, we have provided a $Vir \oplus \overline{Vir}$ algebra of diffeomorphisms. Using this choice, the entropy counting procedure in the context of the proposed Schwarzschild/CFT correspondence is discussed. The main idea in the construction of these $Vir \oplus \overline{Vir}$ diffeomorphisms was to still follow Carlip’s construction [8] closely. But whereas Carlip singles out Vir diffeomorphisms in the presence of a local Killing horizon, we insist in treating both the future and past event horizons of the black hole on the same level. This then leads to two copies of Witt-algebra diffeomorphisms.

Our purpose here is to report further on our investigations as to whether this choice of $Vir \oplus \overline{Vir}$ diffeomorphisms is a proper one. This question is in principle independent of the issues discussed in Ref. [20]. Our $Vir \oplus \overline{Vir}$ diffeomorphisms are of interest as they provide a novel choice of diffeomorphisms to be used in Carlip’s approach. Inspecting the algebra of Hamiltonian generators, we will

in this work infer central charges and Virasoro zero modes (or equivalently CFT temperatures) that reproduce via the Cardy formula the expected Bekenstein-Hawking entropy. There appears no need to correct the canonically derived Hamiltonian generators by any counterterms. Furthermore, the derived CFT temperatures are in agreement with the temperatures obtained directly from the $Vir \oplus \overline{Vir}$ vector fields by thermodynamic considerations.

At this point, we briefly want to clarify a potential misconception. By proposing the mentioned $Vir \oplus \overline{Vir}$ diffeomorphisms, we are specifying the actions of the would-be conformal generators in Hamiltonian phase space at the black hole state under consideration. However, these actions should not be confused with the black hole microstate excitations. The microstate excitations that are responsible for black hole entropy and that are to be counted via the Cardy formula are not given by the Virasoro descendants. In this paper, we do not determine the microstate excitations. Therefore, we also do not know the spectrum of the would-be CFT that may govern the black hole’s phase space. Consequently, as already discussed, for instance, in Ref. [8], the applicability of the Cardy formula in entropy counting arguments following Carlip’s approach is merely an assumption if no further information on the CFT spectrum is available. Hence, the observation that the Cardy formula is able to reproduce the correct entropy should be rather viewed as a remarkable consistency check for the proposed conformal generators. This point should also be kept in mind in our analysis that is presented in the following.

Concerning the applicability of the Cardy formula, it should be furthermore noted that the conformal weights and central charges typically found for black holes via Carlip’s approach are of the same order, which is in general not the regime of applicability of the Cardy formula. However, as shown in Ref. [21], the applicability of the Cardy formula extends to this regime under some additional constraints on the light spectrum of a CFT. Therefore, the Cardy formula might be applicable in Carlip’s approach, and the observation of its agreement with the correct entropy provides a substantial consistency check. Nevertheless, since the CFT spectrum is not known in this approach, the applicability of the Cardy formula remains an assumption.

The recent work by Haco *et al.* [22] also follows Carlip’s approach to black hole entropy counting. There, an alternative choice of $Vir \oplus \overline{Vir}$ diffeomorphisms for the case of a Kerr black hole is proposed. We comment on that choice throughout our investigations.

The paper is organized as follows. In Sec. II, we briefly review Carlip’s approach to black hole entropy counting. To inspect the algebra of the Hamiltonian generators of diffeomorphisms, we derive the relevant formulas.

Especially, we explain how the CFT data needed in the Cardy formula are derived once a choice of $Vir \oplus \overline{Vir}$ diffeomorphisms has been made. In Sec. III, for the example of a Schwarzschild black hole, we explain how to single out a “preferred” $Vir \oplus \overline{Vir}$ algebra of spacetime diffeomorphisms. Alternatively to the considerations in Sec. II, we fix the associated CFT temperatures by some thermodynamic considerations. The role of the Casimir operators of the global conformal algebra for scattering off a black hole is briefly discussed. In Sec. IV, we use the derived $Vir \oplus \overline{Vir}$ diffeomorphisms in the framework of Sec. II to infer the relevant CFT data for entropy counting. We find agreement with the Bekenstein-Hawking entropy. In Sec. V, we connect these findings with Ref. [20].

In the following, we use units in which we set the speed of light to 1, but we keep Newton’s constant G and Planck’s constant $\hbar$ explicit. Latin letters $a, b, \dots = 0, \dots, 3$ denote spacetime indices.

II. GENERAL ARGUMENT AND REALIZATION

A. General argument

Here, we give a brief review of Carlip’s approach to explain the statistical mechanical origin and counting of the black hole entropy especially in dimensions higher than 3. For a more detailed discussion and references, we refer to the original papers [7–10]. The interpretation of this approach in light of a recently proposed Schwarzschild/CFT correspondence [20] was already given in that reference [20] and will also be discussed in Sec. V.

Consider an arbitrary diffeomorphism-invariant theory of gravity given by some action, which can contain black hole solutions. For a diffeomorphism given by some vector field ξ over the spacetime manifold, we denote by H_ξ the associated Hamiltonian generator. The Hamiltonian generator H_ξ —if it exists—is a function over the phase space Γ of the theory and implements the diffeomorphism ξ . The generator H_ξ is given as the sum of a bulk integral over suited gauge constraints and a suited boundary integral. On-shell H_ξ is therefore given by a boundary integral. If H_ξ is nonconstant over phase space, the diffeomorphism ξ constitutes a physical excitation; i.e., H_ξ implements a shift in the Hamiltonian phase space, otherwise a gauge redundancy.

The algebra of the Hamiltonian generators H_ξ with respect to the Poisson bracket forms on shell a representation of the algebra of the associated diffeomorphisms with respect to the ordinary Lie bracket of vector fields over the spacetime manifold up to central extensions. That is, for spacetime vector fields ξ_1 and ξ_2 , we have on shell the relation

$$\{H_{\xi_1}, H_{\xi_2}\} = H_{[\xi_1, \xi_2]} + K_{\xi_1, \xi_2}, \quad (1)$$

where K_{ξ_1, ξ_2} are constant c numbers.

So far, we have reviewed general statements of the Hamiltonian mechanics for gravity theories. What happens if we have a black hole solution? In this case, the idea is to treat the event horizon as a boundary of the spacetime manifold. The presence of such a boundary can render some diffeomorphisms H_ξ from would-be gauge redundancies to physical excitations which could be important for the statistical mechanics of the black hole. The presence of the boundary can furthermore give rise to nonvanishing central extensions in the algebra (1) of the aforementioned generators H_ξ . Carlip’s observation was that for black hole event horizons there are “natural” ways to find diffeomorphisms ξ_n ($n \in \mathbb{Z}$) which form a Witt algebra,

$$[\xi_m, \xi_n] = -i(m-n)\xi_{m+n}. \quad (2)$$

The subalgebra of (1) of the associated generators H_{ξ_n} then forms a Virasoro algebra.

Virasoro algebras constitute the symmetry algebras of two-dimensional conformal field theories. The assumption then is that there is a 2D CFT which describes the part of the phase space that is responsible for black hole microstates and of which the conformal generators are provided by the H_{ξ_n} . At this stage, it is of course not clear *whether* such a theory exists or *what* this theory is. However, accepting this assumption, one fortunately has the luxury that a lot of information about a given 2D CFT can be gained from its Virasoro algebra—for our black hole case, this would then be the algebra of the generators H_{ξ_n} .

For instance, the degeneracy of states in a 2D CFT is (often) fixed through the Cardy formula by the central charge which is read directly from the Virasoro algebra. Therefore, it is tempting to perform the following sort of consistency check of the aforementioned assumption. One can determine the central charge of the Virasoro algebra formed by the H_{ξ_n} by calculating the extensions K_{ξ_m, ξ_n} in (1). The associated degeneracy of states of the would-be CFT is then compared with the black hole’s Bekenstein-Hawking entropy. Carlip’s result is that both of them agree.

As already mentioned, there are still several remaining open questions. For example, why is it appropriate to treat the event horizon as a boundary in the evaluation of the boundary integrals that determine the extensions K_{ξ_m, ξ_n} ? To put it differently, this can be phrased as *what* the CFT governing the black hole microstates in phase space is and *how* to obtain it. To answer these questions is the overlying goal of our investigations, and we will briefly come back to these issues in Sec. V, in which we emphasize the connection with previous work.

However, our point in this work is to revisit Carlip's approach to entropy counting in several directions. Why is such a revision necessary?

The symmetry algebra of a 2D CFT contains *two* mutually commuting copies of Virasoro algebras. The associated chiral and antichiral central charges are usually equal (CFTs in curved background with different chiral and antichiral central charges are even known to be inconsistent [23]). In Carlip's approach [8] instead, the entire contribution to black hole entropy comes from a chiral half of a would-be CFT.

In this work, we therefore want to present a way to construct diffeomorphisms ξ_n and $\bar{\xi}_n$ ($n \in \mathbb{Z}$) satisfying two copies of the Witt algebra

$$\begin{aligned} [\xi_m, \xi_n] &= -i(m-n)\xi_{m+n} \\ [\bar{\xi}_m, \bar{\xi}_n] &= -i(m-n)\bar{\xi}_{m+n} \\ [\xi_m, \bar{\xi}_n] &= 0. \end{aligned} \quad (3)$$

The choice of diffeomorphisms should be such that the associated Hamiltonian generators H_{ξ_n} and $H_{\bar{\xi}_n}$ are reasonable candidates for the symmetry algebra of a possible CFT governing the statistical mechanics of the black hole under consideration. In addition to the comments made already in the Introduction, we provide in Sec. III A further arguments why the entire 2D local conformal algebra seems necessary to provide reasonable candidates for the conformal generators.

B. Realization

To be as simple and as concrete as possible, we consider the case of a Schwarzschild black hole in pure Einstein gravity. We denote spacetime coordinates by $x^a = (x^0, x^1, x^A)$ with angular coordinates indexed by $A, B, \dots = 2, 3$. In ingoing Eddington-Finkelstein coordinates $(v, r, x^A) = (v, r, \theta, \phi)$, the Schwarzschild metric reads

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b \\ &= -\left(1 - \frac{r_S}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{r_S}{r}\right)dv^2 + 2dvdr + r^2\gamma_{AB}dx^A dx^B. \end{aligned} \quad (4)$$

This coordinates are related to the ordinary Schwarzschild coordinates (t, r, θ, ϕ) by

$$v = t + r^*, \quad (5)$$

where the tortoise coordinate is given by

$$r^* = r + r_S \ln \left| \frac{r}{r_S} - 1 \right|. \quad (6)$$

We work here in Eddington-Finkelstein coordinates because then the metric (4) satisfies the (advanced) Bondi gauge just to be compatible with our previous conventions in Ref. [20]. However, for what follows, this choice is arbitrary, and we could work in any coordinates that cover the event horizon. γ_{AB} denotes the metric on the unit 2-sphere. In what follows, we consider the metric (4) as a fixed reference point $g_{ab} \in \Gamma$ in Hamiltonian phase space describing a Schwarzschild black hole with mass parameter $\frac{r_S}{2G}$. Our task is then to look at the behavior of Hamiltonian generators of suited diffeomorphisms in the vicinity of this point.

To study the behavior of these generators, we have to refer to Hamiltonian mechanics. To do so, several approaches exist. One way would be to use the direct Hamiltonian approach to general relativity as was done in Ref. [7]. However, we will use the covariant phase space formalism [24,25] which is manifestly covariant as used in Ref. [8]. The formalism is, for example, reviewed in Refs. [26–28], and we will use some formulas collected there.

In the covariant phase space formalism, the Hamiltonian phase space Γ is given by the solution space of the theory under consideration (i.e., the set of all field configurations satisfying the equations of motion). In principle, one has to divide out symplectic zero modes by appropriately fixing the gauge, but in our present context, this step is not relevant. For a diffeomorphism ξ , the infinitesimal change of the associated Hamiltonian generator between points $g_{ab} + h_{ab} \in \Gamma$ and $g_{ab} \in \Gamma$ in phase space is determined by

$$\delta H_\xi[h_{ab}; g_{ab}] = -\frac{1}{16\pi G} \oint_{\partial\Sigma} *F. \quad (7)$$

Here, Σ is a Cauchy surface in the spacetime manifold, and the 2-form F_{ab} is well known [26–28],

$$\begin{aligned} F_{ab} &= \frac{1}{2}(\nabla_a \xi_b - \nabla_b \xi_a)h^c{}_c + (\nabla_a h^c{}_b - \nabla_b h^c{}_a)\xi_c \\ &\quad + (\nabla_c \xi_a h^c{}_b - \nabla_c \xi_b h^c{}_a) - (\nabla_c h^c{}_b \xi_a - \nabla_c h^c{}_a \xi_b) \\ &\quad - (\nabla_a h^c{}_c \xi_b - \nabla_b h^c{}_c \xi_a). \end{aligned} \quad (8)$$

We will take $\partial\Sigma$ for our case of a Schwarzschild background to be a cross section of the event horizon, so it is given by the coordinates $(v = \text{const.}, r = \text{const.}, x^A)$ and thus has topology of S^2 parametrized by the angular coordinates x^A . In that case, Eq. (7) takes the form

$$\delta H_\xi[h_{ab}; g_{ab}] = -\frac{r^2}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} F_{rv}. \quad (9)$$

Evaluating (8) for our Schwarzschild metric g_{ab} in (4), one obtains for (9)

$$\begin{aligned}
\delta H_\xi[h_{ab}; g_{ab}] = & -\frac{r^2}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} \left(\xi^v \left(-\frac{r_S}{2r^2} h^A_A - r^{-2} D^A h_{Av} - \frac{2}{r} h_{vv} - \frac{4}{r} \left(1 - \frac{r_S}{r} \right) h_{vr} + \partial_v h^A_A \right. \right. \\
& - \left(1 - \frac{r_S}{r} \right) r^{-2} D^A h_{Ar} - \frac{2}{r} \left(1 - \frac{r_S}{r} \right)^2 h_{rr} + \frac{1}{r} \left(1 - \frac{r_S}{r} \right) h^A_A + \left(1 - \frac{r_S}{r} \right) \partial_r h^A_A \Big) \\
& + \partial_r \xi^v \left(\frac{1}{2} \left(1 - \frac{r_S}{r} \right)^2 h_{rr} - \frac{1}{2} \left(1 - \frac{r_S}{r} \right) h^A_A + h_{vv} + \left(1 - \frac{r_S}{r} \right) h_{vr} \right) \\
& + \partial_r \xi^r \left(-\frac{1}{2} \left(1 - \frac{r_S}{r} \right) h_{rr} + \frac{1}{2} h^A_A \right) + \partial_v \xi^v \left(\frac{1}{2} \left(1 - \frac{r_S}{r} \right) h_{rr} - \frac{1}{2} h^A_A \right) \\
& + \xi^r \left(\frac{r_S}{2r^2} h_{rr} + r^{-2} D^A h_{Ar} + \frac{2}{r} h_{vr} + \frac{2}{r} \left(1 - \frac{r_S}{r} \right) h_{rr} - \frac{1}{r} h^A_A - \partial_r h^A_A \right) \\
& + \xi^A \left(\partial_r h_{Av} - \frac{2}{r} h_{Av} - \partial_v h_{Ar} \right) - \partial_v \xi^r h_{rr} + r^{-2} D^A \xi^v \left(h_{Av} + \left(1 - \frac{r_S}{r} \right) h_{Ar} \right) - r^{-2} D^A \xi^r h_{Ar} \Big). \quad (10)
\end{aligned}$$

D_A and D^A denote the covariant derivative on the unit 2-sphere where the index is raised and lowered with γ_{AB} .

Now, we have derived the theoretical grounds to accomplish our task. If we can find natural diffeomorphisms satisfying the two copies of Witt algebra (3), we are able to provide candidates for the Virasoro generators of the black hole at the point $g_{ab} \in \Gamma$ in phase space.

Before proceeding, we comment on some issues that are further discussed in more detail in Sec. V but that might lead to a potential confusion at this place. The covariant phase space formalism endows the phase space Γ with a symplectic form Ω that is defined relative to a fixed codimension-2 surface B_1 in the spacetime manifold [see, for instance, Eq. (54)]. The latter form Ω dictates which excitations are symplectic zero modes and thus redundancies and which excitations constitute physical shifts in phase space. Moreover, Ω determines the gradient of Hamiltonian generators like, for instance, Eq. (7), which then also depend on the choice of B_1 . Usually, B_1 is then taken to be a cross section of the boundary of the spacetime manifold at infinity (as, for instance, in Ref. [25]). However, in the study of black holes (see, for instance, Refs. [7–10] or [29] and [22] and references therein), it became apparent that it seems to be useful to push B_1 into the bulk, e.g., as a cross section of a black hole event horizon. We provide further justification for this step in Sec. V by showing that the different generators obtained by varying the location of B_1 (and hence by varying Ω) can be uniquely obtained from one another.

As mentioned, we want to analyze the phase space in the vicinity of the fixed reference point $g_{ab} \in \Gamma$ given by (4). For this purpose, we choose to define Ω with respect to B_1 being a cross section of the event horizon of g_{ab} , in the hope that this makes the analysis especially simple at the particular point $g_{ab} \in \Gamma$ under consideration. The assumption underlying the analysis of the present paper is that there is a CFT describing the part of phase space

responsible at least for the microstates of the black hole (4). This assumption means that there are Hamiltonian generators over this part of phase space that play the role of the Virasoro generators. For a consistency check of this assumption, we are asking if, after all, suited diffeomorphisms (3), which could give rise to the action (with respect to the chosen symplectic form Ω) of possible Virasoro generators at the reference point $g_{ab} \in \Gamma$, exist.

Our main result here will be that suited (suited in a sense that we will have to explain) diffeomorphisms do indeed exist, and we will show how to single them out.

Contrary to Carlip's approach, we allow the diffeomorphisms generated by the conformal generators to be field dependent; i.e., they can be different at different points in phase space Γ . In Carlip's approach [8], a subset of Γ is specified by a set of horizon boundary conditions. The role of these boundary conditions is to determine the field-independent diffeomorphisms for which integrable Hamiltonian generators exist, i.e., for which (7) can be integrated over this subset in Γ . The observation is then that the resulting diffeomorphisms contain a chiral Virasoro algebra that could give rise to the chiral conformal generators.

Herein, we do not impose any boundary conditions (that might result from integrability constraints) as we allow the conformal generators—if existent—to generate field-dependent diffeomorphisms. However, we will seek for those diffeomorphisms only at the reference point $g_{ab} \in \Gamma$. Fortunately, this information is sufficient to study the generator algebra at $g_{ab} \in \Gamma$ and to infer the needed CFT data at this point. However, note that, since here we do not know the behavior of the conformal generators away from $g_{ab} \in \Gamma$, we cannot determine the Virasoro zero modes [for instance, Eq. (42) or temperatures (51)] away from $g_{ab} \in \Gamma$ [30].

Since the Hamiltonian generators H_ξ have the dimension of an action, we can define dimensionless generators by

$$\begin{aligned} H_{\xi_n} &=: \hbar L_n \\ H_{\bar{\xi}_n} &=: \hbar \bar{L}_n \end{aligned} \quad (11)$$

at $g_{ab} \in \Gamma$ and for $n \in \mathbb{Z}$. Since the generators satisfy according to (1) and (3) two centrally extended Witt algebras,

$$\begin{aligned} \{H_{\xi_m}, H_{\xi_n}\} &= -i(m-n)H_{\xi_{m+n}} + K_{\xi_m, \xi_n} \\ \{H_{\bar{\xi}_m}, H_{\bar{\xi}_n}\} &= -i(m-n)H_{\bar{\xi}_{m+n}} + K_{\bar{\xi}_m, \bar{\xi}_n} \\ \{H_{\xi_m}, H_{\bar{\xi}_n}\} &= 0, \end{aligned} \quad (12)$$

we find after canonical quantization $\{\cdot, \cdot\} \rightarrow \frac{1}{i\hbar}[\cdot, \cdot]$ of (12) that the Virasoro generators L_n and $\bar{L}_n$ fulfill two copies of the Virasoro algebra in the standard form

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n} \\ [L_m, \bar{L}_n] &= 0. \end{aligned} \quad (13)$$

$\delta_{m+n} = \delta_{m+n,0}$ denotes the Kronecker delta. Thus, the central charges c and $\bar{c}$ and Virasoro generators $L_0[g_{ab}], \bar{L}_0[g_{ab}]$ can be inferred from

$$\begin{aligned} \delta_{\xi_{-m}} H_{\xi_m}|_{g_{ab}} &= -2i\hbar m L_0[g_{ab}] - i\frac{\hbar c}{12}m(m^2-1) \\ \delta_{\bar{\xi}_{-m}} H_{\bar{\xi}_m}|_{g_{ab}} &= -2i\hbar m \bar{L}_0[g_{ab}] - i\frac{\hbar \bar{c}}{12}m(m^2-1) \end{aligned} \quad (14)$$

for $m \in \mathbb{Z}$.

Equation (14) already fixes the data needed for the counting of state degeneracy in a CFT. The computation of the left-hand side of (14) can be done by (10).

To summarize, the task left for the next section is to find a preferred Witt algebra of diffeomorphisms (3). We will have to argue in what sense these diffeomorphisms will be preferred. But if this can be accomplished, the associated Hamiltonian generators will provide natural Virasoro generators of a possible CFT describing part of the phase space responsible for the microstates of the black hole.

Nevertheless, the choice of diffeomorphisms is at this stage only a guess. It might be that chosen diffeomorphisms have nothing to do with the symmetry generators of the aforementioned CFT. Within this approach, it is even not clear that such a CFT exists. However, in this section, we have shown that the choice of diffeomorphisms (3) fixes via Eqs. (14) and (10) the central charges and conformal weights (or equivalently the temperatures) of a would-be CFT. This is already enough data to determine the degeneracy of states in this CFT in order to see whether it agrees with the Bekenstein-Hawking entropy of the black hole.

III. SEARCHING FOR VIRASORO ALGEBRA

The goal of this section is to find out whether the presence of a black hole event horizon singles out a Witt algebra of diffeomorphisms (3) in a natural way and what natural in this context might mean. In the last section, we explained how such diffeomorphisms could be related to the generators of a conformal symmetry governing the black hole's phase space and provided formulas to extract information of this CFT directly from the diffeomorphisms.

A. Virasoro vector fields

Given the Schwarzschild black hole $g_{ab} \in \Gamma$ in (4), what diffeomorphisms forming a Witt algebra might be singled out? Remember that a Witt algebra (2) is isomorphic to the algebra $\mathfrak{diff} S^1$ of all diffeomorphisms on S^1 . The question can thus be rephrased as to whether there are preferred directions in a Schwarzschild spacetime. If so, periodic reparametrizations along these directions provide a Witt algebra $\mathfrak{diff} S^1$, and the periodicities would then fix the temperatures of a possible CFT. In Ref. [8], Carlip provided for the general case of a local Killing horizon a candidate for such a preferred direction. Although the associated Hamiltonian generators can be shown to generate central extensions and to give rise to the correct entropy, only one copy of a Virasoro algebra is found.

Instead, we want to give a somewhat different proposal for constructing diffeomorphisms forming the algebra (3). We will see that the associated Hamiltonian generators will indeed form two copies of Virasoro algebra with equal central charges as one would expect for the symmetry algebra of a CFT. The diffeomorphisms we are going to construct were already given in Ref. [20] up to cosmetic changes. We now explain how they are singled out.

We keep the philosophy of Ref. [8] that a local Killing horizon singles out a preferred direction which provides the basis for the construction of a $\mathfrak{diff} S^1$ algebra. However, a Schwarzschild geometry has in its maximal extension two event horizons, and we propose treating both on the same footing in the search for an algebra (3).

In the well-known Kruskal coordinates (U, V, x^A) which cover the entire maximal extension of the Schwarzschild spacetime, the metric takes the form

$$ds^2 = -\frac{4r_S^3}{r} e^{-\frac{r}{r_S}} dU dV + r^2 \gamma_{AB} dx^A dx^B. \quad (15)$$

The future (past) event horizon is located at $U = 0$ ($V = 0$). Indeed, the Schwarzschild geometry (15) provides the *two* preferred lightlike directions ∂_U and ∂_V .

However, the Cardy formula provides the degeneracy of states $S = S(L_0, \bar{L}_0)$ in a CFT at particular values of the Virasoro generators L_0 and $\bar{L}_0$. Since the black hole entropy gives the state degeneracy at fixed mass and angular momentum, the candidates for the Virasoro zero

modes H_{ξ_0} and $H_{\bar{\xi}_0}$ should therefore “measure” the mass and angular momentum parameter of the black hole [31]. Therefore, it seems that reparametrizations along the direction U (V) are not enough. The diffeomorphisms ξ_n and $\bar{\xi}_n$ to form (3) should also contain components in the direction ∂_ϕ , which is conjugated to the angular momentum [32].

Thus, in Kruskal coordinates (U, V, θ, ϕ) , we make the ansatz for the vector fields (3)

$$\begin{aligned}\xi_n &= f_n \partial_V + g_n \partial_\phi \\ \bar{\xi}_n &= \bar{f}_n \partial_U + \bar{g}_n \partial_\phi\end{aligned}\quad (16)$$

for $n \in \mathbb{Z}$. Here, the functions

$$\begin{aligned}f_n &= f_n(V, \phi) \\ g_n &= g_n(V, \phi) \\ \bar{f}_n &= \bar{f}_n(U, \phi) \\ \bar{g}_n &= \bar{g}_n(U, \phi)\end{aligned}\quad (17)$$

need to be determined from the requirement (3). The first equation in (3) yields two conditions on the functions (17), namely,

$$f_m \partial_V f_n + g_m \partial_\phi f_n - (m \leftrightarrow n) = -i(m - n) f_{m+n} \quad (18)$$

and

$$f_m \partial_V g_n + g_m \partial_\phi g_n - (m \leftrightarrow n) = -i(m - n) g_{m+n} \quad (19)$$

for $m, n \in \mathbb{Z}$. Analogous equations follow from the second equation in (3) for the antichiral functions $\bar{f}_n$ and $\bar{g}_n$. The last equation of (3) then yields the conditions

$$\begin{aligned}\partial_\phi \bar{f}_n &= 0 \\ \partial_\phi f_n &= 0 \\ g_m \partial_\phi \bar{g}_n - \bar{g}_n \partial_\phi g_m &= 0.\end{aligned}\quad (20)$$

The last conditions of (20) can be fulfilled by choosing the product ansatz

$$\begin{aligned}g_m(V, \phi) &= \Phi(\phi) G_m(V) \\ \bar{g}_m(U, \phi) &= \Phi(\phi) \bar{G}_m(U).\end{aligned}\quad (21)$$

With these restrictions on f_n and g_n , Eq. (18) becomes

$$f_m \partial_V f_n - (m \leftrightarrow n) = -i(m - n) f_{m+n}, \quad (22)$$

and (19) yields

$$f_m \partial_V G_n - (m \leftrightarrow n) = -i(m - n) G_{m+n}. \quad (23)$$

Choosing $f_n = f_n(V)$ to satisfy (22), Eq. (23) is fulfilled with the choice

$$G_n = \partial_V f_n. \quad (24)$$

Therefore, the vector fields

$$\begin{aligned}\xi_n &= f_n(V) \partial_V + \Phi(\phi) \partial_V f_n \partial_\phi \\ \bar{\xi}_n &= \bar{f}_n(U) \partial_U + \Phi(\phi) \partial_U \bar{f}_n \partial_\phi\end{aligned}\quad (25)$$

provide an algebra (3) if the functions $f_n = f_n(V)$ are chosen to satisfy (22) and $\bar{f}_n$ are chosen analogously. $\Phi = \Phi(\phi)$ is at this stage arbitrary. A legal choice is then

$$\begin{aligned}f_n(V) &= \frac{1}{A} V^{1+inA} \\ \bar{f}_n(U) &= \frac{1}{B} U^{1+inB},\end{aligned}\quad (26)$$

where $A, B \in \mathbb{R} \setminus \{0\}$ are so far arbitrary parameters.

Unfortunately, the constructed vector fields (25) are still not satisfactory. To give rise to independent Virasoro generators L_n , the ξ_n have to be linearly independent functions of the angular coordinates. One possibility is that a factor $e^{in\phi}$ appears in (25) instead of a fixed function $\Phi(\phi)$. However, this is now easy to achieve. Since (25) satisfies a Witt algebra (3), we can generate such vector fields out of (25) by applying an active coordinate transformation. The new vector fields then still satisfy (3). We choose the coordinate transformation [33]

$$\begin{aligned}U' &= U e^{\frac{1}{B}\phi} \\ V' &= V e^{-\frac{1}{A}\phi} \\ \theta &= \theta \\ \phi &= \phi.\end{aligned}\quad (27)$$

The procedure yields

$$\xi_n^a = \begin{pmatrix} \xi_n^U \\ \xi_n^V \\ \xi_n^\theta \\ \xi_n^\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{AB} U(1+inA) V^{inA} e^{in\phi} \Phi(\phi) \\ \frac{1}{A} V^{1+inA} e^{in\phi} - \frac{1}{A^2} (1+inA) V^{1+inA} e^{in\phi} \Phi(\phi) \\ 0 \\ \frac{1}{A} (1+inA) V^{inA} e^{in\phi} \Phi(\phi) \end{pmatrix} \quad (28)$$

and

$$\bar{\xi}_n^a = \begin{pmatrix} \frac{1}{B} U^{1+inB} e^{-in\phi} + \frac{1}{B^2} (1 + inB) U^{1+inB} e^{-in\phi} \Phi(\phi) \\ -\frac{1}{AB} V(1 + inB) U^{inB} e^{-in\phi} \Phi(\phi) \\ 0 \\ \frac{1}{B} (1 + inB) U^{inB} e^{-in\phi} \Phi(\phi) \end{pmatrix}. \quad (29)$$

To meet the conventions of the last section, we formulate (28) and (29) in Eddington-Finkelstein coordinates (v, r, θ, ϕ) , getting

$$\xi_n^a = \begin{pmatrix} \xi_n^v \\ \xi_n^r \\ \xi_n^\theta \\ \xi_n^\phi \end{pmatrix} = \begin{pmatrix} 2r_S \left(\frac{1}{A} - \frac{1}{A^2} (1 + inA) \Phi(\phi) \right) \\ \left(1 - \frac{r_S}{r} \right) r_S \left(\frac{1}{A} + \frac{1}{A} \left(-\frac{1}{A} + \frac{1}{B} \right) (1 + inA) \Phi(\phi) \right) \\ 0 \\ \frac{1}{A} (1 + inA) \Phi(\phi) \end{pmatrix} e^{inA \frac{r}{2r_S}} e^{in\phi} \quad (30)$$

and

$$\bar{\xi}_n^a = \begin{pmatrix} -2r_S \frac{1}{AB} (1 + inB) \Phi(\phi) \\ r_S \left(1 - \frac{r_S}{r} \right) \left(\frac{1}{B} + \frac{1}{B} \left(-\frac{1}{A} + \frac{1}{B} \right) (1 + inB) \Phi(\phi) \right) \\ 0 \\ \frac{1}{B} (1 + inB) \Phi(\phi) \end{pmatrix} (-1)^{inB} e^{inB \frac{r}{r_S}} e^{-inB \frac{r}{2r_S}} e^{-in\phi} \quad (31)$$

for $n \in \mathbb{Z}$ and with r^* from (6). These vector fields fulfill $(\xi_n^a)^* = \xi_{-n}^a$ and $(\bar{\xi}_n^a)^* = \bar{\xi}_{-n}^a$ and form two copies of Witt algebra (3) as required. Formulas (30) and (31) are the main result of this section and will be used in the next section for the entropy counting within the framework developed in Sec. II. In what follows, we will try to fix the remaining arbitrary function $\Phi(\phi)$ and $A, B \in \mathbb{R} \setminus \{0\}$. On the way, we will also comment on different approaches taken to find such Virasoro vector fields.

That other choices of a $Vir \oplus \overline{Vir}$ algebra of diffeomorphisms do exist follows directly from (30) and (31). We can already obtain infinitely many $Vir \oplus \overline{Vir}$ diffeomorphisms by applying arbitrary diffeomorphisms on (30) and (31). However, not all Virasoro diffeomorphisms are suited as to correspond to the action of conformal generators. Most of them will be gauge redundancies. Even out of the physical ones, the Virasoro zero modes will in general not correspond to the black hole's mass and angular momentum parameter, which are the macrovariables to be fixed in the microcanonical ensemble set in the definition of black hole entropy. As described in the beginning of this section, we give a prescription to single out a $Vir \oplus \overline{Vir}$ algebra by motivating the ansatz (16) out of which (30) and (31) are derived. Nevertheless, we have not shown that (30) and (31) provide the unique choice that leads to reasonable

candidates for the conformal generators at the point $g_{ab} \in \Gamma$ in (4) under consideration. However, it is after all remarkable that one choice for a prescription to construct reasonable candidates for the conformal generators can be found.

Even with our choice of $Vir \oplus \overline{Vir}$ diffeomorphisms, we will be left with a remaining parameter A that we are not able to fix in this analysis. This parameter will enter the inferred CFT data [see, for instance, Eq. (42)] but will cancel out of entropy counting. It might be that this parameter is fixed at a deeper level—provided that (30) and (31) after all indeed lead to proper conformal generators. However, it might also be the case that the parameter A reflects the ambiguity in the definition of the symplectic form Ω that is discussed in the paragraphs following Eq. (10).

B. Temperatures

Although our construction of (30) and (31) is motivated by Ref. [8], we note that our final result is really different. Our vector fields violate the horizon boundary conditions proposed in Ref. [8], and thus the construction is genuinely different. This is mainly due to the appearance of a ∂_ϕ component in our choice of diffeomorphisms. Indeed, we

wanted this component to appear in order for the Virasoro zero modes to measure the black hole's mass and angular momentum parameter. The Virasoro zero modes are, according to Sec. II, induced by the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$ subalgebra of (3) spanned by ξ_0 and $\bar{\xi}_0$. These vector fields hence should be—in Schwarzschild coordinates (t, r, θ, ϕ) —linear combinations of ∂_t and ∂_ϕ . For this to be fulfilled, in (30) and (31), the r component has to vanish for $n = 0$, which requires the choice

$$\Phi(\phi) = \frac{AB}{B-A}. \quad (32)$$

With this choice, the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$ subalgebra is given by

$$\begin{aligned} \xi_0 &= -2r_s \frac{1}{B-A} \partial_t + \frac{B}{B-A} \partial_\phi \\ \bar{\xi}_0 &= -2r_s \frac{1}{B-A} \partial_t + \frac{A}{B-A} \partial_\phi. \end{aligned} \quad (33)$$

By some thermodynamic considerations, Eq. (33) provides enough information to fix the temperatures of the CFT described in Sec. II that would be associated with the full Witt algebra (3). However, note that these considerations provide rather a consistency check as the temperatures are already fixed through (14) by $L_0[g_{ab}]$ and $\bar{L}_0[g_{ab}]$; the temperatures inferred from the algebra (14) can be compared to the temperatures that are thermodynamically obtained from (33) by a procedure to be explained in the following.

In Ref. [22], a different approach to find Virasoro vector fields was made, but the latter consistency check was not done. There, the *im* terms which would determine the temperature are displayed neither in Eq. (5.15) nor in the counterterm correction (5.16). It would be interesting to see whether in Ref. [22] the temperatures obtained algebraically via (14) agree with the temperatures inferred thermodynamically.

To thermodynamically infer the CFT temperatures, we introduce a scalar field to be put in thermal contact with the black hole. Consider a free massless Klein-Gordon field on the Schwarzschild background. Its eigenmodes are of the form $F(r, \theta) e^{-i\omega t + im\phi}$ with frequency ω and angular momentum m . These are then also eigenfunctions of (33) with eigenvalues

$$\begin{aligned} \xi_0 &= -in = -i \left(\frac{2r_s \omega}{A-B} - m \frac{B}{B-A} \right) \\ \bar{\xi}_0 &= -i\bar{n} = -i \left(\frac{2r_s \omega}{A-B} - m \frac{A}{B-A} \right). \end{aligned} \quad (34)$$

If we would allow for backreaction, the scalar field can exchange energy and angular momentum with the gravitational field under the constraint that both are conserved in total. In thermal equilibrium, for the Schwarzschild black

hole, the scalar eigenmodes (ω, m) are thermally distributed weighted by a Boltzmann factor,

$$e^{-\frac{\hbar\omega}{T_H}},$$

with the Hawking temperature $T_H = \frac{\hbar}{4\pi r_s}$. This is rewritten in terms of the eigenfrequencies $(n, \bar{n})$ as $e^{-\frac{n}{T} - \frac{\bar{n}}{\bar{T}}}$ with the temperatures

$$\begin{aligned} T &= \frac{1}{2\pi A} \\ \bar{T} &= -\frac{1}{2\pi B}. \end{aligned} \quad (35)$$

Because of the zeroth law of thermodynamics, Eqs. (35) are also the temperatures of the CFT governing the black hole if associated to the diffeomorphisms (30) and (31).

C. $SL(2; \mathbb{R})$ Casimir and conformal symmetry in scattering

In this section, we have presented one particular way to single out Virasoro vector fields (30), (31), which we will use for entropy counting via Carlip's approach in the next section. The recent work [22] also follows Carlip's approach, but a different philosophy is used to find a conformal algebra of vector fields [34].

The Witt algebra (3) has a $\mathfrak{sl}(2, \mathbb{R}) \oplus \overline{\mathfrak{sl}(2, \mathbb{R})}$ subalgebra spanned by ξ_{-1}, ξ_0, ξ_1 and its antichiral counterpart. Associated to this global conformal algebra are the Casimir operators

$$\mathcal{H}^2 = -\mathcal{L}_{\xi_0} \mathcal{L}_{\xi_0} + \frac{1}{2} (\mathcal{L}_{\xi_1} \mathcal{L}_{\xi_{-1}} + \mathcal{L}_{\xi_{-1}} \mathcal{L}_{\xi_1}) \quad (36)$$

and an analogous antichiral expression. Now, one can try to find a preferred $\mathfrak{sl}(2, \mathbb{R})$ algebra of diffeomorphisms by studying the form of the associated differential operator (36). In Ref. [36], vector fields forming an $\mathfrak{sl}(2, \mathbb{R}) \oplus \overline{\mathfrak{sl}(2, \mathbb{R})}$ algebra are given. It is further shown there that for a free massless Klein-Gordon field in a Kerr background—in a suited regime—eigenfunctions of (36) give rise to eigenmodes of the Klein-Gordon equation. As a consequence of the $\mathfrak{sl}(2, \mathbb{R}) \oplus \overline{\mathfrak{sl}(2, \mathbb{R})}$ invariance of (36), scattering—in a suited regime—behaves as being invariant under a “hidden” 2D global conformal symmetry (see Ref. [36] for further details).

The idea of Ref. [22] is then to find a full local conformal $Vir \oplus \overline{Vir}$ algebra (3) of diffeomorphisms which realizes the latter hidden conformal symmetry and then can be used for entropy counting.

However, to our understanding, the $Vir \oplus \overline{Vir}$ vector fields proposed in Ref. [22] form only an enhancement of the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$ algebra given in Ref. [36] (spanned by $\xi_0, \bar{\xi}_0$). They seem not to contain the global conformal

$\mathfrak{sl}(2, \mathbb{R}) \oplus \overline{\mathfrak{sl}(2, \mathbb{R})}$ algebra of Ref. [36] (spanned by $\xi_n, \bar{\xi}_n$ for $n = -1, 0, 1$).

But then, we do not understand in what way the Kerr geometry singles out the $Vir \oplus \overline{Vir}$ vector fields of Ref. [22] so that their Hamiltonian generators could govern a possible CFT of the Kerr black hole. Indeed, in Ref. [22], the expected central charges are only obtained from the generator algebra after noncanonical counterterm corrections.

Nevertheless, it might still be useful to have the $\mathfrak{sl}(2, \mathbb{R})$ Casimir (36) in mind. What is its meaning for our $Vir \oplus \overline{Vir}$ choice in (30) and (31)? Inspired by Ref. [36] and our construction of these diffeomorphisms, a natural expectation would be that $\mathcal{H}^2$ and $\bar{\mathcal{H}}^2$ possibly govern the scalar scattering in the vicinity of the bifurcation of the horizons in a suited regime of parameters. We do not enter an analysis of these questions further at this place. However, we want to note that a computation reveals that the expression for (36) with the vector fields (30) and (31) indeed greatly simplifies for the choice

$$B = -A. \quad (37)$$

This could be a hint that the Vir algebra (30) and $\overline{Vir}$ algebra (31) can belong to the Virasoro algebra of the same

$$\begin{aligned} \delta_{\xi-m} H_{\xi_m}|_{g_{ab}} = & -im \frac{r^2}{4G} \left(\frac{r_S^2}{r^2} \frac{1}{A} \Phi \left(\frac{2}{A} - \frac{2}{A} \left(\frac{1}{A} + \frac{1}{B} \right) \Phi \right) + \frac{r_S}{r} \left(1 - \frac{r_S}{r} \right) \left(-\frac{4}{A} - \frac{4}{AB} \Phi + \frac{4}{A^2} \Phi + \frac{8}{A^2 B} \Phi^2 \right) \right) \\ & - im^3 \frac{r^2}{4G} \left(2\Phi - 2 \frac{r_S^2}{r^2} \Phi^2 \left(\frac{1}{A} + \frac{1}{B} \right) + \frac{r_S}{r} \left(1 - \frac{r_S}{r} \right) \frac{8}{B} \Phi^2 \right) + im^3 \frac{r_S^2}{8G} \int_0^\pi d\theta \sin^{-1}(\theta) \left(1 - \frac{r_S}{r} \right) \frac{8}{AB} \Phi. \end{aligned} \quad (38)$$

In the limit $r \rightarrow r_S$, the expression is well defined, and we get from (14)

$$\begin{aligned} L_0[g_{ab}] - \frac{c}{24} &= \frac{r_S^2}{4\hbar G} \left(\frac{1}{A} \Phi \left(\frac{1}{A} - \frac{1}{A} \left(\frac{1}{A} + \frac{1}{B} \right) \Phi \right) \right) \\ c &= \frac{3r_S^2}{\hbar G} \left(2\Phi - 2\Phi^2 \left(\frac{1}{A} + \frac{1}{B} \right) \right). \end{aligned} \quad (39)$$

Using (32) and applying the Cardy formula [37],

$$S_{\text{chiral}} = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)} = \frac{\pi r_S^2}{\hbar G} \frac{-2AB}{(B-A)^2}. \quad (40)$$

Consistently, the canonical version of the Cardy formula with the temperatures (35) yields the same result,

CFT only with the choice (37). However, we will leave the parameters A and B unspecified. We will then see further evidence for this conjecture from the fact that the Cardy entropy will get extremized precisely for the choice (37).

Using (32) and (37), the last unspecified parameter in (30) and (31) is then A . We will see that it will cancel out of the entropy counting. Such an ambiguity parameter was already present in Ref. [7]. As explained there, Euclidean quantum gravity suggests the choice $A = 1$ together with (37) since the wave number for v is then given by the surface gravity $\kappa = \frac{1}{2r_S}$. However, we will leave A unspecified since this ambiguity can have a mathematical meaning as can be seen in Sec. V.

IV. ENTROPY COUNTING

With the vector fields (30) and (31) of the last section, we are now ready to apply the framework of Sec. II for entropy counting.

A. Schwarzschild entropy

Our goal is to determine the data on the right-hand side of (14). To this end, we evaluate the left-hand side of (14) using the vector fields determined in (30) and (31). A computation yields

$$S_{\text{chiral}} = \frac{\pi^2}{3} c T = \frac{\pi r_S^2}{\hbar G} \frac{-2AB}{(B-A)^2}. \quad (41)$$

As conjectured in Sec. III C, these expressions are maximized if (37) holds. In this case, one has

$$\begin{aligned} L_0[g_{ab}] - \frac{c}{24} &= \frac{r_S^2}{8\hbar G} \frac{1}{A} \\ c &= \frac{3r_S^2}{\hbar G} A \end{aligned} \quad (42)$$

and

$$S_{\text{chiral}} = \frac{1}{2} \frac{\pi r_S^2}{\hbar G}. \quad (43)$$

The antichiral contribution is determined in the same way,

$$\begin{aligned} \delta_{\xi-m} H_{\xi m} \Big|_{g_{ab}} &= -im \frac{r^2}{4G} \left(\frac{r_S^2}{r^2} \left(\frac{2}{B^2} \Phi + 2 \left(\frac{1}{A} + \frac{1}{B} \right) \frac{1}{B^2} \Phi^2 \right) - \frac{r_S}{r} \left(1 - \frac{r_S}{r} \right) \left(2\Phi \left(\frac{2}{AB} - \frac{2}{B^2} \right) - \frac{4}{B} + \Phi^2 \frac{8}{AB^2} \right) \right) \\ &\quad - im^3 \frac{r^2}{4G} \left(2\Phi + 2 \frac{r_S^2}{r^2} \left(\frac{1}{A} + \frac{1}{B} \right) \Phi^2 - \frac{r_S}{r} \left(1 - \frac{r_S}{r} \right) \Phi^2 \frac{8}{A} \right) + im^3 \frac{r_S^2}{8G} \int_0^\pi d\theta \sin^{-1}(\theta) \left(1 - \frac{r_S}{r} \right) \frac{8\Phi}{AB}. \end{aligned} \quad (44)$$

In the limit $r \rightarrow r_S$, the expression is well defined, and we get from (14)

$$\begin{aligned} \bar{L}_0[g_{ab}] - \frac{\bar{c}}{24} &= \frac{r_S^2}{4\hbar G} \left(\frac{1}{B^2} \Phi + \left(\frac{1}{A} + \frac{1}{B} \right) \frac{1}{B^2} \Phi^2 \right) \\ \bar{c} &= \frac{3r_S^2}{\hbar G} \left(2\Phi + 2 \left(\frac{1}{A} + \frac{1}{B} \right) \Phi^2 \right). \end{aligned} \quad (45)$$

Using (32) and applying the Cardy formula,

$$S_{\text{anti-chiral}} = 2\pi \sqrt{\frac{\bar{c}}{6} \left(\bar{L}_0 - \frac{\bar{c}}{24} \right)} = \frac{\pi r_S^2}{\hbar G} \frac{-2AB}{(B-A)^2}. \quad (46)$$

Consistently, the canonical version of the Cardy formula with the temperatures (35) yields the same result,

$$S_{\text{anti-chiral}} = \frac{\pi^2}{3} \bar{c} \bar{T} = \frac{\pi r_S^2}{\hbar G} \frac{-2AB}{(B-A)^2}. \quad (47)$$

These expressions coincide with the chiral contribution and are also maximized if (37) is fulfilled. For that case, one has

$$\begin{aligned} \bar{L}_0[g_{ab}] - \frac{\bar{c}}{24} &= \frac{r_S^2}{8\hbar G A} \\ \bar{c} &= \frac{3r_S^2}{\hbar G} A \end{aligned} \quad (48)$$

and

$$S_{\text{antichiral}} = \frac{1}{2} \frac{\pi r_S^2}{\hbar G}. \quad (49)$$

Thus, the total Cardy entropy is

$$S = S_{\text{chiral}} + S_{\text{antichiral}} = \frac{\pi r_S^2}{\hbar G}. \quad (50)$$

This matches precisely the Bekenstein-Hawking entropy of a Schwarzschild black hole.

B. Extrapolation to the general case

The CFT data derived in (42), (48), and (35) can be written in the form

$$\begin{aligned} c &= \bar{c} = \frac{3\mathcal{A}}{4\pi\hbar G} A \\ T &= \bar{T} = \frac{1}{2\pi A} \end{aligned} \quad (51)$$

with the horizon area $\mathcal{A}$. In Ref. [9], one copy of a Witt algebra of vector fields was presented to reproduce central charges and temperatures similar to the chiral half of (51) for the general case of a stationary black hole of dimension $3+1$. However, there are some differences. In Ref. [9], these quantities contain divergences which cancel out in entropy counting, and the temperature is derived only by thermodynamic considerations and not from a computation of Virasoro zero modes. In addition, the chiral Virasoro algebra in Ref. [9] is only able to account for half of the expected Bekenstein-Hawking entropy.

For a Schwarzschild black hole, we have managed to provide the missing second copy of Virasoro algebra accounting for the second missing half of the entropy. In addition, our choice of $Vir \oplus \bar{Vir}$ vector fields leads to well-defined quantities (51) that contain no divergences. Furthermore, the temperatures in (51) are consistently in agreement with their derivation from Virasoro zero modes $L_0[g_{ab}]$ and $\bar{L}_0[g_{ab}]$ using covariant phase space methods.

We derived (51) for the case of a Schwarzschild black hole. However, our methods employed allow for canonical generalization. The strategy to pick out a $Vir \oplus \bar{Vir}$ algebra of vector fields can be analogously applied in the general case. Because of the similarities of (51) and the general analysis of Ref. [9], we conjecture (51) to apply also in this general case, leading to the entropy

$$S_{\text{Cardy}} = \frac{\pi^2}{3} cT + \frac{\pi^2}{3} \bar{c}\bar{T} = \frac{\mathcal{A}}{4\hbar G} \quad (52)$$

as required. Note that we have provided a proof of (51) and (52) only for the Schwarzschild case and left the general case as a conjecture. Checking the conjecture would now require a straightforward computation that we do not enter at this place.

V. DISCUSSION AND INTERPRETATION

In the preceding sections, we revisited Carlip's approach to entropy counting. We provided a $Vir \oplus \bar{Vir}$ algebra of diffeomorphisms and analyzed the algebra of the associated Hamiltonian generators. We found that the latter give rise to a Virasoro algebra such that counting the state degeneracy

of the would-be CFT is in agreement with Bekenstein-Hawking entropy.

So far, this approach does not tell much about this would-be CFT that possibly governs the part of phase space responsible for black hole microstates. What is needed is to analyze the Hamiltonian phase space in the vicinity of a black hole state in a systematic fashion. In Ref. [20], we proposed a systematic way to analyze the Hamiltonian phase space of general relativity and to find a dual theory describing the relevant part of phase space responsible for black hole microstates.

Here, we want to briefly sketch how and why such a systematic treatment works in order to show how the entropy counting presented here fits into this procedure. An application of this treatment including the role of Carlip's entropy counting was already given in Ref. [20]. A more detailed description of the treatment itself including applications to theories simpler than gravity will be provided somewhere else. Instead, here we will just sketch the main ideas.

A. Holography in covariant phase space

Consider an arbitrary field theory over some n -dimensional manifold M given by an action $S = S[\Phi]$. We denote the fields in the theory collectively by Φ . The goal is to analyze the Hamiltonian phase space in a structured way. Because of its flexibility, we use the covariant phase space approach for our explanations [24,25] (see Ref. [26] for a review). The main idea of the covariant phase space approach is the observation that the Hamiltonian phase space is isomorphic to the set of all field configurations satisfying the field equations. This solution space $\tilde{\mathcal{F}}$ is equipped with a suited (pre)symplectic form and (after dividing out symplectic zero modes through suitable gauge fixing) then gives rise to the covariant phase space Γ equivalent to the Hamiltonian phase space.

We denote the coordinates on the covariant phase space by $[\Phi]^A$ with $A, B, \dots$ being the indices. The action S is assigned a differential form $\omega = \omega[\delta_1\Phi, \delta_2\Phi; \Phi]$ of degree $n-1$ over M , which is in addition a closed 2-form over the space of all field configurations. On shell, that is for $\Phi \in \tilde{\mathcal{F}}$ and $\delta_1\Phi, \delta_2\Phi \in T_\Phi\tilde{\mathcal{F}}$, ω is exact $\omega = dk$ for a form $k = k[\delta_1\Phi, \delta_2\Phi; \Phi]$ of degree $n-2$ over M .

Let $\Sigma \subseteq M$ be a hypersurface with the boundary $\partial\Sigma = B_1 \cup B_2$, where B_1 and B_2 are disconnected codimension-2 surfaces. The symplectic flow passing through Σ is then on shell given by a boundary integral,

$$\int_\Sigma \omega[\delta_1\Phi, \delta_2\Phi; \Phi] = \oint_{B_1} k[\delta_1\Phi, \delta_2\Phi; \Phi] - \oint_{B_2} k[\delta_1\Phi, \delta_2\Phi; \Phi]. \quad (53)$$

Each of the boundary integrals in (53) can be used to define a symplectic form over Γ ;

$$\oint_{B_1} k[\delta_1\Phi, \delta_2\Phi; \Phi] = \Omega_{AB}[\delta_1\Phi]^A[\delta_2\Phi]^B \quad (54)$$

defines the symplectic form $\Omega_{AB} = \Omega_{AB}^{(B_1)}$ over Γ (relative to B_1). This can then be used to define the Poisson bracket in the usual way. We denote quantities sometimes with the superscript (B_1) to remember that they are defined with B_1 as the chosen reference. For a vector field X over Γ corresponding to field variations $\delta_X\Phi$, the expression

$$\delta H_X[\delta\Phi; \Phi] = \oint_{B_1} k[\delta\Phi, \delta_X\Phi; \Phi] \quad (55)$$

defines a 1-form over the phase space. If X is a symplectic symmetry $\mathcal{L}_X\Omega_{AB}^{(B_1)} = 0$, the 1-form (55) is exact and can be integrated over phase space to provide the scalar $H_X = H_X^{(B_1)}[\Phi]$. The role of this scalar is to generate $\delta_X\Phi$ via the Poisson bracket.

Because of the expression (55), the value $H_X[\Phi]$ contains information about the field configuration Φ over the surface B_1 . In fact, for the linearly independent symplectic symmetries X , we can think of $H_X[\Phi]$ as part of the Cauchy data required to specify $\Phi \in \Gamma$. These values $H_X[\Phi]$ can therefore be thought of as forming part of the coordinates of a chart for the phase space Γ . Because of their holographic nature (55), we termed them in Ref. [20] as boundary Cauchy data (BCD).

The BCD are defined with respect to the codimension-2 surface B_1 . What would have been if we had wanted to define it with respect to a different surface B_2 of codimension 2? In that case, we would have to choose a hypersurface Σ connecting B_1 and B_2 and correct the BCD relative to B_1 by the symplectic current passing through Σ . Because of (53) and (55), the BCD of a symplectic symmetry X are related by

$$\delta H_X^{(B_1)}[\delta\Phi; \Phi] - \delta H_X^{(B_2)}[\delta\Phi; \Phi] = \int_\Sigma \omega[\delta\Phi, \delta_X\Phi; \Phi]. \quad (56)$$

That is, the change of the BCD from a surface B_1 to a surface B_2 is dictated by the symplectic current $\omega[\delta\Phi, \delta_X\Phi; \Phi]$ passing through the hypersurface in between. The specification of these symplectic currents along an entire Cauchy surface Σ forms the remaining Cauchy data (in addition to the BCD for a particular codimension-2 cross section of Σ) that uniquely determine a point in phase space Γ .

For the case of four-dimensional Einstein gravity, one can push Σ toward null infinity. In that case, the latter currents reduce to the Bondi news, whereas the BCD are essentially given by the mass aspect, angular momentum aspect, and additional functions on S^2 that provide the Cauchy data for the solution space (for a review of the solution space in that case, see Ref. [11]). This example is meant to illustrate the way of thinking. As already

mentioned, more detailed explanations and examples in simpler settings will be provided somewhere else.

In summary, so far, we have said that the phase space Γ can be parametrized by the BCD associated to the symplectic symmetries over a codimension-2 surface together with their associated symplectic currents. While the BCD will be of our main interest in the following, we want briefly explain that already at this point we are able to learn something.

Equation (56) describes the change of the BCD from B_1 to B_2 caused by the symplectic current passing through a hypersurface Σ connecting them. In this way, Eq. (56) reflects a *memory effect*. The independence of the particular choice of Σ connecting B_1 and B_2 in (56) is a consequence of the *constraint* $d\omega[\delta\Phi, \delta_X\Phi; \Phi] = 0$. In this way, each *symplectic symmetry* X gives rise to a memory effect along an arbitrary hypersurface in M and also to a constraint, which altogether reflect the equations of motion. The relation between the concepts symmetry, memory, and constraints was recently emphasized in a variety of examples starting with Refs. [38–41] and references thereof. Here we see that in the covariant phase space language the equivalence between these concepts becomes obvious and is just reflecting the equations of motion.

In the remaining part of the section, we will explain that the particular way to parametrize the phase space Γ can actually be indeed useful to approach various problems.

Choose a particular point $\Phi \in \Gamma$ by specifying its coordinates, i.e., the BCD $H_X[\Phi] = H_X^{(B_1)}[\Phi]$ for the symplectic symmetries X and their associated symplectic currents. Now, take the latter fixed and vary the BCD. This spans an entire subspace $S \subseteq \Gamma$ on which the BCD then can be seen as coordinates. Thus, S is a submanifold in the phase space Γ . However, S has an additional structure. The Poisson-bracket algebra of the generators $H_X = H_X^{(B_1)}[\Phi]$ forms a representation of the Lie-bracket algebra of symplectic symmetries up to central extension. That means for symplectic symmetries X, Y one has

$$\{H_X, H_Y\} = H_{[X,Y]} + K_{X,Y} \quad (57)$$

for c numbers $K_{X,Y} = K_{X,Y}^{(B_1)}$. Therefore, the submanifold S is a symplectic manifold on its own. Its coordinates are given by the BCD H_X , and their Poisson-bracket algebra is given by (57). The part S in phase space Γ can therefore be described by a theory on its own right, a “holographic dual” associated to the chosen codimension-2 surface $B_1 \subseteq M$.

To summarize, we see that to a chosen codimension-2 surface $B_1 \subseteq M$ a holographic dual theory describing a suited part $S \subseteq \Gamma$ can be associated. Choosing a different surface $B_2 \subseteq M$ or different gauge will in general affect the form of (57) describing the same part $S \subseteq \Gamma$. Choosing a different $B_2 \subseteq M$ can also lead to a different submanifold in phase space.

The hope is that the construction of these submanifolds is useful to approach some problems. Usually, the above constructed submanifold $S \subseteq \Gamma$ is too large. However, subalgebras of the algebra of symplectic symmetries will, due to (57), lead to lower-dimensional submanifolds S' in S . Choosing this S' small enough, one is left with a theory that covers a small part of the phase space that might be of interest for a particular problem under consideration. Can this be useful?

B. Microscopic theory for the Schwarzschild black hole

To apply the ideas of Sec. V A to a Schwarzschild black hole in Einstein gravity was essentially the content of Ref. [20]. We recap very briefly the steps. Working in Bondi gauge, the Schwarzschild metric fixes a particular point $g_{ab} \in \Gamma$ in covariant phase space. The goal is to find the part of phase space that is responsible for the microstates. The hope is that the submanifolds constructed in the last section are candidates for this. For the codimension-2 surface B_1 , it is natural to take a cross section of the event horizon in the hope that the algebra (57) will get especially simple.

The next step is then to study closed algebras of symplectic symmetries and their associated submanifolds in Γ . For the Schwarzschild black hole g_{ab} , there is a simplest choice to start with. Because of the black hole uniqueness theorems, one expects microstate excitations to have the form of residual gauge transformations $\delta g_{ab} = \mathcal{L}_\xi g_{ab} \in T_{g_{ab}}\Gamma$ for suited vector fields ξ . Therefore, one is interested in symplectic symmetries X , which at $g_{ab} \in \Gamma$ take the form of a residual gauge transformation $X|_{g_{ab}} = \delta_\xi \in T_{g_{ab}}\Gamma$. Symplectic symmetries of such form and their associated BCD H_X were called gauge aspects in Ref. [20]. *Under the assumption* that the symplectic symmetries due to the gauge aspects cover enough of the phase space relevant for the Schwarzschild black hole microstates, the BCD parametrizing this submanifold $S \subseteq \Gamma$ as well as its algebra (57) were determined in Ref. [20]. As explained in the last section, this symplectic submanifold $S \subseteq \Gamma$ provides a theory in its own right and is a candidate for the holographic dual theory of the Schwarzschild black hole. Since this procedure determines the BCD, one is able to infer the form of the residual gauge transformations at g_{ab} which are the candidates for the black hole microstates.

C. Counting degrees of freedom

In the last section, we have explained the construction of a symplectic submanifold $S \subseteq \Gamma$ that is a candidate for the part of the phase space relevant for the microstates of a Schwarzschild black hole. Its coordinates given by the BCD provide observables with the Poisson-bracket algebra (57). In this way, we have an explicit theory that provides a candidate for the dual theory governing the Schwarzschild

black hole. How can we check whether our candidate theory is correct?

The first check would be to see whether one can deduce the correct black hole entropy from $S \subseteq \Gamma$. As explained in Ref. [20], there are arguments from several directions indicating that the part of phase space responsible for black hole microstates should possess a 2D local conformal symmetry. Therefore, one is tempted to ask whether $S \subseteq \Gamma$ is compatible with this conformal invariance. If so, the observables of S must give rise to a 2D stress tensor such that its Virasoro generators fulfill a Virasoro algebra. Since we know the algebra of observables (57), we can search for a Sugawara construction of these Virasoro generators out of the BCD over S . This is precisely Carlip's approach to entropy counting in disguise as we will explain in the following.

In Ref. [20], a projection operator $T_{g_{ab}} \tilde{\mathcal{F}} \rightarrow T_{g_{ab}} S$ was given, and it maps an arbitrary (possibly not gauge fixed) excitation of a Schwarzschild black hole g_{ab} onto the relevant microstate excitation of g_{ab} . In this way, an arbitrary gauge excitation $\mathcal{L}_{\xi} g_{ab}$ (that in general also contains components that are not tangential to S at g_{ab}) is mapped to the relevant symplectic symmetry X tangential to S . It is this mapping $\xi \mapsto X$ from spacetime diffeomorphisms to the vector fields over S that makes the connection with Carlip's approach clear. In the above-mentioned Sugawara construction, we are searching for symplectic symmetries, i.e., vector fields $X_n, \bar{X}_n$ over S such that their generators H_{X_n} and $H_{\bar{X}_n}$ satisfy via (57) a $Vir \oplus \overline{Vir}$ algebra. Instead, we can look for diffeomorphisms ξ_n and $\bar{\xi}_n$ giving rise to a $Vir \oplus \overline{Vir}$ algebra with respect to the spacetime Lie bracket. This is precisely what we did in Secs. II–IV. We furthermore inspected the algebra of the Hamiltonian generators of ξ_n and $\bar{\xi}_n$ to see that the emerged Virasoro algebra indeed gives rise to the expected entropy. However, in Secs. II–IV, we did not employ the mapping $\xi \mapsto X$. Therefore, so far, we only know that the Hamiltonian generators of the $Vir \oplus \overline{Vir}$ diffeomorphisms provide candidates for the Virasoro generators of a possible would-be CFT governing the black hole microstates. The approach is not sensitive to the details of what this CFT might be.

As already proposed in Ref. [20], the situation is different once we have figured out our candidate theory $S \subseteq \Gamma$. We can use the projection operator $\xi_n \mapsto X_n$ and

$\bar{\xi}_n \mapsto \bar{X}_n$ to obtain with H_{X_n} and $H_{\bar{X}_n}$ candidates for the Virasoro generators in S . Precisely this step is sensitive to the choice of S . That means X_n and $\bar{X}_n$ and their generators would change if the space S were different. Since we project on S , we are directly probing the degrees of freedom (d.o.f.) covered by S . Up from here, we can proceed the same way as in the indirect approach. Inspecting the Virasoro algebra formed by H_{X_n} and $H_{\bar{X}_n}$ via (57), we can count the degeneracy of states and compare it to the expected entropy. If the result were to agree, this would provide a substantial consistency check that the theory given by $S \subseteq \Gamma$ is correct and covers all d.o.f. of the Schwarzschild black hole. Furthermore, it would support that S is indeed conformally invariant, thus providing a concrete realization of a Schwarzschild/CFT correspondence. In case that disagreement is found, one has to enlarge S successively by allowing larger algebras of symplectic symmetries in its construction, up until the procedure is going to converge.

To summarize, with (30) and (31), we have given the needed $Vir \oplus \overline{Vir}$ algebra of diffeomorphisms that is needed in the above procedure of projecting directly onto black hole d.o.f. and counting entropy. These vector fields were already given in Ref. [20]. Here, we have given their systematic construction. Furthermore, we have shown what singles out the presented $Vir \oplus \overline{Vir}$ diffeomorphisms. Most importantly, we have seen that by inspecting the Hamiltonian generators (without projecting directly onto d.o.f.), we were able to show that the Poisson-bracket algebra consistently leads to the expected Bekenstein-Hawking entropy. Therefore, the $Vir \oplus \overline{Vir}$ diffeomorphisms seem to be the right candidates for the approach described in Ref. [20] and reviewed here. We leave performing this approach for future investigations. The purpose of this work here was to provide convincing arguments that the $Vir \oplus \overline{Vir}$ vector fields are the appropriate diffeomorphisms to use.

ACKNOWLEDGMENTS

We thank Alexander Gussmann for many discussions on this and other topics in physics and proofreading the work. We thank Malcolm Perry for comments on a preview of this work. We further thank Gia Dvali and Dieter Lüster for discussion during a presentation of the work.

-
- [1] J. D. Bekenstein, Black holes and entropy, *Phys. Rev. D* **7**, 2333 (1973).
 - [2] S. W. Hawking, Black hole explosions, *Nature (London)* **248**, 30 (1974).

- [3] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, 199 (1975); **46**, 206 (1976).
- [4] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, New York, 1996).

- [5] J. D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, *Commun. Math. Phys.* **104**, 207 (1986).
- [6] A. Strominger, Black hole entropy from near horizon microstates, *J. High Energy Phys.* **02** (1998) 009.
- [7] S. Carlip, Black hole entropy from conformal field theory in any dimension, *Phys. Rev. Lett.* **82**, 2828 (1999).
- [8] S. Carlip, Entropy from conformal field theory at Killing horizons, *Classical Quantum Gravity* **16**, 3327 (1999).
- [9] S. Carlip, Extremal and nonextremal Kerr/CFT correspondences, *J. High Energy Phys.* **04** (2011) 076; Erratum, *J. High Energy Phys.* **01** (2012) 8.
- [10] S. Carlip, Effective conformal descriptions of black hole entropy, *Entropy* **13**, 1355 (2011).
- [11] G. Barnich and C. Troessaert, Aspects of the BMS/CFT correspondence, *J. High Energy Phys.* **05** (2010) 062.
- [12] S. W. Hawking, M. J. Perry, and A. Strominger, Soft Hair on Black Holes, *Phys. Rev. Lett.* **116**, 231301 (2016).
- [13] The simplest choice, the $\mathfrak{bms}_4$ supertranslations contained in the asymptotic symmetry algebra of spacetimes that are asymptotically flat at null infinity [11], does not work. It was already explained in Ref. [14] that they cannot be responsible for microstates. Instead, it was proposed there that the presence of an event horizon enhances the asymptotic symmetry algebra, and it is the *enhancement* that is responsible for microstates and entropy counting [15]. This provides the resolution to the criticism on the soft hair proposal stated later correctly in Refs. [16–19].
- [14] A. Averin, G. Dvali, C. Gomez, and D. Lust, Gravitational black hole hair from event horizon supertranslations, *J. High Energy Phys.* **06** (2016) 088.
- [15] A. Averin, G. Dvali, C. Gomez, and D. Lust, Goldstone origin of black hole hair from supertranslations and criticality, *Mod. Phys. Lett. A* **31**, 1630045 (2016).
- [16] M. Mirbabayi and M. Porrati, Dressed Hard States and Black Hole Soft Hair, *Phys. Rev. Lett.* **117**, 211301 (2016).
- [17] R. Bousso and M. Porrati, Soft hair as a soft wig, *Classical Quantum Gravity* **34**, 204001 (2017).
- [18] R. Bousso and M. Porrati, Observable supertranslations, *Phys. Rev. D* **96**, 086016 (2017).
- [19] B. Gabai and A. Sever, Large gauge symmetries and asymptotic states in QED, *J. High Energy Phys.* **12** (2016) 095.
- [20] A. Averin, Schwarzschild/CFT from soft black hole hair?, *J. High Energy Phys.* **01** (2019) 092.
- [21] T. Hartman, C. A. Keller, and B. Stoica, Universal spectrum of 2d conformal field theory in the large c limit, *J. High Energy Phys.* **09** (2014) 118.
- [22] S. Haco, S. W. Hawking, M. J. Perry, and A. Strominger, Black hole entropy and soft hair, *J. High Energy Phys.* **12** (2018) 098.
- [23] J. D. Qualls, Lectures on conformal field theory, [arXiv:1511.04074](#).
- [24] J. Lee and R. M. Wald, Local symmetries and constraints, *J. Math. Phys. (N.Y.)* **31**, 725 (1990).
- [25] R. M. Wald and A. Zoupas, A general definition of ‘conserved quantities’ in general relativity and other theories of gravity, *Phys. Rev. D* **61**, 084027 (2000).
- [26] A. Seraj, Conserved charges, surface degrees of freedom, and black hole entropy, [arXiv:1603.02442](#).
- [27] S. W. Hawking, M. J. Perry, and A. Strominger, Superrotation charge and supertranslation hair on black holes, *J. High Energy Phys.* **05** (2017) 161.
- [28] G. Compère and A. Fiorucci, Advanced lectures on general relativity, [arXiv:1801.07064](#).
- [29] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, *Phys. Rev. D* **80**, 124008 (2009).
- [30] For example, suppose that the CFT that was going to cover the microstates of the black hole $g_{ab} \in \Gamma$ given by (4) (and which was defined using the chosen symplectic form Ω relative to the horizon of this particular black hole above) is also going to describe a black hole of some larger mass $\tilde{g}_{ab} \in \Gamma$. Then, we make here no statement about $L_0[\tilde{g}_{ab}]$.
- [31] A chiral Virasoro algebra alone would not be able to fix the black hole’s mass and angular momentum parameter in the microcanonical ensemble as required by the definition of the black hole entropy. This provides further evidence for the necessity of the entire 2D local conformal algebra instead of a chiral one as already discussed in Secs. I and II A.
- [32] One can also check that, although reparametrizations along U (V) form the algebra (3), the Hamiltonian generators do not develop a central charge in (14).
- [33] The vector fields of Ref. [20] are obtained by putting an additional minus sign in the exponential of the first equation in (27). This was omitted here in order to make the frequencies and temperatures that are going to appear later positive. In addition, we replaced A and B with their inverse values as compared to Ref. [20].
- [34] The analysis there is done for a Kerr black hole (Kerr-Newman in Ref. [35]) and diverges in the Schwarzschild limit, but this will be not important here.
- [35] S. Haco, M. J. Perry, and A. Strominger, Kerr-Newman black hole entropy and soft hair, [arXiv:1902.02247](#).
- [36] A. Castro, A. Maloney, and A. Strominger, Hidden conformal symmetry of the Kerr black hole, *Phys. Rev. D* **82**, 024008 (2010).
- [37] We take the convention $A > 0$ and $B < 0$ in the following in order for the temperatures (35) to be positive.
- [38] A. Strominger, On BMS invariance of gravitational scattering, *J. High Energy Phys.* **07** (2014) 152.
- [39] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg’s soft graviton theorem, *J. High Energy Phys.* **05** (2015) 151.
- [40] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$ -matrix, *J. High Energy Phys.* **08** (2014) 058.
- [41] S. Pasterski, A. Strominger, and A. Zhiboedov, New gravitational memories, *J. High Energy Phys.* **12** (2016) 053.