Fragileness of exact *I*-ball/oscillon

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An *I*-ball/oscillon is a solitonlike oscillating configuration of a real scalar field, and defined as a minimum energy state for a given adiabatic invariant. It lasts for a long time because of the approximate conservation of their adiabatic invariant. In this paper, we examine the stability of a special type of *I*-ball/oscillon, the “exact” *I*-ball/oscillon, whose adiabatic invariant is exactly conserved. We show that the exact *I*-ball/oscillon is generally stable in classical field theory, but one with a certain adiabatic invariant is unstable against small perturbations because fluctuations around the exact *I*-ball/oscillon profile have instability bands. Accordingly, the exact *I*-ball/oscillon breaks up in the presence of the fluctuations of corresponding instability modes. We also confirm the fragileness of the exact *I*-ball/oscillon by the classical lattice simulation.

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I. INTRODUCTION

An *I*-ball/oscillon [1–3] is a nontopological solitonlike solution of a real scalar field coherently oscillating around the (local) minimum of the potential. A lot of studies on *I*-balls/oscillons indicate that the lifetime of *I*-balls/oscillons is extremely long [4–27]. This longevity is not only theoretically interesting but also motivates phenomenological study (See, e.g., Refs. [28–30] for gravitational waves from *I*-balls/oscillons and also Ref. [31].).

This lump of a real scalar field is theoretically defined as a minimum energy state of a given adiabatic invariant [13,32]. The adiabatic invariant is an analog to the phase space volume of periodic motion in classical mechanics. The reason for the longevity of *I*-balls/oscillons is the approximate conservation of the adiabatic invariant *I*, which is also regarded as the particle number conservation [33].

In this paper, we study the stability of the *I*-ball/oscillon in a particular potential [13] with a theoretical interest. The adiabatic invariant of this type of *I*-balls/oscillons is exactly conserved, and hence we name them the “exact” *I*-ball/oscillon. We show that the exact *I*-ball/oscillon is stable in classical field theory, but not against small perturbations depending on the adiabatic invariant of the *I*-ball/oscillon.

Accordingly, the exact *I*-ball/oscillon breaks up in the presence of the fluctuations of corresponding instability modes. Although it is generally difficult to carry out the Floquet analysis around the oscillating background which also has a spatial dependence, only for the exact *I*-ball/oscillon case, we can reveal the nonperturbative phenomenon. We confirm this decay process of the exact *I*-ball/oscillon by the classical lattice simulation.

In Sec. II, we introduce the exact *I*-ball/oscillon which exactly conserves the adiabatic invariant. In Sec. III, we show the instability of the exact *I*-ball/oscillon, which results from resonance bands the perturbations around certain *I*-ball/oscillon profiles have. In Sec. IV, we show the setup and the result of our lattice simulation to confirm the fragileness of the exact *I*-ball/oscillon. Finally, in Sec. V, we conclude the results.

II. EXACT *I*-BALL/OSCILLON

A. Exact conservation of adiabatic invariant

I-ball/oscillon is a localized oscillating scalar field configuration that minimizes the energy for a given value of the adiabatic invariant *I*. The adiabatic invariant of a real scalar field ϕ is defined by

$$I = \frac{1}{\omega} \int d^3x \overline{\phi^2}, \quad (1)$$

where ω is the angular frequency of the oscillating field and the overbar denotes the average over one period of the oscillation.

The adiabatic invariant Eq. (1) is approximately conserved when the scalar field oscillates in the potential

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dominated by the quadratic term ($\sim \phi^2$). In other words, the I -ball/oscillon is not an exact periodic motion in time, and hence, the time average over one period of the oscillation in Eq. (1) is not exact. Because of this approximate conservation of I , the I -ball/oscillon is generally quasi-stable and eventually decays by emitting scalar waves [25].

However, the adiabatic invariant is exactly conserved when the solution of the equation of motion is completely separable into the time and the spatial dependent parts [13].

Let us assume an I -ball/oscillon solution with the separated form as

$$\phi(t, \vec{x}) = f(t)\psi(\vec{x}), \quad (2)$$

where $\psi(\vec{x})$ is the theoretical oscillon profile and $f(t)$ is a time periodic function normalized as $\max\{f(t)\} = 1$. The fact that the time and spatial dependencies are determined separately is crucial when we consider the I -ball/oscillon stability. In this case, the adiabatic invariant I is evaluated as

$$I = \frac{\overline{\dot{f}^2}}{\omega} \int d^3x \psi(\vec{x})^2, \quad (3)$$

where the overbar denotes the time average over the one period of oscillation. Because $f(t)$ is exactly periodic, the adiabatic invariant I is constant in time, and hence, is conserved exactly.

Such a separated solution like Eq. (2) is possible only when the scalar potential of ϕ takes the form of

$$V = \frac{1}{2}m^2\phi^2 + \Delta V = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\kappa m^2\phi^2 \log \frac{\phi^2}{M^2}, \quad (4)$$

where $\kappa < 0$ is a dimensionless constant and m and M are mass parameters¹ [13].

For a later purpose, we redefine the parameters by

$$\tilde{m}^2 = m^2 + \kappa m^2 \log \frac{\tilde{m}^2}{M^2}, \quad (5)$$

$$\tilde{\kappa} \tilde{m}^2 = \kappa m^2, \quad (6)$$

with which

$$V = \frac{1}{2}\tilde{m}^2\phi^2 + \frac{1}{2}\tilde{\kappa}\tilde{m}^2\phi^2 \log \frac{\phi^2}{\tilde{m}^2}. \quad (7)$$

In what follows, we use the latter expression with $\tilde{m}^2 \rightarrow m^2$ and $\tilde{\kappa} \rightarrow \kappa$. Eventually, the potential depends only on two parameters, m and κ .

Substituting the solution Eq. (2) into the equation of motion

$$\ddot{\phi} - \nabla^2\phi + m^2\phi + \Delta V' = 0, \quad (8)$$

we obtain

$$[\ddot{f} + m^2f + \kappa m^2(1 + \log f^2)f]\psi - \left[\nabla^2\psi - \kappa m^2\psi \log \frac{\psi^2}{m^2} \right]f = 0. \quad (9)$$

This leads to the following two equations:

$$\ddot{f} + m^2f + \kappa m^2(1 + \log f^2)f = \zeta m^2f, \quad (10)$$

$$\nabla^2\psi - \left(\kappa m^2 \log \frac{\psi^2}{m^2} \right) \psi = \zeta m^2\psi, \quad (11)$$

where ζ is a constant. As we will see in the next section, ζ determines the adiabatic invariant of the I -ball/oscillon for given potential parameters.

Therefore, because the solutions of equations of motions under the potential Eq. (4) are independently determined by Eqs. (10) and (11) the adiabatic invariant is exactly conserved² as shown in Ref. [13].

B. The exact I -ball/oscillon solution

As the I -ball/oscillon corresponds to the minimum energy states for a given value of I , the I -ball/oscillon profile ψ is expected to be spherical, $\psi(\vec{x}) = \psi(r)$. Then, Eq. (11) has an exact solution,

$$\psi(r) = \psi_c \exp(-r^2/R^2), \quad (13)$$

where

$$R = \sqrt{\frac{2}{|\kappa|} \frac{1}{m}}, \quad (14)$$

$$\psi_c^2 = m^2 \exp\left(3 - \frac{\zeta}{\kappa}\right). \quad (15)$$

ψ_c and R are the central value and the radius of the I -ball/oscillon. It should be noted that the spatial size of R does not depend on ζ , and hence, does not depend on I for given potential parameters.

The period of the oscillation can be obtained as follows. By multiplying $\dot{f}$ to Eq. (10) and integrating over time t , we obtain

²From Eq. (10), we can evaluate $\overline{\dot{f}^2}$ as

$$\overline{\dot{f}^2} = m^2(1 + \kappa - \zeta)\overline{f^2} + \kappa m^2\overline{f^2 \log f^2}. \quad (12)$$

¹The parameter M corresponds to the renormalization scale.

FIG. 1. (Left) The relation of ζ and ω for $\kappa = -0.3$. (Right) Same plot for $\kappa = -0.1$.

$$\frac{1}{2}\dot{f}^2 + \frac{1}{2}m^2(1-\zeta)f^2 + \frac{1}{2}\kappa m^2 f^2 \log f^2 = C_f, \quad (16)$$

with C_f being a constant. Because the potential of f is an even function of f , f oscillates in $f = [-1, 1]$, and hence, C_f is required to be $C_f > 0$ to allow $\dot{f}^2 > 0$ at $f = 0$. As we defined $f = \pm 1$ at the turning points of the motion, C_f is represented by

$$C_f = \frac{1}{2}m^2(1-\zeta), \quad (17)$$

with which $\dot{f} = 0$ at the turning points. As a result, we find a period of the oscillation to be

$$T \equiv \frac{2\pi}{\omega} = 2 \int_{-1}^1 \frac{df}{\sqrt{m^2(1-\zeta)(1-f^2) - \kappa m^2 f^2 \log f^2}}, \quad (18)$$

which is determined by ζ for given potential parameters. We plot the frequencies for the potential parameter $\kappa = -0.3$ and $\kappa = -0.1$ in Fig. 1. The upper bound for the range of ζ is determined from Eq. (18) with $T = \text{finite}$,

$$\zeta_{\max} = 1 + \kappa \leq 1. \quad (19)$$

Besides, the region of $\omega \geq m$ is not physically attractive,³ and hence, we consider only a closed range of ζ .

Similarly, the energy of the *I*-ball/oscillon is given by

$$E = \left(\overline{\dot{f}^2} - \frac{1}{2}\kappa m^2 \overline{f^2} \right) \int d^3x \psi(r)^2, \quad (20)$$

$$= \left(1 - \frac{1}{2}\kappa m^2 \frac{\overline{f^2}}{\overline{\dot{f}^2}} \right) \omega I, \quad (21)$$

³A more precise lower bound on ζ is determined by $dE/dI < m$.

where

$$I = \frac{\overline{\dot{f}^2}}{\omega} \int d^3x \psi(r)^2 = \frac{\overline{\dot{f}^2}}{m\omega} \left(\frac{\pi}{|\kappa|} \right)^{3/2} \frac{\psi_c^2}{m^2}. \quad (22)$$

Here, we used Eqs. (10), (3), and (12). Notice that $\overline{\dot{f}^2}$, ω , and ψ_c depend on ζ through Eqs. (12), (15), and (18). Thus, the adiabatic invariant is determined only by ζ for given model parameters κ and m .

III. STABILITY AND INSTABILITY OF THE EXACT *I*-BALL/OSCILLON

A. Stability

As shown in Ref. [25], the nonexact *I*-ball/oscillon decays by emitting relativistic radiation of the scalar field. Here, let us discuss whether the exact *I*-ball/oscillon obeys this decay process.

To see whether a relativistic radiation is emitted from the exact *I*-ball/oscillon, we consider a small perturbation around the *I*-ball/oscillon, $\phi(x) = \phi_I(x) + \xi(x)$. Here, $\phi_I(x)$ is the *I*-ball/oscillon solution obtained in the previous section. Then, the equation of motion of the perturbation $\xi(x)$ is given by

$$(\square + V'')|_{\phi=\phi_I} \xi(x) = \mathcal{O}(\xi(x)^2), \quad (23)$$

where

$$V'' = m^2 \left(1 + 3\kappa + \kappa \log \frac{\phi^2}{m^2} \right). \quad (24)$$

Equation (23) shows that the perturbation does not have source terms, and the *I*-ball/oscillon solution is stable if $\xi = 0$ initially. Thus, unlike the case of the nonexact *I*-ball/oscillon, the exact *I*-ball/oscillon does not decay by emitting relativistic radiations.

The absence of the source term of ξ stems from the fact that the equation for the *I*-ball/oscillon profile is the same as the equation of motion Eqs. (10) and (11) without

averaging over one period of the oscillation. This contrasts with generic *I*-ball/oscillon solutions. In general cases, the *I*-ball/oscillon solutions satisfy the equation of motion only after averaging over a period, and hence, the perturbation has source terms coming from the potential that disappear if averaged. This leads to the decay of the *I*-ball/oscillon as shown in Ref. [25]. These observations are consistent with the exactness of the conservation of the adiabatic invariant.

B. Fragileness

In this subsection, we discuss the instability of the exact *I*-ball/oscillon against small perturbations and derive the growth index (the Floquet exponent) of the instability. The remarkable point is that we can calculate the Floquet exponent with the oscillating background which has a nontrivial spatial configuration.

As $\kappa < 0$, the perturbation around the vacuum, i.e., $\phi = 0$, has an infinite mass, i.e., $m_\xi^2 = \infty > 0$.⁴ Thus, the perturbation around the vacuum is never excited. Around the *I*-ball/oscillon solution,

$$\phi_I(x) = f(t)\psi(r), \quad \psi(r) = \psi_c e^{-r^2/R^2}; \quad (25)$$

on the other hand, the perturbation has a finite nonderivative kernel,

$$\begin{aligned} \ddot{\xi}(x) - \nabla^2 \xi(x) + \left(2|\kappa|m^2 \frac{r^2}{R^2} + (1+3\kappa)m^2 \right. \\ \left. + \kappa m^2 \log \frac{\psi_c^2}{m^2} + F(t) \right) \xi(x) = 0, \end{aligned} \quad (26)$$

where $F(t) = \kappa m^2 \log f(t)^2$, and we neglected the $\mathcal{O}(\xi^2)$ term in Eq. (23). Thus, there could be instability modes around the *I*-ball/oscillon solution.

To analyze Eq. (26), let us remember that the eigenequation

$$(-\nabla^2 + \omega_\xi^2 r^2) \lambda_{\mathbf{n}}(\mathbf{x}) = 2E_{\mathbf{n}} \lambda_{\mathbf{n}}(\mathbf{x}) \quad (27)$$

has the eigenfunctions and eigenvalues

$$\lambda_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^3 \left(\frac{1}{n_i! 2^{n_i}} \sqrt{\frac{\omega_\xi}{\pi}} \right)^{1/2} H_{n_i}(\sqrt{\omega_\xi} x_i) e^{-\omega_\xi x_i^2/2}, \quad (28)$$

$$2E_{\mathbf{n}} = 2\omega_\xi(n_1 + n_2 + n_3 + 3/2), \quad (29)$$

⁴The mass at the origin of this potential is divergent because

$$V''(\phi) = m^2 \left[(1+3\kappa) + \kappa \log \left(\frac{\phi}{m} \right)^2 \right] \rightarrow \infty. \quad (\phi \rightarrow \pm 0)$$

where H_n denotes the Hermite polynomial of order n , and the eigenfunctions satisfy $\lambda_{\mathbf{n}}(|\mathbf{x}| \rightarrow \infty) = 0$.

By expanding $\xi(x)$ by

$$\xi(t, \mathbf{x}) = \sum_{\mathbf{n}} q_{\mathbf{n}}(t) \lambda_{\mathbf{n}}(\mathbf{x}) \quad (30)$$

we find that each $q_{\mathbf{n}}(t)$ satisfies

$$[\partial_t^2 + 2\omega_\xi(n_1 + n_2 + n_3) + \Lambda + F(t)] q_{\mathbf{n}}(t) = 0. \quad (31)$$

Here, we defined

$$\omega_\xi^2 = 2|\kappa| \frac{m^2}{R^2} = \kappa^2 m^4, \quad (32)$$

$$\Lambda = (1+3\kappa)m^2 + \kappa m^2 \log \frac{\psi_c^2}{m^2} + 3\omega_\xi, \quad (33)$$

$$= (1+3\kappa-\zeta)m^2, \quad (34)$$

where we used Eq. (15). The exponential factor in Eq. (28) is identical to that of the *I*-ball/oscillon, i.e.,

$$e^{-\omega_\xi(x_1^2+x_2^2+x_3^2)/2} = e^{-r^2/R^2}. \quad (35)$$

If Eq. (31) has growing modes with nontrivial Floquet exponents, the *I*-ball/oscillon solution can be unstable.

It should be noted that the perturbation of the zero mode, $\mathbf{n} = 0$, is redundant. As the radius of the *I*-ball/oscillon does not depend on ζ but is determined by κ and m , the addition of the zero mode perturbation just enhances the value of I , which leads to an *I*-ball/oscillon with a slightly larger I .

Because we confine ourselves to the spherical configuration in our numerical simulation, we consider the spherical perturbations to analyze the result of our simulation.

Let us rewrite Eq. (27) in the spherical coordinate

$$\begin{aligned} \left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + \omega_\xi r^2 \right) R_{n_r, \ell}(r) \\ = 2E_{n_r, \ell} R_{n_r, \ell}(r), \end{aligned} \quad (36)$$

where the eigenfunction is given by $R_{n_r, \ell}(r) Y_m^\ell(\Omega)$ with $Y_m^\ell(\Omega)$ being the spherical harmonics. By introducing $\tilde{r} = \sqrt{\omega_\xi} r$, the eigenfunction of the radial direction and the corresponding energy are given by

$$\begin{aligned} R_{n_r, \ell}(\tilde{r}) = \frac{2^{1/2} \omega_\xi^{3/2} \Gamma(n_r + \ell + 3/2)^{1/2}}{\Gamma(n_r + 1)^{1/2} \Gamma(\ell + 3/2)} \\ \times \tilde{r}^\ell e^{-\tilde{r}^2/2} {}_1F_1(-n_r; 3/2 + \ell; \tilde{r}^2), \end{aligned} \quad (37)$$

$$2E_{n_r, \ell} = 2(2n_r + \ell + 3/2), \quad (38)$$

FIG. 2. (Left) The growing mode around the I -ball with $\kappa = -0.3$ and $\zeta = 0.4$ when $q_{n_r=3}(t=0) = 0.01$. (Right) The corresponding wave function $R_{n_r=3,0}(r)$.

for $(n_r = 0, 1, 2 \dots)$ and $(\ell = 0, 1, 2 \dots)$. Here, ${}_1F_1(a; b; z)$ denotes the confluent hypergeometric function, and the wave function is normalized so that

$$\int_0^\infty dr r^2 R_{n_r, \ell}(\sqrt{\omega_\xi} r) R_{n_r', \ell}(\sqrt{\omega_\xi} r) = \delta_{n_r, n_r'}. \quad (39)$$

By expanding the perturbation by

$$\xi(t, \mathbf{x}) = \sum_{n_r, \ell, m_z} q_{n_r, \ell, m_z}(t) R_{n_r, \ell}(r) Y_\ell^{m_z}(\Omega), \quad (40)$$

the $q_{n_r, \ell, m_z}(t)$ satisfies

$$[\partial_t^2 + 2\omega_\xi(2n_r + \ell) + \Lambda + F(t)]q_{n_r, \ell, m_z}(t) = 0. \quad (41)$$

Since we are interested in the effects of the spherical perturbation, we hereafter take $\ell = 0$ and denote $q_{n_r}(t) = q_{n_r, 0, 0}(t)$.

In Fig. 2, we show an example of the instability growth of the mode with $n_r = 3$ and $q_{n_r=3}(t=0) = 0.01$ for $\kappa = -0.3$ and $\zeta = 0.4$. The figure shows that the perturbation grows for $\mathcal{O}(100)/m$. We also show the corresponding wave function $R_{n_r=3,0}(r)$. Once the perturbation grows around the I -ball/oscillon, it is expected to be dissociated and breaks up into smaller configurations.

In Fig. 3, we also show the Floquet exponent μ ,

$$\xi(t+T, \mathbf{x}) = e^{\mu T} \xi(t, \mathbf{x}), \quad (42)$$

where $T = 2\pi/\omega$ is a period of one oscillation. The figure shows that there are some resonance bands. This means that the exact I -ball/oscillon is not always a stable solution. Thus, in the presence of tiny fluctuations around the exact I -ball/oscillon, the instability modes exhibit the exponential growth in the timescale of $\mathcal{O}(\mu^{-1})$. Once the exponential growth happens, the exact I -ball/oscillon cannot keep its configuration anymore and is expected to be broken up. We will confirm these behaviors by the classical lattice simulation in the next section.

FIG. 3. (Upper) Plot of the Floquet exponent μ for the variables n_r and ζ for $\kappa = -0.3$. (Lower) Same plot for $\kappa = -0.1$. The instability bands for the spherical modes are at integer values of n_r . The instability bands for the nonspherical modes can be read off by renaming the horizontal axis from n_r to $n_r + \ell/2$.

Here, we only focus on the spherical modes, that is, $\ell = 0$. However, Fig. 2 is also applicable to the nonspherical modes when we change n_r to $n_r + \ell/2$. Because $n_r + \ell/2$ can take a noninteger value unlike the spherical modes, the *I*-ball/oscillon should be more fragile.

IV. NUMERICAL SIMULATION

A. Setup

In this subsection we briefly explain the setup of the simulation. The procedure of the simulation is similar to that of Ref. [25]. Because the lowest energy configuration of the scalar field ϕ is spherically symmetric in three-dimensional space, the equation of motion of ϕ is represented by

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \frac{\partial V}{\partial \phi}, \quad (43)$$

where

$$V(\phi) = m^2 \left[1 + \kappa \log \left(\epsilon + \frac{\phi^2}{m^2} \right) \right] \phi^2. \quad (44)$$

Here, we introduced a small parameter ϵ to avoid the numerical instability caused by the singularity of the effective mass at $\phi = 0$. We have confirmed that the simulation results are independent of this small regularization term ϵ .

For the boundary condition, we use the two following conditions.

- (i) At the origin $r = 0$, to avoid the divergence of the second term of the right-hand side of Eq. (43), we impose

$$\frac{1}{r} \frac{d\phi}{dr} \Big|_{r=0} = 0. \quad (45)$$

- (ii) At the other boundary $r = L (\gg R)$, we impose the absorbing boundary condition (see the Appendix for details). Under this condition, the radiation of the real scalar field emitted from the *I*-ball/oscillon is absorbed at the boundary so that we correctly calculate the time evolution of the *I*-ball/oscillon.

As the initial condition of ϕ , we use the theoretical *I*-ball/oscillon configuration Eqs. (13)–(15) for a given ζ_{ini} with 1% random fluctuations. We also set

$$\dot{\phi}(t = 0, r) = 0, \quad (46)$$

as an initial condition of $\dot{\phi}$. We have confirmed that the exact *I*-ball/oscillon is completely stable in the absence of the random fluctuations.

TABLE I. Simulation parameters. ζ_{ini} is changed in every simulation to set the appropriate initial profile of the exact *I*-ball/oscillon.

ζ_{ini}	Varying
κ	−0.3
ϵ	10^{-10}
Box size L	64
Grid size N	1024
Initial time	0
Final time	1.0×10^6
Time step	2.0×10^{-3}

The other simulation parameters are shown in Table I. Here, the units of the field, time, space, etc., are taken to be m^{-1} , that is,

$$\phi \rightarrow m\phi, \quad t \rightarrow \frac{t}{m}, \quad x \rightarrow \frac{x}{m}, \dots \text{etc.} \quad (47)$$

We utilize the same lattice simulation code in Ref. [25], in which the time evolution is calculated by the fourth-order symplectic integration scheme and the spatial derivatives are calculated by the fourth-order central difference scheme.

B. Result

We numerically calculate the time evolution of the exact *I*-ball/oscillon ϕ and its energy E from

$$E(t) = \int_0^L dr 4\pi r^2 \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + V \right], \quad (48)$$

to see how it behaves in the presence of the instability modes derived in Sec. III.

Stability.—First, we show the result of the stable exact *I*-ball/oscillon which has no strong instability resonance bands. The result is given in Fig. 4, which shows that the exact *I*-ball/oscillation does not break up even in the presence of the tiny fluctuation. This result is consistent with the fact the exact *I*-ball/oscillon for $\zeta_{\text{ini}} = 0.3, 0.2$ does not have the instability modes.

Let us briefly comment on the nonspherical modes here. In the nonspherical configuration, the Floquet exponent is the same as Fig. 2 with taking n_r to $n_r + \ell/2$. In this case, for example, there is an instability band at $n_r + \ell/2 = 1.5$, i.e., $(n_r, \ell) = (1, 1)$ for $\kappa = -0.3$ and $\zeta = 0.3$. Such an instability bound should result in a dissociation of the *I*-ball/oscillon with lifetime of $\mathcal{O}(100 - 1000)/m$.

Fragileness.—Next, we show the result of the unstable exact *I*-ball/oscillon. The results of the simulations are shown in Fig. 5.⁵ Comparing the result with our analytical

⁵We have confirmed that the exact *I*-ball/oscillon for the given ζ_{ini} without initial fluctuations is stable within simulation time.

FIG. 4. The result of our simulations for the stable exact *I*-ball/oscillon. The blue and the orange lines show the exact *I*-ball/oscillon energy [defined by Eq. (48)] for $\zeta_{\text{ini}} = 0.3$ and $\zeta_{\text{ini}} = 0.2$, respectively. The energy suddenly changes at $mt \simeq 60$ because the radiated fluctuations have reached the boundary and start to be absorbed at that time. We find that the exact *I*-ball/oscillon energy is conserved in $mt \lesssim 10^6$ when the exact *I*-ball/oscillon does not have instability modes around it.

calculation (see Fig. 3), we find that the energy of the exact *I*-ball/oscillon strikingly decreases at around instability bands shown as green color bands in Fig. 5. This can be interpreted that the initial fluctuations appended to the exact *I*-ball/oscillon grow exponentially and deform the exact *I*-ball/oscillon profile.⁶

The figure also shows that the timescale of the dissociation is given by

$$t \sim 1/\mu. \quad (49)$$

This confirms that the dissociation process is caused by the exponential growth of the instability mode discussed in Sec. IV. It should be also noted that the dependence of the dissociation timescale on the initial amplitude of the fluctuations around the *I*-ball/oscillon is logarithmic. Thus, the dissociation timescale is not sensitive to the initial amplitude for $\mu = \mathcal{O}(10^{-3}) \times m$.

Our results also suggest that the exact *I*-ball/oscillon ends up with another exact *I*-ball/oscillon profile with a smaller *I* after the decay process. In fact, both the cases in Fig. 5 converge to the smaller but finite energy at $mt \gtrsim 10^6$. The case with $\zeta_{\text{ini}} = 0.4$ is particularly suggestive. In this case, the *I*-ball/oscillon first decays when the instability modes in the narrower band around $\zeta = 0.4$ and $n_r = 3$

⁶Instability bands should be wide and strong enough for the oscillon decay.

FIG. 5. The result of our simulations for the unstable exact *I*-ball/oscillon, which hit the instability bands. The blue and the orange lines show the exact *I*-ball/oscillon energy [defined by Eq. (48)] for $\zeta_{\text{ini}} = 0.4$ and $\zeta_{\text{ini}} = 0.15$, respectively. We also plot the typical instability bands as the Floquet exponent μ exhibited in Fig. 3 by green thick lines. The upper dark green line and the lower bright green line correspond to the instability bands at $\zeta \simeq 0.4$, $n_r = 3$ and $\zeta \simeq 0.1$, $n_r = 2$ in Fig. 3. The sudden change of the energy at $mt \simeq 60$ is for the same reason as the case of Fig. 4. We find that the exact *I*-ball/oscillon energy strikingly decreases around the instability bands.

grows, and ends up with the *I*-ball/oscillon with $\zeta \simeq 0.15$. However, there is a broader instability band for $\zeta \simeq 0.15$ and $n_r = 2$, with which the exact *I*-ball/oscillon further decays very quickly. As a result, the exact *I*-ball/oscillon profile for $\zeta_{\text{ini}} = 0.4$ converges to the smaller *I*-ball/oscillon solution.

In our analysis, we only consider the radial modes of the fluctuation. The exact *I*-ball/oscillon can have more instability bands in the three-dimensional case [see Eqs. (31) and (41)].

V. CONCLUSIONS

In this paper, we examine the stability of the exact *I*-ball/oscillon. The exact *I*-ball/oscillon has been considered to be stable since it has the exactly conserved adiabatic invariant. Its stability is also expected because the perturbations around the exact *I*-ball/oscillon obey the field equation without the source terms.

However, we have found that the exact *I*-ball/oscillon is not always a stable solution and the perturbation around it has growth modes depending on the size of the adiabatic invariant. Thus, in the presence of the fluctuations with the corresponding instability modes, the exact *I*-ball/oscillon cannot keep its configuration anymore and breaks up eventually. Note that the Floquet analysis is generally very

difficult in a background oscillating in time and varying in space.

The mechanism of the exact I -ball/oscillon decay in this paper is completely different from the previous study [25], in which the I -ball/oscillon decays by emitting relativistic radiations of the scalar field. Our results in this paper suggest that it is necessary to consider both decay processes, the decay by radiation and the decay by the instability modes, when we estimate the lifetime of a generic I -ball/oscillon.

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APPENDIX ABSORBING BOUNDARY CONDITION

In this Appendix, we briefly explain the Absorbing boundary condition (ABC) based on Refs. [20,34]. Note that the result of this section is slightly different from the one in Ref. [20] because Eq. (A9) in Ref. [20] may contain a typo of the sign.

We consider the single real scalar field ϕ for simplicity. The equation of motion of ϕ is written as

$$\ddot{\phi} - \Delta\phi + \frac{\partial V}{\partial\phi} = 0, \quad (\text{A1})$$

where V is an arbitrary potential of ϕ . Assuming that the field ϕ can be expanded around the minimum $\phi = v$ as

$$\phi = v + \varphi, \quad (\varphi \ll v), \quad (\text{A2})$$

where

$$\left. \frac{\partial V}{\partial\phi} \right|_{\phi=v} = 0, \quad (\text{A3})$$

the equation of motion of ϕ is rewritten as

$$\ddot{\varphi} - \Delta\varphi + m_{\text{eff}}^2\varphi \simeq 0, \quad (\text{A4})$$

where

$$\left. \frac{\partial^2 V}{\partial\phi^2} \right|_{\phi=v} \equiv m_{\text{eff}}^2. \quad (\text{A5})$$

We ignore the higher terms of φ because we consider φ is much smaller than v .⁷

Assuming the spherical symmetry in three dimensions $\varphi = \varphi(r)$, the equation of motion becomes

$$\ddot{\varphi} - \frac{\partial^2 \varphi}{\partial r^2} - \frac{2}{r} \frac{\partial \varphi}{\partial r} + m_{\text{eff}}^2 \varphi \simeq 0. \quad (\text{A9})$$

Performing Fourier transformation of φ as t ,

$$\varphi(t, r) = \int d\omega \varphi(\omega, r) e^{-i\omega t}, \quad (\text{A10})$$

the equations of motion becomes

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + (\omega^2 - m_{\text{eff}}^2) \right] \varphi = 0, \quad (\text{A11})$$

$$\Leftrightarrow \left[\frac{\partial}{\partial r} - \left(i\sqrt{\omega^2 - m_{\text{eff}}^2} - \frac{1}{r} \right) \right] \times \left[\frac{\partial}{\partial r} + \left(i\sqrt{\omega^2 - m_{\text{eff}}^2} - \frac{1}{r} \right) \right] \varphi = 0. \quad (\text{A12})$$

The special solution of this equation at $m_{\text{eff}}r \gg 1$ is the Bessel function and

$$\varphi(t, r) = \frac{1}{r} \exp \left[i \left(\pm \sqrt{\omega^2 - m_{\text{eff}}^2} r + \omega t \right) \right]. \quad (\text{A13})$$

The reflected wave must be disappeared at the boundary $r = r_b$ ($m_{\text{eff}}r \gg r_b$),

$$\left[\frac{\partial}{\partial r} - \left(i\sqrt{\omega^2 - m_{\text{eff}}^2} - \frac{1}{r} \right) \right] \Big|_{r=r_b} \varphi = 0. \quad (\text{A14})$$

Expanding the square root,

$$\left[\frac{\partial}{\partial r} - i\omega \left(1 - \frac{1}{2} \frac{m_{\text{eff}}^2}{\omega^2} + \mathcal{O} \left(\left(\frac{\omega}{m_{\text{eff}}} \right)^4 \right) \right) + \frac{1}{r} \right] \varphi = 0, \quad (\text{A15})$$

$$\Leftrightarrow \left[i\omega \frac{\partial}{\partial r} + \left(\omega^2 - \frac{1}{2} m_{\text{eff}}^2 \right) + \frac{i\omega}{r} \right] \varphi \simeq 0. \quad (\text{A16})$$

⁷The potential of ϕ is expanded as

$$V(\phi) = V(v + \varphi), \quad (\text{A6})$$

$$= V(v) + \left. \frac{\partial V}{\partial\phi} \right|_{\phi=v} \varphi + \frac{1}{2} \left. \frac{\partial^2 V}{\partial\phi^2} \right|_{\phi=v} \varphi^2 + \mathcal{O}(\varphi^3), \quad (\text{A7})$$

$$= V(v) + \frac{1}{2} \left. \frac{\partial^2 V}{\partial\phi^2} \right|_{\phi=v} \varphi^2 + \mathcal{O}(\varphi^3). \quad (\text{A8})$$

Substituting $i\omega \rightarrow -\frac{\partial}{\partial t}$

$$\left[\frac{\partial}{\partial r} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} + \frac{1}{2} m_{\text{eff}}^2 + \frac{1}{r} \frac{\partial}{\partial t} \right] \varphi = 0, \quad (\text{A17})$$

Therefore, $\pi = \dot{\varphi}$ obeys at the boundary $r = r_b$

$$\dot{\pi}(t, r_b) = -\left. \frac{\partial \pi(t, r)}{\partial r} \right|_{r=r_b} - \frac{1}{2} m_{\text{eff}}^2 \varphi(t, r_b) - \frac{1}{r_b} \pi(t, r_b). \quad (\text{A18})$$

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- [1] I. L. Bogolyubsky and V. G. Makhankov, *Pis'ma Zh. Eksp. Teor. Fiz.* **24**, 15 (1976).
 - [2] M. Gleiser, *Phys. Rev. D* **49**, 2978 (1994).
 - [3] E. J. Copeland, M. Gleiser, and H. R. Muller, *Phys. Rev. D* **52**, 1920 (1995).
 - [4] J. McDonald, *Phys. Rev. D* **66**, 043525 (2002).
 - [5] M. A. Amin and D. Shirokoff, *Phys. Rev. D* **81**, 085045 (2010).
 - [6] M. A. Amin, R. Easter, H. Finkel, R. Flauger, and M. P. Hertzberg, *Phys. Rev. Lett.* **108**, 241302 (2012).
 - [7] M. A. Amin, *Phys. Rev. D* **87**, 123505 (2013).
 - [8] N. Takeda and Y. Watanabe, *Phys. Rev. D* **90**, 023519 (2014).
 - [9] K. D. Lozanov and M. A. Amin, *Phys. Rev. Lett.* **119**, 061301 (2017).
 - [10] F. Hasegawa and J.-P. Hong, *Phys. Rev. D* **97**, 083514 (2018).
 - [11] S. Antusch, F. Cefala, S. Krippendorff, F. Muia, S. Orani, and F. Quevedo, *J. High Energy Phys.* **01** (2018) 083.
 - [12] J.-P. Hong, M. Kawasaki, and M. Yamazaki, *Phys. Rev. D* **98**, 043531 (2018).
 - [13] M. Kawasaki, F. Takahashi, and N. Takeda, *Phys. Rev. D* **92**, 105024 (2015).
 - [14] G. Fodor, P. Forgacs, P. Grandclement, and I. Racz, *Phys. Rev. D* **74**, 124003 (2006).
 - [15] G. Fodor, P. Forgacs, Z. Horvath, and M. Mezei, *Phys. Rev. D* **79**, 065002 (2009).
 - [16] M. Gleiser and D. Sicilia, *Phys. Rev. Lett.* **101**, 011602 (2008).
 - [17] G. Fodor, P. Forgacs, Z. Horvath, and M. Mezei, *Phys. Lett. B* **674**, 319 (2009).
 - [18] M. Gleiser and D. Sicilia, *Phys. Rev. D* **80**, 125037 (2009).
 - [19] M. P. Hertzberg, *Phys. Rev. D* **82**, 045022 (2010).
 - [20] P. Salmi and M. Hindmarsh, *Phys. Rev. D* **85**, 085033 (2012).
 - [21] P. M. Saffin, P. Tognarelli, and A. Tranberg, *J. High Energy Phys.* **08** (2014) 125.
 - [22] M. Kawasaki and M. Yamada, *J. Cosmol. Astropart. Phys.* **02** (2014) 001.
 - [23] K. Mukaida, M. Takimoto, and M. Yamada, *J. High Energy Phys.* **03** (2017) 122.
 - [24] J. Eby, K. Mukaida, M. Takimoto, L. C. R. Wijewardhana, and M. Yamada, *Phys. Rev. D* **99**, 123503 (2019).
 - [25] M. Ibe, M. Kawasaki, W. Nakano, and E. Sonomoto, *J. High Energy Phys.* **04** (2019) 030.
 - [26] J. Ollé, O. Pujolàs, and F. Rompineve, *arXiv:1906.06352*.
 - [27] S. Antusch, F. Cefalà, and F. Torrentí, *J. Cosmol. Astropart. Phys.* **10** (2019) 002.
 - [28] S.-Y. Zhou, E. J. Copeland, R. Easter, H. Finkel, Z.-G. Mou, and P. M. Saffin, *J. High Energy Phys.* **10** (2013) 026.
 - [29] S. Antusch, F. Cefala, and S. Orani, *Phys. Rev. Lett.* **118**, 011303 (2017); **120**, 219901(E) (2018).
 - [30] K. D. Lozanov and M. A. Amin, *Phys. Rev. D* **99**, 123504 (2019).
 - [31] J. Eby, M. Leembruggen, L. Street, P. Suranyi, and L. C. R. Wijewardhana, *Phys. Rev. D* **100**, 063002 (2019).
 - [32] S. Kasuya, M. Kawasaki, and F. Takahashi, *Phys. Lett. B* **559**, 99 (2003).
 - [33] K. Mukaida and M. Takimoto, *J. Cosmol. Astropart. Phys.* **08** (2014) 051.
 - [34] B. Engquist and A. Majda, *Math. Comput.* **31**, 629 (1977).