Seiberg-Witten geometry of four-dimensional $\mathcal{N} = 2$ SO–USp quiver gauge theories

Xinyu Zhang [*]

New High Energy Theory Center and Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08854-8019, USA

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We apply the instanton counting method to study a class of four-dimensional $\mathcal{N} = 2$ supersymmetric quiver gauge theories with alternating SO and USp gauge groups. We compute the partition function in the Ω-background and express it as functional integrals over density functions. Applying the saddle point method, we derive the limit shape equations which determine the dominant instanton configurations in the flat space limit. The solution to the limit shape equations gives the Seiberg-Witten geometry of the low energy effective theory. As an illustrating example, we work out explicitly the Seiberg-Witten geometry for linear quiver gauge theories. Our result matches the Seiberg-Witten solution obtained previously using the method of brane constructions in string theory.

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I. INTRODUCTION

Four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are an extremely interesting playground for studying nonperturbative dynamics of quantum field theories. Following the paradigm of Seiberg and Witten [1,2], we can solve exactly the topological sector of the theory, including the Wilsonian low energy effective prepotential $\mathcal{F}$ and the correlation functions of gauge-invariant chiral operators. These quantities receive perturbative corrections only at one-loop order, while the nonperturbative corrections are entirely from instantons. The solution is encoded in the data of a family of complex algebraic curves Σ_u fibered over the Coulomb moduli space $\mathcal{B}$, with a meromorphic differential λ.

When the theory admits a microscopic Lagrangian description, a purely field theoretical algorithm for the derivation of the Seiberg-Witten solution using the multi-instanton calculus was proposed in [3], and no conjectured dualities are assumed. In order to introduce a supersymmetric regulator of the infinite volume of spacetime and also to simplify the evaluation of the path integral, the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory is formulated in the Ω-background, which is a particular supergravity background with two deformation parameters ε_1, ε_2. The Poincaré symmetry of $\mathbb{R}^4$ is broken in a rotationally covariant way. Applying the technique of equivariant localization, the partition function $\mathcal{Z}$ in the Ω-background can be written as a statistical sum over special instanton configurations. In the flat space limit $\varepsilon_1, \varepsilon_2 \to 0$, where the theory in the Ω-background approaches the original flat space theory, the partition function $\mathcal{Z}$ is dominated by a particular instanton configuration, the so-called limit shape, with the instanton charge $\sim \frac{1}{\varepsilon_1 \varepsilon_2}$. Based on field theoretical arguments [3,4], the Seiberg-Witten prepotential $\mathcal{F}$ of the low energy effective theory can be extracted from the partition function $\mathcal{Z}$ in the following limit,

$$\mathcal{F} = -\lim_{\varepsilon_1,\varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log \mathcal{Z}(\varepsilon_1, \varepsilon_2). \tag{1}$$

This approach has hitherto been used to derive the Seiberg-Witten solution for gauge theories with a simple classical gauge group [4–6], and SU(N) quiver gauge theories with hypermultiplets in the fundamental, adjoint or bifundamental representations [7]. However, these are far from all the $\mathcal{N} = 2$ supersymmetric gauge theories for which we are able to compute the partition function in the Ω-background. It is certainly interesting to cover all the possible cases and test the validity of this approach.

In this paper, we will be dealing with mass-deformed four-dimensional $\mathcal{N} = 2$ superconformal quiver gauge theories with alternating SO and USp gauge groups. Restricting ourselves to superconformal theories does not result in a loss of generality, since asymptotically free theories can always be obtained from superconformal theories by taking proper scaling limits and decoupling a number of fundamental hypermultiplets. Generalizing previous computations for a single SO or USp gauge group

[*] zhangxinyuphysics@gmail.com

[5,8–10], we can compute the partition function in the Ω-background for SO − USp quiver gauge theories. More precisely, we write the partition function in terms of contour integrals, and it is not necessary to give the prescription for choosing the correct poles. One important ingredient in the computation is the treatment of half-hypermultiplets. Although we cannot compute their contributions to the partition function directly, we will follow the conjecture made in [11,12] that the contribution of a half-hypermultiplet is given by the square-root of the contribution of a massless full hypermultiplet composed of a pair of half-hypermultiplets. Similar to the SU quiver gauge theories, the limit shape equations give the gluing conditions of the amplitude functions. The Seiberg-Witten geometry is finally derived by constructing the functions invariant under the instanton Weyl group [7]. As a representative example, we will describe in great detail the Seiberg-Witten geometry for linear quiver gauge theories. Our result matches the Seiberg-Witten solutions obtained previously [13–19].

The rest of the paper is organized as follows. In Sec. II we describe the four-dimensional $\mathcal{N}=2$ SO − USp superconformal quiver gauge theories we are dealing with. We introduce a biparticle quiver diagram to represent the theory. In Sec. III we compute explicitly the partition function in the Ω-background. In the flat space limit, we rewrite the partition function as a functional integral over density functions. In Sec. IV, we apply the saddle point method to determine the special instanton configuration which dominates the partition function in the flat space limit ε_1, $\varepsilon_2 \to 0$. We solve the limit shape equations by constructing the characters invariant under the instanton Weyl group. We write down the Seiberg-Witten curve using the characters. In Sec. V, we describe explicitly the Seiberg-Witten geometry for linear SO − USp quiver gauge theories. In Sec. VI we sketch some possible further developments of our work. In Appendix, we review the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instanton moduli space for all classical gauge groups.

II. SO − USp QUIVER GAUGE THEORIES

A major subtlety of SO − USp quiver gauge theories compared with SU quiver gauge theories is the appearance of half-hypermultiplets. Recall that the $\mathcal{N}=2$ hypermultiplet in the representation R of the gauge group G consists of a pair of $\mathcal{N}=1$ chiral multiplets, one in the representation R and another in the conjugate representation R^* of G. If the representation R is pseudoreal, a single $\mathcal{N}=1$ chiral superfield forms a consistent $\mathcal{N}=2$ multiplet, the half-hypermultiplet. From the representation theory of Lie groups $\mathrm{SO}(N)$ and $\mathrm{USp}(2N)$, we know that the fundamental representation of $\mathrm{SO}(N)$ is strictly real, while the fundamental representation of $\mathrm{USp}(2N)$ is pseudoreal. The bifundamental representation of $\mathrm{SO}(N_1)\times\mathrm{USp}(2N_2)$, which is the tensor product of the fundamental representation of $\mathrm{SO}(N_1)$ and the fundamental representation of $\mathrm{USp}(2N_2)$, is also pseudoreal. When we couple an $\mathrm{SO}(N)$ vector multiplet to N_f fundamental hypermultiplets, the flavor symmetry group is $\mathrm{USp}(2N_f)$, and the gauge coupling constant is marginal when $N_f = N-2$. Meanwhile, when we couple an $\mathrm{USp}(2N)$ vector multiplet to N_f fundamental half-hypermultiplets, the flavor symmetry group is $\mathrm{SO}(N_f)$, and the gauge coupling constant is marginal when $N_f = 4N+4$. Therefore, there is a natural way to construct superconformal quiver gauge theories with alternating SO and USp gauge groups. We certainly cannot avoid half-hypermultiplets in such SO − USp quiver gauge theories.

We represent such an SO − USp quiver gauge theory by a bipartite quiver diagram γ, which consists of vertices which are colored either black or white and edges connecting vertices of different colors. The set of vertices is denoted by $V_\gamma = V_\gamma^{\bigcirc} \cup V_\gamma^{\bullet}$, where each vertex $i \in V_\gamma^{\bigcirc/\bullet}$ is associated with a vector multiplet with SO/USp gauge group G_i. The total gauge group of the quiver gauge theory is

$$G = \prod_{i\in V_\gamma} G_i = \cdots \times \mathrm{SO}(v_i = 2n_i + \mu_i) \times \mathrm{USp}(v_{i+1} - 2 = 2n_{i+1}) \times \cdots, \tag{2}$$

with $\mu_i \in \{0,1\}$. We also define $\mu_i = 0$ for all $i \in V_\gamma^{\bullet}$. The microscopic gauge coupling constant g_i and the theta angle ϑ_i are combined into the complexified gauge couplings,

$$\tau_i = \frac{\vartheta_i}{2\pi} + \frac{4\pi\mathrm{i}}{g_i^2}. \tag{3}$$

We denote the collection of instanton counting parameters by

$$\underline{\mathfrak{q}} = \bigcup_{i\in V_\gamma} \{\mathfrak{q}_i = e^{2\pi\mathrm{i}\tau_i}\}. \tag{4}$$

An edge $e = \langle i,j\rangle$ connecting a vertex $i \in V_\gamma^{\bigcirc}$ with a vertex $j \in V_\gamma^{\bullet}$ represents a half-hypermultiplet in the bifundamental representation of $\mathrm{SO}(v_i)\times\mathrm{USp}(v_{j+1}-2)$. The set of all edges is denoted by E_γ. Unlike the SU quiver gauge theories, the edges are not oriented. For simplicity, we assume that there is no edge connecting a vertex to itself. In particular, in our analysis we disregard the $\mathcal{N}=2^*$ theory, which can be treated separately.

We also couple $w_i \in \mathbb{Z}_{\geq 0}$ fundamental hypermultiplets to the gauge group G_i, and additionally $\xi_i \in \{0,1\}$ fundamental half-hypermultiplets to the gauge group $\mathrm{USp}(2n_i)$. The vanishing of the one-loop beta functions of coupling constants leads to

$$2v_i = 2w_i + (4 - 2\sigma_i) + \sum_{\langle i,j\rangle\in E_\gamma} v_j, \qquad i \in V_\gamma^{\bigcirc},$$
$$2v_j = 2w_j + \xi_j + \sum_{\langle i,j\rangle\in E_\gamma} v_i, \qquad j \in V_\gamma^{\bullet}, \tag{5}$$

where σ_i is the number of edges $\langle i,j\rangle \in E_\gamma$ for the fixed $i \in V_\gamma^{\bigcirc}$. Since we always make sure that the number of half-hypermultiplets for each USp gauge group is even, the theory is safe under Witten's global anomaly [20]. The condition (5) can be solved in a similar way as the SU quiver gauge theories. Following the notation in [7], we have the following classification:

(1) Class I theories. The quiver γ is the simply laced Dynkin diagram of the Lie algebra A_r, D_r or $E_{6,7,8}$. For A_r-type quivers, we have two possible choices of coloring for the biparticle diagram, depending on whether the first vertex is SO or USp gauge group. For the D_r-type and $E_{6,7,8}$-type quivers, we also have two different choices of coloring, depending on the gauge group at the trivalent vertex.

(2) Class II theories. The quiver γ is the simply laced affine Dynkin diagram of the affine Lie algebra $\hat{A}_r$, $\hat{D}_r$ or $\hat{E}_{6,7,8}$. For $\hat{A}_r$-type quivers, we have the consistency condition which requires that r should be an odd positive integer. There is no preferred choice of the first vertex, and we can always fix the first vertex to be a SO gauge group. For the $\hat{D}_r$-type and the $\hat{E}_{6,7,8}$-type quivers, depending on the choices of coloring, we again have two sub-types according to the gauge groups at the trivalent vertices. Notice that there is no class II* theories as in the SU quiver gauge theories (except for the $N=2^*$ theories which we neglect).

(3) Class III theories. There are some extra bizarre theories with non-Dynkin type quivers. Such theories have to be discussed case by case. See [21] for a complete list.

We consider the low energy effective theory on the Coulomb branch $\mathcal{B}$ of the moduli space. The coordinates on the Coulomb branch $\mathcal{B}$ are given by the vacuum expectation values of the gauge-invariant polynomials of the scalars ϕ_i in the vector multiplet,

$$\underline{u} = \bigcup_{i\in V_\gamma} \{u_{i,s}, s=1,\ldots,n_i\}. \tag{6}$$

In the weakly coupled regime, the vacuum expectation values of ϕ_i parametrize the Coulomb branch $\mathcal{B}$ locally,

$$\begin{aligned}\underline{a} = &\bigcup_{i\in V_\gamma^{\bigcirc}} \{a_i = \langle\phi_i\rangle = \mathrm{diag}\{a_{i,1}, -a_{i,1}, \ldots, a_{i,n_i}, -a_{i,n_i}, (0)\}\} \\ &\cup \bigcup_{j\in V_\gamma^{\bullet}} \{a_j = \langle\phi_j\rangle = \mathrm{diag}\{a_{j,1},\ldots,a_{j,n_j}, -a_{j,1},\ldots,-a_{j,n_i}\}\},\end{aligned} \tag{7}$$

where (0) is absent for $\mu_i = 0$. We also turn on generic mass deformations for the fundamental hypermultiplets. Notice that a single half-hypermultiplet does not allow a gauge invariant mass term and must be massless. We collectively denote the set of masses as

$$\underline{m} = \bigcup_{i\in V_\gamma} \{m_i = \mathrm{diag}\{m_{i,1},\ldots,m_{i,w_i}\}\}. \tag{8}$$

It is convenient to encode the Coulomb moduli a_i and masses m_i in the characters of two vector space $\mathbf{N}_i$ and $\mathbf{M}_i$ assigned for each vertex $i\in V_\gamma$,

$$\mathcal{N}_i = \mathrm{ch}(\mathbf{N}_i) = \sum_\alpha (e^{\mathrm{i}a_{i,\alpha}} + e^{-\mathrm{i}a_{i,\alpha}}) + \mu_i, \tag{9}$$

$$\mathcal{M}_i = \mathrm{ch}(\mathbf{M}_i) = \sum_f e^{\mathrm{i}m_{i,f}}. \tag{10}$$

III. PARTITION FUNCTION OF QUIVER GAUGE THEORIES IN THE Ω-BACKGROUND

In this section, we compute the partition function of SO − USp quiver gauge theories in the Ω-background. The partition function contains not only the information of the Seiberg-Witten low energy effective action on $\mathbb{R}^4$, but also the low energy effective couplings of the theory to supergravity background. For the purpose of this paper, we keep only the relevant information of the partition function in the flat space limit and rewrite the partition function as functional integrals over density functions.

It is useful to introduce the following notations. The conversion operator ϵ is defined to map characters into weights,

$$\epsilon\left\{\sum_i n_i e^{x_i}\right\} = \prod_i x_i^{n_i}. \tag{11}$$

When the number of terms is infinite, we adopt the regularization via the analytic continuation,

$$\begin{aligned}\epsilon\left\{\sum_i n_i e^{x_i}\right\} = \exp\Bigg(-\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)}\int_0^\infty \frac{d\beta}{\beta}\beta^s \\ \times\left(\sum_i n_i e^{-\beta x_i}\right)\Bigg).\end{aligned} \tag{12}$$

The dual operator $\vee$ is used to flip of sign of the weights

$$\left(\sum_i n_i e^{x_i}\right)^\vee = \left(\sum_i n_i e^{-x_i}\right). \tag{13}$$

The scaling operator $[p]$ is used to scale the weights,

$$\left(\sum\nolimits_i n_i e^{x_i}\right)^{[p]} = \left(\sum_i n_i e^{p x_i}\right). \tag{14}$$

We denote the Ω-deformation parameters as ε_1, ε_2, and define

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \qquad \varepsilon_\pm = \frac{\varepsilon_1 \pm \varepsilon_2}{2}. \tag{15}$$

We also introduce

$$q_1 = e^{\mathrm{i}\varepsilon_1}, \qquad q_2 = e^{\mathrm{i}\varepsilon_2}, \qquad q_\pm = e^{\mathrm{i}\varepsilon_\pm},$$
$$\mathcal{P} = (q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})(q_2^{\frac{1}{2}} - q_2^{-\frac{1}{2}}). \tag{16}$$

Notice that the definition of $\mathcal{P}$ is different from the standard definition in the SU quiver gauge theories.

A. Instanton partition function

A four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in the Ω-background preserves a supercharge $\mathcal{Q}$, with $\mathcal{Q}^2$ being a sum of the constant gauge transformation acting on the framing at infinity, the automorphism transformation of hypermultiplets, and the spacetime rotation. Hence in the twisted formulation of the theory $\mathcal{Q}$ becomes the equivariant differential, with the equivariant group being the product of the gauge group G, the flavor group G_F, and the rotation group SO(4). Let $\mathbb{T}$ be the maximal torus of the equivariant group, with $(\underline{a}, \underline{m}; \varepsilon_1, \varepsilon_2)$ being the coordinates on the complexified Lie algebra of $\mathbb{T}$. It can be shown using the supersymmetric localization principle that the infinite-dimensional path integral is reduced to finite-dimensional equivariant integrals over the moduli space of framed instantons,

$$\mathfrak{M} = \{A \in \mathcal{A} | F^+ = 0\}/\mathcal{G}_\infty, \tag{17}$$

where $\mathcal{A}$ is the space of connections of principal bundles over $\mathbb{R}^4$, and $\mathcal{G}_\infty$ denotes the group of frame-preserving gauge transformations. For the quiver gauge theories, $\mathfrak{M}$ is factorized as

$$\mathfrak{M} = \bigsqcup_{\underline{k}} \mathfrak{M}_{G,\underline{k}} = \bigsqcup_{\underline{k}} \left(\prod_{i \in V_\gamma} \mathfrak{M}_{G_i,k_i}\right), \tag{18}$$

where we label the instanton charges by

$$\underline{k} = \left\{k_i = \begin{Bmatrix} \kappa_i, & i \in V_\gamma^{\bigcirc} \\ 2\kappa_i + \nu_i, & i \in V_\gamma^{\bullet} \end{Bmatrix}\right\} \in \mathbb{Z}_{\geq 0}^{|\gamma|}, \tag{19}$$

and $\mathfrak{M}_{G_i,k_i}$ is the moduli spaces of framed G_i-instantons with instanton charge k_i (see the Appendix for a review). Then the instanton partition function can be written as

$$\mathcal{Z}^{\text{inst}}(\underline{\mathsf{q}}; \underline{a}, \underline{m}; \varepsilon_1, \varepsilon_2) = \sum_{\underline{k}} \underline{\mathsf{q}}^{\underline{k}} \int_{\mathfrak{M}_{G,\underline{k}}} \mathrm{e}_{\mathbb{T}}(\mathcal{E}_\gamma), \tag{20}$$

where

$$\underline{\mathsf{q}}^{\underline{k}} = \prod_{i \in V_\gamma^{\bigcirc}} \mathsf{q}_i^{\kappa_i} \prod_{i \in V_\gamma^{\bullet}} \mathsf{q}_i^{\kappa_i + \nu_i/2}, \tag{21}$$

and the integration measure $\mathrm{e}_{\mathbb{T}}(\mathcal{E}_\gamma)$ is the $\mathbb{T}$-equivariant Euler class of the matter bundle $\mathcal{E}_\gamma \to \mathfrak{M}_{G,\underline{k}}$ whose fiber is the space of the virtual zero modes for the Dirac operator in the instanton background. According to the Atiyah-Singer equivariant index formula, we have

$$\mathrm{e}_{\mathbb{T}}(\mathcal{E}_\gamma) = \epsilon\left\{-\int_{\mathbb{C}^2} \mathrm{ch}_{\mathbb{T}}(\mathcal{E}_\gamma)\hat{A}(\mathbb{C}^2)\right\} = \epsilon\left\{-\frac{\iota_0^* \mathrm{ch}_{\mathbb{T}}(\mathcal{E}_\gamma)}{\mathcal{P}}\right\}, \tag{22}$$

where ι_0^* is the pull-back homomorphism induced by the inclusion $\iota_0\colon 0 \times \mathfrak{M}_{G,\underline{k}} \to \mathbb{C}^2 \times \mathfrak{M}_{G,\underline{k}}$. Applying the Atiyah-Bott equivariant localization formula we can further reduce the equivariant integration over $\mathfrak{M}_{G,\underline{k}}$ to a discrete sum over the set $\mathfrak{M}_{G,\underline{k}}^{\mathbb{T}}$ of $\mathbb{T}$-fixed points on $\mathfrak{M}_{G,\underline{k}}$,

$$\mathcal{Z}^{\text{inst}}(\underline{\mathsf{q}}; \underline{a}, \underline{m}; \varepsilon_1, \varepsilon_2) = \sum_{\underline{k}} \underline{\mathsf{q}}^{\underline{k}} \sum_{p \in \mathfrak{M}_{G,\underline{k}}^{\mathbb{T}}} \frac{1}{\mathrm{e}_{\mathbb{T}}(T_p \mathfrak{M}_{G,\underline{k}})} \epsilon\left\{-\frac{\iota_{(0,p)}^* \mathrm{ch}_{\mathbb{T}}(\mathcal{E}_\gamma)}{\mathcal{P}}\right\}, \tag{23}$$

where $\iota_{(0,p)}^*$ is the pull-back homomorphism induced by the inclusion $\iota_{(0,p)}\colon 0 \times p \to \mathbb{C}^2 \times \mathfrak{M}_{G,\underline{k}}$. For the SO − USp quiver gauge theories, if we denote $\mathcal{E}_i$ to be the ith universal bundle over $\mathbb{C}^2 \times \mathfrak{M}_{G,\underline{k}}$, whose fiber over an element $A \in \mathfrak{M}_{G_i,k_i} \subset \mathfrak{M}_{G,\underline{k}}$ is the total space of the bundle E with connection A, then $\mathcal{E}_\gamma$ is given by

$$\mathcal{E}_\gamma = \left[\bigoplus_{i \in V_\gamma^{\bigcirc}} \mathbf{M}_i \otimes \mathcal{E}_i\right] \oplus \left[\bigoplus_{i \in V_\gamma^{\bullet}} \mathbf{M}_i \otimes \mathcal{E}_i \oplus \xi_i \mathcal{E}_i^{(\frac{1}{2})}\right] \oplus \left[\bigoplus_{\langle i,j \rangle \in E_\gamma} (\mathcal{E}_i \otimes \mathcal{E}_j)^{(\frac{1}{2})}\right], \tag{24}$$

where the superscript $(\frac{1}{2})$ means half-hypermultiplets.

The classification of fixed points $\mathfrak{M}^{\mathbb{T}}_{G,\underline{k}}$ for SO/USp gauge group is a complicated problem [8–11]. Nevertheless, it is sufficient for us to represent the instanton partition function as contour integrals,

$$\mathcal{Z}^{\text{inst}}(\underline{\mathsf{q}};\underline{a},\underline{m};\varepsilon_1,\varepsilon_2)=\sum_{\underline{k}}\underline{\mathsf{q}}^{\underline{k}}\left(\prod_{i\in V_\gamma}\frac{1}{|W_i|}\int\prod_s\frac{d\phi_{i,s}}{2\pi}\mathcal{Z}^{\text{inst,vec}}_i\mathcal{Z}^{\text{inst,fund}}_i\right)\left(\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Z}^{\text{inst,bif}}_{\langle i,j\rangle}\right), \tag{25}$$

where $|W_i|$ is the order of the dual Weyl group of G_i, and the factors $\mathcal{Z}^{\text{inst,vec}}_i$, $\mathcal{Z}^{\text{inst,fund}}_i$ and $\mathcal{Z}^{\text{inst,bif}}_{\langle i,j\rangle}$ are contributions to the instanton partition function from the vector multiplet, the fundamental matter, and the bifundamental half-hyper-multiplet. The variables $\phi_{i,s}$ in the integral are the weights of the $\mathbb{T}$-action on the space $\mathbb{K}_i$,

$$\begin{aligned}&i\in V^{\bigcirc}_\gamma:\ \text{diag}\{e^{\mathrm{i}\phi_1},\ldots,e^{\mathrm{i}\phi_{\kappa_i}},e^{-\mathrm{i}\phi_1},\ldots,e^{-\mathrm{i}\phi_{\kappa_i}}\},\\&i\in V^{\bullet}_\gamma:\ \text{diag}\{e^{\mathrm{i}\phi_1},e^{-\mathrm{i}\phi_1},\ldots,e^{\mathrm{i}\phi_{\kappa_i}},e^{-\mathrm{i}\phi_{\kappa_i}},(1)\},\end{aligned} \tag{26}$$

where (1) is absent for $\nu_i=0$. The equivariant character of the universal bundle $\mathcal{E}_i$ evaluated at the origin is given by

$$\mathcal{E}_i=\iota_0\text{ch}_{\mathbb{T}}(\mathcal{E}_i)=\mathcal{N}_i-\mathcal{P}\mathcal{K}_i, \tag{27}$$

where $\mathcal{N}_i$ is given in (9), and

$$\mathcal{K}_i=\sum_{r=1}^{\kappa_i}(e^{\mathrm{i}\phi_{i,r}}+e^{-\mathrm{i}\phi_{i,r}})+\nu_i. \tag{28}$$

We will compute explicitly the factors $\mathcal{Z}^{\text{inst,vec}}_i$, $\mathcal{Z}^{\text{inst,fund}}_i$ and $\mathcal{Z}^{\text{inst,bif}}_{\langle i,j\rangle}$ in the following. We also compute their expansion around the flat space limit.

1. Vector multiplets

In order to compute $\mathcal{Z}^{\text{inst,vec}}_i$, we use the basic fact that the character of the tangent space $T\mathfrak{M}_{G,\underline{k}}$ is dual to the index of Dirac operator in the adjoint representation twisted by the square-root of the canonical bundle. From the representation theory, the adjoint representation is isomorphic to the rank-two antisymmetric or symmetric representation for SO or USp group, respectively. Therefore, the equivariant character for the vector multiplet G_i, $i\in V^{\bigcirc}_\gamma$ is

$$\chi^{\text{vec}}_i=\frac{q_+}{\mathcal{P}}\left(\frac{\mathcal{E}^2_i-\mathcal{E}^{[2]}_i}{2}\right)=\frac{q_+}{\mathcal{P}}\left(\frac{\mathcal{N}^2_i-\mathcal{N}^{[2]}_i}{2}\right)-q_+\mathcal{N}_i\mathcal{K}_i+(1+q)\left(\frac{\mathcal{K}^2_i+\mathcal{K}^{[2]}_i}{2}\right)-(q_1+q_2)\left(\frac{\mathcal{K}^2_i-\mathcal{K}^{[2]}_i}{2}\right), \tag{29}$$

where the first term is the perturbative contribution, and the contribution to the instanton partition function from the vector multiplet at the vertex $i\in V^{\bigcirc}_\gamma$ is

$$\begin{aligned}\mathcal{Z}^{\text{inst,vec}}_i&=\epsilon\left\{-q_+\mathcal{N}_i\mathcal{K}_i+(1+q)\left(\frac{\mathcal{K}^2_i+\mathcal{K}^{[2]}_i}{2}\right)-(q_1+q_2)\left(\frac{\mathcal{K}^2_i-\mathcal{K}^{[2]}_i}{2}\right)\right\}\\&=\left(\frac{\varepsilon}{\varepsilon_1\varepsilon_2}\right)^{k_i}\left(\prod_{r=1}^{k_i}\frac{4\phi^2_{i,r}(4\phi^2_{i,r}-\varepsilon^2)}{\mathbb{A}_i(\phi_{i,r}+\varepsilon_+)\mathbb{A}_i(\phi_{i,r}-\varepsilon_+)}\right)\left(\frac{\Delta_i(0)\Delta_i(\varepsilon)}{\Delta_i(\varepsilon_1)\Delta_i(\varepsilon_2)}\right),\end{aligned} \tag{30}$$

where

$$\mathbb{A}_i(x)=x^{\mu_i}\prod_\alpha(x^2-a^2_{i,\alpha}), \tag{31}$$

$$\Delta_i(x)=\prod_{r<s}[(\phi_{i,r}+\phi_{i,s})^2-x^2][(\phi_{i,r}-\phi_{i,s})^2-x^2]. \tag{32}$$

Similarly, for $i\in V^{\bullet}_\gamma$, we have

$$\chi^{\text{vec}}_i=\frac{q_+}{2\mathcal{P}}(\mathcal{E}^2_i+\mathcal{E}^{[2]}_i)=\frac{q_+}{\mathcal{P}}\left(\frac{\mathcal{N}^2_i+\mathcal{N}^{[2]}_i}{2}\right)-q_+\mathcal{N}_i\mathcal{K}_i+(1+q)\left(\frac{\mathcal{K}^2_i-\mathcal{K}^{[2]}_i}{2}\right)-(q_1+q_2)\left(\frac{\mathcal{K}^2_i+\mathcal{K}^{[2]}_i}{2}\right). \tag{33}$$

Again, the first term is the perturbative contribution, and the contribution to the instanton partition function from the vector multiplet at the vertex $i\in V^{\bullet}_\gamma$ is

$$\mathcal{Z}_i^{\text{inst,vec}} = \epsilon\left\{-q_+\mathcal{N}_i\mathcal{K}_i + (1+q)\left(\frac{\mathcal{K}_i^2-\mathcal{K}_i^{[2]}}{2}\right) - (q_1+q_2)\left(\frac{\mathcal{K}_i^2+\mathcal{K}_i^{[2]}}{2}\right)\right\}$$
$$= \left(\frac{\varepsilon}{\varepsilon_1\varepsilon_2}\right)^{\kappa_i}\left[\prod_{r=1}^{\kappa_i}\mathbb{A}_i(\phi_{i,r}+\varepsilon_+)\mathbb{A}_i(\phi_{i,r}-\varepsilon_+)(4\phi_{i,r}^2-\varepsilon_1^2)(4\phi_{i,r}^2-\varepsilon_2^2)\right]^{-1}$$
$$\times\left[\frac{1}{\varepsilon_1\varepsilon_2\mathbb{A}_i(\varepsilon_+)}\prod_{r=1}^{\kappa_i}\frac{\phi_{i,r}^2(\phi_{i,r}^2-\varepsilon^2)}{(\phi_{i,r}^2-\varepsilon_1^2)(\phi_{i,r}^2-\varepsilon_2^2)}\right]^{\nu_i}\left(\frac{\Delta_i(0)\Delta_i(\varepsilon)}{\Delta_i(\varepsilon_1)\Delta_i(\varepsilon_2)}\right). \tag{34}$$

In the flat space limit, the dominant instanton configuration contributing to the instanton partition function has the instanton charge of the order $\sim\frac{1}{\varepsilon_1\varepsilon_2}$. Therefore, we should take the limit $\varepsilon_1,\varepsilon_2\to 0$, $\kappa_i\to\infty$ while keeping $\varepsilon_1\varepsilon_2\kappa_i\sim\mathcal{O}(1)$ fixed. Using the expansion

$$\log\frac{x^2(x^2-\varepsilon^2)}{(x^2-\varepsilon_1^2)(x^2-\varepsilon_2^2)} = \frac{-2\varepsilon_1\varepsilon_2}{x^2} + \mathcal{O}((\varepsilon_1,\varepsilon_2)^4), \tag{35}$$

we have for $i\in V_\gamma^{\bigcirc}$

$$F_i^{\text{inst,vec}} = -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}_i^{\text{inst,vec}}$$
$$= 2\varepsilon_1\varepsilon_2\sum_{r=1}^{k_i}\left[\left(\frac{1}{2}\mu_i-1\right)\log(\phi_{i,r}^2) + \sum_\alpha\log(\phi_{i,r}^2-a_{i,\alpha}^2)\right] + 2(\varepsilon_1\varepsilon_2)^2\sum_{r<s}^{\kappa_i}\left[\frac{1}{(\phi_{i,r}+\phi_{i,s})^2}+\frac{1}{(\phi_{i,r}-\phi_{i,s})^2}\right], \tag{36}$$

and for $i\in V_\gamma^{\bullet}$

$$F_i^{\text{inst,vec}} = -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}_i^{\text{inst,vec}}$$
$$= 2\varepsilon_1\varepsilon_2\sum_{r=1}^{k_i}\left[\log(\phi_{i,r}^2) + \sum_\alpha\log(\phi_{i,r}^2-a_{i,\alpha}^2)\right] + 2(\varepsilon_1\varepsilon_2)^2\sum_{r<s}^{\kappa_i}\left[\frac{1}{(\phi_{i,r}+\phi_{i,s})^2}+\frac{1}{(\phi_{i,r}-\phi_{i,s})^2}\right]. \tag{37}$$

Notice that the ν_i-dependent term drops out because it behaves as

$$(\varepsilon_1\varepsilon_2)^2\sum_{r=1}^{k_i}\frac{1}{\phi_{i,r}^2}, \tag{38}$$

which vanishes in the flat space limit.

2. Fundamental matter

The equivariant index of a fundamental hypermultiplet is

$$\chi_i^{\text{fund,hyper}} = -\frac{\iota_0^*\text{ch}_{\mathbb{T}}(\mathbf{M}_i\otimes\mathcal{E}_i)}{\mathcal{P}}$$
$$= -\frac{1}{\mathcal{P}}\mathcal{M}_i\mathcal{E}_i = -\frac{1}{\mathcal{P}}\mathcal{M}_i\mathcal{N}_i + \mathcal{M}_i\mathcal{K}_i, \tag{39}$$

where the first term is the perturbative contribution, and the second term gives

$$\mathcal{Z}_i^{\text{inst,fund,hyper}} = \epsilon\{\mathcal{M}_i\mathcal{K}_i\} = \prod_{f=1}^{w_i}\left[m_{i,f}^{\nu_i}\prod_{r=1}^{\kappa_i}(\phi_{i,r}^2-m_{i,f}^2)\right]. \tag{40}$$

For $i\in V_\gamma^{\bullet}$, we also need to consider the half-hypermultiplet, which must be massless. We take the contribution of a fundamental half-hypermultiplet to be the square-root of a massless fundamental hypermultiplet [11],

$$\mathcal{Z}_i^{\text{inst,fund,hh}} = (\epsilon\{\mathcal{K}_i\})^{\frac{1}{2}} = \zeta_i\prod_{r=1}^{\kappa_i}\phi_{i,r}, \tag{41}$$

where $\zeta_i=\pm$. Combining the fundamental hypermultiplet with possible half-hypermultiplet, we can write the contribution of the fundamental matter to the instanton partition function as

$$\mathcal{Z}_i^{\text{inst,fund}} = \zeta_i^{\xi_i}\left(\prod_{f=1}^{w_i}m_{i,f}^{\nu_i}\right)\prod_{r=1}^{\kappa_i}\left[\phi_{i,r}^{\xi_i}\prod_{f=1}^{w_i}(\phi_{i,r}^2-m_{i,f}^2)\right]. \tag{42}$$

In the flat space limit,

$$F_i^{\text{inst,fund}} = -\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 \log \mathcal{Z}_i^{\text{inst,fund}} = -\epsilon_1\epsilon_2\sum_{r=1}^{\kappa_i}\left[\frac{\varphi_i}{2}\log(\phi_{i,r}^2)+\sum_{f=1}^{w_i}\log(\phi_{i,r}^2-m_{i,f}^2)\right], \tag{43}$$

where the dependence on the overall sign ζ_i disappears.

3. Bifundamental half-hypermultiplet

The equivariant index of a massless bifundamental hypermultiplet composed of a pair of bifundamental half-hypermultiplets is given by

$$\chi_{\langle i,j\rangle}^{\text{bif,hyper}} = -\frac{1}{\mathcal{P}}(\mathcal{N}_i-\mathcal{P}\mathcal{K}_i)(\mathcal{N}_j-\mathcal{P}\mathcal{K}_j) = -\frac{1}{\mathcal{P}}\mathcal{N}_i\mathcal{N}_j+\mathcal{N}_i\mathcal{K}_j+\mathcal{N}_j\mathcal{K}_i-\mathcal{P}\mathcal{K}_i\mathcal{K}_j, \tag{44}$$

where the first term is the perturbative contribution, and the instanton contribution given by the remaining terms is a complete square,

$$\mathcal{Z}_{\langle i,j\rangle}^{\text{inst,bif,hyper}} = \left[\left(\prod\nolimits_\alpha a_{i,\alpha}\right)\left(\prod_{r=1}^{\kappa_i}\frac{\phi_{i,r}^2-\epsilon_-^2}{\phi_{i,r}^2-\epsilon_+^2}\right)\right]^{2\nu_j} \times\left[\prod_{r=1}^{\kappa_i}\mathbb{A}_j(\phi_{i,r})\right]^2\left[\prod_{s=1}^{\kappa_j}\mathbb{A}_i(\phi_{j,s})\right]^2\left(\frac{\Delta_{i,j}(\epsilon_-)}{\Delta_{i,j}(\epsilon_+)}\right)^2, \tag{45}$$

where

$$\Delta_{i,j}(x) = \prod_{r=1}^{\kappa_i}\prod_{s=1}^{\kappa_j}[(\phi_{i,r}+\phi_{j,s})^2-x^2][(\phi_{i,r}-\phi_{j,s})^2-x^2]. \tag{46}$$

We identify the contribution to the instanton partition function from the bifundamental half-hypermultiplet as [11]

$$\mathcal{Z}_{\langle i,j\rangle}^{\text{inst,bif}} = \zeta_{\langle i,j\rangle}\left[\left(\prod\nolimits_\alpha a_{i,\alpha}\right)\left(\prod_{r=1}^{\kappa_i}\frac{\phi_{i,r}^2-\epsilon_-^2}{\phi_{i,r}^2-\epsilon_+^2}\right)\right]^{\nu_j} \times\left[\prod_{r=1}^{\kappa_i}\mathbb{A}_j(\phi_{i,r})\right]\left[\prod_{s=1}^{\kappa_j}\mathbb{A}_i(\phi_{j,s})\right]\left(\frac{\Delta_{i,j}(\epsilon_-)}{\Delta_{i,j}(\epsilon_+)}\right), \tag{47}$$

with the overall sign $\zeta_{\langle i,j\rangle}=\pm$. Using the expansion

$$\log\frac{x^2-\epsilon_-^2}{x^2-\epsilon_+^2} = \frac{\epsilon_1\epsilon_2}{x^2}+\mathcal{O}((\epsilon_1,\epsilon_2)^4), \tag{48}$$

we can compute the flat space limit,

$$\begin{aligned}F_{\langle i,j\rangle}^{\text{inst,bif}} &= -\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}_{\langle i,j\rangle}^{\text{inst,bif}} \\ &= -\epsilon_1\epsilon_2\sum_{r=1}^{\kappa_i}\sum_\alpha\log(\phi_{i,r}^2-a_{j,\alpha}^2) \\ &\quad -\epsilon_1\epsilon_2\sum_{s=1}^{\kappa_j}\left[\frac{\mu_i}{2}\log(\phi_{j,s}^2)+\sum_\alpha\log(\phi_{j,s}^2-a_{i,\alpha}^2)\right] \\ &\quad -(\epsilon_1\epsilon_2)^2\sum_{r=1}^{\kappa_i}\sum_{s=1}^{\kappa_j}\left[\frac{1}{(\phi_{i,r}+\phi_{j,s})^2}+\frac{1}{(\phi_{i,r}-\phi_{j,s})^2}\right].\end{aligned} \tag{49}$$

We see that the ν_j-dependent term and the overall sign $\zeta_{\langle i,j\rangle}$ drop out again.

B. Full partition function

After deriving the instanton partition function, we would like to combine it with the classical and the perturbative contributions to form the full partition function,

$$\mathcal{Z}(\underline{\mathsf{q}};\underline{a},\underline{m};\epsilon_1,\epsilon_2) = \mathcal{Z}^{\text{cl}}\mathcal{Z}^{\text{pert}}\mathcal{Z}^{\text{inst}} = \mathcal{Z}^{\text{cl}}\sum_{\underline{k}}\underline{\mathsf{q}}^{\underline{k}}\left(\prod_{i\in V_\gamma}\frac{1}{|W_i|}\int\prod_s\frac{d\phi_{i,s}}{2\pi}\mathcal{Z}_i^{\text{pert,vec}}\mathcal{Z}_i^{\text{pert,fund}}\mathcal{Z}_i^{\text{inst,vec}}\mathcal{Z}_i^{\text{inst,fund}}\right)\left(\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Z}_{\langle i,j\rangle}^{\text{pert,bif}}\mathcal{Z}_{\langle i,j\rangle}^{\text{inst,bif}}\right). \tag{50}$$

The classical partition function is simply given by

$$\mathcal{Z}^{\text{cl}}(\underline{\mathsf{q}};\underline{a};\epsilon_1,\epsilon_2) = \prod_{i\in V_\gamma}\mathsf{q}_i^{-\frac{1}{2\epsilon_1\epsilon_2}\sum_\alpha a_{i,\alpha}^2}$$

whose flat space limit is

$$F^{\text{cl}} = -\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}^{\text{cl}} = \frac{1}{2}\sum_{i\in V_\gamma}\log(\mathsf{q}_i)\sum_\alpha a_{i,\alpha}^2. \tag{51}$$

The perturbative contribution to the partition function in the Ω-background is one-loop exact. In fact, we have already obtained them as the byproduct of our derivation of the instanton partition function,

$$\mathcal{Z}^{\text{pert,vec}}_{i\in V^{\circ}_{\gamma}} = \epsilon\left\{\frac{q_+}{\mathcal{P}}\left(\frac{\mathcal{N}_i^2 - \mathcal{N}_i^{[2]}}{2}\right)\right\}, \tag{52}$$

$$\mathcal{Z}^{\text{pert,vec}}_{i\in V^{\bullet}_{\gamma}} = \epsilon\left\{\frac{q_+}{\mathcal{P}}\left(\frac{\mathcal{N}_i^2 + \mathcal{N}_i^{[2]}}{2}\right)\right\}, \tag{53}$$

$$\mathcal{Z}^{\text{pert,fund}}_{i} = \epsilon\left\{-\frac{1}{\mathcal{P}}\left(\mathcal{M}_i + \frac{1}{2}\varphi_i\right)\mathcal{N}_i\right\}, \tag{54}$$

$$\mathcal{Z}^{\text{pert,bif}}_{\langle i,j\rangle} = \epsilon\left\{-\frac{1}{2\mathcal{P}}\mathcal{N}_i\mathcal{N}_j\right\}. \tag{55}$$

We set the cutoff energy scale $\Lambda_{\text{UV}} = 1$. Using the regularization (12), we can write the perturbative contributions in terms of Barnes' double Gamma function $\Gamma_2(x|\varepsilon_1,\varepsilon_2)$. Apply the relation

$$-\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2 \log\Gamma_2(x|\varepsilon_1,\varepsilon_2) = \frac{x^2}{4}\log(x^2) - \frac{3x^2}{4} = \mathcal{K}(x), \tag{56}$$

we get the perturbative contributions in the flat space limit,

$$\begin{aligned} F^{\text{pert,vec}}_{i\in V^{\circ}_{\gamma}} &= -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}^{\text{pert,vec}}_i \\ &= -\sum_{\alpha<\beta}\mathcal{K}(a_\alpha + a_\beta) - \sum_{\alpha<\beta}\mathcal{K}(a_\alpha - a_\beta) \\ &\quad + 2\mu_i\sum_{\alpha}\mathcal{K}(a_\alpha), \end{aligned} \tag{57}$$

$$\begin{aligned} F^{\text{pert,vec}}_{i\in V^{\bullet}_{\gamma}} &= -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}^{\text{pert,vec}}_i \\ &= -\sum_{\alpha\le\beta}\mathcal{K}(a_\alpha + a_\beta) - \sum_{\alpha<\beta}\mathcal{K}(a_\alpha - a_\beta), \end{aligned} \tag{58}$$

$$\begin{aligned} F^{\text{pert,fund}}_{i} &= -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}^{\text{pert,fund}}_i \\ &= \sum_{\alpha}\sum_{f=1}^{w_i}[\mathcal{K}(a_{i,\alpha} + m_{i,f}) + \mathcal{K}(a_{i,\alpha} - m_{i,f})] \\ &\quad + \xi_i\sum_{\alpha}\mathcal{K}(a_{i,\alpha}) + \mu_i\sum_{f=1}^{w_i}\mathcal{K}(m_{i,f}), \end{aligned} \tag{59}$$

$$\begin{aligned} F^{\text{pert,bif}}_{\langle i,j\rangle} &= -\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}^{\text{pert,bif}}_{\langle i,j\rangle} \\ &= \sum_{\alpha,\beta}[\mathcal{K}(a_{i,\alpha} + a_{j,\beta}) + \mathcal{K}(a_{i,\alpha} - a_{j,\beta})] \\ &\quad + \mu_i\sum_{\beta}\mathcal{K}(a_{j,\beta}). \end{aligned} \tag{60}$$

Therefore, the full partition function in the flat space limit can be written as

$$\mathcal{Z}(\underline{\mathfrak{q}};\underline{a},\underline{m};\varepsilon_1,\varepsilon_2) = \sum_{\underline{k}}\underline{\mathfrak{q}}^{\underline{k}}\int\prod_{i\in V_\gamma}\prod_{s}\frac{d\phi_{i,s}}{2\pi}\exp\left(-\frac{F}{\varepsilon_1\varepsilon_2}\right), \tag{61}$$

where

$$\begin{aligned} F = F^{\text{cl}} &+ \sum_{i\in V_\gamma}(F^{\text{pert,vec}}_i + F^{\text{pert,fund}}_i + F^{\text{inst,vec}}_i + F^{\text{inst,fund}}_i) \\ &+ \sum_{\langle i,j\rangle\in E_\gamma}(F^{\text{pert,bif}}_{\langle i,j\rangle} + F^{\text{inst,bif}}_{\langle i,j\rangle}) + \mathcal{O}(\varepsilon_1,\varepsilon_2). \end{aligned} \tag{62}$$

C. The functional integrals over density functions

It is useful to rewrite F in terms of functional integrals of density functions. We introduce the instanton density functions

$$\varrho_i(z) = \varepsilon_1\varepsilon_2\sum_{s=1}^{\kappa_i}[\delta(z-\phi_{i,s}) + \delta(z+\phi_{i,s})], \tag{63}$$

with the normalization ensuring the finiteness in the flat space limit. They are even functions,

$$\varrho_i(z) = \varrho_i(-z). \tag{64}$$

Using the standard rule

$$\varepsilon_1\varepsilon_2\sum_{r=1}^{\kappa_i}[f(\phi_{i,r}) + f(-\phi_{i,r})] \to \int dz\varrho_i(z)f(z), \tag{65}$$

we can rewrite $F^{\text{inst,vec}}_i$, $F^{\text{inst,fund}}_i$, and $F^{\text{inst,bif}}_{\langle i,j\rangle}$ in terms of $\varrho_i(z)$ as

$$\begin{aligned} F^{\text{inst,vec}}_{i\in V^{\circ}_{\gamma}} &= 2\int dz\varrho_i(z)\left[\left(\frac{1}{2}\mu_i - 1\right)\log(z) + \sum_{\alpha}\log(z - a_{i,\alpha})\right] + \frac{1}{2}⨍ dz dz'\frac{\varrho_i(z)\varrho_i(z')}{(z-z')^2} \\ &= 2\int dz\varrho''_i(z)\left[\left(\frac{1}{2}\mu_i - 1\right)\mathcal{K}(z) + \sum_{\alpha}\mathcal{K}(z - a_{i,\alpha})\right] - \frac{1}{2}⨍ dz dz'\varrho''_i(z)\varrho''_i(z')\mathcal{K}(z-z'), \end{aligned} \tag{66}$$

$$
\begin{aligned}
F^{\text{inst,vec}}_{i\in V^{\bullet}_{\gamma}} &= 2\int dz\varrho_i(z)\left[\log(z)+\sum_{\alpha}\log(z-a_{i,\alpha})\right]+\frac{1}{2}\fint dzdz'\frac{\varrho_i(z)\varrho_i(z')}{(z-z')^2}\\
&= 2\int dz\varrho_i''(z)\left[\mathcal{K}(z)+\sum_{\alpha}\mathcal{K}(z-a_{i,\alpha})\right]-\frac{1}{2}\fint dzdz'\varrho_i''(z)\varrho_i''(z')\mathcal{K}(z-z'),
\end{aligned}
\tag{67}
$$

$$
F^{\text{inst,fund}}_i = -\int dz\varrho_i(z)\left[\frac{\xi_i}{2}\log(z)+\sum_{f=1}^{w_i}\log(z-m_{i,f})\right] = -\int dz\varrho_i''(z)\left[\frac{\xi_i}{2}\mathcal{K}(z)+\sum_{f=1}^{w_i}\mathcal{K}(z-m_{i,f})\right],
\tag{68}
$$

$$
\begin{aligned}
F^{\text{inst,bif}}_{\langle i,j\rangle} &= -\int dz\varrho_i(z)\sum_{\alpha}\log(z-a_{j,\alpha})-\int dz\varrho_j(z)\left[\frac{\mu_i}{2}\log(z)+\sum_{\alpha}\log(z-a_{i,\alpha})\right]-\frac{1}{2}\fint dzdz'\frac{\varrho_i(z)\varrho_j(z')}{(z-z')^2}\\
&= -\int dz\varrho_i''(z)\sum_{\alpha}\mathcal{K}(z-a_{j,\alpha})-\int dz\varrho_j''(z)\left[\frac{\mu_i}{2}\mathcal{K}(z)+\sum_{\alpha}\mathcal{K}(z-a_{i,\alpha})\right]+\frac{1}{2}\fint dzdz'\varrho_i''(z)\varrho_j''(z')\mathcal{K}(z-z'),
\end{aligned}
\tag{69}
$$

where $\fint$ denotes the principal value of the improper integral.

In order to combine the instanton contribution with the perturbative contribution, we introduce the full density functions

$$
\rho_i(z) = \begin{cases} \mu_i\delta(z)+\sum_{\alpha}[\delta(z-a_{i,\alpha})+\delta(z+a_{i,\alpha})]-\varrho''(z), & i\in V^{\circ}_{\gamma}\\ 2\delta(z)+\sum_{\alpha}[\delta(z-a_{i,\alpha})+\delta(z+a_{i,\alpha})]-\varrho''(z), & i\in V^{\bullet}_{\gamma} \end{cases},
\tag{70}
$$

so that

$$
F^{\text{vec}}_i = F^{\text{pert,vec}}_i + F^{\text{inst,vec}}_i = \begin{cases} -\frac{1}{2}\fint dzdz'\rho_i(z)\rho_i(z')\mathcal{K}(z-z')+2\int dz\rho_i(z)\mathcal{K}(z), & i\in V^{\circ}_{\gamma}\\ -\frac{1}{2}\fint dzdz'\rho_i(z)\rho_i(z')\mathcal{K}(z-z'), & i\in V^{\bullet}_{\gamma} \end{cases},
\tag{71}
$$

$$
F^{\text{fund}}_i = F^{\text{pert,fund}}_i + F^{\text{inst,fund}}_i = \sum_{f=1}^{w_i}\int dz\rho_i(z)\mathcal{K}(z-m_{i,f})+\frac{\xi_i}{2}\int dz\rho_i(z)\mathcal{K}(z),
\tag{72}
$$

$$
F^{\text{bif}}_{\langle i,j\rangle} = F^{\text{pert,bif}}_{\langle i,j\rangle} + F^{\text{inst,bif}}_{\langle i,j\rangle} = \frac{1}{2}\fint dzdz'\rho_i(z)\rho_j(z')\mathcal{K}(z-z')-\int dz\rho_i(z)\mathcal{K}(z),
\tag{73}
$$

where we have used the fact that $\mathcal{K}(0)=0$. Clearly the expressions (71)–(73) are much simpler than the perturbative and instanton contributions separately, and the explicit dependence of the Coulomb moduli $a_{i,\alpha}$ disappears.

For the classical contribution and the factor $\underline{\mathsf{q}}^{\underline{k}}$, we can evaluate that

$$
\int z^2\rho_i(z)dz = 2\sum_{\alpha}a^2_{i,\alpha}-4\varepsilon_1\varepsilon_2\kappa_i,
\tag{74}
$$

which leads to

$$
\begin{aligned}
&-\varepsilon_1\varepsilon_2\log(\mathcal{Z}^{\text{cl}}\underline{\mathsf{q}}^{\underline{k}})\\
&= \sum_{i\in V_{\gamma}}\log(\mathsf{q}_i)\left(\frac{1}{2}\sum_{\alpha}a^2_{i,\alpha}-\varepsilon_1\varepsilon_2\left(\kappa_i+\frac{1}{2}\nu_i\right)\right)\\
&= \frac{1}{4}\sum_{i\in V_{\gamma}}\log(\mathsf{q}_i)\int z^2\rho_i(z)dz+\mathcal{O}(\varepsilon_1,\varepsilon_2).
\end{aligned}
\tag{75}
$$

Therefore, we can rewrite the full partition function in terms of functional integrals over $\rho=\{\rho_i\}_{i\in V_{\gamma}}$,

$$
\mathcal{Z} = \int\prod_{i\in V_{\gamma}}d\rho_i\exp\left(-\frac{F[\rho]}{\varepsilon_1\varepsilon_2}+\mathcal{O}(\varepsilon_1,\varepsilon_2)\right),
\tag{76}
$$

where $F[\rho]$ is given by

$$
\begin{aligned}
F[\rho] = & -\frac{1}{2}\sum_{i\in V_\gamma} \fint dz dz' \rho_i(z)\rho_i(z')\mathcal{K}(z-z') + \sum_{i\in V_\gamma^{\bigcirc}} \int dz \rho_i(z)\left[2\mathcal{K}(z) + \frac{1}{4}\log(\mathsf{q}_i)z^2 + \sum_{f=1}^{w_i}\mathcal{K}(z-m_{i,f})\right] \\
& + \sum_{i\in V_\gamma^{\bullet}} \int dz\rho_i(z)\left[\frac{1}{4}\log(\mathsf{q}_i)z^2 + \sum_{f=1}^{w_i}\mathcal{K}(z-m_{i,f}) + \frac{\xi_i}{2}\mathcal{K}(z)\right] \\
& + \sum_{\langle i,j\rangle\in E_\gamma}\left[\frac{1}{2}\fint dz dz'\rho_i(z)\rho_j(z')\mathcal{K}(z-z') - \int dz\rho_i(z)\mathcal{K}(z)\right]. \qquad (77)
\end{aligned}
$$

IV. THE LIMIT SHAPE EQUATIONS

Now we are ready to perform the saddle-point evaluation following the approach in [4–7,9] to determine the limit shape of the instanton configuration which dominates the partition function (61) in the flat space limit $\varepsilon_1, \varepsilon_2 \to 0$.

A. Saddle point analysis

In the limit $\varepsilon_1, \varepsilon_2 \to 0$, the distribution $\rho_i(z)$ becomes a function with compact support $\mathcal{C}_i$. In an appropriate domain of the parameter space, $\mathcal{C}_i$ for different i are widely separated, and each $\mathcal{C}_i$ is a union of disjoint intervals along the real axis,

$$
\begin{aligned}
&\mathcal{C}_i = \bigcup_\ell \mathcal{I}_{i,\ell} = \bigcup_\ell [a^-_{i,\ell}, a^+_{i,\ell}], \\
&\cdots < a^-_{i,\ell} < a^+_{i,\ell} < a^-_{i,\ell+1} < a^+_{i,\ell+1} < \cdots. \qquad (78)
\end{aligned}
$$

Here

$$
\ell = \begin{cases} \pm1, \pm2, \ldots, \pm n_i, & i \in V_\gamma^{\bigcirc} \\ 0, \pm1, \pm2, \ldots, \pm n_i, & i \in V_\gamma^{\bullet} \end{cases},
$$

with $\pm a_{i,\alpha} \in \mathcal{I}_{i,\pm\alpha}$ and $0 \in \mathcal{I}_{i,0}$. The function $\rho_i(z)$ is normalized according to

$$
\int_{\mathcal{I}_{i,\ell}} \rho_i(z)dz = \begin{cases} 2, & i \in V_\gamma^{\bullet}, \ell = 0 \\ 1, & \text{otherwise} \end{cases}. \qquad (79)
$$

The Coulomb moduli enter the variational problem via the additional constraints

$$
\int_{\mathcal{I}_{i,\pm\alpha}} z\rho_i(z)dz = \pm a_{i,\alpha}. \qquad (80)
$$

After incorporating the constraints via Lagrangian multipliers $b_{i,\alpha}$ and $a^D_{i,\alpha}$, our task is to find the limit shape $\rho_\star$ which extremizes the following effective free energy,

$$
\begin{aligned}
F^{\mathrm{eff}}[\rho] = F[\rho] + \sum_{i\in V_\gamma}\sum_\ell \Bigg[& b_{i,\ell}\left(1 - \int_{\mathcal{I}_{i,\ell}}\rho_i(z)dz\right) \\
& + a^D_{i,\ell}\left(a_{i,\ell} - \int_{\mathcal{I}_{i,\ell}} z\rho_i(z)dz\right)\Bigg], \qquad (81)
\end{aligned}
$$

where $a^D_{i,\alpha}$ is the dual special coordinate of $a_{i,\alpha}$, and the low energy effective prepotential is given in terms of $\rho_\star$ as

$$
\mathcal{F} = F^{\mathrm{eff}}[\rho_\star]. \qquad (82)
$$

For any $i \in V_\gamma^{\bigcirc}$ and $x \in \mathcal{I}_{i,\ell}$, the variation of $F^{\mathrm{eff}}[\rho]$ with respect to $\rho_i(x)$ leads to the following linear integral equation,

$$
\begin{aligned}
0 = & -\int dz\rho_i(z)\mathcal{K}(z-x) + 2\mathcal{K}(x) + \frac{1}{4}\log(\mathsf{q}_i)x^2 \\
& + \frac{1}{2}\sum_{f=1}^{w_i}[\mathcal{K}(x-m_{i,f}) + \mathcal{K}(x+m_{i,f})] \\
& + \sum_{\langle i,j\rangle\in E_\gamma}\left[\frac{1}{2}\int dz\rho_j(z)\mathcal{K}(x-z) - \mathcal{K}(x)\right] - b_{i,\ell} - xa^D_{i,\ell}. \qquad (83)
\end{aligned}
$$

Keep in mind that we must preserve the symmetry $\rho_i(x) = \rho_i(-x)$ in the variation. Similarly, for any $j \in V_\gamma^{\bullet}$ and $x \in \mathcal{I}_{j,\ell}$, we have

$$
\begin{aligned}
0 = & -\int dz\rho_j(z)\mathcal{K}(z-x) + \frac{1}{4}\log(\mathsf{q}_j)x^2 \\
& + \frac{1}{2}\sum_{f=1}^{w_j}[\mathcal{K}(x-m_{j,f}) + \mathcal{K}(x+m_{j,f})] + \frac{\xi_j}{2}\mathcal{K}(x) \\
& + \frac{1}{2}\sum_{\langle i,j\rangle\in E_\gamma}\int dz\rho_i(z)\mathcal{K}(z-x) - b_{j,\ell} - xa^D_{j,\ell}. \qquad (84)
\end{aligned}
$$

Taking the second derivative with respect to x, we obtain the limit shape equations

$$
\begin{aligned}
0 = & -\fint dz\rho_i(z)\log(x-z) + 2\log x + \frac{1}{2}\log(\mathsf{q}_i) \\
& + \frac{1}{2}\sum_{f=1}^{w_i}\log(x^2 - m_{i,f}^2) \\
& + \sum_{\langle i,j\rangle\in E_\gamma}\left[\frac{1}{2}\fint dz\rho_j(z)\log(x-z) - \log x\right], \qquad i\in V_\gamma^{\bigcirc}, \\
0 = & -\fint dz\rho_j(z)\log(x-z) + \frac{1}{2}\log(\mathsf{q}_j) \\
& + \frac{1}{2}\sum_{f=1}^{w_j}\log(x^2 - m_{i,f}^2) \\
& + \frac{\xi_j}{2}\log x + \frac{1}{2}\sum_{\langle i,j\rangle\in E_\gamma}\fint dz\rho_i(z)\log(x-z), \\
& j\in V_\gamma^{\bullet}.
\end{aligned} \tag{85}
$$

B. Analytic continuation and the instanton Weyl group

The limit shape equations (85) can be solved in terms of the amplitude function, which is the generating function of the vacuum expectation values of all the gauge invariant local observables commuting with the supercharge $\mathcal{Q}$ [7],

$$\mathcal{Y}_i(x) = \exp\int dz\rho_i(z)\log(x-z). \tag{86}$$

We can expand $\mathcal{Y}_i(x)$ as a Laurant series in x,

$$\mathcal{Y}_i(x) = x^{v_i} + \sum_{j=-\infty}^{v_i-2}\mathcal{Y}_{i,j}x^j. \tag{87}$$

The function $\mathcal{Y}_i(x)$ is analytic on $\mathbb{C}\backslash\mathcal{C}_i$, and has branch cuts on $\mathcal{C}_i$. According to Sokhotsky's formula, for $x\in\mathcal{C}_i$, the principal value

$$\fint dz\rho_i(z)\log(x-z) = \mathcal{Y}_i(x+\mathtt{i}0)\mathcal{Y}_i(x-\mathtt{i}0), \tag{88}$$

and the discontinuity across $\mathcal{C}_i$

$$\frac{\mathcal{Y}_i(x+\mathtt{i}0)}{\mathcal{Y}_i(x-\mathtt{i}0)} = \exp\left(-2\pi\mathtt{i}\int_{-\infty}^{x}dz\rho_i(z)\right), \tag{89}$$

where $\mathcal{Y}_i(x\pm\mathtt{i}0)$ are the limit values at the top and the bottom of $\mathcal{I}_{i,\ell}$. If $\mathcal{A}_{i,\ell}$ is a small cycle surrounding the cut $[a_{i,\ell}^-, a_{i,\ell}^+]$, then from (80) we know that

$$\pm a_{i,\alpha} = \frac{1}{2\pi\mathtt{i}}\oint_{\mathcal{A}_{i,\pm\alpha}} x\,d\log\mathcal{Y}_i. \tag{90}$$

Alternatively, we can view $\mathcal{Y}_i(x)$ as a single-valued holomorphic function living on a Riemann surface, which is the double cover of the complex plane, glued together along the cuts.

The limit shape equations (85) in terms of ρ_i is the same as the nonlinear difference equations on $\mathcal{Y}_i(x)$,

$$
\begin{aligned}
& \mathcal{Y}_i(x+\mathtt{i}0)\mathcal{Y}_i(x-\mathtt{i}0) = \mathsf{q}_i x^{4-2\sigma_i}\prod_{f=1}^{w_i}(x^2-m_{i,f}^2)\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Y}_j(x), \\
& \qquad i\in V_\gamma^{\bigcirc}, \\
& \mathcal{Y}_j(x+\mathtt{i}0)\mathcal{Y}_j(x-\mathtt{i}0) = \mathsf{q}_j x^{\xi_j}\prod_{f=1}^{w_j}(x^2-m_{j,f}^2)\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Y}_i(x), \\
& \qquad j\in V_\gamma^{\bullet}.
\end{aligned} \tag{91}
$$

Recall that for SU quiver gauge theories, the limit shape equations can be written as [7]

$$
\begin{aligned}
\mathcal{Y}_i(x+\mathtt{i}0)\mathcal{Y}_i(x-\mathtt{i}0) = \mathsf{q}_i & \prod_{f=1}^{w_i}(x-m_{i,f}) \\
& \times \prod_{e:t(e)=i}\mathcal{Y}_{s(e)}(x+m_e) \\
& \times \prod_{e:s(e)=i}\mathcal{Y}_{t(e)}(x-m_e),
\end{aligned} \tag{92}
$$

where $\mathcal{Y}_i(x)$ is the ith amplitude function in the SU quiver gauge theory, m_e is the mass of the bifundamental hypermultiplet associated with the oriented edge e whose source and target vertices are $s(e)$ and $t(e)$ respectively. We find that (91) and (92) are very similar. The differences arise because we have unoriented bipartite quiver diagrams for SO − USp quiver gauge theories and the bifundamental matter fields are half-hypermultiplets rather than full hypermultiplets.

We can analytically continue $\mathcal{Y}_i(x)$ across the cuts according to (91), leading to a multivalued function on the complex plane. We define the following reflections on a single vertex,

$$
\begin{aligned}
& s_i:\ \mathcal{Y}_i(x)\mapsto x^{4-2\sigma_i}P_i(x)\mathcal{Y}_i(x)^{-1}\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Y}_j(x), \\
& \qquad i\in V_\gamma^{\bigcirc}, \\
& s_j:\ \mathcal{Y}_j(x)\mapsto P_j(x)\mathcal{Y}_j(x)^{-1}\prod_{\langle i,j\rangle\in E_\gamma}\mathcal{Y}_i(x), \\
& \qquad j\in V_\gamma^{\bullet}.
\end{aligned} \tag{93}
$$

It is easy to check that $s_i^2 = 1$ and $s_is_j = s_js_i$ if $\langle i,j\rangle\notin E_\gamma$. These reflections generate a group, called the instanton Weyl group ${}^i\mathcal{W}$. It is the finite Weyl group $\mathcal{W}(\mathfrak{g})$ of the ADE simple Lie algebra $\mathfrak{g}$ for the Class I theories of type $\mathfrak{g}$, and is the affine Weyl group $\mathcal{W}(\hat{\mathfrak{g}})$ of the affine Lie algebra

$\hat{\mathfrak{g}}$ for the Class II theories of type $\hat{\mathfrak{g}}$ [7]. For class III theories, it needs to be worked out case by case.

The instanton Weyl group ${}^i\mathcal{W}$ is useful due to the following reason. Notice that although $\mathcal{Y}_i(x)$ has discontinuity across the cut $\mathcal{C}_i$, the combination $\mathcal{Y}_i(x) + s_i[\mathcal{Y}_i(x)]$ is invariant under the reflection s_i, making it continuous across the cut $\mathcal{C}_i$. There are new discontinuities across other cuts $\mathcal{C}_j$ introduced by $s_i[\mathcal{Y}_i(x)]$. Again these discontinuities can be canceled by acting on other reflections s_j. The iteration process will close in a finite or infinite number of steps and produces an ${}^i\mathcal{W}$-orbit. The resulting function is manifestly analytic on $\mathbb{C}\backslash(\cup_i \mathcal{C}_i)$ and is also continuous across all the cuts due to the ${}^i\mathcal{W}$-invariance. Therefore, it must be a single-valued analytic function on the whole complex plane. Our solution to the limit shape equations (91) can then be given in terms of a set of ${}^i\mathcal{W}$-invariants.

C. Characters and Seiberg-Witten geometry

Among all the ${}^i\mathcal{W}$-invariants, we are particularly interested in the characters $\chi_i(\mathcal{Y}(x))$ of the ${}^i\mathcal{W}$-orbits containing $\mathcal{Y}_i(x)$,

$$\chi_i(\mathcal{Y}(x)) = \mathcal{Y}_i(x) + \cdots = \mathrm{Tr} L_i(x), \qquad i \in V_\gamma, \tag{94}$$

where $L_i(x)$ is a diagonal matrix with entries the components of χ_i,

$$L_i(x) = \mathrm{diag}\{L_{i,1}, L_{i,2}, \ldots\}, \tag{95}$$

which is a finite matrix for class I theories, and is an infinite matrix for class II theories. Each term in $\cdots$ are Laurent polynomials in $\mathcal{Y}_j(x)$ and $\mathcal{Y}_i(x)^{-1}$ and the asymptotic power of x is the same as $\mathcal{Y}_i(x)$ near $x = \infty$. In the weakly coupled limit $\underline{\mathfrak{q}} \to 0$, $\chi_i(x) \to \mathcal{Y}_i(x)$. According to the asymptotic behavior near $x = \infty$, we know that they must be polynomials in x,

$$\chi_i(\mathcal{Y}(x)) = T_i(x), \tag{96}$$

where the coefficients of $T_i(x)$ are functions of the couplings $\underline{\mathfrak{q}}$, the masses $\underline{m}$, and the coordinate $\underline{u}$ on the Coulomb branch $\mathcal{B}$.

The Seiberg-Witten curve of the theory can be uniformly written as

$$\Sigma_u\colon\ \det\left(1 - t^{-1}\zeta(x) L_i(x)\right)\big|_{\chi_j = T_j} = 0, \tag{97}$$

where $\zeta(x)$ is a normalization factor to be determined, and the meromorphic differential takes the canonical form

$$\lambda = x\frac{dt}{t}. \tag{98}$$

V. SEIBERG-WITTEN GEOMETRY OF LINEAR QUIVER GAUGE THEORIES

In this section, we shall carefully describe the Seiberg-Witten geometry of linear quiver gauge theories as an illustrative example.

A. Linear quiver gauge theories

We consider the class I theory of A_r-type. The set of vertices is $V_\gamma = \{1, 2, \ldots, r\}$, and the edges connect vertices of nearest neighbors. The total gauge group has the structure [16,21]

$$G = \prod_{i=1}^{r} G_i = \cdots \times \mathrm{SO}(v_i) \times \mathrm{USp}(v_{i+1} - 2) \times \cdots, \tag{99}$$

where

$$v_1 < v_2 < \cdots < v_{\heartsuit-1} < v_\heartsuit = \cdots = v_\spadesuit > v_{\spadesuit+1} > \cdots v_r. \tag{100}$$

We refer to the parts to the left of $v_\heartsuit$ and to the right of $v_\spadesuit$ as the two tails of the quiver. We also define $v_0 = v_{r+1} = 0$. The condition (5) becomes

$$2v_i - v_{i-1} - v_{i+1} = 2w_i + \xi_i + 2\delta^{\bigcirc}_{i,1} + 2\delta^{\bigcirc}_{i,r}, \tag{101}$$

where $\delta^{\bigcirc}_{i,1} = 1$ if $i = 1 \in V^{\bigcirc}_\gamma$ and vanishes otherwise, and similarly for $\delta^{\bigcirc}_{i,r}$. It is convenient to express v_i in the following way,

$$v_i = \begin{cases} \sum_{j=1}^{i} d_j, & 1 \le i \le \heartsuit - 1 \\ 2N, & \heartsuit \le i \le \spadesuit, \\ \sum_{j=i}^{r} d_j & \spadesuit + 1 \le i \le r, \end{cases} \tag{102}$$

where

$$d_i = \begin{cases} v_i - v_{i-1}, & 1 \le i \le \heartsuit \\ 0, & \heartsuit + 1 \le i \le \spadesuit - 1\,, \\ v_i - v_{i+1}, & \spadesuit \le i \le r \end{cases} \tag{103}$$

satisfying

$$d_1 \ge d_2 \ge \cdots \ge d_\heartsuit \ge 0, \qquad 0 \le d_\spadesuit \le d_{\spadesuit+1} \le \cdots \le d_r, \tag{104}$$

and

$$\sum_{i=1}^{\heartsuit} d_i = \sum_{i=\heartsuit}^{\spadesuit} d_i = v_\heartsuit. \tag{105}$$

Therefore, it is natural to associate each tail with a Young tableau [16]. The Young tableau associated with the left tail has row lengths being nonincreasing integers $d_1-2\delta^{\bigcirc}_{1,1}, d_2, \ldots, d_{\heartsuit}$, and the difference between the ith and the $(i+1)$th row lengths gives $2w_i+\xi_i$. We also have a similar Young tableau associated with the right tail.

B. Seiberg-Witten curve

The instanton Weyl group ${}^i\mathcal{W}$ for the quiver of A_r-type is the symmetric group $\mathcal{S}_{r+1}$, which is generated by

$$\mathcal{Y}_i \mapsto P_i \mathcal{Y}_i^{-1}\mathcal{Y}_{i-1}\mathcal{Y}_{i+1}, \qquad i=1,\ldots,r, \tag{106}$$

where

$$P_i(x) = \mathsf{q}_i x^{2\delta^{\bigcirc}_{i,1}+2\delta^{\bigcirc}_{i,r}+\xi_i} \prod_{f=1}^{w_i}(x^2-m_{i,f}^2). \tag{107}$$

This is very similar to the case for the SU linear quiver gauge theories. Under a chain of Weyl reflections, we get a ${}^i\mathcal{W}$-orbit starting from $\mathcal{Y}_1$,

$$\mathcal{Y}_1 \xrightarrow{s_1} P^{[1]}\mathcal{Y}_1^{-1}\mathcal{Y}_2 \xrightarrow{s_2} P^{[2]}\mathcal{Y}_2^{-1}\mathcal{Y}_3 \xrightarrow{s_3} \cdots \xrightarrow{s_{r-1}} P^{[r-1]}\mathcal{Y}_{r-1}^{-1}\mathcal{Y}_r \xrightarrow{s_r} P^{[r]}\mathcal{Y}_r^{-1}, \tag{108}$$

where

$$P^{[i]} = \prod_{j=1}^{i} P_j, \qquad i=1,\ldots,r. \tag{109}$$

The instanton Weyl group ${}^i\mathcal{W}$ acts by permuting the eigenvalues of the matrix $L_1(x)$,

$$L_1(x) = \mathrm{diag}\{\mathcal{Y}_1, P^{[1]}\mathcal{Y}_1^{-1}\mathcal{Y}_2, P^{[2]}\mathcal{Y}_2^{-1}\mathcal{Y}_3, \ldots, P^{[r-1]}\mathcal{Y}_{r-1}^{-1}\mathcal{Y}_r, P^{[r]}\mathcal{Y}_r^{-1}\}. \tag{110}$$

We can check that the ith character is given by

$$\chi_i = (P^{[1,i-1]})^{-1}\mathsf{e}_i(\mathcal{Y}_1, P^{[1]}\mathcal{Y}_1^{-1}\mathcal{Y}_2, P^{[2]}\mathcal{Y}_2^{-1}\mathcal{Y}_3, \ldots, P^{[r-1]}\mathcal{Y}_{r-1}^{-1}\mathcal{Y}_r, P^{[r]}\mathcal{Y}_r^{-1}), \tag{111}$$

where we introduce the notation

$$P^{[i,j]} = \prod_{n=i}^{j} P^{[n]}, \tag{112}$$

and e_i is the ith elementary symmetric polynomial

$$\begin{aligned}
&\mathsf{e}_0(x_1,\ldots,x_n) = 1,\\
&\mathsf{e}_i(x_1,\ldots,x_n) = \sum_{1\le j_1<\cdots<j_i\le n} x_{j_1}\cdots x_{j_i},\\
&1\le i\le n.
\end{aligned} \tag{113}$$

In fact, the elementary symmetric polynomials are totally symmetric in the variables, and $\mathcal{Y}_i$ is one term in χ_i (111),

$$(P^{[1,i-1]})^{-1}(\mathcal{Y}_1)(P^{[1]}\mathcal{Y}_1^{-1}\mathcal{Y}_2)\cdots(P^{[i-1]}\mathcal{Y}_{i-1}^{-1}\mathcal{Y}_i) = \mathcal{Y}_i. \tag{114}$$

Therefore, (111) is the characters of the ${}^i\mathcal{W}$-orbits starting from $\mathcal{Y}_i$ for $i=1,\ldots,r$. We also define

$$\chi_0 = \chi_{r+1} = 1. \tag{115}$$

We expand the character polynomial in terms of the elementary symmetric polynomials

$$\begin{aligned}
\det(t-\zeta(x)L_1(x)) &= \sum_{i=0}^{r+1}(-1)^i\zeta(x)^i\mathsf{e}_i(L_1(x))t^{r+1-i}\\
&= \sum_{i=0}^{r+1}(-1)^i\zeta(x)^i P^{[1,i-1]}\chi_i(x)t^{r+1-i}.
\end{aligned} \tag{116}$$

Substituting $\chi_i(x)$ by $T_i(x)$, we get explicitly the Seiberg-Witten curve,

$$\begin{aligned}
\Sigma_u\colon\ & t^{r+1} + \sum_{i=1}^{r}(-1)^i\zeta(x)^i P^{[1,i-1]}T_i(x)t^{r+1-i}\\
&+ (-1)^r\zeta(x)^r P^{[1,r]} = 0,
\end{aligned} \tag{117}$$

where the polynomial $T_i(x)$ takes the form

$$\begin{aligned}
T_i(x) &= x^{v_i}(T_{i,0} + T_{i,1}x^{-2} + T_{i,2}x^{-4} + \cdots + T_{i,\lfloor\frac{v_i}{2}\rfloor}x^{-2\lfloor\frac{v_i}{2}\rfloor}),\\
&\quad i=1,\ldots,r,
\end{aligned} \tag{118}$$

with $\lfloor\alpha\rfloor$ being the greatest integer less than or equal to α. Notice that only even powers of x can appear in the bracket. The leading coefficient depends only on the couplings

$$T_{i,0} = \left(\prod_{j=1}^{i-1} \mathsf{q}_j^{j-i}\right) e_i(1, \mathsf{q}_1, \mathsf{q}_1\mathsf{q}_2, \ldots, \mathsf{q}_1\mathsf{q}_2\cdots\mathsf{q}_r). \quad (119)$$

The next-to-leading coefficient $T_{i,1}$ is a function of the couplings and the masses. The remaining coefficients encode the information of vacuum expectation values of Coulomb branch operators. The Seiberg-Witten curves (117) with different normalization factor $\zeta(x)$ contain the same physical information and are related to each other by a change of variables (t, x). The Seiberg-Witten curve obtained in [14,15] by lifting a system of D4/NS5/D6-branes with orientifolds in type IIA string theory to M-theory matches (117) with $\zeta(x) = -1$.

VI. FURTHER DEVELOPMENTS

In this paper, we derive the Seiberg-Witten geometry of SO − USp quiver gauge theories using the instanton counting method, with an emphasize on linear quiver gauge theories. Our discussion can be straightforwardly lifted to five-dimensional $\mathcal{N} = 1$ theories compactified on S^1 or six-dimensional $\mathcal{N} = (1, 0)$ theories compactified on T^2. For the partition function, the equivariant cohomology should be replaced by corresponding K-theoretical or elliptic version. The corresponding Seiberg-Witten geometry can be derived in the same way.

After solving the linear quiver gauge theories, it is very natural to also work out the other quiver gauge theories. Indeed, if the quiver is one of the ADE or affine ADE Dynkin diagrams, the analysis would be very similar to the corresponding SU quiver gauge theories [7]. However, it is more interesting to consider the non-Dynkin type quivers. Even the Seiberg-Witten solutions to most of them are unknown so far. The instanton counting method seems to be the most promising approach to solve them. We will discuss all these cases in Part II of our article.

There are many other open questions that will be studied in the future. We can study the Bethe/gauge correspondence between the supersymmetric gauge theories and quantum integrable systems by sending only $\varepsilon_2 \to 0$ while keeping $\varepsilon_1 = \hbar$ finite [22], generalizing the derivation for SU quiver gauge theories in [23,24]. The effective twisted superpotential can be obtained from the partition function via

$$\begin{aligned}\widetilde{W}^{\mathrm{eff}}&(\underline{\mathsf{q}}; \underline{a}, \underline{m}; \hbar) \\ &= -\lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}(\underline{\mathsf{q}}; \underline{a}, \underline{m}; \varepsilon_1 = \hbar, \varepsilon_2) + \widetilde{W}^{\infty}(\underline{a}, \underline{m}; \hbar),\end{aligned} \quad (120)$$

where $\widetilde{W}^{\infty}$ is the possible perturbative contribution from the boundary conditions at infinity, and $\widetilde{W}^{\mathrm{eff}}$ is identified with the Yang-Yang function of some quantum integrable system [22,25–29].

We can go one step further and consider the situation where both ε_1 and ε_2 are finite. In this case we shall introduce an interesting class of gauge-invariant observables, $\mathcal{Y}_i(x)$, whose vacuum expectation values are $\mathcal{Y}_i(x)$. We should be able to define the so-called qq-characters $\mathcal{X}_i(x)$, which are composite operators built from $\mathcal{Y}_i(x)$ and satisfy the nonperturbative Dyson-Schwinger equations [30,31]. The theory of qq-characters play an important role in the study of SU quiver gauge theories. For example, they can be used to derive the Belavin-Polyakov-Zamolodchikov equations from the field theory point of view [32,33]. It is also interesting to study the relations among the vacuum expectation values of chiral operators in the Ω-background [34]. A closely related issue is the study of the gauge origami [35] in the presence of the orientifold plane.

In recent years much of the investigation of supersymmetric gauge theories has involved the presence of nonlocal operators. We can naturally study surface operators in the Ω-background [36–44]. Unfortunately, almost nothing has been said about surface operators in the Ω-background when the gauge group is SO/USp.

Finally, we would like to emphasize that the analysis of SO − USp quiver gauge theories is not as rigorous as that of SU quiver gauge theories. The hazards come not only from the treatment of the half-hypermultiplets, but also the noncompactness of the moduli space of SO/USp instantons. When we take the flat space limit, we neglect a lot of information of the partition function in the Ω-background, and many potentially problematic issues are avoided. Hence we cannot say that we fully understand the Ω-background before we have completed the above generalizations from SU to SO − USp quiver gauge theories.

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APPENDIX: INSTANTON MODULI SPACE

In this appendix, we review some properties of the moduli spaces $\mathfrak{M}_{G,k}$ of framed G-instantons on $\mathbb{C}^2$ with instanton charge k,

$$\mathfrak{M}_{G,k} = \left\{ A \in \mathcal{A}_G | F + *F = 0, k = \frac{1}{16\pi^2 h^\vee} \int_{\mathbb{C}^2} \mathrm{Tr}_{\mathrm{adj}} F \wedge F \right\} \Big/ \mathcal{G}_\infty, \quad (A1)$$

where $\mathcal{A}_G$ is the space of G-connections, $h^\vee$ is the dual Coxeter number for the Lie algebra of G, $\mathrm{Tr}_{\mathrm{adj}}$ is the trace in the adjoint representation, and $\mathcal{G}_\infty$ is the group of gauge transformations that are identity at infinity. For G being a classical group, Atiyah, Drinfeld, Hitchin and Manin found a description of $\mathfrak{M}_{G,k}$ in terms of solutions to quadratic equations for certain finite-dimensional matrices [45].

It is useful to introduce the following notations. Let $S^\pm$ be the positive and negative spin bundles, and the line bundle $L = \mathcal{K}_{\mathbb{C}^2}^{1/2}$ be the half canonical bundle of $\mathbb{C}^2$. The group of rotations on $\mathbb{C}^2$ with a fixed translationally invariant symplectic form is $G_R = \mathrm{U}(2) \simeq \mathrm{SU}(2)_- \times \mathrm{U}(1)_+ \subset \mathrm{Spin}(4)$, under which S^+ splits as $L \oplus L^{-1}$.

1. U(n) instantons

We start with the moduli space of U(n) instantons. We introduce a quartet of linear operators

$$(B_1, B_2, I, J) \in \mathrm{Hom}(\mathbf{K}, \mathbf{K}) \otimes \mathbb{C}^2 \bigoplus \mathrm{Hom}(\mathbf{N}, \mathbf{K}) \bigoplus \mathrm{Hom}(\mathbf{K}, \mathbf{N}), \tag{A2}$$

where $\mathbf{K}$ and $\mathbf{N}$ are two complex vector spaces of dimension k and n, respectively. The moduli space of framed U(n) instantons on $\mathbb{C}^2$ with instanton charge k is given by the regular locus of the hyperkahler quotient of the space of operators (B_1, B_2, I, J) by the U(k) action,

$$\mathfrak{M}_{\mathrm{U}(n),k} \cong \{(B_1, B_2, I, J) | \mu_{\mathbb{C}} = 0, \mu_{\mathbb{R}} = 0\}^{\mathrm{reg}}/\mathrm{U}(k), \tag{A3}$$

where the ADHM moment maps are

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ, \tag{A4}$$

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \tag{A5}$$

the action of $g \in \mathrm{U}(k)$ is

$$g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}), \quad g \in \mathrm{U}(k), \tag{A6}$$

and the regularity requires that the group action of U(k) is free on the solution (B_1, B_2, I, J).

The ADHM construction can be represented by the following complex,

$$0 \to \mathbf{K} \otimes \mathcal{L}^{-1} \xrightarrow{\alpha} \mathbf{K} \otimes \mathcal{S}^- \bigoplus \mathbf{N} \xrightarrow{\beta} \mathbf{K} \otimes \mathcal{L} \to 0, \tag{A7}$$

where $\mathcal{L}$ and $\mathcal{S}^-$ are the fibers of L and S^-, respectively, and

$$\alpha = \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ J \end{pmatrix}, \qquad \beta = (\, -B_2 + z_2, \quad B_1 - z_1, \quad I \,). \tag{A8}$$

From the middle cohomology of the complex (A7), we can form the virtual universal bundle $\mathcal{E}$ on $\mathbb{C}^2 \times \mathfrak{M}_{\mathrm{U}(n),k}$ by

$$\mathcal{E} = \mathbf{N} \bigoplus \mathbf{K} \otimes (S^- \ominus S^+). \tag{A9}$$

The moduli space $\mathfrak{M}_{\mathrm{U}(n),k}$ has singularities due to pointlike instantons. In order to make the localization computations appropriate, we may work with another space

$$\mathfrak{M}^\zeta_{\mathrm{U}(n),k} \cong \{(B_1, B_2, I, J) | \mu_{\mathbb{C}} = 0, \mu_{\mathbb{R}} = \zeta \cdot \mathbb{I}_{\mathbf{K}}\}/\mathrm{U}(k), \tag{A10}$$

where $\zeta > 0$ is a constant. The moduli space $\mathfrak{M}^\zeta_{\mathrm{U}(n),k}$ is a $4nk$ dimensional smooth manifold, with the metric inherited from the flat metric on (B_1, B_2, I, J), and we can again form the universal sheaf $\mathcal{E}$ on $\mathbb{C}^2 \times \mathfrak{M}_{\mathrm{U}(n),k}$ similar to (A9). An equivalent description of $\mathfrak{M}^\zeta_{\mathrm{U}(n),k}$ can be given as

$$\mathfrak{M}^\zeta_{\mathrm{U}(n),k} \cong \{(B_1, B_2, I, J) | \mu_{\mathbb{C}} = 0, \mathbb{C}[B_1, B_2]I(\mathbf{N}) = \mathbf{K}\}/\mathrm{GL}(\mathbf{K}). \tag{A11}$$

It was shown in [46] that $\mathfrak{M}^\zeta_{\mathrm{U}(n),k}$ describes the moduli space of framed U(n) instantons on noncommutative $\mathbb{C}^2$ with instanton charge k. Mathematically, $\mathfrak{M}^\zeta_{\mathrm{U}(n),k}$ is the moduli space of framed torsion free sheaves $(E, \Phi\colon\ E|_{\mathbb{CP}^1_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus n}_{\mathbb{CP}^1_\infty})$ of rank N on $\mathbb{CP}^2 = \mathbb{C} \cup \mathbb{CP}^1_\infty$, with $\langle \mathrm{ch}_2(E), [\mathbb{CP}^2] \rangle = k$ [47].

There is a natural GL($\mathbf{N}$) action acting on the moduli space,

$$\rho \cdot (B_1, B_2, I, J) = (B_1, B_2, I\rho^{-1}, \rho J), \qquad \rho \in \mathrm{GL}(\mathbf{N}). \tag{A12}$$

The central GL($1, \mathbb{C}$) subgroup acts trivially due to the equivalence under $\mathrm{GL}(1, \mathbb{C}) \subset \mathrm{GL}(\mathbf{K})$. Meanwhile, the rotation symmetry of $\mathbb{C}^2$ induces a $(\mathbb{C}^*)^2$-action on the moduli space via

$$(B_1, B_2, I, J) \mapsto (q_1 B_1, q_2 B_2, I, q_1 q_2 J), \qquad q_1, q_2 \in \mathbb{C}^*. \tag{A13}$$

2. SO/USp instantons

The ADHM construction for SO/USp instantons can be obtained by a projection of the U(n) instantons. Here we follow the description given in [48]. We define $\mathrm{SO}(n)$ to be the special unitary transformations on $\mathbb{C}^n$ that preserve its real structure Φ_r, and define $\mathrm{USp}(2n)$ to be the special unitary transformations on $\mathbb{C}^{2n}$ that preserve its symplectic structure Φ_s.

For $\mathrm{SO}(n)$, we consider linear operators

$$(B_1, B_2, J) \in \mathrm{Hom}(\mathbf{K}, \mathbf{K}) \otimes \mathbb{C}^2 \bigoplus \mathrm{Hom}(\mathbf{K}, \mathbf{N}), \tag{A14}$$

where $\mathbf{K}$ and $\mathbf{N}$ are two complex vector spaces of dimension $2k$ and n, respectively, together with a symplectic structure Φ_s on $\mathbf{K}$ and a real structure Φ_r on $\mathbf{N}$. The moduli space of framed $\mathrm{SO}(n)$ instantons is given by

$$\begin{aligned}\mathfrak{M}_{\mathrm{SO}(n),k} = \{(B_1, B_2, J) | \Phi_s B_1, \Phi_s B_2 \in \wedge^2 \mathbf{K}^*,\\ \Phi_s[B_1, B_2] - J^* \Phi_r J = 0\}^{\mathrm{reg}}/\mathrm{USp}(2k).\end{aligned} \tag{A15}$$

Similarly, for $\mathrm{USp}(2n)$, we consider linear operators

$$(B_1, B_2, J) \in \mathrm{Hom}(\mathbf{K}, \mathbf{K}) \otimes \mathbb{C}^2 \bigoplus \mathrm{Hom}(\mathbf{K}, \mathbf{N}), \tag{A16}$$

where $\mathbf{K}$ and $\mathbf{N}$ are two complex vector spaces of dimension k and $2n$, respectively, together with a real structure Φ_r on $\mathbf{K}$ and a symplectic structure Φ_s on $\mathbf{N}$. The moduli space of framed $\mathrm{USp}(2n)$ instantons is given by

$$\begin{aligned}\mathfrak{M}_{\mathrm{USp}(2n),k} = \{(B_1, B_2, J) | \Phi_r B_1, \Phi_r B_2 \in S^2 \mathbf{K}^*,\\ \Phi_r[B_1, B_2] - J^* \Phi_s J = 0\}^{\mathrm{reg}}/\mathrm{O}(k).\end{aligned} \tag{A17}$$

For both $\mathrm{SO}(n)$ and $\mathrm{USp}(2n)$ instantons, we do not know the compactification of the moduli space of framed instantons which admits a universal bundle with the universal instanton connection over $\mathfrak{M}_{G,k} \times \mathbb{C}^2$. Nevertheless, we still have the ADHM complex

$$0 \to \mathbf{K} \otimes \mathcal{L}^{-1} \xrightarrow{\alpha} \mathbf{K} \otimes \mathcal{S}^- \bigoplus \mathbf{N} \xrightarrow{\alpha^* \beta^*} \mathbf{K}^* \otimes \mathcal{L} \to 0, \tag{A18}$$

where

$$\begin{aligned}\alpha &= \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ J \end{pmatrix},\\ \beta^{\mathrm{SO}} &= \begin{pmatrix} 0 & \Phi_s & 0 \\ -\Phi_s & 0 & 0 \\ 0 & 0 & -\Phi_r \end{pmatrix},\\ \beta^{\mathrm{Sp}} &= \begin{pmatrix} 0 & \Phi_r & 0 \\ -\Phi_r & 0 & 0 \\ 0 & 0 & -\Phi_s \end{pmatrix}.\end{aligned} \tag{A19}$$

The induced $(\mathbb{C}^*)^2$-action on the moduli space is given by

$$(B_1, B_2, J) \mapsto (q_1 B_1, q_2 B_2, q_+ J). \tag{A20}$$

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