Neutrino mass and mixing with A_5 modular symmetry

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We present a comprehensive analysis of neutrino mass and lepton mixing in theories with A_5 modular symmetry. We construct the weight 2, weight 4, and weight 6 modular forms of level 5 in terms of Dedekind eta-functions and Klein forms, and their decomposition into irreducible representation of A_5 . We construct all the simplest models based on A_5 modular symmetry, including scenarios of models with and without flavons in the charged lepton sectors. For each case, the neutrino masses can be generated through either the Weinberg operator or the type I seesaw mechanism. We perform an exhaustive numerical analysis, organizing our results in an extensive set of figures and tables.

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I. INTRODUCTION

The flavor puzzle, in particular the origin of neutrino mass and lepton mixing, is a major unresolved problem of the Standard Model (SM). The large mixing angles in the lepton sector can be explained using some discrete non-Abelian family symmetry [1,2]. Such a symmetry could either originate from a continuous non-Abelian gauge symmetry [3–9], or from extra dimensions [10–21]. In the case of extra dimensions, the discrete non-Abelian family symmetry could either arise as an accidental symmetry of the orbifold fixed points, or it could appear as a subgroup of the symmetry of the extra dimensional lattice vectors, commonly referred to as modular symmetry [22].

Some time ago it was suggested that a finite subgroup of the modular symmetry group, when interpreted as a family symmetry, might help to provide a possible explanation for the neutrino mass matrices [23,24]. Recently it has been suggested that neutrino masses might be modular forms [25], with constraints on the Yukawa couplings. This has led to a revival of the idea that modular symmetries are symmetries of the extra dimensional spacetime with Yukawa couplings determined by their modular weights [26]. The modular groups $\Gamma(2)$ [27,28], $\Gamma(3)$ [25,26,28–31], $\Gamma(4)$ [32,33], and $\Gamma(5)$ [34] and the phenomenological predictions for lepton mixing have all been discussed in the literature.

In the present work, we focus on the finite modular group $\Gamma_5 \cong A_5$. In particular, we construct the weight 2, weight 4, and weight 6 modular forms of level 5 in terms of the Dedekind eta-function and Klein forms, and their decomposition into irreducible representation of A_5 . This is complementary to [34] where only weight 2 and weight 4 modular forms are constructed using the Jacobi theta function. Since the results overlap, we have checked that the same q -expansions of the modular forms are obtained up to an overall irrelevant constant. We construct all the simplest models based on A_5 modular symmetry, both with and without flavons in the charged lepton sectors. For each case, the neutrino mass is generated through both the Weinberg operator or the type I seesaw mechanism. The model building discussed here goes beyond that in [34] where only two models were constructed under the assumption of diagonal charged lepton mass matrix, one of which is not viable. For each model, we give an extensive numerical analysis, organizing our results in a comprehensive set of figures and tables.

We highlight the new and original parts of our analysis in the following

- (i) We plot the fundamental domain of $\Gamma(5)$ in Fig. 1, this result is new as far as we know. When numerically analyzing any given A_5 modular model, one only needs to consider the values of the modulus τ in the fundamental domain such that the numerical calculations can be simplified considerably.
- (ii) Our paper overlaps with [34] in the weights 2, 4, 6 modular forms of $\Gamma(5)$. However, we use completely different mathematical tools: Dedekind eta-function and Klein forms versus Jacobi theta function.
- (iii) In our paper, scenarios both with and without flavons in the charged lepton sector are studied, and the possible realizations and corresponding predictions

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FIG. 1. Fundamental domain for $\Gamma(5)$.

for the charged lepton mass matrix are presented. By contrast, the charged lepton mass matrix is assumed to be diagonal in [34], and the possible models for the charged lepton sector are not discussed.

- (iv) In our paper, the neutrino masses are generated through either the Weinberg operator or the type I seesaw mechanism. By contrast, in [34] the neutrino masses are assumed to originate exclusively from the Weinberg operator.
- (v) We have constructed all the simplest models based on A_5 modular symmetry, and we propose 16 possible A_5 modular models, as summarized in Table IV, while only two models are built in [34].
- (vi) We have performed an exhaustive numerical analysis for each model, the best fitting values and the predicted regions of the mixing parameters are listed, and we present the results in the form of extensive sets of figures and tables. In comparison, only the numerical results at a benchmark point for each model are reported in [34].

The layout of the remainder of the paper is as follows. In Sec. II we review the basics of modular symmetry and modular forms of level $N = 5$ corresponding to A_5 . We also construct the modular space through an infinite product expansion, construct an eleven dimensional basis for the lowest weight modular forms, and decompose this space into the two irreducible triplets and one quintuplet representations of A_5 , from which the higher weight representations may be obtained. In Sec. III we systematically construct all the simplest models based on $\Gamma_5 \cong A_5$. The charged lepton sector is analysed both with and without flavons. The neutrino sector is analyzed without flavons using both the Weinberg operator and the type I seesaw mechanism. In Sec. IV we perform a comprehensive numerical analysis for each of the models discussed in the previous section. We give the best fit values of the parameters of each model and the corresponding predictions. We also give the possible regions

of the parameters and ranges of the predictions in graphical form. In Sec. V we summarize the main results of our paper. In the Appendix A we present the group theory of A_5 . In Appendix B we show that the charged lepton sector with flavons leads to a diagonal charged lepton mass matrix when the flavons are aligned appropriately.

II. MODULAR SYMMETRY AND MODULAR FORMS OF LEVEL $N = 5$

The *full modular group* $SL(2, \mathbb{Z})$ is the group of 2-by-2 matrices with integer entries and determinant one [35,36],

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}. \quad (1)$$

The modular group can be regarded as the linear fraction transformations of the upper half complex plane $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im}\tau > 0\}$, and it has the following form

$$\tau \mapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (2)$$

It is easy to see that γ and $-\gamma$ lead to the same linear fractional transformation. Therefore the modular group of the linear fraction transformations is isomorphic to the projective special linear group $PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z}) / \{1, -1\}$, which is the quotient of $SL(2, \mathbb{Z})$ over its center $\{1, -1\}$. Here 1 refers to the two-dimensional unit matrix. The modular group can be generated by two elements S and T [35]

$$\begin{aligned} S: \tau &\mapsto -\frac{1}{\tau}, & S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ T: \tau &\mapsto \tau + 1, & T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (3)$$

We see that the generators S and T obey the relations:

$$S^2 = (ST)^3 = 1. \quad (4)$$

There are some important subgroups of $SL(2, \mathbb{Z})$, i.e.,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \right. \\ \left. b \equiv c \equiv 0, a \equiv d \equiv 1 \pmod{N} \right\}, \quad (5)$$

where N is a positive integer, and $\Gamma(N)$ is usually called *principal congruence subgroup of level N* . Obviously we have $\Gamma(1) \cong SL(2, \mathbb{Z})$, and $\Gamma(N > 1)$ is the normal subgroup of $\Gamma(1)$. Taking the quotient of $\Gamma(N)$ over $\{1, -1\}$ or identifying γ with $-\gamma$, we obtain the group $\bar{\Gamma}(N) = \Gamma(N)/\{1, -1\}$. To be more specific, $\bar{\Gamma}(1) = \Gamma(1)/\{1, -1\}$ and $\bar{\Gamma}(2) = \Gamma(2)/\{1, -1\}$, and $\bar{\Gamma}(N > 2) = \Gamma(N)$ since the element -1 is not in $\Gamma(N)$ for $N > 2$. The finite modular group $\Gamma_N = \bar{\Gamma}(1)/\bar{\Gamma}(N)$ has been used as flavor symmetry to explain quark and lepton flavor mixing in past years [24]. Γ_N is isomorphic to permutation groups for smaller values of N . For example, $\Gamma_2 \cong S_3$, $\Gamma_3 \cong A_4$, $\Gamma_4 \cong S_4$ and $\Gamma_5 \cong A_5$. The multiplication rules of the group Γ_N can be obtained by extending Eq. (4) with the condition $T^N = 1$, such that the generators S and T of Γ_N satisfy

$$S^2 = (ST)^3 = T^N = 1. \quad (6)$$

The modular forms $f(\tau)$ of weight $2k$ and level N are holomorphic functions of the complex variable τ with well-defined transformation properties under the group $\Gamma(N)$:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} f(\tau) \\ \text{for } \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), \quad (7)$$

where $k \geq 0$ is an integer. We shall only consider modular forms of even weight in this work. The function $f(\tau)$ is required to be holomorphic in $\mathcal{H}$ and at all the cusps. In particular, the element T^N is in $\Gamma(N)$ and we have

$$f(\tau + N) = f(\tau). \quad (8)$$

In the present work, we shall study the $\Gamma(5)$ modular group and construct some models to explain neutrino masses and mixing. The quotient space $\mathcal{H}/\Gamma(5)$ can be described by a fundamental domain $\mathcal{F}_5$ for $\Gamma(5)$, that is a connected region of $\mathcal{H}$ such that every $z \in \mathcal{H}$ can be mapped into $\mathcal{F}_5$ by a $\Gamma(5)$ transformation, but no two distinct points in the interior of $\mathcal{F}_5$ are related under $\Gamma(5)$. The space $\mathcal{H}/\Gamma(5)$ is simply $\mathcal{F}_5$ with certain boundary points identified. A fundamental domain for $\Gamma(5)$ is shown in Fig. 1. It can be constructed by the fundamental domain $\mathcal{F} = \{\tau \in \mathcal{H} | -1/2 \leq \text{Re}\tau \leq 1/2 \text{ and } |\tau| \geq 1\}$ of $SL(2, \mathbb{Z})$ [37]. The cusps are $i\infty$, -2 ,

-1 , $-\frac{3}{2}$, $-\frac{1}{2}$, 0 , $\frac{2}{5}$, $\frac{1}{2}$, 1 , $\frac{3}{2}$, 2 , and $\frac{5}{2}$. $\mathcal{H}/\Gamma(5)$ can be made compact by adding these cusp points. The compactified space $\mathcal{H}/\Gamma(5)$ has genus zero and can be thought of as an icosahedron whose vertices are the cusps. Indeed the cusps are related by transformations of $\Gamma_5 = \bar{\Gamma}(1)/\bar{\Gamma}(5)$, which is icosahedral group, given the isomorphism between Γ_5 and A_5 . It is sufficient to focus on the values of the modulus τ in the above fundamental region when performing numerical analysis of a given A_5 modular symmetry model.

The modular forms of weight $2k$ and level $N = 5$ form a linear space $\mathcal{M}_{2k}(\Gamma(5))$, and its dimension turns out to be $10k + 1$ [25,38]. For the lowest nontrivial weight $2k = 2$, the dimension is equal to 11. The modular space $\mathcal{M}_{2k}(\Gamma(5))$ can be constructed from the Dedekind eta-function and the Klein form. In the following, we present the definition and basic properties of the Dedekind eta-function and the Klein form which are the building blocks to construct modular forms of level 5.

For any complex number τ with $\text{Im}\tau > 0$, the Dedekind eta-function $\eta(\tau)$ is defined to be the infinite product [35,36,39],

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q \equiv e^{i2\pi\tau}. \quad (9)$$

This function plays an important role in constructing various modular forms of integral or half-integral weight. It is well known that $\eta(\tau)$ satisfies the following identities [35,36,39]

$$\eta(\tau + 1) = e^{i\pi/12} \eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (10)$$

Then we turn to the Klein form. Let $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice in $\mathbb{C}$ and any rational pair $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ with $z = r_1\tau + r_2$, the Klein form $\mathfrak{k}_{(r_1, r_2)}(\tau)$ is defined by [39–41]

$$\mathfrak{k}_{(r_1, r_2)}(\tau) = -2\pi i z e^{\pi i r_1 z} \prod_{\omega \in \Lambda'} \left(1 + \frac{z}{\omega}\right) e^{-\frac{z}{\omega}}, \quad (11)$$

where $\Lambda' = \Lambda \setminus \{0\}$, and the product over $\omega = m\tau + n$ are performed over n first and then m . Notice that there is an extra factor $i/2\pi$ in the original definition of [39–41]. From the definition of Eq. (11) we know that the Klein form $\mathfrak{k}_{(r_1, r_2)}(\tau)$ is a holomorphic function which has no zeros and poles on $\mathcal{H}$. The Klein form has the following infinite product expansion [39–42]:

$$\mathfrak{k}_{(r_1, r_2)}(\tau) = q_z^{(r_1-1)/2} (1 - q_z) \\ \times \prod_{n=1}^{\infty} (1 - q^n q_z) (1 - q^n q_z^{-1}) (1 - q^n)^{-2}, \quad (12)$$

where $q_z = e^{2\pi i z}$. It is proved that the Klein form satisfies the following transformation formulas [39–41]

$$\begin{aligned}\mathfrak{f}_{(-r_1, -r_2)}(\tau) &= -\mathfrak{f}_{(r_1, r_2)}(\tau), \\ \mathfrak{f}_{(r_1, r_2)}(\gamma(\tau)) &= (c\tau + d)^{-1} \mathfrak{f}_{(r_1, r_2) \cdot \gamma}(\tau) \text{ for } \gamma \in \bar{\Gamma}(1), \\ \mathfrak{f}_{(r_1+s_1, r_2+s_2)}(\tau) &= \epsilon(\vec{r}, \vec{s}) \mathfrak{f}_{(r_1, r_2)}(\tau) \text{ for } (s_1, s_2) \in \mathbb{Z}^2, \quad (13)\end{aligned}$$

with $\epsilon(\vec{r}, \vec{s}) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i(s_1 r_2 - s_2 r_1)}$. Using Eq. (13), we find the Klein form $\mathfrak{f}_{(r_1, r_2)}(\tau)$ fulfills

$$\begin{aligned}\mathfrak{f}_{(r_1, r_2)}(\tau + 1) &= \mathfrak{f}_{(r_1, r_1 + r_2)}(\tau), \\ \mathfrak{f}_{(r_1, r_2)}\left(-\frac{1}{\tau}\right) &= -\frac{1}{\tau} \mathfrak{f}_{(-r_2, r_1)}(\tau).\end{aligned} \quad (14)$$

The modular space $\mathcal{M}_{2k}(\Gamma(5))$ has been explicitly constructed through η function and Klein form as follow [43]

$$\mathcal{M}_{2k}(\Gamma(5)) = \bigoplus_{a+b=10k, a, b \geq 0} \mathbb{C} \frac{\eta^{30k}(5\tau)}{\eta^{6k}(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^a(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^b(5\tau). \quad (15)$$

Obviously the dimension of $\mathcal{M}_{2k}(\Gamma(5))$ is $10k + 1$. For the lowest weight 2 modular forms, we could choose the basis vectors to be

$$\begin{aligned}\hat{e}_1(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^{10}(5\tau), \\ \hat{e}_2(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^9(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^0(5\tau), \\ \hat{e}_3(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^2(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^8(5\tau), \\ \hat{e}_4(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^3(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^7(5\tau), \\ \hat{e}_5(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^4(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^6(5\tau), \\ \hat{e}_6(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^5(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^5(5\tau), \\ \hat{e}_7(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^6(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^4(5\tau), \\ \hat{e}_8(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^7(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^3(5\tau), \\ \hat{e}_9(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^8(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^2(5\tau), \\ \hat{e}_{10}(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^9(5\tau) \mathfrak{f}_{\frac{2}{5}, \frac{0}{5}}^1(5\tau), \\ \hat{e}_{11}(\tau) &= \frac{\eta^{30}(5\tau)}{\eta^6(\tau)} \mathfrak{f}_{\frac{1}{5}, \frac{0}{5}}^{10}(5\tau).\end{aligned} \quad (16)$$

The above basis vectors are linearly independent and they span the whole weight 2 modular space $\mathcal{M}_2(\Gamma(5))$. Any modular function of weight 2 and level 5 can be expressed as a linear combination of the basis elements $\hat{e}_i$ with $i = 1, 2, \dots, 11$. Under the action of the generator T , they transform as

$$\begin{aligned}\hat{e}_1(\tau) &\rightarrow \hat{e}_1(\tau), \quad \hat{e}_2(\tau) \rightarrow e^{i\frac{2\pi}{5}} \hat{e}_2(\tau), \quad \hat{e}_3(\tau) \rightarrow e^{i\frac{4\pi}{5}} \hat{e}_3(\tau), \\ \hat{e}_4(\tau) &\rightarrow e^{i\frac{6\pi}{5}} \hat{e}_4(\tau), \quad \hat{e}_5(\tau) \rightarrow e^{i\frac{8\pi}{5}} \hat{e}_5(\tau), \quad \hat{e}_6(\tau) \rightarrow \hat{e}_6(\tau), \\ \hat{e}_7(\tau) &\rightarrow e^{i\frac{2\pi}{5}} \hat{e}_7(\tau), \quad \hat{e}_8(\tau) \rightarrow e^{i\frac{4\pi}{5}} \hat{e}_8(\tau), \quad \hat{e}_9(\tau) \rightarrow e^{i\frac{6\pi}{5}} \hat{e}_9(\tau), \\ \hat{e}_{10}(\tau) &\rightarrow e^{i\frac{8\pi}{5}} \hat{e}_{10}(\tau), \quad \hat{e}_{11}(\tau) \rightarrow \hat{e}_{11}(\tau).\end{aligned} \quad (17)$$

Furthermore, we find the following transformation properties under another generator S ,

$$\begin{aligned}\hat{e}_1(\tau) &\rightarrow \tau^2 \left\{ -\left(\frac{1}{10} + \frac{11}{50\sqrt{5}}\right) \hat{e}_1(\tau) - \left(\frac{3}{5} + \frac{7}{5\sqrt{5}}\right) \hat{e}_2(\tau) \right. \\ &\quad - \left(\frac{9}{5} + \frac{18}{5\sqrt{5}}\right) \hat{e}_3(\tau) - \left(\frac{12}{5} + \frac{36}{5\sqrt{5}}\right) \hat{e}_4(\tau) \\ &\quad - \left(\frac{21}{5} + \frac{21}{5\sqrt{5}}\right) \hat{e}_5(\tau) - \frac{252}{25\sqrt{5}} \hat{e}_6(\tau) \\ &\quad + \frac{21}{25}(-5 + \sqrt{5}) \hat{e}_7(\tau) + \frac{12}{25}(5 - 3\sqrt{5}) \hat{e}_8(\tau) \\ &\quad + \frac{9}{25}(-5 + 2\sqrt{5}) \hat{e}_9(\tau) + \left(\frac{3}{5} - \frac{7}{5\sqrt{5}}\right) \hat{e}_{10}(\tau) \\ &\quad \left. + \frac{1}{250}(-25 + 11\sqrt{5}) \hat{e}_{11}(\tau) \right\}.\end{aligned} \quad (18)$$

$$\begin{aligned}\hat{e}_2(\tau) &\rightarrow \tau^2 \left\{ -\left(\frac{3}{50} + \frac{7}{50\sqrt{5}}\right) \hat{e}_1(\tau) - \left(\frac{13}{50} + \frac{1}{2\sqrt{5}}\right) \hat{e}_2(\tau) \right. \\ &\quad - \left(\frac{9}{50} + \frac{9}{10\sqrt{5}}\right) \hat{e}_3(\tau) + \left(-\frac{6}{25} + \frac{6}{5\sqrt{5}}\right) \hat{e}_4(\tau) \\ &\quad + \frac{42}{25} \hat{e}_5(\tau) + \frac{126}{25\sqrt{5}} \hat{e}_6(\tau) + \frac{42}{25} \hat{e}_7(\tau) \\ &\quad + \frac{6}{25}(1 + \sqrt{5}) \hat{e}_8(\tau) + \frac{9}{50}(-1 + \sqrt{5}) \hat{e}_9(\tau) \\ &\quad \left. + \frac{1}{50}(13 - 5\sqrt{5}) \hat{e}_{10}(\tau) + \frac{1}{250}(-15 + 7\sqrt{5}) \hat{e}_{11}(\tau) \right\}.\end{aligned} \quad (19)$$

$$\begin{aligned}\hat{e}_3(\tau) &\rightarrow \tau^2 \left\{ -\left(\frac{1}{25} + \frac{2}{25\sqrt{5}}\right) \hat{e}_1(\tau) - \left(\frac{1}{25} + \frac{1}{5\sqrt{5}}\right) \hat{e}_2(\tau) \right. \\ &\quad + \left(-\frac{1}{50} + \frac{1}{2\sqrt{5}}\right) \hat{e}_3(\tau) + \frac{16}{25} \hat{e}_4(\tau) \\ &\quad + \left(-\frac{7}{25} + \frac{7}{5\sqrt{5}}\right) \hat{e}_5(\tau) - \frac{28}{25\sqrt{5}} \hat{e}_6(\tau) \\ &\quad - \frac{7}{25}(1 + \sqrt{5}) \hat{e}_7(\tau) - \frac{16}{25} \hat{e}_8(\tau) \\ &\quad + \frac{1}{50}(-1 - 5\sqrt{5}) \hat{e}_9(\tau) + \frac{1}{25}(1 - \sqrt{5}) \hat{e}_{10}(\tau) \\ &\quad \left. + \frac{1}{125}(-5 + 2\sqrt{5}) \hat{e}_{11}(\tau) \right\}.\end{aligned} \quad (20)$$

$$\begin{aligned} \hat{e}_4(\tau) \rightarrow \tau^2 \left\{ -\left(\frac{1}{50} + \frac{3}{50\sqrt{5}}\right)\hat{e}_1(\tau) + \left(-\frac{1}{50} + \frac{1}{10\sqrt{5}}\right)\hat{e}_2(\tau) + \frac{6}{25}\hat{e}_3(\tau) + \left(-\frac{9}{50} + \frac{1}{2\sqrt{5}}\right)\hat{e}_4(\tau) - \left(\frac{7}{50} + \frac{7}{10\sqrt{5}}\right)\hat{e}_5(\tau) \right. \\ \left. - \frac{21}{25\sqrt{5}}\hat{e}_6(\tau) + \frac{7}{50}(-1 + \sqrt{5})\hat{e}_7(\tau) + \frac{1}{50}(9 + 5\sqrt{5})\hat{e}_8(\tau) + \frac{6}{25}\hat{e}_9(\tau) + \frac{1}{50}(1 + \sqrt{5})\hat{e}_{10}(\tau) + \frac{1}{250}(-5 + 3\sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (21)$$

$$\begin{aligned} \hat{e}_5(\tau) \rightarrow \tau^2 \left\{ -\left(\frac{1}{50} + \frac{1}{50\sqrt{5}}\right)\hat{e}_1(\tau) + \frac{2}{25}\hat{e}_2(\tau) + \left(-\frac{3}{50} + \frac{3}{10\sqrt{5}}\right)\hat{e}_3(\tau) - \left(\frac{2}{25} + \frac{2}{5\sqrt{5}}\right)\hat{e}_4(\tau) + \left(\frac{3}{50} - \frac{1}{2\sqrt{5}}\right)\hat{e}_5(\tau) \right. \\ \left. + \frac{18}{25\sqrt{5}}\hat{e}_6(\tau) + \frac{1}{50}(3 + 5\sqrt{5})\hat{e}_7(\tau) - \frac{2}{25}(-1 + \sqrt{5})\hat{e}_8(\tau) - \frac{3}{50}(1 + \sqrt{5})\hat{e}_9(\tau) - \frac{2}{25}\hat{e}_{10}(\tau) + \frac{1}{250}(-5 + \sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (22)$$

$$\begin{aligned} \hat{e}_6(\tau) \rightarrow \tau^2 \left\{ -\frac{1}{25\sqrt{5}}\hat{e}_1(\tau) + \frac{1}{5\sqrt{5}}\hat{e}_2(\tau) - \frac{1}{5\sqrt{5}}\hat{e}_3(\tau) - \frac{2}{5\sqrt{5}}\hat{e}_4(\tau) + \frac{3}{5\sqrt{5}}\hat{e}_5(\tau) + \frac{11}{25\sqrt{5}}\hat{e}_6(\tau) \right. \\ \left. - \frac{3}{5\sqrt{5}}\hat{e}_7(\tau) - \frac{2}{5\sqrt{5}}\hat{e}_8(\tau) + \frac{1}{5\sqrt{5}}\hat{e}_9(\tau) + \frac{1}{5\sqrt{5}}\hat{e}_{10}(\tau) + \frac{1}{25\sqrt{5}}\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (23)$$

$$\begin{aligned} \hat{e}_7(\tau) \rightarrow \tau^2 \left\{ \frac{1}{250}(-5 + \sqrt{5})\hat{e}_1(\tau) + \frac{2}{25}\hat{e}_2(\tau) - \left(\frac{3}{50} + \frac{3}{10\sqrt{5}}\right)\hat{e}_3(\tau) + \left(-\frac{2}{25} + \frac{2}{5\sqrt{5}}\right)\hat{e}_4(\tau) + \left(\frac{3}{50} + \frac{1}{2\sqrt{5}}\right)\hat{e}_5(\tau) \right. \\ \left. - \frac{18}{25\sqrt{5}}\hat{e}_6(\tau) + \frac{1}{50}(3 - 5\sqrt{5})\hat{e}_7(\tau) + \frac{2}{25}(1 + \sqrt{5})\hat{e}_8(\tau) - \frac{3}{50}(-1 + \sqrt{5})\hat{e}_9(\tau) - \frac{2}{25}\hat{e}_{10}(\tau) - \frac{1}{250}(5 + \sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (24)$$

$$\begin{aligned} \hat{e}_8(\tau) \rightarrow \tau^2 \left\{ \left(\frac{1}{50} - \frac{3}{50\sqrt{5}}\right)\hat{e}_1(\tau) + \left(\frac{1}{50} + \frac{1}{10\sqrt{5}}\right)\hat{e}_2(\tau) - \frac{6}{25}\hat{e}_3(\tau) + \left(\frac{9}{50} + \frac{1}{2\sqrt{5}}\right)\hat{e}_4(\tau) + \left(\frac{7}{50} - \frac{7}{10\sqrt{5}}\right)\hat{e}_5(\tau) - \frac{21}{25\sqrt{5}}\hat{e}_6(\tau) \right. \\ \left. + \frac{7}{50}(1 + \sqrt{5})\hat{e}_7(\tau) + \frac{1}{50}(-9 + 5\sqrt{5})\hat{e}_8(\tau) - \frac{6}{25}\hat{e}_9(\tau) + \frac{1}{50}(-1 + \sqrt{5})\hat{e}_{10}(\tau) + \frac{1}{250}(5 + 3\sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (25)$$

$$\begin{aligned} \hat{e}_9(\tau) \rightarrow \tau^2 \left\{ \left(-\frac{1}{25} + \frac{2}{25\sqrt{5}}\right)\hat{e}_1(\tau) + \left(-\frac{1}{25} + \frac{1}{5\sqrt{5}}\right)\hat{e}_2(\tau) - \left(\frac{1}{50} + \frac{1}{2\sqrt{5}}\right)\hat{e}_3(\tau) + \frac{16}{25}\hat{e}_4(\tau) - \left(\frac{7}{25} + \frac{7}{5\sqrt{5}}\right)\hat{e}_5(\tau) \right. \\ \left. + \frac{28}{25\sqrt{5}}\hat{e}_6(\tau) + \frac{7}{25}(-1 + \sqrt{5})\hat{e}_7(\tau) - \frac{16}{25}\hat{e}_8(\tau) + \frac{1}{50}(-1 + 5\sqrt{5})\hat{e}_9(\tau) + \frac{1}{25}(1 + \sqrt{5})\hat{e}_{10}(\tau) + \frac{1}{125}(-5 - 2\sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (26)$$

$$\begin{aligned} \hat{e}_{10}(\tau) \rightarrow \tau^2 \left\{ \left(\frac{3}{50} - \frac{7}{50\sqrt{5}}\right)\hat{e}_1(\tau) + \left(\frac{13}{50} - \frac{1}{2\sqrt{5}}\right)\hat{e}_2(\tau) + \left(\frac{9}{50} - \frac{9}{10\sqrt{5}}\right)\hat{e}_3(\tau) + \left(\frac{6}{25} + \frac{6}{5\sqrt{5}}\right)\hat{e}_4(\tau) - \frac{42}{25}\hat{e}_5(\tau) + \frac{126}{25\sqrt{5}}\hat{e}_6(\tau) \right. \\ \left. - \frac{42}{25}\hat{e}_7(\tau) + \frac{6}{25}(-1 + \sqrt{5})\hat{e}_8(\tau) + \frac{9}{50}(1 + \sqrt{5})\hat{e}_9(\tau) + \frac{1}{50}(-13 - 5\sqrt{5})\hat{e}_{10}(\tau) + \frac{1}{250}(15 + 7\sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (27)$$

$$\begin{aligned} \hat{e}_{11}(\tau) \rightarrow \tau^2 \left\{ \left(-\frac{1}{10} + \frac{11}{50\sqrt{5}}\right)\hat{e}_1(\tau) + \left(-\frac{3}{5} + \frac{7}{5\sqrt{5}}\right)\hat{e}_2(\tau) + \left(-\frac{9}{5} + \frac{18}{5\sqrt{5}}\right)\hat{e}_3(\tau) + \left(-\frac{12}{5} + \frac{36}{5\sqrt{5}}\right)\hat{e}_4(\tau) \right. \\ \left. + \left(-\frac{21}{5} + \frac{21}{5\sqrt{5}}\right)\hat{e}_5(\tau) + \frac{252}{25\sqrt{5}}\hat{e}_6(\tau) - \frac{21}{25}(5 + \sqrt{5})\hat{e}_7(\tau) + \frac{12}{25}(5 + 3\sqrt{5})\hat{e}_8(\tau) - \frac{9}{25}(5 + 2\sqrt{5})\hat{e}_9(\tau) \right. \\ \left. + \left(\frac{3}{5} + \frac{7}{5\sqrt{5}}\right)\hat{e}_{10}(\tau) - \frac{1}{250}(25 + 11\sqrt{5})\hat{e}_{11}(\tau) \right\}. \end{aligned} \quad (28)$$

We see that the basis vectors $\hat{e}_i$ are closed under S and T up to multiplicative factors, and each element is exactly mapped into itself under the action of S^2 , $(ST)^3$, and T^5 . Therefore we conclude that $\hat{e}_i$ really span the whole

modular space $\mathcal{M}_2(\Gamma(5))$. As shown in [25], the modular space of weight $2k$ and level N can always be decomposed into different irreducible representations of Γ_N . As a consequence, there should exist modular forms $f_i(\tau)$ of weight 2 and level 5 which transform under $\Gamma_5 \cong A_5$ as

$$f_i(\gamma(\tau)) = (c\tau + d)^2 \rho(\gamma)_{ij} f_j(\tau), \quad (29)$$

$$Y_3 \equiv \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ e_3(\tau) \end{pmatrix}, \quad Y_{3'} \equiv \begin{pmatrix} e'_1(\tau) \\ e'_2(\tau) \\ e'_3(\tau) \end{pmatrix}, \quad Y_5 \equiv \begin{pmatrix} \tilde{e}_1(\tau) \\ \tilde{e}_2(\tau) \\ \tilde{e}_3(\tau) \\ \tilde{e}_4(\tau) \\ \tilde{e}_5(\tau) \end{pmatrix}, \quad (32)$$

with

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (30)$$

is a representative element of Γ_N and $\rho(\gamma)$ is a unitary representation matrix of γ in Γ_N . It is sufficient that Eq. (29) is satisfied for the generators S and T [25], i.e.,

$$f_i(-1/\tau) = \tau^2 \rho(S)_{ij} f_j(\tau), \quad f_i(\tau + 1) = \rho(T)_{ij} f_j(\tau), \quad (31)$$

where the representation matrices $\rho(S)$ and $\rho(T)$ are given in Eq. (A3). Solving the constraints in Eq. (31), we can construct modular form Y_3 , $Y_{3'}$, and Y_5 which transform as **3**, **3'** and **5** of A_5 respectively,

$$\begin{aligned} e_1(\tau) &= \hat{e}_1(\tau) - 36\hat{e}_6(\tau) - \hat{e}_{11}(\tau), \\ e_2(\tau) &= 5\sqrt{2}\hat{e}_2(\tau) - 15\sqrt{2}\hat{e}_7(\tau), \\ e_3(\tau) &= 15\sqrt{2}\hat{e}_5(\tau) + 5\sqrt{2}\hat{e}_{10}(\tau), \\ e'_1(\tau) &= \hat{e}_1(\tau) + 14\hat{e}_6(\tau) - \hat{e}_{11}(\tau), \\ e'_2(\tau) &= -5\sqrt{2}\hat{e}_3(\tau) - 10\sqrt{2}\hat{e}_8(\tau), \\ e'_3(\tau) &= -10\sqrt{2}\hat{e}_4(\tau) + 5\sqrt{2}\hat{e}_9(\tau), \\ \tilde{e}_1(\tau) &= \hat{e}_1(\tau) + \hat{e}_{11}(\tau), \\ \tilde{e}_2(\tau) &= -\sqrt{6}\hat{e}_2(\tau) - 7\sqrt{6}\hat{e}_7(\tau), \\ \tilde{e}_3(\tau) &= -3\sqrt{6}\hat{e}_3(\tau) + 4\sqrt{6}\hat{e}_8(\tau), \\ \tilde{e}_4(\tau) &= -4\sqrt{6}\hat{e}_4(\tau) - 3\sqrt{6}\hat{e}_9(\tau), \\ \tilde{e}_5(\tau) &= -7\sqrt{6}\hat{e}_5(\tau) + \sqrt{6}\hat{e}_{10}(\tau). \end{aligned} \quad (33)$$

The q -expansion of the above linearly independent modular forms e_i , e'_i , and $\tilde{e}_i$ are given by

$$\begin{aligned} e_1(\tau) &= 1 - 30q - 20q^2 - 40q^3 - 90q^4 - 130q^5 - 60q^6 - 120q^7 - 100q^8 + \dots, \\ e_2(\tau) &= 5\sqrt{2}q^{\frac{1}{5}}(1 + 2q + 12q^2 + 11q^3 + 12q^4 + 12q^5 + 32q^6 + 21q^7 + 42q^8 + \dots), \\ e_3(\tau) &= 5\sqrt{2}q^{\frac{4}{5}}(3 + 7q + 6q^2 + 20q^3 + 10q^4 + 30q^5 + 16q^6 + 24q^7 + 36q^8 + \dots), \\ e'_1(\tau) &= 1 + 20q + 30q^2 + 60q^3 + 60q^4 + 120q^5 + 40q^6 + 180q^7 + 150q^8 + \dots, \\ e'_2(\tau) &= -5\sqrt{2}q^{\frac{3}{5}}(1 + 6q + 6q^2 + 16q^3 + 12q^4 + 20q^5 + 21q^6 + 36q^7 + 12q^8 + \dots), \\ e'_3(\tau) &= -5\sqrt{2}q^{\frac{3}{5}}(2 + 5q + 12q^2 + 7q^3 + 22q^4 + 18q^5 + 24q^6 + 20q^7 + 42q^8 + \dots), \\ \tilde{e}_1(\tau) &= 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 6q^5 + 72q^6 + 48q^7 + 90q^8 + \dots, \\ \tilde{e}_2(\tau) &= -\sqrt{6}q^{\frac{1}{5}}(1 + 12q + 12q^2 + 31q^3 + 32q^4 + 42q^5 + 32q^6 + 91q^7 + 42q^8 + \dots), \\ \tilde{e}_3(\tau) &= -\sqrt{6}q^{\frac{3}{5}}(3 + 8q + 28q^2 + 18q^3 + 36q^4 + 40q^5 + 63q^6 + 38q^7 + 96q^8 + \dots), \\ \tilde{e}_4(\tau) &= -\sqrt{6}q^{\frac{3}{5}}(4 + 15q + 14q^2 + 39q^3 + 24q^4 + 56q^5 + 48q^6 + 60q^7 + 44q^8 + \dots), \\ \tilde{e}_5(\tau) &= -\sqrt{6}q^{\frac{4}{5}}(7 + 13q + 24q^2 + 20q^3 + 60q^4 + 30q^5 + 54q^6 + 56q^7 + 84q^8 + \dots). \end{aligned} \quad (34)$$

The weight 2 modular forms of $\Gamma(5)$ have been constructed in [34] in terms of Jacobi theta functions, and it is also found that these eleven modular forms transform as the representations **3**, **3'**, and **5** of A_5 . In the present work, we have constructed the $\Gamma(5)$ modular forms by using the Dedekind eta-function and Klein forms, which is an independent method to the Jacobi theta function approach, and we obtain the same q -expansions of the modular forms as those given in [34], up to irrelevant factors.

A. Weights 4 and 6 modular forms of level $N=5$

The weight 4 modular forms can be generated by the tensor products of weight 2 modular forms, and they can be arranged into different A_5 irreducible representations as well:

$$Y_{1,I}^{(4)} = (Y_3 Y_3)_1 = e_1^2 + 2e_2 e_3, \quad Y_{1,II}^{(4)} = (Y_3 Y_3)_1 = e_1'^2 + 2e_2' e_3', \quad Y_{1,III}^{(4)} = (Y_5 Y_5)_1 = \tilde{e}_1^2 + 2\tilde{e}_2 \tilde{e}_5 + 2\tilde{e}_3 \tilde{e}_4. \quad (35)$$

$$Y_{3,I}^{(4)} = (Y_3 Y_5)_3 = (-2e_1 \tilde{e}_1 + \sqrt{3}(e_2 \tilde{e}_5 + e_3 \tilde{e}_2), \sqrt{3}e_1 \tilde{e}_2 + e_2 \tilde{e}_1 - \sqrt{6}e_3 \tilde{e}_3, \sqrt{3}e_1 \tilde{e}_5 - \sqrt{6}e_2 \tilde{e}_4 + e_3 \tilde{e}_1)^T, \\ Y_{3,II}^{(4)} = (Y_3 Y_5)_3 = (\sqrt{3}e_1' \tilde{e}_1 + e_2' \tilde{e}_4 + e_3' \tilde{e}_3, e_1' \tilde{e}_2 - \sqrt{2}(e_2' \tilde{e}_5 + e_3' \tilde{e}_4), e_1' \tilde{e}_5 - \sqrt{2}(e_2' \tilde{e}_3 + e_3' \tilde{e}_2))^T. \quad (36)$$

$$Y_{3',I}^{(4)} = (Y_3 Y_5)_{3'} = (\sqrt{3}e_1 \tilde{e}_1 + e_2 \tilde{e}_5 + e_3 \tilde{e}_2, e_1 \tilde{e}_3 - \sqrt{2}(e_2 \tilde{e}_2 + e_3 \tilde{e}_4), e_1 \tilde{e}_4 - \sqrt{2}(e_2 \tilde{e}_3 + e_3 \tilde{e}_5))^T, \\ Y_{3',II}^{(4)} = (Y_3 Y_5)_{3'} = (-2e_1' \tilde{e}_1 + \sqrt{3}(e_2' \tilde{e}_4 + e_3' \tilde{e}_3), \sqrt{3}e_1' \tilde{e}_3 + e_2' \tilde{e}_1 - \sqrt{6}e_3' \tilde{e}_5, \sqrt{3}e_1' \tilde{e}_4 - \sqrt{6}e_2' \tilde{e}_2 + e_3' \tilde{e}_1)^T. \quad (37)$$

$$Y_{4,I}^{(4)} = \frac{1}{2\sqrt{3}}(Y_5 Y_5)_{4_s} = (\sqrt{6}\tilde{e}_1 \tilde{e}_2 - \tilde{e}_3 \tilde{e}_5 + 2\tilde{e}_4^2, \sqrt{6}\tilde{e}_1 \tilde{e}_3 + 2\tilde{e}_2^2 - \tilde{e}_4 \tilde{e}_5, \sqrt{6}\tilde{e}_1 \tilde{e}_4 - \tilde{e}_2 \tilde{e}_3 + 2\tilde{e}_5^2, \sqrt{6}\tilde{e}_1 \tilde{e}_5 - \tilde{e}_2 \tilde{e}_4 + 2\tilde{e}_3^2)^T, \\ Y_{4,II}^{(4)} = (Y_3 Y_3)_4 = (\sqrt{2}e_2 e_1' + e_3 e_2', -\sqrt{2}e_1 e_2' - e_3 e_3', -\sqrt{2}e_1 e_3' - e_2 e_2', e_2 e_3' + \sqrt{2}e_3 e_1')^T, \\ Y_{4,III}^{(4)} = (Y_3 Y_5)_4 = (2\sqrt{2}e_1 \tilde{e}_2 - \sqrt{6}e_2 \tilde{e}_1 + e_3 \tilde{e}_3, -\sqrt{2}e_1 \tilde{e}_3 + 2e_2 \tilde{e}_2 - 3e_3 \tilde{e}_4, \sqrt{2}e_1 \tilde{e}_4 + 3e_2 \tilde{e}_3 - 2e_3 \tilde{e}_5, \\ -2\sqrt{2}e_1 \tilde{e}_5 - e_2 \tilde{e}_4 + \sqrt{6}e_3 \tilde{e}_1)^T, \\ Y_{4,IV}^{(4)} = (Y_3 Y_5)_4 = (\sqrt{2}e_1' \tilde{e}_2 + 3e_2' \tilde{e}_5 - 2e_3' \tilde{e}_4, 2\sqrt{2}e_1' \tilde{e}_3 - \sqrt{6}e_2' \tilde{e}_1 + e_3' \tilde{e}_5, -2\sqrt{2}e_1' \tilde{e}_4 - e_2' \tilde{e}_2 \\ + \sqrt{6}e_3' \tilde{e}_1, -\sqrt{2}e_1' \tilde{e}_5 + 2e_2' \tilde{e}_3 - 3e_3' \tilde{e}_2)^T. \quad (38)$$

$$Y_{5,I}^{(4)} = (Y_3 Y_3)_5 = (2(e_1^2 - e_2 e_3), -2\sqrt{3}e_1 e_2, \sqrt{6}e_2^2, \sqrt{6}e_3^2, -2\sqrt{3}e_1 e_3)^T, \\ Y_{5,II}^{(4)} = (Y_3 Y_3)_5 = (2(e_1'^2 - e_2' e_3'), \sqrt{6}e_1'^2, -2\sqrt{3}e_1' e_2', -2\sqrt{3}e_1' e_3', \sqrt{6}e_2'^2)^T, \\ Y_{5,III}^{(4)} = \frac{1}{2}(Y_5 Y_5)_{5_{s,1}} = \left(\tilde{e}_1^2 + \tilde{e}_2 \tilde{e}_5 - 2\tilde{e}_3 \tilde{e}_4, \tilde{e}_1 \tilde{e}_2 + \sqrt{6}\tilde{e}_3 \tilde{e}_5, \sqrt{\frac{3}{2}}\tilde{e}_2^2 - 2\tilde{e}_1 \tilde{e}_3, \sqrt{\frac{3}{2}}\tilde{e}_5^2 - 2\tilde{e}_1 \tilde{e}_4, \tilde{e}_1 \tilde{e}_5 + \sqrt{6}\tilde{e}_2 \tilde{e}_4 \right)^T, \\ Y_{5,IV}^{(4)} = \frac{1}{2}(Y_5 Y_5)_{5_{s,2}} = \left(\tilde{e}_1^2 - 2\tilde{e}_2 \tilde{e}_5 + \tilde{e}_3 \tilde{e}_4, \sqrt{\frac{3}{2}}\tilde{e}_4^2 - 2\tilde{e}_1 \tilde{e}_2, \tilde{e}_1 \tilde{e}_3 + \sqrt{6}\tilde{e}_4 \tilde{e}_5, \tilde{e}_1 \tilde{e}_4 + \sqrt{6}\tilde{e}_2 \tilde{e}_3, \sqrt{\frac{3}{2}}\tilde{e}_3^2 - 2\tilde{e}_1 \tilde{e}_5 \right)^T, \\ Y_{5,V}^{(4)} = (Y_3 Y_3)_5 = (\sqrt{3}e_1 e_1', e_2 e_1' - \sqrt{2}e_3 e_2', e_1 e_2' - \sqrt{2}e_3 e_3', e_1 e_3' - \sqrt{2}e_2 e_2', e_3 e_1' - \sqrt{2}e_2 e_3')^T, \\ Y_{5,VI}^{(4)} = (Y_3 Y_5)_5 = (\sqrt{3}(e_2 \tilde{e}_5 - e_3 \tilde{e}_2), -e_1 \tilde{e}_2 - \sqrt{3}e_2 \tilde{e}_1 - \sqrt{2}e_3 \tilde{e}_3, -2e_1 \tilde{e}_3 - \sqrt{2}e_2 \tilde{e}_2, 2e_1 \tilde{e}_4 + \sqrt{2}e_3 \tilde{e}_5, \\ e_1 \tilde{e}_5 + \sqrt{2}e_2 \tilde{e}_4 + \sqrt{3}e_3 \tilde{e}_1)^T, \\ Y_{5,VII}^{(4)} = (Y_3 Y_5)_5 = (\sqrt{3}(e_2' \tilde{e}_4 - e_3' \tilde{e}_3), 2e_1' \tilde{e}_2 + \sqrt{2}e_3' \tilde{e}_4, -e_1' \tilde{e}_3 - \sqrt{3}e_2' \tilde{e}_1 - \sqrt{2}e_3' \tilde{e}_5, e_1' \tilde{e}_4 + \sqrt{2}e_2' \tilde{e}_2 + \sqrt{3}e_3' \tilde{e}_1, \\ -2e_1' \tilde{e}_5 - \sqrt{2}e_2' \tilde{e}_3)^T, \quad (39)$$

where the upper index is the weight and the lower index refers to the representation under A_5 . We notice that not all the above weight 4 modular functions are linearly independent. From the expressions of the q -expansion of e_i , e_i' , and $\tilde{e}_i$ given in Eq. (34), we find the following relations are satisfied,

$$Y_{1,I}^{(4)} = Y_{1,II}^{(4)} = Y_{1,III}^{(4)}, \quad Y_{3,II}^{(4)} = -\frac{\sqrt{3}}{2}Y_{3,I}^{(4)}, \quad Y_{3',I}^{(4)} = -\frac{\sqrt{3}}{2}Y_{3',II}^{(4)}, \quad (40)$$

$$Y_{4,I}^{(4)} = -\frac{3}{5}Y_{4,II}^{(4)}, \quad Y_{4,III}^{(4)} = -\frac{7\sqrt{3}}{5}Y_{4,II}^{(4)}, \quad Y_{4,IV}^{(4)} = -\frac{\sqrt{3}}{5}Y_{4,II}^{(4)}, \quad (41)$$

$$Y_{5,III}^{(4)} = \frac{1}{10}Y_{5,I}^{(4)} + \frac{2}{5}Y_{5,II}^{(4)}, \quad Y_{5,IV}^{(4)} = -\frac{1}{5}Y_{5,I}^{(4)} + \frac{7}{10}Y_{5,II}^{(4)}, \quad (42)$$

$$Y_{5,V}^{(4)} = -\frac{1}{2\sqrt{3}}Y_{5,I}^{(4)} + \frac{2}{\sqrt{3}}Y_{5,II}^{(4)}, \quad Y_{5,VI}^{(4)} = \frac{2}{5}Y_{5,I}^{(4)} - \frac{2}{5}Y_{5,II}^{(4)}, \quad Y_{5,VII}^{(4)} = 2Y_{5,III}^{(4)}. \quad (43)$$

Therefore the singlet modular forms $Y_{1,I}^{(4)}$, $Y_{1,II}^{(4)}$, and $Y_{1,III}^{(4)}$ are exactly identical, the triplet modular functions $Y_{3,II}^{(4)}$ and $Y_{3',II}^{(4)}$ are parallel to $Y_{3,I}^{(4)}$ and $Y_{3',I}^{(4)}$ respectively. Analogously $Y_{4,I}^{(4)}$, $Y_{4,III}^{(4)}$, and $Y_{4,IV}^{(4)}$ are proportional to $Y_{4,II}^{(4)}$. Moreover, Eqs. (42), (43) imply that all the seven **5**-plet modular functions in Eq. (39) can be expressed in terms of the linear combinations of $Y_{5,I}^{(4)}$ and $Y_{5,II}^{(4)}$. As a result, there are totally 21 independent weight 4 modular functions which can be chosen as $Y_{1,I}^{(4)}$, $Y_{3,I}^{(4)}$, $Y_{3',II}^{(4)}$, $Y_{4,II}^{(4)}$, $Y_{5,I}^{(4)}$, and $Y_{5,II}^{(4)}$. Note the linear space of modular forms of weight $2k$ and level 5 is of dimension $10k + 1$ which is equal to 21 for $k = 2$. Thus we have constructed an explicit basis for the weight 4 modular

forms at level 5. The weight 4 modular forms are also built from the tensor products of Y_3 , $Y_{3'}$, and Y_5 in [34], and constraints similar to those in Eqs. (40)–(43) are obtained.

As regards the weight 6 modular forms for $k = 3$, the dimension of the modular space is equal to $10 \times 3 + 1 = 31$. That is to say, there should be 31 independent combination $Y_i Y_j Y_k$ where $i, j, k = \mathbf{3}, \mathbf{3}', \mathbf{5}$. In the same fashion as the $k = 2$ case, all the basis vectors and the corresponding constraints can be straightforwardly found although the involved algebra is lengthy and tedious. First we consider the weight 6 modular forms which are A_5 singlet. We can construct the following four singlet modular forms of weight 6 through the tensor products of weight 2 and weight 4 modular forms listed above,

$$\begin{aligned} Y_{1,I}^{(6)} &= (Y_3 Y_{3,I}^{(4)})_1 = [Y_3(Y_3 Y_5)_3]_1, & Y_{1,II}^{(6)} &= (Y_{3'} Y_{3',II}^{(4)})_1 = [Y_{3'}(Y_{3'} Y_5)_{3'}]_1, \\ Y_{1,III}^{(6)} &= (Y_5 Y_{5,I}^{(4)})_1 = [Y_5(Y_3 Y_3)_5]_1, & Y_{1,IV}^{(6)} &= (Y_5 Y_{5,II}^{(4)})_1 = [Y_5(Y_{3'} Y_{3'})_5]_1, \end{aligned} \quad (44)$$

which fulfill the relations

$$Y_{1,II}^{(6)} = -Y_{1,III}^{(6)} = -Y_{1,IV}^{(6)} = Y_{1,I}^{(6)}. \quad (45)$$

Hence there is only one singlet modular form of weight 6 and it can be taken as

$$Y_{1,I}^{(6)} = (Y_3 Y_{3,I}^{(4)})_1 = -2e_1^2 \tilde{e}_1 - \sqrt{6}e_2^2 \tilde{e}_4 - \sqrt{6}e_3^2 \tilde{e}_3 + 2\sqrt{3}e_1 e_2 \tilde{e}_5 + 2\sqrt{3}e_1 e_3 \tilde{e}_2 + 2e_2 e_3 \tilde{e}_1. \quad (46)$$

Analogously we can construct the following twelve weight 6 modular forms transforming as **3** under A_5 ,

$$\begin{aligned} Y_3 Y_{1,I}^{(4)} &= Y_3(Y_3 Y_3)_1, & (Y_3 Y_{3,I}^{(4)})_3 &= [Y_3(Y_3 Y_5)_3]_3, & (Y_3 Y_{5,I}^{(4)})_3 &= [Y_3(Y_3 Y_3)_5]_3, \\ (Y_3 Y_{5,II}^{(4)})_3 &= [Y_3(Y_{3'} Y_{3'})_5]_3, & (Y_{3'} Y_{4,II}^{(4)})_3 &= [Y_{3'}(Y_3 Y_{3'})_4]_3, & (Y_{3'} Y_{5,I}^{(4)})_3 &= [Y_{3'}(Y_3 Y_3)_5]_3, \\ (Y_{3'} Y_{5,II}^{(4)})_3 &= [Y_{3'}(Y_{3'} Y_{3'})_5]_3, & (Y_5 Y_{3,I}^{(4)})_3 &= [Y_5(Y_3 Y_5)_3]_3, & (Y_5 Y_{3',II}^{(4)})_3 &= [Y_5(Y_{3'} Y_5)_{3'}]_3, \\ (Y_5 Y_{4,II}^{(4)})_3 &= [Y_5(Y_3 Y_{3'})_4]_3, & (Y_5 Y_{5,I}^{(4)})_3 &= [Y_5(Y_3 Y_3)_5]_3, & (Y_5 Y_{5,II}^{(4)})_3 &= [Y_5(Y_{3'} Y_{3'})_5]_3. \end{aligned} \quad (47)$$

Furthermore, we find only two of them are linearly independent and they could be chosen to be

$$\begin{aligned} Y_{3,I}^{(6)} &= Y_3 Y_{1,I}^{(4)} = (e_1^2 + 2e_2 e_3)(e_1, e_2, e_3)^T, \\ Y_{3,II}^{(6)} &= (Y_{3'} Y_{5,II}^{(4)})_3 = (2\sqrt{3}e_1'^3 - 6\sqrt{3}e_1' e_2' e_3', 3\sqrt{6}e_1' e_3'^2 - 2\sqrt{3}e_2'^3, 3\sqrt{6}e_1' e_2'^2 - 2\sqrt{3}e_3'^3)^T. \end{aligned} \quad (48)$$

In a similar manner, we can construct the following twelve A_5 triplet **3'** modular forms of weight 6,

$$\begin{aligned} (Y_3 Y_{4,II}^{(4)})_{3'} &= [Y_3(Y_3 Y_{3'})_4]_{3'}, & (Y_3 Y_{5,I}^{(4)})_{3'} &= [Y_3(Y_3 Y_3)_5]_{3'}, & (Y_3 Y_{5,II}^{(4)})_{3'} &= [Y_3(Y_{3'} Y_{3'})_5]_{3'}, \\ Y_{3'} Y_{1,I}^{(4)} &= Y_{3'}(Y_3 Y_3)_1, & (Y_{3'} Y_{3',II}^{(4)})_{3'} &= [Y_{3'}(Y_{3'} Y_5)_{3'}]_{3'}, & (Y_{3'} Y_{5,I}^{(4)})_{3'} &= [Y_{3'}(Y_3 Y_3)_5]_{3'}, \\ (Y_{3'} Y_{5,II}^{(4)})_{3'} &= [Y_{3'}(Y_{3'} Y_{3'})_5]_{3'}, & (Y_5 Y_{3,I}^{(4)})_{3'} &= [Y_5(Y_3 Y_5)_3]_{3'}, & (Y_5 Y_{3',II}^{(4)})_{3'} &= [Y_5(Y_{3'} Y_5)_{3'}]_{3'}, \\ (Y_5 Y_{4,II}^{(4)})_{3'} &= [Y_5(Y_3 Y_{3'})_4]_{3'}, & (Y_5 Y_{5,I}^{(4)})_{3'} &= [Y_5(Y_3 Y_3)_5]_{3'}, & (Y_5 Y_{5,II}^{(4)})_{3'} &= [Y_5(Y_{3'} Y_{3'})_5]_{3'}. \end{aligned} \quad (49)$$

Only two out of the above twelve modular forms are linearly independent and they can be taken as

$$\begin{aligned}
Y_{3,I}^{(6)} &= Y_3 Y_{1,I}^{(4)} = (e_1^2 + 2e_2 e_3)(e'_1, e'_2, e'_3)^T, \\
Y_{3,II}^{(6)} &= (Y_3 Y_{5,II}^{(4)})_3 = (2\sqrt{3}e_1^3 - 6\sqrt{3}e_1 e_2 e_3, 3\sqrt{6}e_1 e_2^2 - 2\sqrt{3}e_3^3, 3\sqrt{6}e_1 e_3^2 - 2\sqrt{3}e_2^3)^T.
\end{aligned} \tag{50}$$

Eventually we find that there are two linearly independent four-dimensional and five-dimensional modular forms of weight 6,

$$\begin{aligned}
Y_{4,I}^{(6)} &= (Y_3 Y_{5,I}^{(4)})_4 / 6\sqrt{6} = (-2e_1^2 e_2 + e_2^2 e_3, -e_3^3 - \sqrt{2}e_1 e_2^2, e_2^3 + \sqrt{2}e_1 e_3^2, 2e_1^2 e_3 - e_2 e_3^2)^T, \\
Y_{4,II}^{(6)} &= (Y_3 Y_{5,II}^{(4)})_4 / 6\sqrt{6} = (e_2^3 + \sqrt{2}e_1 e_3^2, -2e_1^2 e_2 + e_2^2 e_3, 2e_1^2 e_3 - e_2 e_3^2, -e_3^3 - \sqrt{2}e_1 e_2^2)^T, \\
Y_{5,I}^{(6)} &= Y_5 Y_{1,I}^{(4)} = (e_1^2 + 2e_2 e_3)(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5)^T, \\
Y_{5,II}^{(6)} &= (Y_3 Y_{4,II}^{(4)})_5 = (\sqrt{6}e_2^2 e'_3 - \sqrt{6}e_3^2 e'_2, 4e_1 e_2 e'_1 - 2e_3^2 e'_3, 2e_1^2 e'_2 + \sqrt{2}e_2^2 e'_1 - 2\sqrt{2}e_1 e_3 e'_3 \\
&\quad - 2e_2 e_3 e'_2, -2e_1^2 e'_3 - \sqrt{2}e_3^2 e'_1 + 2\sqrt{2}e_1 e_2 e'_2 + 2e_2 e_3 e'_3, 2e_2^2 e'_2 - 4e_1 e_3 e'_1)^T.
\end{aligned} \tag{51}$$

In short, a linearly independent basis of modular forms of weight 6 and level $N = 5$ can be chosen to be $Y_{1,I}^{(6)}, Y_{3,I}^{(6)}, Y_{3,II}^{(6)}, Y_{3',I}^{(6)}, Y_{3',II}^{(6)}, Y_{4,I}^{(6)}, Y_{4,II}^{(6)}, Y_{5,I}^{(6)}$, and $Y_{5,II}^{(6)}$. These weight 6 modular forms are quite useful for constructing modular symmetry models in the charged lepton sector, as shown in the following section.

III. MODELS BASED ON Γ_5

In this section, we shall construct some modular-invariant supersymmetric models based on Γ_5 . Adopting the $N = 1$ global supersymmetry, the most general form of the action is

$$\begin{aligned}
\mathcal{S} &= \int d^4x d^2\theta d^2\bar{\theta} K(\Phi_I, \bar{\Phi}_I; \tau, \bar{\tau}) \\
&\quad + \int d^4x d^2\theta W(\Phi_I, \tau) + \text{H.c.},
\end{aligned} \tag{52}$$

where the Kähler potential $K(\Phi_I, \bar{\Phi}_I; \tau, \bar{\tau})$ is a real gauge invariant function of the set of chiral superfields Φ_I and complex modulus τ . The action is required to be invariant under the finite modular group Γ_N . The supermultiplets Φ_I are assumed to transform in a representation ρ_I of the quotient group Γ_N and its modular weight is $-k_I$,

$$\begin{cases} \tau \rightarrow \gamma(\tau) = \frac{a\tau+b}{c\tau+d}, \\ \Phi_I \rightarrow (c\tau+d)^{-k_I} \rho_I(\gamma) \Phi_I, \end{cases} \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N, \tag{53}$$

where $\rho_I(\gamma)$ is the representation matrix of the element γ and k_I are integers. Under the modular transformation of Eq. (53), the Kähler potential should be invariant up to Kähler transformations. Thus one can determine the simplest Kähler potential is

$$K(\Phi_I, \bar{\Phi}_I; \tau, \bar{\tau}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\Phi_I|^2, \tag{54}$$

where the constant $h > 0$. After the modulus τ develops a vacuum expectation value (VEV), the above Kähler potential leads to the following kinetic term for the scalar components of the supermultiplets Φ_I and τ ,

$$\frac{h}{\langle -i\tau + i\bar{\tau} \rangle^2} \partial_\mu \bar{\tau} \partial^\mu \tau + \sum_I \frac{\partial_\mu \bar{\phi}_I \partial^\mu \phi_I}{\langle -i\tau + i\bar{\tau} \rangle^{k_I}}. \tag{55}$$

For each given value of the VEV of τ , the kinetic term of ϕ_I can be made into canonical form by rescaling the fields ϕ_I . This effect amounts to a redefinition of the superpotential parameters in a concrete model.

On the other hand, the superpotential $W(\Phi_I, \tau)$ should be invariant under the group Γ_N , and the total weight of $W(\Phi_I, \tau)$ should be vanishing. In general, the modular invariant superpotential $W(\Phi_I, \tau)$ can be expanded in power series of the supermultiplets Φ_I ,

$$W(\Phi_I, \tau) = \sum_n Y_{I_1 \dots I_n}(\tau) \Phi_{I_1} \dots \Phi_{I_n}. \tag{56}$$

For the transformation rule of the field Φ_I in Eq. (53), the functions $Y_{I_1 \dots I_n}$ should be modular forms of weight k_Y and transform in the representation ρ_Y of Γ_N ,

$$\begin{cases} \tau \rightarrow \gamma(\tau) = \frac{a\tau+b}{c\tau+d}, \\ Y(\tau) \rightarrow Y(\gamma(\tau)) = (c\tau+d)^{k_Y} \rho_Y(\gamma) Y(\tau). \end{cases} \tag{57}$$

The modular invariance requires

$$k_Y = k_{I_1} + \dots + k_{I_n}, \quad \rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \ni \mathbf{1}. \tag{58}$$

In the following, we shall construct some simple models of lepton masses and mixing based on the Γ_5 modular symmetry. The neutrinos are assumed to be Majorana particles. We shall formulate our models in the framework of supersymmetry, i.e., the minimal supersymmetric standard model (MSSM) and its extension with right-handed neutrinos. In order to be as simple as possible, we suppose

TABLE I. The transformation properties of the MSSM chiral superfields under the SM gauge group $SU(2)_L \times U(1)_Y$ and modular group Γ_5 , where $-k_I$ denotes the modular weight of the fields.

	N^c	$E_{1,2,3}^c$	L	H_d	H_u
$SU(2)_L \times U(1)_Y$	(1,0)	(1,1)	(2, -1/2)	(2, -1/2)	(2, +1/2)
Γ_5	ρ_N	$\rho_{E_{1,2,3}}$	ρ_L	1	1
k_I	k_N	$k_{E_{1,2,3}}$	k_L	0	0

the Higgs doublets $H_{u,d}$ are singlets of A_5 and their modular weights are zero. There are freedoms for the assignments of irreducible representation and modular weight to the matter fields. We assume the three left-handed lepton doublets transform as an irreducible three-dimensional representation **3** or **3'** of the modular group Γ_5 . The generic assignments of representations and modular weights to the MSSM fields are listed in Table I. Notice that the modular group Γ_5 has two triplet irreducible representations **3** and **3'** while the A_4 modular symmetry only has one triplet representation **3**, hence Γ_5 should provide more freedom and possibilities than A_4 to construct lepton mass models. In the following, we proceed to discuss the possible structures of the models in the charged lepton and neutrino sectors.

A. Charged lepton sector

The most general superpotential for the charged lepton masses can be written as

$$W_e = \alpha(E^c L H_d f_E(Y, \varphi))_1, \quad (59)$$

where φ denotes a generic flavon field which is a SM singlet, and $f_E(Y, \varphi)$ is the most general function of the modular forms $Y(\tau)$ and the flavon field φ . Notice that the total weight of each term of W_e should be vanishing. In the present work, for simplicity we shall consider the case

$$(i) (\rho_{E_{1,2,3}^c}, \rho_L) = (\mathbf{1}, \mathbf{3})$$

$$\begin{aligned} W_e &= \alpha E_1^c (LY_3)_1 H_d + \beta E_2^c (LY_{3,I}^{(4)})_1 H_d + \gamma_1 E_3^c (LY_{3,I}^{(6)})_1 H_d + \gamma_2 E_3^c (LY_{3,II}^{(6)})_1 H_d \\ &= \alpha E_1^c (L_1 e_1 + L_2 e_3 + L_3 e_2) H_d + \beta E_2^c [L_1 (-2e_1 \tilde{e}_1 + \sqrt{3}e_2 \tilde{e}_5 + \sqrt{3}e_3 \tilde{e}_2) + L_2 (\sqrt{3}e_1 \tilde{e}_5 - \sqrt{6}e_2 \tilde{e}_4 + e_3 \tilde{e}_1) \\ &\quad + L_3 (\sqrt{3}e_1 \tilde{e}_2 + e_2 \tilde{e}_1 - \sqrt{6}e_3 \tilde{e}_3)] H_d + \gamma_1 E_3^c [L_1 (e_1^2 + 2e_2 e_3) e_1 + L_2 (e_1^2 + 2e_2 e_3) e_3 + L_3 (e_1^2 + 2e_2 e_3) e_2] H_d \\ &\quad + \gamma_2 E_3^c [L_1 (2\sqrt{3}e_1^3 - 6\sqrt{3}e_1' e_2' e_3') + L_2 (3\sqrt{6}e_1' e_2'^2 - 2\sqrt{3}e_3'^3) + L_3 (3\sqrt{6}e_1' e_2'^2 - 2\sqrt{3}e_3'^3)] H_d. \end{aligned} \quad (62)$$

$$(ii) (\rho_{E_{1,2,3}^c}, \rho_L) = (\mathbf{1}, \mathbf{3}')$$

$$\begin{aligned} W_e &= \alpha E_1^c (LY_{3'})_1 H_d + \beta E_2^c (LY_{3',II}^{(4)})_1 H_d + \gamma_1 E_3^c (LY_{3',I}^{(6)})_1 H_d + \gamma_2 E_3^c (LY_{3',II}^{(6)})_1 H_d \\ &= \alpha E_1^c (L_1 e_1' + L_2 e_3' + L_3 e_2') H_d + \beta E_2^c [L_1 (-2e_1' \tilde{e}_1 + \sqrt{3}e_2' \tilde{e}_4 + \sqrt{3}e_3' \tilde{e}_3) + L_2 (\sqrt{3}e_1' \tilde{e}_4 - \sqrt{6}e_2' \tilde{e}_2 + e_3' \tilde{e}_1) \\ &\quad + L_3 (\sqrt{3}e_1' \tilde{e}_3 + e_2' \tilde{e}_1 - \sqrt{6}e_3' \tilde{e}_5)] H_d + \gamma_1 E_3^c [L_1 (e_1'^2 + 2e_2 e_3) e_1' + L_2 (e_1'^2 + 2e_2 e_3) e_3' + L_3 (e_1'^2 + 2e_2 e_3) e_2'] H_d \\ &\quad + \gamma_2 E_3^c [L_1 (2\sqrt{3}e_1'^3 - 6\sqrt{3}e_1 e_2 e_3) + L_2 (3\sqrt{6}e_1 e_2^2 - 2\sqrt{3}e_3^3) + L_3 (3\sqrt{6}e_1 e_2^2 - 2\sqrt{3}e_3^3)] H_d. \end{aligned} \quad (63)$$

that $f_E(Y, \varphi)$ only depends on either the flavon field φ or the modular forms Y . For the scenario that only φ is involved in W_e , we can choose the modular weights k_I to forbid the dependence of W_e on the modulus τ . As shown in Appendix B, the flavon multiplets can develop the VEVs along certain directions such that the charged lepton mass matrix is diagonal. Then the lepton flavor mixing completely arises from the neutrino sector.

In the second scenario that the function $f_E(Y, \varphi)$ only depends on the modulus τ , the three generations of left-handed lepton doublets are assigned to the triplet representation **3** or **3'** while all the right-handed charged leptons E_i^c transform as **1** under Γ_5 . In order to avoid a charged lepton mass matrix with rank less than 3, we assume that E_1^c , E_2^c , and E_3^c carry different modular weights such that modular forms of weights 2, 4, 6 (i.e., Y , $Y^{(4)}$, $Y^{(6)}$) are involved. Thus the superpotential for the charged lepton masses is of the following form,

$$W_e = \alpha(E_1^c LY)_1 H_d + \beta(E_2^c LY^{(4)})_1 H_d + \gamma(E_3^c LY^{(6)})_1 H_d, \quad (60)$$

where the condition of weight cancellation requires

$$\begin{cases} 2 = k_{E_1} + k_L, \\ 4 = k_{E_2} + k_L, \Rightarrow k_{E_3} = 6 - k_L = k_{E_2} + 2 = k_{E_1} + 4. \\ 6 = k_{E_3} + k_L. \end{cases} \quad (61)$$

Notice that Y , $Y^{(4)}$, and $Y^{(6)}$ should transform in the same way as L under A_5 in order to form A_5 singlet. All the linearly independent modular forms of weights 2, 4, and 6 have been found out in Sec. II. The explicit form of the superpotential W_e is different for $\rho_L \sim \mathbf{3}$ and $\rho_L \sim \mathbf{3}'$. To be more specific, using the Clebsch-Gordan coefficients given in Appendix A, we can expand W_e in Eq. (60) as follow.

TABLE II. The prediction for the charged lepton mass matrix in the models without flavon fields, where the charged lepton mass matrix is given in the right-left basis $E^c M_e L$ with $v_d \equiv \langle H_d^0 \rangle$.

Models		Mass matrices M_e
Cases	(ρ_{E^c}, ρ_L)	
C1	$(\mathbf{1}, \mathbf{3})$	$\begin{pmatrix} \alpha e_1 & \alpha e_3 & \alpha e_2 \\ \beta(-2e_1\tilde{e}_1 + \sqrt{3}e_2\tilde{e}_5 + \sqrt{3}e_3\tilde{e}_2) & \beta(\sqrt{3}e_1\tilde{e}_5 - \sqrt{6}e_2\tilde{e}_4 + e_3\tilde{e}_1) & \beta(\sqrt{3}e_1\tilde{e}_2 + e_2\tilde{e}_1 - \sqrt{6}e_3\tilde{e}_3) \\ \gamma_1(e_1^2 + 2e_2e_3)e_1 & \gamma_1(e_1^2 + 2e_2e_3)e_3 & \gamma_1(e_1^2 + 2e_2e_3)e_2 \\ +\gamma_2(2\sqrt{3}e_1^3 - 6\sqrt{3}e_1'e_2'e_3) & +\gamma_2(3\sqrt{6}e_1'e_2'^2 - 2\sqrt{3}e_3'^3) & +\gamma_2(3\sqrt{6}e_1'e_2'^2 - 2\sqrt{3}e_3'^2) \end{pmatrix} v_d$
C2	$(\mathbf{1}, \mathbf{3}')$	$\begin{pmatrix} \alpha e'_1 & \alpha e'_3 & \alpha e'_2 \\ \beta(-2e'_1\tilde{e}_1 + \sqrt{3}e'_2\tilde{e}_4 + \sqrt{3}e'_3\tilde{e}_3) & \beta(\sqrt{3}e'_1\tilde{e}_4 - \sqrt{6}e'_2\tilde{e}_2 + e'_3\tilde{e}_1) & \beta(\sqrt{3}e'_1\tilde{e}_3 + e'_2\tilde{e}_1 - \sqrt{6}e'_3\tilde{e}_5) \\ \gamma_1(e_1^2 + 2e_2e_3)e'_1 & \gamma_1(e_1^2 + 2e_2e_3)e'_3 & \gamma_1(e_1^2 + 2e_2e_3)e'_2 \\ +\gamma_2(2\sqrt{3}e_1^3 - 6\sqrt{3}e_1e_2e_3) & +\gamma_2(3\sqrt{6}e_1e_3^2 - 2\sqrt{3}e_2^3) & +\gamma_2(3\sqrt{6}e_1e_2^2 - 2\sqrt{3}e_3^3) \end{pmatrix} v_d$

After electroweak symmetry breaking, we can read out the charged lepton mass matrices which are summarized in the Table II. In short, we have presented the possible minimal constructions of the charged lepton sectors in both the scenarios with and without flavon fields. The present analysis extends that of [34], where the various models in the charged lepton sector are not studied and it is simply assumed that charged lepton mass matrix is diagonal.

B. Neutrino sector

We shall assume neutrinos are Majorana particles and the left-handed leptons are assigned to a triplet representation $\mathbf{3}$ or $\mathbf{3}'$ of the modular group Γ_5 . We shall consider two different scenarios: the neutrinos masses are described by the effective Weinberg operator or are generated by the type-I seesaw mechanism. If the neutrino masses originate from the Weinberg operator, the most general form of the superpotential responsible for neutrino masses is,

$$W_\nu = \frac{1}{\Lambda} (H_u H_u LL f_W(Y))_1, \quad (64)$$

where $f_W(Y)$ is a generic function of the modular form Y . For the sake of simplicity, we are concerned with the case that $f_W(Y)$ is the modular form of lowest weight 2. Thus the neutrino superpotential reads

$$W_\nu = \frac{1}{\Lambda} (H_u H_u LLY)_1. \quad (65)$$

The modular invariance requires the weight of lepton doublet should be equal to -1 , i.e.,

$$k_L = 1. \quad (66)$$

Since the linearly independent modular forms of weight 2 decompose into three $\Gamma_5 \cong A_5$ irreducible representations

$Y_3, Y_{3'}, Y_5$, from the Kronecker products $\mathbf{3} \otimes \mathbf{3} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{5}_S$ and $\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1}_S \oplus \mathbf{3}'_A \oplus \mathbf{5}_S$ we know that only the quintuplet modular form Y_5 contributes to W_ν in Eq. (65). The concrete form of W_ν depends on the transformation property of L under Γ_5 .

(i) $\rho_L = \mathbf{3}$

$$\begin{aligned} W_\nu &= \frac{1}{\Lambda} (LLY)_1 H_u^2 = \frac{1}{\Lambda} ((LL)_5 Y_5)_1 H_u^2 \\ &= \frac{1}{\Lambda} [2(L_1 L_1 - L_2 L_3)\tilde{e}_1 - 2\sqrt{3}L_1 L_2 \tilde{e}_5 \\ &\quad + \sqrt{6}L_2 L_2 \tilde{e}_4 + \sqrt{6}L_3 L_3 \tilde{e}_3 - 2\sqrt{3}L_1 L_3 \tilde{e}_2] H_u^2, \end{aligned} \quad (67)$$

which gives rise to the following light neutrino mass matrix

$$M_\nu = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_5 & -\sqrt{3}\tilde{e}_2 \\ -\sqrt{3}\tilde{e}_5 & \sqrt{6}\tilde{e}_4 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_2 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_3 \end{pmatrix} \frac{v_u^2}{\Lambda}, \quad (68)$$

with $v_u = \langle H_u \rangle$.

(ii) $\rho_L = \mathbf{3}'$

$$\begin{aligned} W_\nu &= \frac{1}{\Lambda} (LLY)_1 H_u^2 = \frac{1}{\Lambda} ((LL)_5 Y_5)_1 H_u^2 \\ &= \frac{1}{\Lambda} [2(L_1 L_1 - L_2 L_3)\tilde{e}_1 + \sqrt{6}L_3 L_3 \tilde{e}_5 \\ &\quad - 2\sqrt{3}L_1 L_2 \tilde{e}_4 - 2\sqrt{3}L_1 L_3 \tilde{e}_3 \\ &\quad + \sqrt{6}L_2 L_2 \tilde{e}_2] H_u^2. \end{aligned} \quad (69)$$

Then the neutrino mass matrix reads

$$M_\nu = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_4 & -\sqrt{3}\tilde{e}_3 \\ -\sqrt{3}\tilde{e}_4 & \sqrt{6}\tilde{e}_2 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_3 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_5 \end{pmatrix} \frac{v_u^2}{\Lambda}. \quad (70)$$

In the second scenario, the neutrino masses are generated via the type-I seesaw mechanism. Three right-handed neutrinos $N^c = (N_1^c, N_2^c, N_3^c)^T$ are introduced and they are assigned to transform as a triplet **3** or **3'** under the action of Γ_5 . The superpotential of the neutrino sector can be generally written as

$$W_\nu = g(N^c L H_u f_N(Y))_1 + \Lambda(N^c N^c f_M(Y))_1, \quad (71)$$

where $f_N(Y)$ and $f_M(Y)$ are generic functions of the modular form Y . In order to build models with minimal number of parameters, we consider the case that both $f_N(Y)$ and $f_M(Y)$ are of weight 2 yet only one of them is involved. If the modulus parameter τ enters into the

neutrino masses through the Yukawa coupling, the neutrino superpotential would be

$$W_\nu^I = g(N^c L H_u Y)_1 + \Lambda(N^c N^c)_1. \quad (72)$$

In this case the weights of N^c and L should be

$$k_N = 0, \quad k_L = 2. \quad (73)$$

If the complex modulus τ only appears in the right-handed neutrino mass term, the neutrino superpotential would be

$$W_\nu^{II} = g(N^c L H_u)_1 + \Lambda(N^c N^c Y)_1, \quad (74)$$

with the weights

$$k_N = 1, \quad k_L = -1. \quad (75)$$

The explicit form of W_ν depends on the assignments of ρ_L and ρ_{N^c} .

(i) $(\rho_L, \rho_{N^c}) = (\mathbf{3}, \mathbf{3})$

$$\begin{aligned} W_\nu^I &= g_1((N^c L)_3 Y_3)_1 H_u + g_2((N^c L)_5 Y_5)_1 H_u + \Lambda(N^c N^c)_1, \\ &= g_1[(N_2^c L_3 - N_3^c L_2)e_1 + (N_1^c L_2 - N_2^c L_1)e_3 + (N_3^c L_1 - N_1^c L_3)e_2]H_u, \\ &\quad + g_2[(2N_1^c L_1 - N_2^c L_3 - N_3^c L_2)\tilde{e}_1 - \sqrt{3}(N_1^c L_2 + N_2^c L_1)\tilde{e}_5 + \sqrt{6}N_2^c L_2\tilde{e}_4 \\ &\quad + \sqrt{6}N_3^c L_3\tilde{e}_3 - \sqrt{3}(N_1^c L_3 + N_3^c L_1)\tilde{e}_2]H_u + \Lambda(N_1^c N_1^c + 2N_2^c N_3^c), \end{aligned} \quad (76)$$

and

$$\begin{aligned} W_\nu^{II} &= g(N^c L)_1 H_u + \Lambda((N^c N^c)_5 Y_5)_1, \\ &= g(N_1^c L_1 + N_2^c L_3 + N_3^c L_2)H_u + \Lambda[2(N_1^c N_1^c - N_2^c N_3^c)\tilde{e}_1 - 2\sqrt{3}N_1^c N_2^c \tilde{e}_5 \\ &\quad + \sqrt{6}N_2^c N_2^c \tilde{e}_4 + \sqrt{6}N_3^c N_3^c \tilde{e}_3 - 2\sqrt{3}N_1^c N_3^c \tilde{e}_2]. \end{aligned} \quad (77)$$

(ii) $(\rho_L, \rho_{N^c}) = (\mathbf{3}', \mathbf{3})$

From the multiplication rule $\mathbf{3} \otimes \mathbf{3}' = \mathbf{4} \oplus \mathbf{5}$ which does not contain a singlet, we know that the Yukawa coupling $N^c L H_u$ is not invariant under the Γ_5 modular symmetry. Consequently the superpotential W_ν^{II} is absent in this case, and we have

$$\begin{aligned} W_\nu^I &= g((N^c L)_5 Y_5)_1 H_u + \Lambda(N^c N^c)_1, \\ &= g[\sqrt{3}N_1^c L_1\tilde{e}_1 + (N_2^c L_1 - \sqrt{2}N_3^c L_2)\tilde{e}_5 + (N_1^c L_2 - \sqrt{2}N_3^c L_3)\tilde{e}_4 \\ &\quad + (N_1^c L_3 - \sqrt{2}N_2^c L_2)\tilde{e}_3 + (N_3^c L_1 - \sqrt{2}N_2^c L_3)\tilde{e}_2]H_u + \Lambda(N_1^c N_1^c + 2N_2^c N_3^c). \end{aligned} \quad (78)$$

(iii) $(\rho_L, \rho_{N^c}) = (\mathbf{3}, \mathbf{3}')$

Similar to the previous case, W_ν^{II} is not allowed by the $\Gamma_5 \cong A_5$ flavor symmetry, and the neutrino superpotential is

$$\begin{aligned} W_\nu^I &= g((N^c L)_5 Y_5)_1 H_u + \Lambda(N^c N^c)_1, \\ &= g[\sqrt{3}N_1^c L_1\tilde{e}_1 + (N_1^c L_2 - \sqrt{2}N_2^c L_3)\tilde{e}_5 + (N_2^c L_1 - \sqrt{2}N_3^c L_3)\tilde{e}_4 \\ &\quad + (N_3^c L_1 - \sqrt{2}N_2^c L_2)\tilde{e}_3 + (N_1^c L_3 - \sqrt{2}N_3^c L_2)\tilde{e}_2]H_u + \Lambda(N_1^c N_1^c + 2N_2^c N_3^c). \end{aligned} \quad (79)$$

(iv) $(\rho_L, \rho_{N^c}) = (\mathbf{3}', \mathbf{3}')$

Analogous to the case of $(\rho_L, \rho_{N^c}) = (\mathbf{3}, \mathbf{3})$, the neutrino superpotential reads

$$\begin{aligned} W_\nu^I &= g_1((N^c L)_3 Y_{\mathbf{3}})_1 H_u + g_2((N^c L)_5 Y_{\mathbf{5}})_1 H_u + \Lambda(N^c N^c)_1, \\ &= g_1[(N_2^c L_3 - N_3^c L_2)e'_1 + (N_1^c L_2 - N_2^c L_1)e'_3 + (N_3^c L_1 - N_1^c L_3)e'_2]H_u \\ &\quad + g_2[(2N_1^c L_1 - N_2^c L_3 - N_3^c L_2)\tilde{e}_1 + \sqrt{6}N_3^c L_3\tilde{e}_5 - \sqrt{3}(N_1^c L_2 + N_2^c L_1)\tilde{e}_4 \\ &\quad - \sqrt{3}(N_1^c L_3 + N_3^c L_1)\tilde{e}_3 + \sqrt{6}N_2^c L_2\tilde{e}_2]H_u + \Lambda(N_1^c N_1^c + 2N_2^c N_3^c), \end{aligned} \quad (80)$$

and

$$\begin{aligned} W_\nu^{II} &= g(N^c L)_1 H_u + \Lambda((N^c N^c)_5 Y_{\mathbf{5}})_1, \\ &= g(N_1^c L_1 + N_2^c L_3 + N_3^c L_2)H_u + \Lambda[2(N_1^c N_1^c - N_2^c N_3^c)\tilde{e}_1 + \sqrt{6}N_3^c N_3^c\tilde{e}_5 \\ &\quad - 2\sqrt{3}N_1^c N_2^c\tilde{e}_4 - 2\sqrt{3}N_1^c N_3^c\tilde{e}_3 + \sqrt{6}N_2^c N_2^c\tilde{e}_2]. \end{aligned} \quad (81)$$

Subsequently we can straightforwardly read out the predictions for the Dirac neutrino mass matrix M_D and the right-handed Majorana neutrino mass matrix M_N for each case above, and the results are summarized in Table III. The finite modular group $\Gamma_5 = A_5$ has two irreducible representations $\mathbf{3}$ and $\mathbf{3}'$, while the $\Gamma_3 = A_4$ modular symmetry only has one triplet representation $\mathbf{3}$, hence A_5 provides more freedom and possibilities than A_4 to construct lepton mass models based on modular invariance. Indeed, as summarized in Table III, it is clear that there are more possible realizations in the neutrino sector than with A_4 modular symmetry. After integrating out the heavy neutrinos N^c , we can obtain the effective neutrino mass matrix given by the well-known seesaw formula,

$$M_\nu = -M_D^T M_N^{-1} M_D. \quad (82)$$

Since there are a few constructions in the charged lepton and neutrino sectors listed in above, we can obtain plenty of possible models with Γ_5 modular symmetry, as summarized in Table IV. We see sixteen models are constructed, eight of them named as $\mathcal{A}i (i = 1, \dots, 8)$ involve flavon fields in the charged lepton mass terms while the remaining eight models $\mathcal{B}i (i = 1, \dots, 8)$ only depend on the modulus τ .

IV. NUMERICAL ANALYSIS

In this section we shall perform a comprehensive numerical analysis for each possible model listed in Table IV. Since some phases can be absorbed through field redefinition, some coupling constants of the models can be taken to be real. We first count the number of independent real free parameters of each model. If flavon fields are used, the resulting charged lepton mass matrix would be diagonal in our models and the observed charged lepton masses can be reproduced for certain values of the coupling constants and the flavon VEVs. If the models make no use of any flavon field other than the complex modulus τ , the charged lepton mass matrix is

summarized in Table II. We can rephase the charged lepton superfields E_1^c, E_2^c, E_3^c to make the parameters α, β, γ_1 real while the phase of γ_2 can not be removed. Thus the charged lepton mass matrix depends on four independent real parameters $\beta/\alpha, \gamma_1/\alpha, |\gamma_2/\alpha|, \arg(\gamma_2/\alpha)$ except the overall scale factor αv_d . If the neutrino masses originate from the Weinberg operator, the effective neutrino mass matrix would be expressed in terms of modular forms as functions of the modulus τ besides the overall factor v_u^2/Λ . If the neutrino masses are generated through the seesaw mechanism, the light neutrino mass matrix has two independent real parameters $|g_1/g_2|, \arg(g_1/g_2)$ and the overall scale factor is $g_2^2 v_u^2/\Lambda$ for the models $S1$ and $S5$. For the cases of $S2, S3, S4$ and $S6$, the neutrino mass matrix uniquely depends on the modulus parameter τ and $g^2 v_u^2/\Lambda$ controls the absolute scale of neutrino masses, as can be seen from Table III. In this way, we can easily read out the independent real input parameters of our models and the results are collected in Table V.

If the modulus τ is changed to $-\tau^*$, from the q -expansion of the modular forms we see that each modular form would become its complex conjugate. If we also set all the coupling constants to be their complex conjugate, i.e.,

$$\tau \rightarrow -\tau^*, \quad g_i \rightarrow g_i^*, \quad (83)$$

the charged lepton and neutrino mass matrices as well as the lepton mixing matrix will become complex conjugate. As a consequence, this transformation leaves lepton masses and mixing angles unchanged while the signs of both Dirac and Majorana CP phases would be flipped. This property is firstly observed in [33]. Hence it is sufficient to limit in the range $\text{Re}\tau > 0$ in the numerical analysis. We call two sets of input parameters are conjugate if they are related through the transformation in Eq. (83).

Moreover we notice that the number of free parameters of each model is less than the number of low energy

TABLE III. The classification of neutrino mass matrices for neutrino mass arising from Weinberg operator or the type-I seesaw. The last column is the name of each possible case.

Models	(ρ_L, ρ_{N^c})	Neutrino mass matrices	Cases
Weinberg operator			
	$(\mathbf{3}, -)$	$M_\nu = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_5 & -\sqrt{3}\tilde{e}_2 \\ -\sqrt{3}\tilde{e}_5 & \sqrt{6}\tilde{e}_4 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_2 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$	W1
	$(\mathbf{3}', -)$	$M_\nu = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_4 & -\sqrt{3}\tilde{e}_3 \\ -\sqrt{3}\tilde{e}_4 & \sqrt{6}\tilde{e}_2 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_3 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_5 \end{pmatrix} \frac{v_u^2}{\Lambda}$	W2
Type I seesaw			
		$M_D = \begin{pmatrix} 2g_2\tilde{e}_1 & g_1e_3 - \sqrt{3}g_2\tilde{e}_5 & -g_1e_2 - \sqrt{3}g_2\tilde{e}_2 \\ -g_1e_3 - \sqrt{3}g_2\tilde{e}_5 & \sqrt{6}g_2\tilde{e}_4 & g_1e_1 - g_2\tilde{e}_1 \\ g_1e_2 - \sqrt{3}g_2\tilde{e}_2 & -g_1e_1 - g_2\tilde{e}_1 & \sqrt{6}g_2\tilde{e}_3 \end{pmatrix} v_u,$	S1
	$(\mathbf{3}, \mathbf{3})$	$M_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda$	
		$M_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_5 & -\sqrt{3}\tilde{e}_2 \\ -\sqrt{3}\tilde{e}_5 & \sqrt{6}\tilde{e}_4 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_2 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_3 \end{pmatrix} \Lambda$	S2
	$(\mathbf{3}', \mathbf{3})$	$M_D = g \begin{pmatrix} \sqrt{3}\tilde{e}_1 & \tilde{e}_4 & \tilde{e}_3 \\ \tilde{e}_5 & -\sqrt{2}\tilde{e}_3 & -\sqrt{2}\tilde{e}_2 \\ \tilde{e}_2 & -\sqrt{2}\tilde{e}_5 & -\sqrt{2}\tilde{e}_4 \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda$	S3
	$(\mathbf{3}, \mathbf{3}')$	$M_D = g \begin{pmatrix} \sqrt{3}\tilde{e}_1 & \tilde{e}_5 & \tilde{e}_2 \\ \tilde{e}_4 & -\sqrt{2}\tilde{e}_3 & -\sqrt{2}\tilde{e}_5 \\ \tilde{e}_3 & -\sqrt{2}\tilde{e}_2 & -\sqrt{2}\tilde{e}_4 \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda$	S4
		$M_D = \begin{pmatrix} 2g_2\tilde{e}_1 & g_1e'_3 - \sqrt{3}g_2\tilde{e}_4 & -g_1e'_2 - \sqrt{3}g_2\tilde{e}_3 \\ -g_1e'_3 - \sqrt{3}g_2\tilde{e}_4 & \sqrt{6}g_2\tilde{e}_2 & g_1e'_1 - g_2\tilde{e}_1 \\ g_1e'_2 - \sqrt{3}g_2\tilde{e}_3 & -g_1e'_1 - g_2\tilde{e}_1 & \sqrt{6}g_2\tilde{e}_5 \end{pmatrix} v_u,$	S5
	$(\mathbf{3}', \mathbf{3}')$	$M_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda$	
		$M_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} 2\tilde{e}_1 & -\sqrt{3}\tilde{e}_4 & -\sqrt{3}\tilde{e}_3 \\ -\sqrt{3}\tilde{e}_4 & \sqrt{6}\tilde{e}_2 & -\tilde{e}_1 \\ -\sqrt{3}\tilde{e}_3 & -\tilde{e}_1 & \sqrt{6}\tilde{e}_5 \end{pmatrix} \Lambda$	S6

observables including three charged lepton masses $m_{e,\mu,\tau}$, three light neutrino masses $m_{1,2,3}$ and three lepton mixing angles θ_{12} , θ_{13} , θ_{23} , one Dirac CP phase δ_{CP} and two Majorana phases α_{21} and α_{31} . In order to quantitatively measure how well the models can describe the experimental data, we define the χ^2 function to serve as a test-statistic for the goodness-of-fit,

$$\chi^2(p_i) = \sum_j \left(\frac{Q_j(p_i) - Q_{j,\text{best-fit}}}{\sigma_j} \right)^2, \quad (84)$$

where p_i denote the free input parameters of a model and they are listed in Table V, and $Q_i \in \{\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, r \equiv \Delta m_{21}^2 / |\Delta m_{3\ell}^2|, m_e/m_\mu, m_\mu/m_\tau\}$ are observable

TABLE IV. Summary of models and the corresponding predictions for neutrino and charged lepton mass matrices. For the models with flavons in the charged lepton sector, the weights of the right-handed charged leptons should satisfy the constraint in Eq. (B4a), i.e., $k_{E_1} = 5k_{E_3} + 4k_L$ and $k_{E_2} = 4k_{E_3} + 3k_L$.

Models	Mass matrices	Assignment		Weight	
		$(\rho_{E^c}, \rho_L, \rho_{N^c})$	$k_{E_{1,2,3}}$	k_L	k_{N^c}
With flavons	$\mathcal{A}1$	$W1$	$(\mathbf{1}, \mathbf{3}, -)$	...	...
	$\mathcal{A}2$	$W2$	$(\mathbf{1}, \mathbf{3}', -)$	...	...
	$\mathcal{A}3$	$S1$	$(\mathbf{1}, \mathbf{3}, \mathbf{3})$	...	0
	$\mathcal{A}4$	$S2$	$(\mathbf{1}, \mathbf{3}, \mathbf{3})$	...	1
	$\mathcal{A}5$	$S3$	$(\mathbf{1}, \mathbf{3}', \mathbf{3})$	...	0
	$\mathcal{A}6$	$S4$	$(\mathbf{1}, \mathbf{3}, \mathbf{3}')$	...	0
	$\mathcal{A}7$	$S5$	$(\mathbf{1}, \mathbf{3}', \mathbf{3}')$	...	0
	$\mathcal{A}8$	$S6$	$(\mathbf{1}, \mathbf{3}', \mathbf{3}')$	...	1
Without flavons	$\mathcal{B}1$	$C1, W1$	$(\mathbf{1}, \mathbf{3}, -)$	1, 3, 5	...
	$\mathcal{B}2$	$C2, W2$	$(\mathbf{1}, \mathbf{3}', -)$	1, 3, 5	...
	$\mathcal{B}3$	$C1, S1$	$(\mathbf{1}, \mathbf{3}, \mathbf{3})$	0, 2, 4	0
	$\mathcal{B}4$	$C1, S2$	$(\mathbf{1}, \mathbf{3}, \mathbf{3})$	3, 5, 7	1
	$\mathcal{B}5$	$C2, S3$	$(\mathbf{1}, \mathbf{3}', \mathbf{3})$	0, 2, 4	0
	$\mathcal{B}6$	$C1, S4$	$(\mathbf{1}, \mathbf{3}, \mathbf{3}')$	0, 2, 4	0
	$\mathcal{B}7$	$C2, S5$	$(\mathbf{1}, \mathbf{3}', \mathbf{3}')$	0, 2, 4	0
	$\mathcal{B}8$	$C2, S6$	$(\mathbf{1}, \mathbf{3}', \mathbf{3}')$	3, 5, 7	1

TABLE V. The input parameters of each model, where the freedom of field redefinition has been used to absorb the physically irrelevant phases. Notice that all the input parameters are real.

Models	Free input parameters p_i	Overall factors
With flavons	$\mathcal{A}1, \mathcal{A}2$	$\{\text{Re}\tau, \text{Im}\tau\}$
	$\mathcal{A}4, \mathcal{A}5, \mathcal{A}6, \mathcal{A}8$	$\{\text{Re}\tau, \text{Im}\tau\}$
	$\mathcal{A}3, \mathcal{A}7$	$\{\text{Re}\tau, \text{Im}\tau, g_1/g_2 , \arg(g_1/g_2)\}$
Without flavons	$\mathcal{B}1, \mathcal{B}2$	$\{\text{Re}\tau, \text{Im}\tau, \beta/\alpha, \gamma_1/\alpha, \gamma_2/\alpha , \arg(\gamma_2/\alpha)\}$
	$\mathcal{B}4, \mathcal{B}5, \mathcal{B}6, \mathcal{B}8$	$\{\text{Re}\tau, \text{Im}\tau, \beta/\alpha, \gamma_1/\alpha, \gamma_2/\alpha , \arg(\gamma_2/\alpha)\}$
	$\mathcal{B}3, \mathcal{B}7$	$\{\text{Re}\tau, \text{Im}\tau, \beta/\alpha, \gamma_1/\alpha, \gamma_2/\alpha , \arg(\gamma_2/\alpha), g_1/g_2 , \arg(g_1/g_2)\}$

quantities derived from the neutrino and charged lepton mass matrices as complex nonlinear functions of the free parameters of the models.¹ The parameters $Q_{j,\text{best-fit}}$ and σ_j refer to the current central values and 1σ deviations respectively of the corresponding observable quantities listed in Table VI. Since the indication of a preferred value of the Dirac CP violation phase δ_{CP} coming from global data analyses is rather weak [44], we do not include any information on δ_{CP} in the χ^2 function. We consider both the normal ordering (NO) neutrino mass spectrum $m_1 < m_2 < m_3$ and the inverted ordering (IO) neutrino masses $m_3 < m_1 < m_2$, where $m_{1,2,3}$ denote three light neutrino masses. We define a quantity $N\sigma \equiv \sqrt{\chi^2_{\min}}$ where $\chi^2_{\min}$ is

¹For the models with flavons, the mass ratios m_e/m_μ and m_μ/m_τ are not included in the χ^2 function since their measured values can be obtained exactly for particular values of the coupling constants and flavon VEVs.

the global minimum of function χ^2 . The free parameters of the models are scanned over the following ranges,

$$\begin{aligned} \beta/\alpha, \quad \gamma_1/\alpha, \quad |\gamma_2/\alpha|, \quad |g_1/g_2| &\in [0, 10^4], \\ \arg(\gamma_2/\alpha), \quad \arg(g_1/g_2) &\in [0, 2\pi]. \end{aligned} \quad (85)$$

The modulus τ is taken from the fundamental domain $\mathcal{F}_5$ shown in Fig. 1, and we restrict the parameter search in the range $\text{Re}\tau \in [0, 0.5]$. The predictions of the mixing parameters in the conjugate region $\text{Re}\tau \in [-0.5, 0]$ can be easily obtained by only shifting the overall signs of the Dirac as well as Majorana CP phases. Hence all the numerical results given in the following should come in pair with opposite CP violation phases, and we show only one of them for the sake of readability. The function χ^2 is numerically minimized by using the minimization algorithms incorporated in the package MINUIT developed by CERN to determine the best fit values of the input

TABLE VI. The best-fit values and 1σ errors of the observable quantities used in the χ^2 analysis. The values of the neutrino oscillation parameters are adapted from NuFIT 4.0 for normal ordering and inverted ordering without SK atmospheric data [44], where $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$, $\Delta m_{3\ell}^2 \equiv m_3^2 - m_\ell^2 > 0$ for normal ordering and $\Delta m_{3\ell}^2 \equiv m_3^2 - m_\ell^2 < 0$ for inverted ordering. The charged lepton mass ratios m_e/m_μ and m_μ/m_τ given at the scale 2×10^{16} GeV are taken from Refs. [46,47].

Observables	Best-fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	Normal ordering	Inverted ordering
$\sin^2 \theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.310^{+0.013}_{-0.012}$
$\sin^2 \theta_{13}$	$0.02241^{+0.00065}_{-0.00065}$	$0.02264^{+0.00066}_{-0.00066}$
$\sin^2 \theta_{23}$	$0.580^{+0.017}_{-0.021}$	$0.584^{+0.016}_{-0.020}$
δ_{CP}/π	$1.194^{+0.222}_{-0.161}$	$1.578^{+0.150}_{-0.161}$
$\Delta m_{21}^2/(10^{-5} \text{ eV}^2)$	$7.39^{+0.21}_{-0.20}$	$7.39^{+0.21}_{-0.20}$
$ \Delta m_{3\ell}^2 /(10^{-3} \text{ eV}^2)$	$2.525^{+0.033}_{-0.032}$	$2.512^{+0.034}_{-0.032}$
$r \equiv \Delta m_{21}^2/ \Delta m_{3\ell}^2 $	$0.02927^{+0.000895}_{-0.000895}$	$0.02942^{+0.000903}_{-0.000903}$

parameters [45]. Requiring all the three mixing angles θ_{12} , θ_{13} , θ_{23} , and r in the 3σ ranges of global data analysis [44] and the mass ratios m_e/m_μ , m_μ/m_τ in the experimentally favored intervals, we can obtain the allowed regions of the

mixing angles, CP violation phases and light neutrino masses. The overall scale factors αv_d and v_u^2/Λ (or $g^2 v_u^2/\Lambda$, $g_2^2 v_u^2/\Lambda$) of the charged lepton and neutrino mass matrices can be fixed by the measured values of the electron mass m_e and the squared mass difference Δm_{21}^2 .

A. Numerical results of the models with flavons

We have extensively scanned over the parameter space of each model. As shown in Table V, the models $\mathcal{A}1$, $\mathcal{A}2$, $\mathcal{A}4$, $\mathcal{A}5$, $\mathcal{A}6$, and $\mathcal{A}8$ only depend on the complex modulus τ besides the overall mass scale. These models are too constrained to give a realistic description of lepton mixing angles as well as neutrino masses. Among these models $\mathcal{A}1$ can approach the experimental data best. We find the best fit values of the neutrino mixing parameters and neutrino masses are

$$\begin{aligned} \sin^2 \theta_{12} &= 0.2937, & \sin^2 \theta_{13} &= 0.0346, & \sin^2 \theta_{23} &= 0.7954, \\ \delta_{CP}/\pi &= 1.3963, & \alpha_{21}/\pi &= 1.6286, & \alpha_{31}/\pi &= 0.1542, \\ m_1 &= 0.04903 \text{ eV}, & m_2 &= 0.04978 \text{ eV}, \\ m_3 &= 0.00106 \text{ eV}, \end{aligned} \quad (86)$$

and the best fit point is

$$\langle \tau \rangle = -0.0222 + 0.9946i, \quad v_u^2/\Lambda = 0.0203 \text{ eV}. \quad (87)$$

FIG. 2. The phenomenologically viable regions of the modular parameter τ in the fundamental domain for the models $\mathcal{A}3$ and $\mathcal{A}7$.

TABLE VII. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}3$ with IO in region 1 and region 2.

Model $\mathcal{A}3$	Region 1		Region 2	
	Best-fit values	Allowed ranges	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.2664	0.0–0.5	0.2872	0.2525–0.2902
$\text{Im}\langle\tau\rangle$	1.6324	1.6060–1.8206	0.3441	0.3409–0.3949
$ g_1/g_2 $	2.0926	2.0568–2.1503	2.1296	2.0854–2.1457
$\arg(g_1/g_2)$	6.2308	6.1825–6.3905	6.2954	6.2028–6.3265
$g_2^2 v_u^2/\Lambda$ [eV]	0.0173	...	0.0008	...
$\sin^2 \theta_{12}$	0.3098	0.2750–0.3500	0.3099	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0226	0.02068–0.02463	0.0226	0.02068–0.02463
$\sin^2 \theta_{23}$	0.5835	0.5521–0.5989	0.5840	0.4230–0.6290
δ_{CP}/π	1.5265	0.4669–1.5453	1.7040	0.1066–1.7877
α_{21}/π	1.9831	0.0149–1.9859	1.3212	0.6335–1.3604
α_{31}/π	1.0485	0.9716–1.0709	1.4525	0.3758–1.6190
r	0.02942	0.0260–0.0332	0.0294	0.0260–0.0332
m_1 [eV]	0.0696	0.0593–0.0909	0.0814	0.0650–0.0908
m_2 [eV]	0.0702	0.0600–0.0913	0.0819	0.0656–0.0913
m_3 [eV]	0.0491	0.0371–0.0741	0.0647	0.0456–0.0741
$\sum_i m_i$ [eV]	0.1889	0.1566–0.2562	0.2280	0.1762–0.2562
$ m_{ee} $ [eV]	0.0693	0.0590–0.0906	0.0479	0.0376–0.0537
Ordering	IO		IO	
$N\sigma$	0.03	...	0.013	...

TABLE VIII. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}3$ with IO in region 3. Notice that the Dirac CP phase $\delta_{CP} \simeq 1.47\pi$ at the conjugate best fit point.

Model $\mathcal{A}3$	Region 3	
	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.4937	0.4829–0.5000
$\text{Im}\langle\tau\rangle$	0.4830	0.4680–0.5285
$ g_1/g_2 $	1.7244	1.7207–1.7365
$\arg(g_1/g_2)$	6.4672	6.0579–6.5125
$g_2^2 v_u^2/\Lambda$ [eV]	0.0022	...
$\sin^2 \theta_{12}$	0.3099	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0226	0.02068–0.02463
$\sin^2 \theta_{23}$	0.5840	0.4230–0.6290
δ_{CP}/π	0.5300	0–2
α_{21}/π	1.2400	0.6920–1.3041
α_{31}/π	0.2211	0–2
r	0.0294	0.0260–0.0332
m_1 [eV]	0.0494	0.0464–0.0526
m_2 [eV]	0.0501	0.0472–0.0533
m_3 [eV]	0.00035	0–0.0011
$\sum_i m_i$ [eV]	0.0998	0.0935–0.1071
$ m_{ee} $ [eV]	0.0245	0.0226–0.0272
Ordering	IO	
$N\sigma$	0.012	...

We see that the observed quantities $\sin^2 \theta_{12}$, $r \equiv \Delta m_{21}^2/|\Delta m_{3\ell}^2|$ and δ_{CP} can be reproduced very well, particularly the Dirac phase is close to $3\pi/2$. However, both reactor mixing angle θ_{13} and atmospheric angle θ_{23} are slightly larger than the 3σ upper bounds, and their pulls dominate the minimum of χ^2 . Similar to Ref. [26], we expect that the discrepancy could be resolved if there is small deviation from the leading order charged lepton flavon alignment such that the charged lepton mass matrix is not diagonal.

For the remaining two models $\mathcal{A}3$ and $\mathcal{A}7$, the neutrino mass matrix m_ν only involves the modulus τ and the parameter g_1/g_2 apart from the overall scale factor $g_2^2 v_u^2/\Lambda$. The model $\mathcal{A}3$ can give realistic values of the observables except θ_{23} in certain parameter space for NO neutrino mass spectrum. We find that a best fit to the experimental data can be obtained at the point

$$\begin{aligned} \langle\tau\rangle &= 0.2497 + 0.1935i, & |g_1/g_2| &= 2.2913, \\ \arg(g_1/g_2) &= 0.00456, & g_2^2 v_u^2/\Lambda &= 0.00021 \text{ eV}. \end{aligned} \quad (88)$$

The neutrino masses and mixing parameters are determined to be

TABLE IX. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}7$ with IO in region 1 and region 2. Notice that the Dirac CP phase is $\delta_{CP} \simeq 1.499\pi$ in region 1 and $\delta_{CP} \simeq 1.29\pi$ in region 2 at the conjugate best-fit points.

Model $\mathcal{A}7$	Region 1		Region 2	
	Best-fit values	Allowed ranges	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.0285	0.0085–0.0490	0.1541	0–0.2154
$\text{Im}\langle\tau\rangle$	2.7418	2.2323–2.7776	0.3319	0.1800–0.4049
$ g_1/g_2 $	2.1867	1.6078–2.1889	1.7603	1.4208–2.2314
$\arg(g_1/g_2)$	3.1478	1.8976–4.3823	0.7635	0–6.2832
$g_2^2 v_u^2/\Lambda$ [eV]	0.0266	...	0.0003	...
$\sin^2 \theta_{12}$	0.3106	0.2750–0.3500	0.3099	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0226	0.02068–0.02463	0.0226	0.02068–0.02463
$\sin^2 \theta_{23}$	0.4714	0.4230–0.5039	0.5840	0.4966–0.6290
δ_{CP}/π	0.5006	0.4965–0.5039	0.7096	0.0024–2
α_{21}/π	0.0119	0.0106–0.0275	1.6433	0–2
α_{31}/π	1.0084	1.0071–1.0208	1.0794	0.0015–2
r	0.02940	0.0260–0.0332	0.02942	0.02601–0.03321
m_1 [eV]	0.1066	0.0669–0.1156	0.0652	0.0522–0.3826
m_2 [eV]	0.1070	0.0675–0.1159	0.0658	0.0529–0.3827
m_3 [eV]	0.0945	0.0482–0.1030	0.0426	0.0238–0.3791
$\sum_i m_i$ [eV]	0.3081	0.1826–0.3345	0.1736	0.1288–1.1444
$ m_{ee} $ [eV]	0.1064	0.0666–0.1154	0.0564	0.0329–0.3262
Ordering	IO		IO	
$N\sigma$	5.63	...	0.005	...

TABLE X. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}7$ with IO in region 3 and region 4.

Model $\mathcal{A}7$	Region 3		Region 4	
	Best-fit values	Allowed ranges	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.3808	0.3734–0.4211	0.4686	0.4411–0.4930
$\text{Im}\langle\tau\rangle$	0.1836	0.1741–0.2165	0.5535	0.5185–0.5600
$ g_1/g_2 $	4.4862	4.2475–4.8215	8.0775	6.5062–17.3445
$\arg(g_1/g_2)$	1.5227	1.4285–1.6106	4.7888	4.6699–4.8307
$g_2^2 v_u^2/\Lambda$ [eV]	0.00005	...	0.00015	...
$\sin^2 \theta_{12}$	0.3100	0.2750–0.3450	0.3086	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0226	0.02068–0.02463	0.0235	0.02068–0.02463
$\sin^2 \theta_{23}$	0.5840	0.4230–0.6290	0.4497	0.4230–0.4593
δ_{CP}/π	1.4709	0.4552–1.4944	1.1137	0.016–1.3366
α_{21}/π	1.9181	0–2	1.8428	0.1181–1.9226
α_{31}/π	0.2260	0–2	1.6057	1.3481–1.7857
r	0.02942	0.02601–0.03321	0.02973	0.02601–0.03321
m_1 [eV]	0.0494	0.0464–0.0527	0.0491	0.0464–0.0526
m_2 [eV]	0.0501	0.0472–0.0534	0.0498	0.0472–0.0533
m_3 [eV]	0.0015	0–0.0035	0.0005	0–0.0006
$\sum_i m_i$ [eV]	0.1010	0.0935–0.1096	0.0995	0.0936–0.1066
$ m_{ee} $ [eV]	0.0481	0.0448–0.0517	0.0469	0.0440–0.0512
Ordering	IO		IO	
$N\sigma$	0.006	...	6.8373	...

TABLE XI. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}7$ with NO in region 1 and region 2. Notice that the Dirac CP phase $\delta_{CP} \simeq 1.515\pi$ at the conjugate best fit point in region 1.

Model $\mathcal{A}7$	Region 1		Region 2	
	Best-fit values	Allowed ranges	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.4691	0.4493–0.4866	0.3921	0–0.4410
$\text{Im}\langle\tau\rangle$	2.1224	1.7957–2.7549	0.3267	0.1704–0.4765
$ g_1/g_2 $	1.9604	1.9494–2.2923	2.0702	1.7430–2.4915
$\arg(g_1/g_2)$	4.4413	1.3466–4.9197	4.3707	0–6.2832
$g_2^2 v_u^2 / \Lambda$ [eV]	0.0125	...	0.0004	...
$\sin^2 \theta_{12}$	0.3099	0.2750–0.3500	0.3099	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0224	0.02045–0.02439	0.0230	0.02045–0.02439
$\sin^2 \theta_{23}$	0.5800	0.5274–0.6270	0.5519	0.4180–0.5873
δ_{CP}/π	0.4847	0.4542–0.4966	1.2307	0–2
α_{21}/π	1.9486	1.8688–1.9881	1.7959	0–2
α_{31}/π	0.9647	0.9038–0.9924	1.2150	0.0018–1.9655
r	0.02927	0.02587–0.03300	0.02928	0.02587–0.03300
m_1 [eV]	0.04930	0.0314–0.1029	0.0260	0.0228–0.3784
m_2 [eV]	0.05004	0.0325–0.1032	0.0274	0.0244–0.3785
m_3 [eV]	0.07039	0.0568–0.1159	0.0566	0.0525–0.3821
$\sum_i m_i$ [eV]	0.1697	0.1207–0.3220	0.1100	0.0997–1.1390
$ m_{ee} $ [eV]	0.0499	0.0318–0.1033	0.0236	0.0189–0.3355
Ordering	NO		NO	
$N\sigma$	0.005	...	1.622	...

TABLE XII. The best-fit values and the allowed ranges of the model parameters and lepton mixing parameters and neutrino masses for the model $\mathcal{A}7$ with NO in region 3.

Model $\mathcal{A}7$	Region 3	
	Best-fit values	Allowed ranges
$\text{Re}\langle\tau\rangle$	0.4186	0.3836–0.500
$\text{Im}\langle\tau\rangle$	0.1171	0.0101–0.1250
$ g_1/g_2 $	2.0955	1.7422–2.4913
$\arg(g_1/g_2)$	4.7360	0–6.2832
$g_2^2 v_u^2 / \Lambda$ [eV]	0.00004	...
$\sin^2 \theta_{12}$	0.3100	0.2750–0.3500
$\sin^2 \theta_{13}$	0.0224	0.02045–0.02439
$\sin^2 \theta_{23}$	0.5800	0.4180–0.6270
δ_{CP}/π	1.0437	0–2
α_{21}/π	0.2403	0–2
α_{31}/π	1.3293	0.0016–1.9925
r	0.02927	0.02587–0.03300
m_1 [eV]	0.0259	0.02278–0.37556
m_2 [eV]	0.0273	0.02435–0.37566
m_3 [eV]	0.0565	0.05252–0.37934
$\sum_i m_i$ [eV]	0.1098	0.0997–1.13057
$ m_{ee} $ [eV]	0.0231	0.0190–0.3343
Ordering	NO	
$N\sigma$	0.003	...

$$\begin{aligned}
\sin^2 \theta_{12} &= 0.3109, & \sin^2 \theta_{13} &= 0.0234, & \sin^2 \theta_{23} &= 0.6875, \\
\delta_{CP}/\pi &= 1.6784, & \alpha_{21}/\pi &= 0.6628, & \alpha_{31}/\pi &= 0.5088, \\
m_1 &= 0.08291 \text{ eV}, & m_2 &= 0.08336 \text{ eV}, \\
m_3 &= 0.09697 \text{ eV}.
\end{aligned} \tag{89}$$

We can see that both $\sin^2 \theta_{12}$ and $r \equiv \Delta m_{21}^2 / |\Delta m_{3e}^2|$ are in the 1σ region [44], while $\sin^2 \theta_{23}$ is a bit larger than its 3σ upper limit. Moreover, from Eq. (89) we know the sum of neutrino masses is

$$\sum_i m_i = 0.2632 \text{ eV}. \tag{90}$$

Under the assumption of Λ CDM cosmology, the latest Planck result on the neutrino mass sum is [48].

$$\sum_i m_i < 0.12 \text{ eV} - 0.60 \text{ eV}. \tag{91}$$

We see that the prediction for neutrino masses is compatible with this Planck result. Since the upper bound of the neutrino mass sum depends on the cosmological model and whether other experimental data such as baryon acoustic oscillation, gravitational lensing of galaxies and the high multipole TT, TE and EE polarization spectra [48] are considered, we shall not seriously take the constraint in Eq. (91).

FIG. 3. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}3$ with IO in region1. The 3σ bounds of the mixing angles are shown by vertical red dashed lines. In the last panel of $|m_{ee}|$ versus $m_{\min}$, the red (blue) dashed lines denote the most general allowed regions for IO (NO) neutrino mass spectrum obtained by varying the mixing parameters over their 3σ ranges [44]. The present most stringent upper limits $|m_{ee}| < 0.120$ eV from EXO-200 [50,51] and KamLAND-ZEN [52] is represented by horizontal grey band. The vertical grey exclusion band denotes the current bound from the cosmological data of $\sum_i m_i < 0.130$ eV at 95% confidence level obtained by the Planck collaboration [48]. Notice that the bound on $\sum_i m_i$ sensitively depends on the cosmological model and whether other experimental data such as BAO and gravitational lensing etc., are included [48].

FIG. 4. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}3$ with IO in region 2. Here we adopt the same conventions as Fig. 3.

The neutrinoless double beta ($0\nu\beta\beta$) decay is the unique probe for the Majorana nature of neutrinos, and it depends on the values of the Majorana CP violation phases. The $0\nu\beta\beta$ decay experiments can provide

valuable information on the neutrino mass spectrum and constrain the Majorana phases. The $0\nu\beta\beta$ decay rate is proportional to the effective Majorana mass $|m_{ee}|$ defined as [49],

FIG. 5. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}3$ with IO in region 3. Here we adopt the same conventions as Fig. 3.

$$|m_{ee}| = |m_1 \cos^2 \theta_{12} \cos^2 \theta_{13} + m_2 \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_{21}} + m_3 \sin^2 \theta_{13} e^{i(\alpha_{31} - 2\delta_{CP})}|. \quad (92)$$

From Eq. (89) we can extract the following predicted value for $|m_{ee}|$ in the $\mathcal{A}3$ model with NO,

$$|m_{ee}| = 0.04643 \text{ eV}, \quad (93)$$

which is within the reach of future $0\nu\beta\beta$ experiments.

For the model $\mathcal{A}3$ with IO and model $\mathcal{A}7$ with either NO or IO, there exist parameter spaces to be completely

FIG. 6. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}7$ with IO in region 1. Here we adopt the same conventions as Fig. 3.

consistent with the experimental data on neutrino masses and mixing angles. We find that the allowed ranges of the coupling constants and theoretical predictions crucially depend on the values of the complex modulus τ . In each case, we find several distinct local minima of χ^2 in the

fundamental domain of τ . We denote the phenomenologically viable regions of τ as region1, region2, region3 and so on, as shown in Fig. 2, where we have focused on $\text{Re}\tau > 0$ and the conjugate regions are not shown for simplicity. We perform an extensive numerical scan over the parameter

FIG. 7. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}7$ with IO in region 2. Here we adopt the same conventions as Fig. 3.

space, and the results of the numerical analysis are summarized in Tables VII–XII. The allowed regions of the input parameters and observables are determined by requiring all the lepton mixing angles and the squared mass splittings Δm_{21}^2 and $\Delta m_{3\ell}^2$ within the 3σ intervals [44].

Since the models depend on few free parameters, we find that the lepton mixing angles, CP violation phases, the lightest neutrino mass $m_{\min}$ and the effective Majorana mass $|m_{ee}|$ are generally correlated with each other. We display the different correlations in Figs. 3–12.

FIG. 8. The predictions for the correlations among the neutrino mixing angles, CP violation phases, and neutrino masses in the model $\mathcal{A}7$ with IO in region 3. Here we adopt the same conventions as Fig. 3.

We see that the Dirac CP violation phase δ_{CP} is predicted to lie in narrow regions in most cases. The forthcoming long-baseline neutrino oscillation experiments will considerably improve the sensitivity to δ_{CP} if running in both the neutrino and the antineutrino modes, we expect the predictions for δ_{CP} in this paper could be tested by future

neutrino facilities. Furthermore, we notice that the values of both $m_{\min}$ and $|m_{ee}|$ vary in quite limited ranges. The next generation experiments searching for $0\nu\beta\beta$ decay will be able to probe almost all the IO region, up to $m_{ee} \simeq 0.02$ eV, thus allowing us to test these modular symmetry models as well.

FIG. 9. The predictions for the correlations among the neutrino mixing angles, CP violation phases, and neutrino masses in the model $\mathcal{A}7$ with IO in region4. Here we adopt the same conventions as Fig. 3.

B. Numerical results of the models without flavons

Taking into account the freedom of field redefinition, the parameters α , β , and γ_1 in the charged lepton mass matrix can be set to be real. However, the coupling constant γ_2 is generally complex. The neutrino mass

matrix m_ν only depends on the modular parameter τ up to an overall factor in the models $\mathcal{B}1$, $\mathcal{B}2$, $\mathcal{B}4$, $\mathcal{B}5$, $\mathcal{B}6$, and $\mathcal{B}8$, and an additional complex parameter g_1/g_2 is involved in the models $\mathcal{B}3$ and $\mathcal{B}7$. The vacuum expectation value of the complex modulus τ is the unique

FIG. 10. The predictions for the correlations among the neutrino mixing angles, CP violation phases, and neutrino masses in the model $\mathcal{A}_7$ with NO in region1. Here we adopt the same conventions as Fig. 3.

source of flavor symmetry breaking in these models. In the same fashion as Sec. IV A, we use the package MINUIT to search for the minimum of the χ^2 function. We find that only the models $B3$ and $B7$ can accommodate

the experimental results, and the neutrino mass spectrum can be either NO or IO. Since the number of the input parameters is larger than that of the models with flavons, it takes a lot of time to comprehensively scan the whole

FIG. 11. The predictions for the correlations among the neutrino mixing angles, CP violation phases, and neutrino masses in the model $\mathcal{A7}$ with NO in region 2. Here we adopt the same conventions as Fig. 3.

parameter space of the models $\mathcal{B3}$ and $\mathcal{B7}$. We report the numerical results of the χ^2 analysis in Table XIII, and the values of the lepton mixing parameters and neutrino masses and charged lepton masses at the best fitting

points are displayed. It is remarkable that all the observables are quite close to their central values, and the Dirac CP phase is approximately maximal $\delta_{CP} \simeq 1.599\pi$ in the $\mathcal{B3}$ model for NO neutrino masses.

FIG. 12. The predictions for the correlations among the neutrino mixing angles, CP violation phases and neutrino masses in the model $\mathcal{A}7$ with NO in region 3. Here we adopt the same conventions as Fig. 3.

V. CONCLUSION

In this paper we have provided a comprehensive analysis of neutrino mass and lepton mixing in theories with $\Gamma_5 \cong A_5$ modular symmetry. We have constructed the weight 2, weight

4 and weight 6 modular forms of level 5 in terms of Dedekind eta-function and Klein forms, and their decomposition into irreducible representation of A_5 . We have provided the necessary mathematical tools which may be useful for

TABLE XIII. The best-fit values of the model parameters and lepton mixing parameters and neutrino masses for the models $\mathcal{B}3$ and $\mathcal{B}7$.

Models	$\mathcal{B}3$		$\mathcal{B}7$	
	Best-fit values			
$\text{Re}\langle\tau\rangle$	0.4972	0.2105	0.0355	0.1686
$\text{Im}\langle\tau\rangle$	0.6833	1.6860	1.2192	1.2519
β/α	172.531	76.55	632.301	349.182
γ_1/α	1003.07	160.105	86.55	49.182
$ \gamma_2/\alpha $	744.12	188.914	1.854	0.6863
$\arg(\gamma_2/\alpha)$	3.340	2.146	5.979	2.342
$ g_1/g_2 $	0.2185	1.180	38.741	1.198
$\arg(g_1/g_2)$	6.241	3.521	4.216	3.086
αv_d [MeV]	0.162	3.049	1.33	2.412
$g_2^2 v_u^2/\Lambda$ [eV]	0.0065	0.0066	0.00003	0.0107
m_e/m_μ	0.0048	0.0048	0.0048	0.0048
m_μ/m_τ	0.0565	0.0565	0.0565	0.0565
$\sin^2\theta_{12}$	0.31009	0.30999	0.31000	0.30999
$\sin^2\theta_{13}$	0.02264	0.02241	0.02264	0.02241
$\sin^2\theta_{23}$	0.58389	0.57999	0.58398	0.58000
δ_{CP}/π	0.808	1.599	1.164	1.933
α_{21}/π	1.814	0.981	0.126	1.088
α_{31}/π	0.823	0.101	1.735	1.710
r	0.02942	0.02927	0.02942	0.02927
m_1 [eV]	0.0504	0.0023	0.0494	0.0020
m_2 [eV]	0.0511	0.0089	0.0501	0.0088
m_3 [eV]	0.0102	0.0503	0.0001	0.0503
$\sum_i m_i$ [eV]	0.1117	0.0615	0.0995	0.0611
$ m_{ee} $ [eV]	0.0475	0.00226	0.04767	0.0013
Ordering	IO	NO	IO	NO
$N\sigma$	0.009	0.001	0.002	0.001

future studies of model building based on A_5 modular symmetry.

The modular group Γ_5 has two triplet irreducible representations $\mathbf{3}$ and $\mathbf{3}'$, the three generations of left-handed leptons can be assigned to transform as either $\mathbf{3}$ or $\mathbf{3}'$. We have constructed all the simplest models based on A_5 modular symmetry, including models with and without flavons in the charged lepton sectors. For each case, the neutrino masses are considered both using the Weinberg operator and the type I seesaw mechanism. This extends the scope of the previous analyses for A_5 modular symmetry in the literature [34]. We highlight the new and original parts of our analysis in the following

- (i) We have plotted the fundamental domain of $\Gamma(5)$ in Fig. 1. When numerically analyzing any given A_5 modular model, one only needs to consider the values of the modulus τ in the fundamental domain such that the numerical calculations can be simplified considerably.
- (ii) We have used completely different mathematical tools to construct modular forms: Dedekind eta-function and Klein forms versus Jacobi theta function. The

same q -expansions of modular forms as [34] are obtained up to irrelevant overall factor.

- (iii) We have considered scenarios both with and without flavons in the charged lepton sector, and the possible realizations and corresponding predictions for the charged lepton mass matrix are presented.
- (iv) The neutrino masses are generated through either the Weinberg operator or the type I seesaw mechanism.
- (v) We have constructed all the simplest models based on A_5 modular symmetry, and we propose 16 possible A_5 modular models, as summarized in Table IV.

We have performed an exhaustive numerical analysis for each model, presenting our results in the form of extensive sets of figures and tables. Surveying the comprehensive results in this paper, we observe that the predictions for lepton mixing strongly depend on the modular parameter τ , in other words the assumed vacuum expectation value of the modulus field. The modular symmetry models are therefore not as predictive as might have been expected. If the value of τ is not fixed by any mechanism, the lepton mixing parameters and neutrino masses can vary widely in different regions of τ parameter space. This observation is even true for benchmark modular symmetry models which only depend on τ and overall couplings. For this reason we divided the modular parameter τ into several regions when performing the numerical analysis. This motivates models in which the modulus τ is fixed, as recently discussed in [53] or in the context of string compactifications and supergravity [54–56]. The hope is that the compendium of models presented here may be a useful guide to future model building directions.

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APPENDIX A: GROUP THEORY OF A_5

A_5 is the group of even permutations of five objects, and it has $5!/2 = 60$ elements. Geometrically it is the symmetry group of a regular icosahedron. A_5 group can be generated by two generators S and T which obey the multiplication rules [57,58]:

$$S^2 = T^5 = (ST)^3 = 1. \quad (\text{A1})$$

The 60 element of A_5 group are divided into 5 conjugacy classes,

$$\begin{aligned} 1C_1: & 1 \\ 15C_2: & ST^2ST^3S, TST^4, T^4(ST^2)^2, T^2ST^3, (T^2S)^2T^3S, ST^2ST, S, T^3ST^2ST^3, \\ & T^3ST^2ST^3S, T^3ST^2, T^4ST^2ST^3S, TST^2S, ST^3ST^2S, T^4ST, (T^2S)^2T^4 \\ 20C_3: & ST, TS, ST^4, T^4S, TST^3, T^2ST^2, T^2ST^4, T^3ST, T^3ST^3, T^4ST^2, TST^3S, T^2ST^3S, \\ & T^3ST^2S, ST^2ST^3, ST^3ST, ST^3ST^2, (T^2S)^2T^2, T^2(T^2S)^2, (ST^2)^2S, (ST^2)^2T^2 \\ 12C_5: & T, T^4, ST^2, T^2S, ST^3, T^3S, STS, TST, TST^2, T^2ST, T^3ST^4, T^4ST^3 \\ 12C'_5: & T^2, T^3, ST^2S, ST^3S, (ST^2)^2, (T^2S)^2, (ST^3)^2, (T^3S)^2, (T^2S)^2T^3, \\ & T^3(ST^2)^2, T^3ST^2ST^4, T^4ST^2ST^3, \end{aligned} \quad (\text{A2})$$

where nC_k denotes a class with n elements which are of order k . The group structure of A_5 has been elaborately analyzed in Refs. [57,58]. The A_5 group has five irreducible representations: one singlet representation **1**, two three-dimensional representations **3** and **3'**, one four-dimensional representation **4** and one five-dimensional representation **5**. In the present work, we choose the same basis as that of Refs. [57,58]. The explicit forms of the generators S and T in the five irreducible representations are as follows

$$\begin{aligned} \mathbf{1}: & S = 1, \quad T = 1, \\ \mathbf{3}: & S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi_g & \phi_g - 1 \\ -\sqrt{2} & \phi_g - 1 & -\phi_g \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_5 & 0 \\ 0 & 0 & \omega_5^4 \end{pmatrix}, \\ \mathbf{3}': & S = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 - \phi_g & \phi_g \\ \sqrt{2} & \phi_g & 1 - \phi_g \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_5^2 & 0 \\ 0 & 0 & \omega_5^3 \end{pmatrix}, \\ \mathbf{4}: & S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \phi_g - 1 & \phi_g & -1 \\ \phi_g - 1 & -1 & 1 & \phi_g \\ \phi_g & 1 & -1 & \phi_g - 1 \\ -1 & \phi_g & \phi_g - 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \omega_5 & 0 & 0 & 0 \\ 0 & \omega_5^2 & 0 & 0 \\ 0 & 0 & \omega_5^3 & 0 \\ 0 & 0 & 0 & \omega_5^4 \end{pmatrix}, \\ \mathbf{5}: & S = \frac{1}{5} \begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & (\phi_g - 1)^2 & -2\phi_g & 2(\phi_g - 1) & \phi_g^2 \\ \sqrt{6} & -2\phi_g & \phi_g^2 & (\phi_g - 1)^2 & 2(\phi_g - 1) \\ \sqrt{6} & 2(\phi_g - 1) & (\phi_g - 1)^2 & \phi_g^2 & -2\phi_g \\ \sqrt{6} & \phi_g^2 & 2(\phi_g - 1) & -2\phi_g & (\phi_g - 1)^2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_5 & 0 & 0 & 0 \\ 0 & 0 & \omega_5^2 & 0 & 0 \\ 0 & 0 & 0 & \omega_5^3 & 0 \\ 0 & 0 & 0 & 0 & \omega_5^4 \end{pmatrix}, \end{aligned} \quad (\text{A3})$$

where $\phi_g = (1 + \sqrt{5})/2$ and $\omega_5 = e^{\frac{2\pi i}{5}}$. The Kronecker products between various irreducible representations are as follows,

$$\begin{aligned} \mathbf{1} \otimes \mathbf{R} &= \mathbf{R} \otimes \mathbf{1} = \mathbf{R}, & \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}, & \mathbf{3}' \otimes \mathbf{3}' &= \mathbf{1} \oplus \mathbf{3}' \oplus \mathbf{5}, & \mathbf{3} \times \mathbf{3}' &= \mathbf{4} \oplus \mathbf{5}, \\ \mathbf{3} \otimes \mathbf{4} &= \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}, & \mathbf{3}' \otimes \mathbf{4} &= \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5}, & \mathbf{3} \otimes \mathbf{5} &= \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}, \\ \mathbf{3}' \otimes \mathbf{5} &= \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}, & \mathbf{4} \otimes \mathbf{4} &= \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}, & \mathbf{4} \otimes \mathbf{5} &= \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}_1 \oplus \mathbf{5}_2, \\ \mathbf{5} \otimes \mathbf{5} &= \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4}_1 \oplus \mathbf{4}_2 \oplus \mathbf{5}_1 \oplus \mathbf{5}_2. \end{aligned} \quad (\text{A4})$$

where $\mathbf{R}$ represents any irreducible representation of A_5 , and $\mathbf{4}_1$, $\mathbf{4}_2$, $\mathbf{5}_1$, and $\mathbf{5}_2$ denote the two $\mathbf{4}$ and two $\mathbf{5}$ representations which appear in the Kronecker products.

Subsequently we list the Clebsch-Gordan coefficients in the chosen basis. We use α_i to denote the elements of the first representation, β_i to indicate these of the second representation of the product. The subscripts “S” and “A” indicate a combination which is symmetric and antisymmetric respectively.

$$(i) \quad \mathbf{3} \otimes \mathbf{3} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{5}_S$$

$$\mathbf{1}_S \sim \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2, \quad (A5)$$

$$\mathbf{3}_A \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}, \quad (A6)$$

$$\mathbf{5}_S \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ -\sqrt{3}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 \\ \sqrt{6}\alpha_2\beta_2 \\ \sqrt{6}\alpha_3\beta_3 \\ -\sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 \end{pmatrix}. \quad (A7)$$

$$(ii) \quad \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1}_S \oplus \mathbf{3}'_A \oplus \mathbf{5}_S$$

$$\mathbf{1}_S \sim \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2, \quad (A8)$$

$$\mathbf{3}'_A \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}, \quad (A9)$$

$$\mathbf{5}_S \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ \sqrt{6}\alpha_3\beta_3 \\ -\sqrt{3}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 \\ -\sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 \\ \sqrt{6}\alpha_2\beta_2 \end{pmatrix}. \quad (A10)$$

$$(iii) \quad \mathbf{3} \otimes \mathbf{3}' = \mathbf{4} \oplus \mathbf{5}$$

$$\mathbf{4} \sim \begin{pmatrix} \sqrt{2}\alpha_2\beta_1 + \alpha_3\beta_2 \\ -\sqrt{2}\alpha_1\beta_2 - \alpha_3\beta_3 \\ -\sqrt{2}\alpha_1\beta_3 - \alpha_2\beta_2 \\ \sqrt{2}\alpha_3\beta_1 + \alpha_2\beta_3 \end{pmatrix}, \quad (A11)$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 \\ \alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_2 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_3\beta_3 \\ \alpha_1\beta_3 - \sqrt{2}\alpha_2\beta_2 \\ \alpha_3\beta_1 - \sqrt{2}\alpha_2\beta_3 \end{pmatrix}. \quad (A12)$$

$$(iv) \quad \mathbf{3} \otimes \mathbf{4} = \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$$

$$\mathbf{3}' \sim \begin{pmatrix} -\sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_3\beta_1 \\ \sqrt{2}\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_3 \\ \sqrt{2}\alpha_1\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_4 \end{pmatrix}, \quad (A13)$$

$$\mathbf{4} \sim \begin{pmatrix} \alpha_1\beta_1 - \sqrt{2}\alpha_3\beta_2 \\ -\alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_1 \\ \alpha_1\beta_3 + \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_3 \end{pmatrix}, \quad (A14)$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{6}\alpha_2\beta_4 - \sqrt{6}\alpha_3\beta_1 \\ 2\sqrt{2}\alpha_1\beta_1 + 2\alpha_3\beta_2 \\ -\sqrt{2}\alpha_1\beta_2 + \alpha_2\beta_1 + 3\alpha_3\beta_3 \\ \sqrt{2}\alpha_1\beta_3 - 3\alpha_2\beta_2 - \alpha_3\beta_4 \\ -2\sqrt{2}\alpha_1\beta_4 - 2\alpha_2\beta_3 \end{pmatrix}. \quad (A15)$$

$$(v) \quad \mathbf{3}' \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5}$$

$$\mathbf{3} \sim \begin{pmatrix} -\sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 \\ \sqrt{2}\alpha_1\beta_1 + \alpha_2\beta_4 - \alpha_3\beta_3 \\ \sqrt{2}\alpha_1\beta_4 - \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix}, \quad (A16)$$

$$\mathbf{4} \sim \begin{pmatrix} \alpha_1\beta_1 + \sqrt{2}\alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_3 + \sqrt{2}\alpha_2\beta_1 \\ -\alpha_1\beta_4 - \sqrt{2}\alpha_2\beta_2 \end{pmatrix}, \quad (A17)$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{6}\alpha_2\beta_3 - \sqrt{6}\alpha_3\beta_2 \\ \sqrt{2}\alpha_1\beta_1 - 3\alpha_2\beta_4 - \alpha_3\beta_3 \\ 2\sqrt{2}\alpha_1\beta_2 + 2\alpha_3\beta_4 \\ -2\sqrt{2}\alpha_1\beta_3 - 2\alpha_2\beta_1 \\ -\sqrt{2}\alpha_1\beta_4 + \alpha_2\beta_2 + 3\alpha_3\beta_1 \end{pmatrix}. \quad (A18)$$

$$(vi) \quad \mathbf{3} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5},$$

$$\mathbf{3} \sim \begin{pmatrix} -2\alpha_1\beta_1 + \sqrt{3}\alpha_2\beta_5 + \sqrt{3}\alpha_3\beta_2 \\ \sqrt{3}\alpha_1\beta_2 + \alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_3 \\ \sqrt{3}\alpha_1\beta_5 - \sqrt{6}\alpha_2\beta_4 + \alpha_3\beta_1 \end{pmatrix}, \quad (A19)$$

$$\mathbf{3}' \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_2 \\ \alpha_1\beta_3 - \sqrt{2}\alpha_2\beta_2 - \sqrt{2}\alpha_3\beta_4 \\ \alpha_1\beta_4 - \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_5 \end{pmatrix}, \quad (\text{A20})$$

$$\mathbf{4} \sim \begin{pmatrix} 2\sqrt{2}\alpha_1\beta_2 - \sqrt{6}\alpha_2\beta_1 + \alpha_3\beta_3 \\ -\sqrt{2}\alpha_1\beta_3 + 2\alpha_2\beta_2 - 3\alpha_3\beta_4 \\ \sqrt{2}\alpha_1\beta_4 + 3\alpha_2\beta_3 - 2\alpha_3\beta_5 \\ -2\sqrt{2}\alpha_1\beta_5 - \alpha_2\beta_4 + \sqrt{6}\alpha_3\beta_1 \end{pmatrix}, \quad (\text{A21})$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{3}\alpha_2\beta_5 - \sqrt{3}\alpha_3\beta_2 \\ -\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_3 \\ -2\alpha_1\beta_3 - \sqrt{2}\alpha_2\beta_2 \\ 2\alpha_1\beta_4 + \sqrt{2}\alpha_3\beta_5 \\ \alpha_1\beta_5 + \sqrt{2}\alpha_2\beta_4 + \sqrt{3}\alpha_3\beta_1 \end{pmatrix}. \quad (\text{A22})$$

$$(\text{vii}) \quad \mathbf{3}' \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5},$$

$$\mathbf{3} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 + \alpha_2\beta_4 + \alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_5 - \sqrt{2}\alpha_3\beta_4 \\ \alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 \end{pmatrix}, \quad (\text{A23})$$

$$\mathbf{3}' \sim \begin{pmatrix} -2\alpha_1\beta_1 + \sqrt{3}\alpha_2\beta_4 + \sqrt{3}\alpha_3\beta_3 \\ \sqrt{3}\alpha_1\beta_3 + \alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_5 \\ \sqrt{3}\alpha_1\beta_4 - \sqrt{6}\alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix}, \quad (\text{A24})$$

$$\mathbf{4} \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2 + 3\alpha_2\beta_5 - 2\alpha_3\beta_4 \\ 2\sqrt{2}\alpha_1\beta_3 - \sqrt{6}\alpha_2\beta_1 + \alpha_3\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4 - \alpha_2\beta_2 + \sqrt{6}\alpha_3\beta_1 \\ -\sqrt{2}\alpha_1\beta_5 + 2\alpha_2\beta_3 - 3\alpha_3\beta_2 \end{pmatrix}, \quad (\text{A25})$$

$$\mathbf{5} \sim \begin{pmatrix} \sqrt{3}\alpha_2\beta_4 - \sqrt{3}\alpha_3\beta_3 \\ 2\alpha_1\beta_2 + \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_3 - \sqrt{3}\alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_5 \\ \alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_2 + \sqrt{3}\alpha_3\beta_1 \\ -2\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 \end{pmatrix}. \quad (\text{A26})$$

$$(\text{viii}) \quad \mathbf{4} \otimes \mathbf{4} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{3}'_A \oplus \mathbf{4}_S \oplus \mathbf{5}_S$$

$$\mathbf{1}_S \sim \alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1, \quad (\text{A27})$$

$$\mathbf{3}_A \sim \begin{pmatrix} -\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1 \\ \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_4\beta_2 \\ \sqrt{2}\alpha_1\beta_3 - \sqrt{2}\alpha_3\beta_1 \end{pmatrix}, \quad (\text{A28})$$

$$\mathbf{3}'_A \sim \begin{pmatrix} \alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 \\ \sqrt{2}\alpha_3\beta_4 - \sqrt{2}\alpha_4\beta_3 \\ \sqrt{2}\alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_1 \end{pmatrix}, \quad (\text{A29})$$

$$\mathbf{4}_S \sim \begin{pmatrix} \alpha_2\beta_4 + \alpha_3\beta_3 + \alpha_4\beta_2 \\ \alpha_1\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix}, \quad (\text{A30})$$

$$\mathbf{5}_S \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{3}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \\ -\sqrt{2}\alpha_2\beta_4 + 2\sqrt{2}\alpha_3\beta_3 - \sqrt{2}\alpha_4\beta_2 \\ -2\sqrt{2}\alpha_1\beta_1 + \sqrt{2}\alpha_3\beta_4 + \sqrt{2}\alpha_4\beta_3 \\ \sqrt{2}\alpha_1\beta_2 + \sqrt{2}\alpha_2\beta_1 - 2\sqrt{2}\alpha_4\beta_4 \\ -\sqrt{2}\alpha_1\beta_3 + 2\sqrt{2}\alpha_2\beta_2 - \sqrt{2}\alpha_3\beta_1 \end{pmatrix}. \quad (\text{A31})$$

$$(\text{ix}) \quad \mathbf{4} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}_1 \oplus \mathbf{5}_2$$

$$\mathbf{3} \sim \begin{pmatrix} 2\sqrt{2}\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_4 + \sqrt{2}\alpha_3\beta_3 - 2\sqrt{2}\alpha_4\beta_2 \\ -\sqrt{6}\alpha_1\beta_1 + 2\alpha_2\beta_5 + 3\alpha_3\beta_4 - \alpha_4\beta_3 \\ \alpha_1\beta_4 - 3\alpha_2\beta_3 - 2\alpha_3\beta_2 + \sqrt{6}\alpha_4\beta_1 \end{pmatrix}, \quad (\text{A32})$$

$$\mathbf{3}' \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5 + 2\sqrt{2}\alpha_2\beta_4 - 2\sqrt{2}\alpha_3\beta_3 - \sqrt{2}\alpha_4\beta_2 \\ 3\alpha_1\beta_2 - \sqrt{6}\alpha_2\beta_1 - \alpha_3\beta_5 + 2\alpha_4\beta_4 \\ -2\alpha_1\beta_3 + \alpha_2\beta_2 + \sqrt{6}\alpha_3\beta_1 - 3\alpha_4\beta_5 \end{pmatrix}, \quad (\text{A33})$$

$$\mathbf{4} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 - \sqrt{2}\alpha_2\beta_5 + \sqrt{2}\alpha_3\beta_4 - 2\sqrt{2}\alpha_4\beta_3 \\ -\sqrt{2}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 + 2\sqrt{2}\alpha_3\beta_5 + \sqrt{2}\alpha_4\beta_4 \\ \sqrt{2}\alpha_1\beta_3 + 2\sqrt{2}\alpha_2\beta_2 - \sqrt{3}\alpha_3\beta_1 - \sqrt{2}\alpha_4\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \end{pmatrix}, \quad (\text{A34})$$

$$\mathbf{5}_1 \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_3\beta_3 + \sqrt{2}\alpha_4\beta_2 \\ -\sqrt{2}\alpha_1\beta_1 - \sqrt{3}\alpha_3\beta_4 - \sqrt{3}\alpha_4\beta_3 \\ \sqrt{3}\alpha_1\beta_2 + \sqrt{2}\alpha_2\beta_1 + \sqrt{3}\alpha_3\beta_5 \\ \sqrt{3}\alpha_2\beta_2 + \sqrt{2}\alpha_3\beta_1 + \sqrt{3}\alpha_4\beta_5 \\ -\sqrt{3}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{2}\alpha_4\beta_1 \end{pmatrix}, \quad (\text{A35})$$

$$\mathbf{5}_2 \sim \begin{pmatrix} 2\alpha_1\beta_5 + 4\alpha_2\beta_4 + 4\alpha_3\beta_3 + 2\alpha_4\beta_2 \\ 4\alpha_1\beta_1 + 2\sqrt{6}\alpha_2\beta_5 \\ -\sqrt{6}\alpha_1\beta_2 + 2\alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_5 + 2\sqrt{6}\alpha_4\beta_4 \\ 2\sqrt{6}\alpha_1\beta_3 - \sqrt{6}\alpha_2\beta_2 + 2\alpha_3\beta_1 - \sqrt{6}\alpha_4\beta_5 \\ 2\sqrt{6}\alpha_3\beta_2 + 4\alpha_4\beta_1 \end{pmatrix}. \quad (\text{A36})$$

$$(\text{x}) \quad \mathbf{5} \otimes \mathbf{5} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{3}'_A \oplus \mathbf{4}_S \oplus \mathbf{4}_A \oplus \mathbf{5}_{S,1} \oplus \mathbf{5}_{S,2} \\ \mathbf{1}_S \sim \alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 + \alpha_5\beta_2, \quad (\text{A37})$$

$$\mathbf{3}_A \sim \begin{pmatrix} \alpha_2\beta_5 + 2\alpha_3\beta_4 - 2\alpha_4\beta_3 - \alpha_5\beta_2 \\ -\sqrt{3}\alpha_1\beta_2 + \sqrt{3}\alpha_2\beta_1 + \sqrt{2}\alpha_3\beta_5 - \sqrt{2}\alpha_5\beta_3 \\ \sqrt{3}\alpha_1\beta_5 + \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_4\beta_2 - \sqrt{3}\alpha_5\beta_1 \end{pmatrix}, \quad (\text{A38})$$

$$\mathbf{3}'_A \sim \begin{pmatrix} 2\alpha_2\beta_5 - \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ \sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 + \sqrt{2}\alpha_4\beta_5 - \sqrt{2}\alpha_5\beta_4 \\ -\sqrt{3}\alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \end{pmatrix}, \quad (\text{A39})$$

$$\mathbf{4}_S \sim \begin{pmatrix} 3\sqrt{2}\alpha_1\beta_2 + 3\sqrt{2}\alpha_2\beta_1 - \sqrt{3}\alpha_3\beta_5 + 4\sqrt{3}\alpha_4\beta_4 - \sqrt{3}\alpha_5\beta_3 \\ 3\sqrt{2}\alpha_1\beta_3 + 4\sqrt{3}\alpha_2\beta_2 + 3\sqrt{2}\alpha_3\beta_1 - \sqrt{3}\alpha_4\beta_5 - \sqrt{3}\alpha_5\beta_4 \\ 3\sqrt{2}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{3}\alpha_3\beta_2 + 3\sqrt{2}\alpha_4\beta_1 + 4\sqrt{3}\alpha_5\beta_5 \\ 3\sqrt{2}\alpha_1\beta_5 - \sqrt{3}\alpha_2\beta_4 + 4\sqrt{3}\alpha_3\beta_3 - \sqrt{3}\alpha_4\beta_2 + 3\sqrt{2}\alpha_5\beta_1 \end{pmatrix}, \quad (\text{A40})$$

$$\mathbf{4}_A \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_1 + \sqrt{3}\alpha_3\beta_5 - \sqrt{3}\alpha_5\beta_3 \\ -\sqrt{2}\alpha_1\beta_3 + \sqrt{2}\alpha_3\beta_1 + \sqrt{3}\alpha_4\beta_5 - \sqrt{3}\alpha_5\beta_4 \\ -\sqrt{2}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 + \sqrt{3}\alpha_3\beta_2 + \sqrt{2}\alpha_4\beta_1 \\ \sqrt{2}\alpha_1\beta_5 - \sqrt{3}\alpha_2\beta_4 + \sqrt{3}\alpha_4\beta_2 - \sqrt{2}\alpha_5\beta_1 \end{pmatrix}, \quad (\text{A41})$$

$$\mathbf{5}_{S,1} \sim \begin{pmatrix} 2\alpha_1\beta_1 + \alpha_2\beta_5 - 2\alpha_3\beta_4 - 2\alpha_4\beta_3 + \alpha_5\beta_2 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \sqrt{6}\alpha_3\beta_5 + \sqrt{6}\alpha_5\beta_3 \\ -2\alpha_1\beta_3 + \sqrt{6}\alpha_2\beta_2 - 2\alpha_3\beta_1 \\ -2\alpha_1\beta_4 - 2\alpha_4\beta_1 + \sqrt{6}\alpha_5\beta_5 \\ \alpha_1\beta_5 + \sqrt{6}\alpha_2\beta_4 + \sqrt{6}\alpha_4\beta_2 + \alpha_5\beta_1 \end{pmatrix}, \quad (\text{A42})$$

$$\mathbf{5}_{S,2} \sim \begin{pmatrix} 2\alpha_1\beta_1 - 2\alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ -2\alpha_1\beta_2 - 2\alpha_2\beta_1 + \sqrt{6}\alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_3\beta_1 + \sqrt{6}\alpha_4\beta_5 + \sqrt{6}\alpha_5\beta_4 \\ \alpha_1\beta_4 + \sqrt{6}\alpha_2\beta_3 + \sqrt{6}\alpha_3\beta_2 + \alpha_4\beta_1 \\ -2\alpha_1\beta_5 + \sqrt{6}\alpha_3\beta_3 - 2\alpha_5\beta_1 \end{pmatrix}. \quad (\text{A43})$$

APPENDIX B: CHARGED LEPTON SECTOR WITH FLAVONS

In this Appendix, we shall show that a diagonal charged lepton mass matrix can be achieved if the VEVs of the flavon fields are aligned in appropriate directions. The left-handed lepton doublets L are assigned to a triplet $\mathbf{3}$ of A_5 , and the right-handed charged leptons E_i^c transform as $\mathbf{1}$. In order to achieve the desired vacuum alignment and to reproduce the observed charged lepton mass hierarchies, the auxiliary symmetry Z_5 is invoked. We use the supersymmetric driving field mechanism to determine the vacuum alignment of the flavons [23], and a continuous $U(1)_R$ symmetry related to the usual R -parity is assumed. The R -charge of the matter superfields is $+1$, the Higgs and flavon fields are uncharged and the so-called driving fields, indicated with the superscript “0” carry two units of R -charge. The field content and the symmetry assignments are shown in Table XIV. The superpotential of the charged lepton sector can be written as

$$w = w_l + w_d, \quad (\text{B1})$$

with

$$w_d = f_1\sigma^0(\varphi\varphi)_1 + f_2(\phi^0(\varphi\phi)_4)_1 + f_3(\phi^0(\varphi\psi)_4)_1 + M_\psi(\psi^0\psi)_1 + f_4(\psi^0(\varphi\varphi)_5)_1, \quad (\text{B2})$$

$$w_l = \frac{y_\tau}{\Lambda} E_3^c(L\varphi)_1 H_d + \frac{y_{\mu_1}}{\Lambda^2} E_2^c(L(\phi\psi)_3)_1 H_d + \frac{y_{\mu_2}}{\Lambda^2} E_2^c(L(\psi\psi)_3)_1 H_d + \frac{y_{e_1}}{\Lambda^3} E_1^c(L\varphi)_1(\phi\phi)_1 H_d \\ + \frac{y_{e_2}}{\Lambda^3} E_1^c((L\varphi)_5(\phi\phi)_5)_1 H_d + \frac{y_{e_3}}{\Lambda^3} E_1^c((L\varphi)_3(\phi\psi)_3)_1 H_d + \frac{y_{e_4}}{\Lambda^3} E_1^c((L\varphi)_5(\phi\psi)_5)_1 H_d \\ + \frac{y_{e_5}}{\Lambda^3} E_1^c(L\varphi)_1(\psi\psi)_1 H_d + \frac{y_{e_6}}{\Lambda^3} E_1^c((L\varphi)_3(\psi\psi)_3)_1 H_d + \frac{y_{e_7}}{\Lambda^3} E_1^c((L\varphi)_5(\psi\psi)_5)_1 H_d + \frac{y_{e_8}}{\Lambda^3} E_1^c((L\varphi)_5(\psi\psi)_5)_1 H_d. \quad (\text{B3})$$

TABLE XIV. The charge assignment of the matter fields, flavon fields, and driving fields under the Γ_5 modular symmetry, Z_5 auxiliary symmetry and $U(1)_R$ in the charged lepton sector.

Fields	L	E_1^c	E_2^c	E_3^c	H_d	φ	ϕ	ψ	σ^0	ϕ^0	ψ^0
$\Gamma_5 \cong A_5$	3	1	1	1	1	3	3'	5	1	4	5
k_I	k_L	k_{E_1}	k_{E_2}	k_{E_3}	k_d	k_φ	k_ϕ	k_ψ	k_{σ^0}	k_{ϕ^0}	k_{ψ^0}
Z_5	1	1	ω_5	ω_5^4	1	ω_5	ω_5^2	ω_5^2	ω_5^3	ω_5^2	ω_5^3
$U(1)_R$	1	1	1	1	0	0	0	0	2	2	2

The modular weight of each term should be zero such that the following constraints should be satisfied,

$$-\frac{2}{5}(k_{E_1} + k_L + k_d) = -\frac{1}{2}(k_{E_2} + k_L + k_d) = -2(k_{E_3} + k_L + k_d) = k_\phi, \quad (\text{B4a})$$

$$k_\phi = k_\psi = 2k_\varphi = -k_{\sigma^0} = -k_{\psi^0} = -\frac{2}{3}k_{\phi^0}. \quad (\text{B4b})$$

In the limit of supersymmetry, the vacuum configuration is determined by the vanishing of the derivative of the driving superpotential w_d with respect to each component of the driving fields,

$$\begin{aligned}
\frac{\partial w_d}{\partial \sigma^0} &= f_1(\varphi_1^2 + 2\varphi_2\varphi_3) = 0, \\
\frac{\partial w_d}{\partial \phi_1^0} &= f_2(\varphi_2\phi_3 + \sqrt{2}\varphi_3\phi_1) - f_3(2\sqrt{2}\varphi_1\psi_5 + \varphi_2\psi_4 - \sqrt{6}\varphi_3\psi_1) = 0, \\
\frac{\partial w_d}{\partial \phi_2^0} &= -f_2(\sqrt{2}\varphi_1\phi_3 + \varphi_2\phi_2) + f_3(\sqrt{2}\varphi_1\psi_4 + 3\varphi_2\psi_3 - 2\varphi_3\psi_5) = 0, \\
\frac{\partial w_d}{\partial \phi_3^0} &= -f_2(\sqrt{2}\varphi_1\phi_2 + \varphi_3\phi_3) - f_3(\sqrt{2}\varphi_1\psi_3 - 2\varphi_2\psi_2 + 3\varphi_3\psi_4) = 0, \\
\frac{\partial w_d}{\partial \phi_4^0} &= f_2(\sqrt{2}\varphi_2\phi_1 + \varphi_3\phi_2) + f_3(2\sqrt{2}\varphi_1\psi_2 - \sqrt{6}\varphi_2\psi_1 + \varphi_3\psi_3) = 0, \\
\frac{\partial w_d}{\partial \psi_1^0} &= M_\psi\psi_1 + 2f_4(\varphi_1^2 - \varphi_2\varphi_3) = 0, \\
\frac{\partial w_d}{\partial \psi_2^0} &= M_\psi\psi_5 - 2\sqrt{3}f_4\varphi_1\varphi_3 = 0, \\
\frac{\partial w_d}{\partial \psi_3^0} &= M_\psi\psi_4 + \sqrt{6}f_4\varphi_3^2 = 0, \\
\frac{\partial w_d}{\partial \psi_4^0} &= M_\psi\psi_3 + \sqrt{6}f_4\varphi_2^2 = 0, \\
\frac{\partial w_d}{\partial \psi_5^0} &= M_\psi\psi_2 - 2\sqrt{3}f_4\varphi_1\varphi_2 = 0.
\end{aligned} \quad (\text{B5})$$

These equations are satisfied by the alignment

$$\begin{aligned}
\langle \varphi \rangle &= v_\varphi(0, 1, 0), & \langle \phi \rangle &= v_\phi(0, 1, 0), \\
\langle \psi \rangle &= v_\psi(0, 0, 1, 0, 0),
\end{aligned} \quad (\text{B6})$$

up to symmetry transformations of A_5 , where the VEVs v_φ , v_ϕ , and v_ψ are related through

$$v_\phi = -\frac{3\sqrt{6}f_3f_4}{M_\psi f_2}v_\varphi^2, \quad v_\psi = -\frac{\sqrt{6}f_4}{M_\psi}v_\varphi^2, \quad v_\varphi \text{ undetermined.} \quad (\text{B7})$$

With the vacuum configuration in Eq. (B6), we find the charged lepton mass matrix is diagonal and the three charged lepton masses are given by,

$$\begin{aligned} m_e &= \sqrt{2} \left| 3y_{e_2} \frac{v_\phi^2 v_\varphi}{\Lambda^3} + (y_{e_3} - \sqrt{3}y_{e_4}) \frac{v_\phi v_\varphi v_\psi}{\Lambda^3} + 3y_{e_8} \frac{v_\phi v_\psi^2}{\Lambda^3} \right| v_d \\ m_\mu &= \sqrt{2} \left| y_{\mu_1} \frac{v_\phi v_\psi}{\Lambda^2} \right| v_d \\ m_\tau &= \left| y_\tau \frac{v_\varphi}{\Lambda} \right| v_d. \end{aligned} \quad (\text{B8})$$

We see that the electron, muon, and tau masses involve three flavons, two flavons and one flavon respectively because of the Z_5 auxiliary symmetry and modular weight

assignments. The observed charged lepton mass hierarchies $m_e:m_\mu:m_\tau \simeq \lambda_c^4:\lambda_c^2:1$ can be reproduced if all the VEVs are of the same order of magnitude

$$\frac{v_\varphi}{\Lambda} \sim \frac{v_\phi}{\Lambda} \sim \frac{v_\psi}{\Lambda} \sim \mathcal{O}(\lambda_c^2), \quad (\text{B9})$$

where $\lambda_c \simeq 0.23$ is the Cabibbo angle. It is easy to see that the model for $\rho_L = \mathbf{3}'$ can be constructed in a similar way. In some modular invariant models, few flavon fields aligned along certain directions are introduced such that the charged lepton mass matrix is diagonal [25]. In order to achieve the desired vacuum alignment, additional fields such as driving fields and shaping symmetry are needed, as shown above. The complexity of vacuum alignment can be avoided in the full modular symmetry models without flavons where the modular invariance plays the role of flavor symmetry and it is uniquely broken by the complex modulus τ [25].

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