Path integral optimization for $T\bar{T}$ deformation

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We use the path integral optimization approach of Caputa, Kundu, Miyaji, Takayanagi, and Watanabe to find the time slice of geometries dual to vacuum, primary, and thermal states in the $T\bar{T}$ -deformed two-dimensional conformal field theories (CFTs). The obtained optimized geometries actually capture the entire bulk, which fits well with the integrability and expected UV completeness of $T\bar{T}$ -deformed CFTs. When the deformation parameter is positive, these optimized solutions can be reinterpreted as geometries at finite bulk radius, in agreement with a previous proposal by McGough, Mezei, and Verlinde. We also calculate the holographic entanglement entropy and quantum state complexity for these solutions. We show that the complexity of formation for the thermofield double state in the deformed theory is UV finite, and it depends on the temperature.

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I. INTRODUCTION

Applications of quantum information concepts to gravity and high energy physics recently led to many wide-spreading developments. In particular, quantum entanglement helps us to understand how gravity emerges from field theories [1,2]. One concrete idea to realize this emergent spacetime is to utilize the connection between tensor networks and a holographic entanglement surface. It is argued that a time slice of AdS spacetime corresponds to a special tensor network called the multiscale entanglement renormalization ansatz (MERA) [3]. However, when one tries to understand the interior of the black hole, the entanglement is not enough [4]. This observation has led to significant interest in quantum state complexity, which can be used as a probe to investigate the growth rate of the Einstein-Rosen bridge [5,6].

Two holographic conjectures are proposed for quantum complexity: the complexity volume (CV) conjecture [6,7] and the complexity action (CA) conjecture [8], which share some similarities but differ in many ways. To better understand the holographic results, the computational complexity for quantum field theory states has recently

been studied [9,10]. In one approach, which is based on the idea of Nielsen and collaborators [11], one associates a geometry to the space of unitaries that connect the desired state to a reference state. Then the complexity of the desired state is defined as a length of a geodesic in this geometry. Up to now, this approach has been well established just for Gaussian states. In another approach, which is applicable for any 2D CFTs (free or interacting) [12–16], one works with the Euclidean path integral description of the quantum state and performs the optimization by changing the structure (or geometry) of lattice regularization. The resulting change in path integral, namely, the Liouville action, was defined as a complexity of the corresponding state. Very interestingly, it has been shown recently that these two seemingly different approaches are surprisingly connected [17,18].

In this paper we apply the path integral optimization [13] to the new class of integrable quantum field theories. These theories were introduced by Smirnov and Zamolodchikov, who studied the deformation of 2D integrable QFTs by a special irrelevant composite operator $T\bar{T}$ and found that the theory remains integrable even after the deformation. Interestingly, the resulting theory is UV complete and nonlocal, and its energy spectrum and the S-matrix can be found exactly. After the proposal of [19], which relates the $T\bar{T}$ deformation of 2D CFTs to 3D AdS gravity with a finite bulk cutoff, these new integrable QFTs have received remarkable attention.

In the following, we show that the outcome of path integral optimization for $T\bar{T}$ deformation of 2D CFTs is geometries that capture the entire bulk, especially the region outside the finite bulk cutoff in [19]. This result

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is in agreement with a recent study [20], where it was shown that the dual gravitational theory to the $T\bar{T}$ -deformed 2D CFTs has mixed boundary conditions for the nondynamical graviton. It is worth mentioning that our result for the positive sign of the deformation parameter can be interpreted as the existence of the hard bulk cutoff in agreement with [19]. Moreover, we calculate the holographic entanglement entropy for the obtained solutions and also the quantum state complexity for some dual boundary QFT states. These states are the $T\bar{T}$ deformation of vacuum, primary, and thermal states. The quantum complexity of these states has the same UV structure as corresponding states in 2D CFTs. Moreover, the complexity of formation [21] for the thermofield double state is finite, and it depends on the temperature. Based on the bulk cutoff interpretation, the holographic complexity (CA) of the thermofield double state in $T\bar{T}$ -deformed CFTs is studied in [22]. It is worth mentioning that the holographic complexity is finite, but as we will show, the path integral one is UV divergent.

II. PATH INTEGRAL OPTIMIZATION

For 2D CFTs, the ground-state wave functional on $\mathbb{R}^2$ is computed by a Euclidean path integral:

$$\Psi_{\text{CFT}}[\bar{\Phi}(x)] = \int \left(\prod_x \prod_{-\infty \leq \tau < \epsilon} \mathcal{D}\Phi \right) e^{-S_{\text{CFT}}(\Phi)} \times \prod_x \delta(\Phi(\epsilon, x) - \bar{\Phi}(x)), \quad (1)$$

where τ is the Euclidean time and ϵ is the UV cutoff (i.e., the lattice constant). It is worth noting that to evaluate this discretized path integral in an optimal way, one can omit any unnecessary lattice sites. To systematically quantify such coarse graining, one might introduce a 2D metric (on which the path integration is performed)

$$ds^2 = \frac{l^2}{\epsilon^2} (d\tau^2 + dx^2), \quad (2)$$

such that one lattice site has a unit area. The optimization procedure can then be described by modifying the background metric for the path integration as

$$ds^2 = g_{\tau\tau}(\tau, x) d\tau^2 + g_{xx}(\tau, x) dx^2 + 2g_{\tau x}(\tau, x) d\tau dx. \quad (3)$$

Now, the key point is that the optimized wave functional, up to a normalization factor, should be proportional to the correct ground-state wave function, i.e., the (1) for the metric (2). This constraint implies that

$$g_{\tau\tau}(\epsilon, x) = g_{xx}(\epsilon, x) = \frac{l^2}{\epsilon^2}, \quad g_{\tau x}(\epsilon, x) = 0. \quad (4)$$

For 2D $T\bar{T}$ -deformed CFTs, which are described by the action

$$S = S_{\text{CFT}} + \mu \int d\tau dx T\bar{T}(\tau, x), \quad (5)$$

the deformation operator is a double-trace operator. According to [23], this double-trace deformation of CFT corresponds to imposing a mixed boundary condition on the bulk metric. Motivated by this fact and noting that the position-dependent coupling μ is related to the bulk matter field, in the following we keep the coupling constant μ independent from (τ, x) coordinates and just allow the metric to vary. Accordingly, the optimization is performed by the following ansatz:

$$ds^2 = e^{2\Omega(\tau, x)} (d\tau^2 + dx^2), \quad (6)$$

where the mixed boundary condition is encoded in two requirements: $e^{2\Omega(\epsilon, x)} = \frac{l^2}{\epsilon^2}$, and the energy of the optimized solution matches the energy spectrum of $T\bar{T}$ -deformed CFT.

On the metric $\hat{g}_{\alpha\beta} = e^{2\Omega(\tau, x)} g_{\alpha\beta}$, the partition function of 2D $T\bar{T}$ -deformed CFTs is given by

$$Z_\mu[\hat{g}] = \int \mathcal{D}\Phi \text{Exp} \left[-S_{\text{CFT}}[\Phi, \hat{g}] - \int d^2\sigma \sqrt{\hat{g}} (\Lambda + \mu T\bar{T}[\Phi, \hat{g}]) \right], \quad (7)$$

where $d^2\sigma = d\tau dx$ and Λ is related to the cosmological counterterm. It is worth noting that this counterterm action is responsible for removing the UV divergence in the presence of $T\bar{T}$ deformation. Under a change of the Weyl factor $\Omega(\tau, x)$ and for small μ , the partition function (7) changes as follows:

$$\frac{\partial \log Z_\mu[\hat{g}]}{\partial \Omega} = \frac{c}{24\pi} \sqrt{\hat{g}} \hat{R}[\hat{g}] - 2\Lambda \sqrt{\hat{g}} - \mu \partial_\Omega \langle T\bar{T} \rangle_{\hat{g}}, \quad (8)$$

where constant c is the central charge of the 2D CFT and, at least for its large values,

$$\langle T\bar{T} \rangle_{\hat{g}} = \int d^2\sigma \sqrt{\hat{g}} (\langle 0|T_{zz}|0 \rangle_{\hat{g}} \langle 0|\bar{T}_{\bar{z}\bar{z}}|0 \rangle_{\hat{g}} - e^{-4\Omega} \langle 0|T_{z\bar{z}}|0 \rangle_{\hat{g}}^2). \quad (9)$$

One can now just treat (8) as a differential equation for the partition function Z_μ and solve it. This allows us to express the partition function $Z_\mu[\hat{g}]$, defined on one metric $\hat{g}$, in terms of $Z[g]$, defined on metric g . The relationship is as follows:

$$Z_\mu[\hat{g}] = e^{S_{GL}[\Omega, g] - S_{GL}[0, g]} Z_\mu[g], \quad (10)$$

where

$$S_{GL}[\Omega, g] = \int d^2\sigma \left(\frac{c}{24\pi} \sqrt{g} (R[g]\Omega - \Omega \nabla^2 \Omega - \bar{\Lambda} e^{2\Omega}) - \mu \langle T\bar{T} \rangle_{\hat{g}} \right), \quad (11)$$

and $\bar{\Lambda} = \frac{24\pi}{c} \Lambda$. To find the above relation, it is enough to note that $R[\hat{g}] = e^{-2\Omega} (R[g] - 2\nabla^2 \Omega)$ and $\sqrt{\hat{g}} = e^{2\Omega} \sqrt{g}$. Then, integrating both sides of (8) leads to the desired result (10). The subscript (GL) in the above equation means the generalized Liouville action since the first term in the above equation is actually the standard Liouville action. For $g_{\alpha\beta} = \delta_{\alpha\beta}$, the last term in (11) can be calculated explicitly using the following known relations,

$$\begin{aligned} \langle 0|T|0 \rangle_{e^{2\Omega}\delta_{\alpha\beta}} &= \frac{c}{12} e^{-2\Omega} (\partial^2 \Omega), \\ \langle 0|\bar{T}|0 \rangle_{e^{2\Omega}\delta_{\alpha\beta}} &= \frac{c}{12} e^{-2\Omega} (\bar{\partial}^2 \Omega), \\ \langle 0|\Theta|0 \rangle_{e^{2\Omega}\delta_{\alpha\beta}} &= \frac{c}{6} e^{-2\Omega} (\partial\bar{\partial}\Omega), \end{aligned} \quad (12)$$

where Θ is the trace of the stress tensor. Accordingly, by defining the complex coordinates ($w = \tau + ix$, $\bar{w} = \tau - ix$) the action (11) simplifies to

$$S_{GL}[\Omega, \delta] = \frac{c}{24\pi} \int d^2w \left[-4\Omega \partial\bar{\partial}\Omega - \bar{\Lambda} e^{2\Omega} - \tilde{\mu} e^{-2\Omega} \left(\partial^2 \Omega \bar{\partial}^2 \Omega - 4(\partial\bar{\partial}\Omega)^2 + \frac{3}{16} \bar{\Lambda}^2 e^{4\Omega} \right) \right], \quad (13)$$

where $\tilde{\mu} = \mu\pi c/6$. In obtaining the above equation, we used $\bar{\Lambda} = \tilde{\Lambda} + \frac{3}{16} \tilde{\mu} \bar{\Lambda}^2$, the reason for which will be clear in the following. Finally, the definition (1), together with (10), implies that the ground-state wave functional $\Psi_{e^{2\Omega}\delta_{\alpha\beta}}$ computed from the path integral for the metric (6) is proportional to the one $\Psi_{\delta_{\alpha\beta}}$ for the flat metric

$$\Psi_{e^{2\Omega}\delta_{\alpha\beta}}[\bar{\Phi}(x)] = e^{S_{GL}[\Omega, \delta] - S_{GL}[0, \delta]} \Psi_{\delta_{\alpha\beta}}[\bar{\Phi}(x)]. \quad (14)$$

It is worth emphasizing that these two states describe the same quantum state if at the UV cutoff, $\Omega(\epsilon, x) = \log(l/\epsilon)$.

III. OPTIMIZING VARIOUS STATES IN $T\bar{T}$ -DEFORMED CFTs

Now, according to [12,13], the optimization is equivalent to minimizing the normalization factor $e^{S_{GL}[\Omega, \delta]}$. In order to find this minimum value, we must vary the action $S_{GL}[\Omega, \delta]$ (13), which gives following equation of motion:

$$\begin{aligned} 4\partial\bar{\partial}\Omega - \bar{\Lambda} e^{2\Omega} + \tilde{\mu} \left(\frac{3}{16} \bar{\Lambda}^2 e^{2\Omega} + 3e^{-2\Omega} \partial^2 \bar{\partial}^2 \Omega \right. \\ \left. + e^{-2\Omega} \bar{\partial}^2 \Omega (-2(\partial\Omega)^2 + 3\partial^2 \Omega) - 2e^{-2\Omega} [3\partial\Omega \partial\bar{\partial}^2 \Omega \right. \\ \left. + (\bar{\partial}\Omega)^2 \partial^2 \Omega + 6(\partial\bar{\partial}\Omega)^2 + \bar{\partial}\Omega (-8\partial\Omega \partial\bar{\partial}\Omega + 3\partial^2 \bar{\partial}\Omega)] \right) \\ = 0. \end{aligned} \quad (15)$$

Since finding the analytic solutions of this equation, in general, is difficult, one can solve it perturbatively in the $\tilde{\mu}$ parameter. Since in this paper we are interested in finding the solutions which only depend on $2\tau = (\omega + \bar{\omega})$, we consider $\Omega(\omega + \bar{\omega}) = \Omega_{\text{CFT}}(\omega + \bar{\omega}) + \tilde{\mu} \Omega_{T\bar{T}}(\omega + \bar{\omega})$. Using this perturbative ansatz, Eq. (15) is simplified to

$$\begin{aligned} - (32\tilde{\mu} \partial\bar{\partial}\Omega_{T\bar{T}} + 9) e^{4\Omega_{\text{CFT}}} + 16\tilde{\mu} \Omega_{T\bar{T}} e^{6\Omega_{\text{CFT}}} \\ - 64\partial^2 \Omega_{\text{CFT}} \bar{\partial}^2 \Omega_{\text{CFT}} + 56e^{2\Omega_{\text{CFT}}} \partial\Omega_{\text{CFT}} \bar{\partial}\Omega_{\text{CFT}} \\ + 16\bar{\partial}^2 \Omega_{\text{CFT}} (3\partial^2 \Omega_{\text{CFT}} - 4\partial\Omega_{\text{CFT}}^2) = 0. \end{aligned} \quad (16)$$

According to [15], a vacuum state in two-dimensional QFT can be constructed from various codimension-one surfaces in the 3D gravity dual. For example, one of them can be a two-dimensional boundary and another can be the time slice of 3D bulk geometry. These two surfaces can be related by the bulk coordinate transformations; until they have the same topology, they give the same state up to the normalization factor. In the following, we represent this bulk codimension-one surface with (z, x) coordinates. Now, we solve (16). One of interesting solutions for $\tilde{\mu} = 0$ is that of excited states created by acting a primary operator $\mathcal{O}_a$ with the conformal dimension $h_a = \bar{h}_a$; its behavior under the Weyl rescaling is expressed as

$$\mathcal{O}_a \sim e^{-2h_a} \mathcal{O}_a. \quad (17)$$

It is shown that in the absence of $\tilde{\mu}$, the geometry dual to this state is given by

$$\Omega_{\text{CFT}} = \frac{1}{2} \log \left(\frac{a^2}{\sinh^2(\frac{az}{l})} \right), \quad a = 1 - \frac{12h_a}{c}. \quad (18)$$

For $a = 1$, this describes the time slice of global AdS_3 . Using this unperturbed solution in (16), the perturbed solution becomes

$$\begin{aligned} ds^2 = a^2 \text{csch}^2 \left(\frac{az}{l} \right) \left(1 + \frac{2\tilde{\mu} c_1}{l^2} \left(1 - \frac{az}{l} \coth \left(\frac{az}{l} \right) \right) \right) \\ \times (dz^2 + dx^2), \end{aligned} \quad (19)$$

where c_1 is a dimensionless constant of integration; also, a coefficient of a term which violates the condition $g_{zz}(\epsilon, x) \sim \frac{1}{\epsilon^2}$ is set to zero. Using the coordinate transformation

$$z = \frac{2l}{a} \left(1 - \frac{\tilde{\mu}c_1}{l^2} \right) \tanh^{-1} \left(\frac{\sqrt{\rho}}{2} \right), \quad (20)$$

the metric (19) takes the Fefferman-Graham expansion

$$ds^2 = \frac{l^2 a^2}{4\rho^2} d\rho^2 + \frac{l^2 a^2}{\rho} \left(1 + \frac{2\tilde{\mu}c_1}{l^2} \right) \left(1 - \frac{1}{2}\rho + \frac{1}{16}\rho^2 \right) dx^2. \quad (21)$$

The energy of this solution matches the corresponding energy of the deformed primary state [24] for $c_1 = -1/32\pi^2$. It is also easy to see that the Poincare AdS₃ remains undeformed.

Up to now, we have focused on the $T\bar{T}$ deformation of a CFT at zero temperature. To extend the analysis to a finite temperature $T = 1/\beta'$ case, one can use the thermofield double representation of the wave functional. In this representation, the wave functional is computed from a path integral on a cylinder with a finite width $-\frac{\beta'}{4} < z < \frac{\beta'}{4}$ accordingly,

$$\begin{aligned} \Psi_{\beta'}[\bar{\Phi}_2(x), \bar{\Phi}_1(x)] &= \int \left(\prod_x \prod_{-\frac{\beta'}{4} \leq z < \frac{\beta'}{4}} \mathcal{D}\Phi \right) e^{-S(\Phi)} \\ &\times \prod_x \delta \left(\Phi \left(-\frac{\beta'}{4}, x \right) - \bar{\Phi}_1(x) \right) \\ &\times \delta \left(\Phi \left(\frac{\beta'}{4}, x \right) - \bar{\Phi}_2(x) \right), \end{aligned} \quad (22)$$

where S is given by (5). In the absence of deformation, it is shown [12] that the minimization procedure gives

$$\Omega_{\text{CFT}}(z) = \frac{1}{2} \log \left(\frac{4\pi^2 l^2 \sec^2 \left(\frac{2\pi z}{\beta} \right)}{\beta^2} \right), \quad (23)$$

which actually describes the time slice of the BTZ black hole. It is worth noting that the temperature for the solution (23) is assumed to be different from the temperature of $1/\beta'$. The reason for this will be clear in the following. Substituting this solution in (16) gives the following equation for $\Omega_{T\bar{T}}$,

$$\frac{\beta^2}{4} \cos^2 \left(\frac{2\pi z}{\beta} \right) \partial_z^2 \Omega_{T\bar{T}}(z) - 2\pi^2 \Omega_{T\bar{T}}(z) = 0, \quad (24)$$

whose solution is

$$\Omega_{T\bar{T}}(z) = \left(\frac{2\pi c_1 z}{\beta} \right) \tan \left(\frac{2\pi z}{\beta} \right) + c_1. \quad (25)$$

In the above expression c_1 is an arbitrary integration constant. Substituting (24) and (25) in (16), up to first order in $\tilde{\mu}$, we have

$$ds^2 = \frac{4\pi^2 l^2}{\beta^2} \sec^2 \left(\frac{2\pi z}{\beta} \right) \left(1 + \frac{2\tilde{\mu}c_1}{l^2} \left[1 + \frac{2\pi z}{\beta} \tan \left(\frac{2\pi z}{\beta} \right) \right] \right) \times (dz^2 + dx^2). \quad (26)$$

To determine the constant c_1 , it is better to transform this metric in the global coordinate. This can be achieved according to the coordinate transformation

$$z = \frac{\beta}{2\pi} \left(1 - \frac{\tilde{\mu}c_1}{l^2} \right) \cos^{-1} \left(\frac{2\pi l^2}{\beta r} \right) - \frac{\tilde{\mu}c_1}{r \sqrt{1 - \frac{4\pi^2 l^4}{\beta^2 r^2}}}, \quad (27)$$

which, using the metric (26), changes to

$$ds^2 = \frac{dr^2}{f(r)} + r^2 dx^2, \quad f(r) = \frac{r^2}{l^2} - \frac{4\pi^2 l^2}{\beta^2} \left(1 + \frac{2\tilde{\mu}}{l^2} c_1 \right). \quad (28)$$

The temperature of this black hole is

$$1/\beta' \equiv T_{T\bar{T}} = \left(1 + \frac{\tilde{\mu}}{l^2} c_1 \right) / \beta. \quad (29)$$

Remember that the time slice of the standard BTZ black hole with the ADM mass M is

$$ds^2 = \frac{dr^2}{f(r)} + r^2 dx^2, \quad f(r) = \frac{r^2}{l^2} - 8GM_{\text{BTZ}}, \quad (30)$$

where the mass and temperature are related according to

$$M_{\text{BTZ}} = \frac{\pi^2 l^2}{2G} \frac{1}{\beta^2}. \quad (31)$$

For $c_1 = GM_{\text{BTZ}}/4\pi^2$ and considering the $T_{T\bar{T}}$ in the right-hand side of (31) instead of $1/\beta$, for $M_{T\bar{T}}$ up to first order in μ , we find

$$M_{T\bar{T}} = M_{\text{BTZ}} \left(1 + \frac{M_{\text{BTZ}}}{8\pi l} \mu \right), \quad (32)$$

which exactly matches with the energy spectrum of the $T\bar{T}$ -deformed CFT [24,25] for small μ . By using

$$r(\rho) = -\frac{l(\beta^2 + \pi^2 l^2 \rho)}{\beta^2 \sqrt{\rho}} \left(1 + c_1 \frac{\tilde{\mu}}{l^2} \right), \quad (33)$$

the metric (28) changes to

$$ds^2 = \frac{l^2 d\rho^2}{4\rho^2} + \frac{l^2 (1 + \frac{2\tilde{\mu}}{l^2} c_1)}{\rho} \left(1 + \frac{2\pi^2 l^2}{\beta^2} \rho + \frac{\pi^4 l^4}{\beta^4} \rho^2 \right) dx^2. \quad (34)$$

To understand this result better, let us remember that the most general solution of the three-dimensional Einstein

equations with a negative cosmological constant can be expressed in Fefferman-Graham gauge by

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} g_{\alpha\beta}(\rho, x) dx^\alpha dx^\beta, \quad (35)$$

$$g_{\alpha\beta}(\rho, x^\alpha) = g^{(0)}(x^\alpha) + g_{\alpha\beta}^{(2)}(x^\alpha)\rho + g_{\alpha\beta}^{(4)}(x^\alpha)\rho^2, \quad (35)$$

where $g^{(4)}$ and $g^{(2)}$ are determined algebraically in terms of $g^{(0)}$ [26]. Based on the variational principle approach to the $T\bar{T}$ deformation of CFTs in [20] [actually, with a mixed boundary condition as we discussed below Eq. (5)], it is argued that in the deformed theory the source for the stress tensor is given by a nonlinear combination of the metric and stress tensor expectation value in the original CFT as follows [27]:

$$g_{\alpha\beta}^{[\mu]} = g_{\alpha\beta}^{(0)} + \frac{\mu}{32\pi G l} g_{\alpha\beta}^{(2)} + \mathcal{O}(\mu^2). \quad (36)$$

For the BTZ black hole, $g_{xx}^{(0)} = 1$ and $g_{xx}^{(2)} = \frac{2\pi^2 l^2}{\beta^2}$, which, together with (31), implies that

$$g_{xx}^{[\mu]} = 1 + \frac{M_{\text{BTZ}}}{8\pi l} \mu + \mathcal{O}(\mu^2). \quad (37)$$

Remarkably, we get the conformal boundary of (34) by setting $c_1 = GM_{\text{BTZ}}/4\pi^2$, which is the same as the previous analysis for the energy of the black hole solution (28). Another interpretation for the geometry (34) is that its conformal boundary corresponds to fixing the induced metric on a constant $\rho = \rho_c$ surface, with

$$\rho_c = \frac{\mu}{32\pi G l}, \quad (38)$$

in agreement with the earlier proposal of [19]. It is worth noting that the path integral optimization solution (28) shows that the entire spacetime should be kept, especially the region outside the would-be cutoff surface, which, in the proposal [19], is removed. Since $T\bar{T}$ -deformed CFTs are conjectured to contain (nonlocal) observables of arbitrarily high energy, this makes sense.

By setting the c_1 value in (28), $c_1 = GM_{\text{BTZ}}/4\pi^2$, one can find the holographic entanglement entropy for the solution (28) easily. Let us remember that the holographic entanglement entropy of the 2D CFT at finite temperature, which is dual to the BTZ black hole, is

$$S_{EE}^{\text{CFT}} = \frac{c}{3} \log \left(\frac{1}{\epsilon} \frac{l}{\sqrt{2GM_{\text{BTZ}}}} \sinh \left(\frac{R\sqrt{2GM_{\text{BTZ}}}}{l} \right) \right), \quad (39)$$

where R is the size of the entangling region. Substituting M_{BTZ} with $M_{T\bar{T}}$ from (31) in (39), the result to first order in μ becomes

$$S_{EE}^{T\bar{T}} = S_{EE}^{\text{CFT}} + \mu \frac{\pi c^2}{144\beta^2} \left(\frac{\pi R}{\beta} \coth \left(\frac{\pi R}{\beta} \right) - 1 \right), \quad (40)$$

where we have also used Eq. (31). The result (40) has an extra term (last term in big parentheses) in comparison with the previous study [28] in which the authors assumed the finite cutoff interpretation. This extra term causes the entanglement entropy (40) to vanish for the zero entangling region. To close this section, we report the final result for the entanglement entropy of the ground state (21) (with $a = 1$) [29],

$$S_{EE}^{T\bar{T}} \simeq \frac{c}{3} \log \left(\frac{R}{\epsilon} \right) + \mu \frac{c^2 R^2}{6912\pi l^4}, \quad (41)$$

which again vanishes in the UV limit, $R \rightarrow 0$, like the result in [30]. It is worth noting that from this latter result, we cannot conclude that the theory flows to a trivial theory since we have assumed μ is small.

IV. PATH INTEGRAL COMPLEXITY

It seems that the similarity between the tensor network representation and the path integral representation of the vacuum state can be utilized to define the computational complexity of the ground state (1) as the minimum value of the $S_{GL}[\Omega, g]$ action. However, $S_{GL}[\Omega, g]$ in (13) not only depends on the final metric $e^{2\Omega}g$ but also on the reference metric g , which means that this action does not provide an absolute quantity that measures the complexity of the optimized state. According to [13], it is convenient to look at the relative quantity $I_{GL}[g_2, g_1]$, which satisfies the following identity:

$$I_{GL}[g_1, g_2] + I_{GL}[g_2, g_3] = I_{GL}[g_1, g_3]. \quad (42)$$

The above relation implies that

$$I_{GL}[e^{2\Omega}g, \tilde{g}] = I_{GL}[e^{2\Omega}g, g] - I_{GL}[\tilde{g}, g], \quad (43)$$

which means that $I_{GL}[g_2, g_1]$ actually measures the difference of complexity between the path integral in g_2 and g_1 . In order to find this quantity, motivated by the form of S_{GL} in (13), one can assume a general action, which by implementing the constraint equation (42), reduces to

$$I_{GL}[\Omega, g] = \int d^2\sigma \sqrt{g} \left(R[g]\Omega + \nabla_\alpha \Omega \nabla^\alpha \Omega - \tilde{\Lambda}(e^{2\Omega} - 1) - \frac{3\tilde{\mu}}{16} e^{-2\Omega} \left[\tilde{\Lambda}^2 e^{2\Omega}(e^{2\Omega} - 1) + \square\Omega(R[g] - \square\Omega) + \frac{1}{4} R^2[g](e^{2\Omega} - 1) \right] \right). \quad (44)$$

To have a well-defined variational principle, we should add a proper generalized Gibbons-Hawking term to the above action, which is given by

$$I_{GGH}[\Omega, \gamma] = 2 \int dx \sqrt{\gamma} \left(K[\gamma] \Omega - \frac{3}{16} \tilde{\mu} e^{-2\Omega} \left[K[\gamma] \square \Omega + \frac{1}{2} K[\gamma] R[g] (e^{2\Omega} - 1) - n^\alpha \nabla_\alpha \Omega (R[g] - \square \Omega) \right] \right), \quad (45)$$

where γ and $K[\gamma]$ are, respectively, the induced metric and its extrinsic curvature. In the above we have set $\tilde{\Lambda} = 1/l^2$. Now we have all the ingredients to calculate the path integral complexity,

$$C_{T\bar{T}} \equiv \frac{c}{24\pi} (I_{GL}[\Omega, g] + I_{GGH}[\Omega, \gamma]), \quad (46)$$

for the solutions we have found in the previous section. The final results for solutions (19) (with $c_1 = -1/32\pi^2$) and (26) (with $c_1 = GM_{\text{BTZ}}/4\pi^2$) are, respectively,

$$\begin{aligned} C_{T\bar{T}}^{\text{PS}} &= \frac{c}{6} \left(1 + \frac{\pi c \mu}{64 l^2} \right) \frac{l}{\epsilon} - \frac{ac}{6} \left(1 + \frac{\pi c \mu}{32 l^2} \left(\frac{1}{3\pi^2} + \frac{1}{2} \right) \right) \\ &\quad + \frac{c}{12} \left((a^2 - 1) - \frac{\pi c \mu}{16 l^2} \left(\frac{a^2}{3\pi^2} + \frac{1}{2} \right) \right) \frac{z_\infty}{l}, \\ C_{T\bar{T}}^{\text{BTZ}} &= \frac{c}{3} \left(1 + \frac{\pi c \mu}{64 l^2} \right) \frac{l}{\epsilon} - \frac{\beta c}{24 l} - \frac{\pi^2 c l}{6 \beta} \\ &\quad - \frac{\pi c \mu}{6} \left(\frac{\beta c}{128 l^3} + \frac{\pi^2 c l}{48 \beta^3} - \frac{c}{192 \beta l} \right), \end{aligned} \quad (47)$$

where $z_\infty (\rightarrow \infty)$ is the IR cutoff in the z integral. This IR divergence might be related to the nonlocal nature of $T\bar{T}$ -deformed CFTs [31]. Remarkably, the complexity of formation, $C_{T\bar{T}}^{\text{BTZ}} - 2C_{T\bar{T}}^{\text{AdS}}$, in these theories is UV finite, which is the same as in 2D CFTs. This result, that the UV divergence of $C_{T\bar{T}}^{\text{BTZ}}$ is independent of its mass, is in agreement with the CA holographic complexity proposal [21,22].

We close this section by pointing out the interesting consistency of the action (44). We show that the stress tensor obtained from the first law of entanglement entropy is the same as the one obtained from the action (44). Let us remember that the change in entanglement entropy under a small variation of a quantum state is captured by the variation in the Weyl factor field, $\Omega(z) \rightarrow \Omega_0(z) + \delta\Omega(z)$. This fact implies that for a small entangling region $A = [-R/2, R/2]$, the change in entanglement entropy becomes

$$\Delta S_A \simeq \frac{c}{6} \int ds e^{\Omega(z)} \delta\Omega(z) = \frac{c R^2}{24} \partial_z^2 \delta\Omega(z), \quad (48)$$

where for the thermofield double solution (26), we have

$$\delta\Omega(z) = \frac{2\pi^2}{3\beta^2} z^2 + \mathcal{O}(z^4), \quad (49)$$

with β' given by (29). Therefore, according to the first law of entanglement entropy [32],

$$\Delta S_A \simeq \frac{\pi R^2}{3} T_{tt}, \quad (50)$$

the energy density becomes

$$T_{tt} = \frac{\pi c}{6\beta^2}. \quad (51)$$

Independently, by taking the metric variation from the action (44), the corresponding tt component of the stress tensor for the solution (26) becomes exactly the energy density (51). This can also be checked easily for deformed primary states (19). Last but not least, the energy density (51) is in complete agreement with the variational principle analysis of [20]. The authors of [20] have shown that if a 2D CFT lives on the conformal boundary $g_{\alpha\beta}$ in (35), the deformed stress tensor becomes

$$T_{\alpha\beta}^{[\mu]} = \frac{1}{8\pi G l} g_{\alpha\beta}^{(2)} + \frac{\mu}{64\pi^2 G^2 l^2} g_{\alpha\beta}^{(4)}. \quad (52)$$

For example, for the BTZ solution, $g_{tt}^{(2)} = 2\pi^2 l^2 / \beta^2$, $g_{tt}^{(4)} = \pi^4 l^4 / \beta^4$, and $c = 3l/2G$, which together imply that the deformed energy density (52) exactly matches with (51).

V. SUMMARY AND OUTLOOK

We have shown that the path integral optimization approach, for $T\bar{T}$ deformation of 2D CFTs, implies that the time slice of optimized geometries indeed captures the entire spacetime. For positive deformation coupling, these optimized solutions can be reinterpreted as a geometry at the finite cutoff radius. In this paper we studied the entanglement entropy and quantum complexity for those optimized solutions. It is worth comparing our results more concretely with previous studies in the literature. The proposal of Verlinde *et al.* [19] for the gravitational dual of $T\bar{T}$ deformation is a geometry with a finite radial cutoff. Following this proposal, all studies from the gravity side assumed the existence of that cutoff and explored physical quantities such as entanglement entropy, complexity, partition functions, correlation functions, and so on. Contrary to all of the previous studies, we started from a different point of view: the tensor network representation of the corresponding wave function (the one that is equal to the path integral representation). Accordingly, the dual geometry that we have found captures the entire spacetime, especially the part that is removed in all previous studies. Interestingly, the energy of the system cannot explain this extra bulk region, but as we have shown, other quantities such as entanglement entropy and state complexity can explain it. Since these quantities are related to the correlation functions, our study may shed light on the structure of correlation functions in the deformed theory. Moreover,

the entanglement wedge for geometry with a finite radial cutoff is different for the geometries we have found, and this means that the bulk reconstruction might be different for them. If correct, this implies that the tensor factorization of Hilbert space (actually, code subspace) will be different. This possible difference can also be checked in the presence of quantum matter on the bulk (equal to adding matter to the generalized Liouville action) by studying the quantum version of the RT formula in addition to checking the equivalence between the bulk relative entropy and boundary relative entropy. Moreover, it can be checked at the level of bulk or boundary modular Hamiltonians. It is also worth mentioning that our results for dual geometries are in precise agreement with the ones obtained by a completely different approach [20]. However, in contrast to [20] in which just the asymptotic form of deformed geometry is found, our results capture the entire spacetime, so our

results are more general. Last but not least, our results also show that the Wess-Zumino-like condition (42) for state complexity leads to the same form for the deformed partition function as (10) and (11). This observation implies that there may exist a deep relation between the condition (42) and the integrability of QFTs.

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