Note on bulk reconstruction in $\text{AdS}_3/\text{warped CFT}_2$

Yu-han Lin¹ and Bin Chen^{1,2,3,*}¹*Department of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, No. 5 Yiheyuan Rd, Beijing 100871, People's Republic China*²*Center for High Energy Physics, Peking University, No. 5 Yiheyuan Rd, Beijing 100871, People's Republic China*³*Collaborative Innovation Center of Quantum Matter, No. 5 Yiheyuan Rd, Beijing 100871, People's Republic China*

(Received 17 May 2019; published 25 October 2019)

The bulk reconstructions in AdS/CFT and their cousins are essential to understand the holographic nature of quantum gravity. In this work, we try to study the bulk reconstruction in the $\text{AdS}_3/\text{WCFT}_2$ correspondence. After deriving the bulk-boundary propagator, which is different from the usual one in AdS/CFT , we define the bulk protoscalar field by using the Virasoro-Kac-Moody symmetry in two different ways. One is to impose the bulk primary conditions on the field and construct the field algebraically. The other is to use the bulk-boundary vacuum operator product expansion block, which can be read by applying the diffeomorphism preserving the Compère-Song-Strominger boundary conditions. Two approaches lead to a consistent picture.

DOI: [10.1103/PhysRevD.100.086018](https://doi.org/10.1103/PhysRevD.100.086018)

I. INTRODUCTION

The AdS/CFT correspondence states that the quantum gravity in anti-de Sitter (AdS) spacetime is dual to a conformal field theory at asymptotical AdS boundary [1]. It has been intensely investigated from various points of view since its proposal. One central question is to understand how the holography works in the correspondence. This inspired people to study the holographic duality to the spacetime beyond the AdS or the quantum field theory without full conformal symmetry. Such a kind of study has resulted in the setup of other correspondences beyond AdS/CFT , including dS/CFT [2], Kerr/CFT [3], warped AdS/CFT [4], AdS/WCFT [5], etc.

It turns out that the choice of asymptotic boundary conditions and the corresponding asymptotic symmetry group (ASG) plays an essential role in setting up the correspondences. In particular, for the AdS_3 gravity, it has been found that besides the well-known Brown-Henneaux boundary conditions [6], which lead to the $\text{AdS}_3/\text{CFT}_2$ correspondence [7], there exist other sets of consistent boundary conditions [5,8]. One set of them is the Compère-Song-Strominger (CSS) boundary conditions, whose ASG

is generated by an Virasoro-Kac-Moody algebra. It was proposed that the AdS_3 gravity with CSS boundary conditions should be dual to a warped CFT , which has the same Virasoro-Kac-Moody symmetry [5]. This so-called $\text{AdS}_3/\text{WCFT}_2$ correspondence¹ has been studied from several points of view, including the microscopic entropy of black holes [11], the entanglement entropy [9,10,12], the one-loop partition function [13] and very recently the Rényi mutual information of two disjoint intervals [14]. In this work, we study the bulk reconstruction in the $\text{AdS}_3/\text{WCFT}_2$ correspondence.

The bulk reconstruction is one of most important questions in the AdS/CFT correspondence. As a concrete realization of holographic principle in quantum gravity, the AdS/CFT correspondence provides a way to probe bulk physics from boundary data. In a narrow sense, the bulk reconstruction is an attempt to express the bulk field in terms of boundary local operators. It allows us to understand bulk processed using CFT techniques [15]. There are several attempts to investigate the bulk reconstruction; for a nice review see [16]. The Hamilton-Kabat-Lifschytz-Lowe (HKLL) global reconstruction [17,18] is the earliest and mostly studied one. In two-dimensional CFT , the existence

*bchen01@pku.edu.cn, 1301110085@pku.edu.cn

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

¹There exists another correspondence that states that the gravity in a warped AdS_3 may be dual to a warped CFT [9,10]. This correspondence relied on the fact that the isometry group of warped AdS_3 spacetime is the same as the warped conformal symmetry. However, the warped spacetime can only appear in the topological gravity theory [4]. In this work, we focus on the AdS_3/WCFT correspondence.

of local conformal algebra allows local bulk reconstruction. Recently an exact AdS_3 protofield [19,20] was proposed to capture the nonperturbative information of gravitational background. The study relied heavily on the symmetry of the theory. From the symmetry considerations, a bulk primary condition is imposed and can be used to solve the protofield exactly.

In this work, we study the bulk reconstruction via the protofield. As the first step, we need to study the global reconstruction in the AdS/WCFT correspondence. We propose a bulk-boundary propagator for the boundary primary operator. Similar to the AdS/CFT case, we start by solving the Klein-Gordon equation for the bulk-boundary propagator. But now as the isometry group is broken by the choice of the boundary condition, we cannot turn on the delta-function source to the Klein-Gordon equation. Instead we have to require that the bulk-boundary propagator matches the two-point function of WCFT as the bulk point approaches the boundary. With the bulk-boundary propagator, we compute the bulk three-point Witten diagrams and find agreement with the three-point functions in WCFT. Using the bulk-boundary propagator obtained, we apply the HKLL proposal to construct the bulk field and express it in terms of the boundary primary and its global descendants, similar to the AdS/CFT case. And using the bulk field we get, we calculate the bulk-bulk propagator. Furthermore, we show how to define a protofield by imposing bulk primary conditions. We solve the protofield exactly. We have also considered the correlator of two bulk protofields, but a compact result has not been found. We take another way to describe the bulk protofield by using the bulk-boundary operator product expansion (OPE) block. Using a diffeomorphism between AdS spaces compatible with the CSS boundary conditions and the vacuum AdS space, we obtain the bulk-boundary vacuum OPE block for AdS/WCFT correspondence.

The rest of the paper goes as follows. In Sec. II we give a brief review on the AdS/WCFT correspondence. In Sec. III, we study the bulk-boundary propagator in the AdS/WCFT correspondence. We discuss the requirement for the bulk-boundary propagator and find the propagator. And then we construct the bulk field following the HKLL proposal. In Sec. IV we study the local reconstruction in AdS/WCFT . We impose the bulk primary condition on the protoscalar field, which can be solved level by level. We also discuss the properties of the correlator between two bulk level n protofields. Moreover, we investigate the bulk protofield by studying its OPE block with the boundary primary. We focus on the bulk-boundary vacuum OPE block, which can be read from the diffeomorphism preserving the CSS boundary conditions and is also most related to the bulk gravitational physics.

II. THE AdS/WCFT CORRESPONDENCE

This section is a brief review on the AdS/WCFT correspondence. Both the bulk and boundary perspectives

are introduced. Some basic concepts and properties are presented for later use.

A two-dimensional translation-invariant theory with a chiral scaling symmetry could be extended to a local algebra [21]. There are two minimal options for this algebra. One is the usual conformal algebras with two copies of Virasoro algebras, and the other is warped conformal algebra with a Virasoro algebra plus a $\text{U}(1)$ Kac-Moody algebra. The latter choice is what we called warped conformal field theory [11]. The local symmetry of a WCFT is of the form

$$x^+ \rightarrow f(x^+), \quad x^- \rightarrow x^- + g(x^+). \quad (2.1)$$

When the theory is put on a cylinder, the conserved charges can be written in terms of the Fourier modes. And the symmetries of WCFT are generated by a copy of Virasoro algebra with central charge c and a copy of Kac-Moody algebra with level k . The commutation relations of the generators are

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}, \\ [L_n, P_m] &= -mP_{m+n}, \\ [P_n, P_m] &= \frac{k}{2}n\delta_{n,-m}, \end{aligned} \quad (2.2)$$

where the Kac-Moody generators transform canonically under the action of the Virasoro generators. After subtracting the Sugawara construction of the $\text{U}(1)$ current from the energy-momentum tensor, the resulting Virasoro algebra and Kac-Moody algebra commute with each other. With respect to the symmetries, the states of WCFT can be described by the representations of the algebra. The primary states are defined to obey the relations

$$\begin{aligned} L_n|\Delta, Q\rangle &= 0, \quad P_n|\Delta, Q\rangle = 0, \quad \forall n > 0 \\ L_0|\Delta, Q\rangle &= \Delta|\Delta, Q\rangle, \quad P_0|\Delta, Q\rangle = Q|\Delta, Q\rangle. \end{aligned} \quad (2.3)$$

Namely, the primary states are characterized by the scaling dimension Δ and the charge Q . And the states in the highest weight representation are the descendants of the primary state by acting on the creation operators. In this paper only the primary states are of our concern.²

Three-dimensional AdS gravity should be defined properly with the asymptotical boundary conditions. When imposing the Brown-Henneaux (BH) boundary conditions [6] on AdS_3 gravity, the asymptotic symmetry group on the boundary is generated by two copies of the Virasoro algebra with central charges $c_L = c_R = \frac{3l}{2G}$. This suggests that the AdS_3 gravity with BH boundary conditions could be dual to a 2D CFT. The corresponding dual conformal

²For other aspects of the warped CFT, see [22–24].

field theory is called holographic CFT, which should have a large central charge and a sparse spectrum of light states [25].

The AdS₃ gravity can be defined with other consistent boundary conditions. Under the CSS boundary conditions, the ASG is generated by the Virasoro-Kac-Moody algebra [5], but not in the canonical form

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}, \\ [L_n, P_m] &= -mP_{m+n} + mP_0\delta_{n,-m}, \\ [P_n, P_m] &= -2nP_0\delta_{n,-m}. \end{aligned} \quad (2.4)$$

The central charge takes the same value as in the Brown-Henneaux case

$$c = 3l/2G. \quad (2.5)$$

The asymptotic symmetry algebra and the canonical WCFT algebra are actually related by a charge redefinition,

$$L_n \rightarrow L_n + 2P_0P_n - P_0^2\delta_n, \quad P_n \rightarrow 2P_0P_n - P_0^2\delta_n. \quad (2.6)$$

This suggests that the AdS₃ gravity with the CSS boundary condition could be dual to a WCFT. The corresponding holographic WCFT is expected to have a large central charge, a negative Kac level, and a sparse spectrum of light states. The regularized on-shell bulk gravity action is now a chiral Liouville gravity [26].

The CSS boundary conditions are chiral boundary conditions with one component of the boundary metric being promoted to a dynamical variable. The most general solution to the Einstein's equations compatible with these boundary conditions can be written in the Fefferman-Graham gauge as follows,

$$\begin{aligned} ds^2 &= \frac{dy^2 + dz(d\bar{z} - B'(z)dz)}{y^2} \\ &\quad - \frac{6}{c}L(z)dz^2 - \frac{6}{c}\bar{L}(d\bar{z} - B'(z)dz)^2 \\ &\quad + \frac{36}{c^2}L(z)\bar{L}y^2dz(d\bar{z} - B'(z)dz), \end{aligned} \quad (2.7)$$

where $\bar{L}$ is a constant and $B'(z) \equiv \partial_z B$ are only the functions of z . When $B' = 0$, this metric can be viewed as a Banados solution with constant antiholomorphic stress tensor.

III. GLOBAL BULK RECONSTRUCTION IN AdS/WCFT

In this section, we study the global bulk reconstruction in AdS/WCFT, which is based mainly on the global symmetry of the correspondence. As discussed above, the global symmetry is now simply $SL(2, R) \times U(1)$ symmetry.

We follow the line of HKLL global reconstruction, but due to the isometry of the AdS₃ spacetime being broken by the asymptotical boundary conditions, we need to pay more attention to the boundary conditions.

A. Bulk-boundary propagator in AdS/WCFT

We start from the construction of bulk-boundary propagator in the AdS/WCFT correspondence. We first study the bulk-boundary propagator for a WCFT primary in the vacuum. In the following, we use the Poincare coordinates to describe the AdS₃ spacetime. The coordinates are $(y, z, \bar{z})$ with the vacuum AdS metric being

$$ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}. \quad (3.1)$$

The asymptotic boundary corresponds to $y \rightarrow 0$.

From the extrapolate dictionary, the bulk field should be related to the boundary primary as it approaches the boundary

$$\phi(y, z)|_{y \rightarrow 0} = y^{2h}\mathcal{O}(z). \quad (3.2)$$

The bulk-boundary propagator should recover a boundary WCFT correlator as the bulk point approaches the boundary. The correlator of two primaries in a WCFT can be determined by the global symmetries [27]

$$G(z_1, z_2) = \langle \mathcal{O}(z_1)\mathcal{O}(z_2) \rangle = e^{iQ(\bar{z}_1 - \bar{z}_2)} \frac{1}{(z_1 - z_2)^{2h}}, \quad (3.3)$$

where the two primaries are required to have the opposite charges and the same scaling dimension h . Without loss of generality, the bulk field could be put at $(y, z, \bar{z})$ and the boundary primary at the origin $(0, 0)$. Then the bulk-boundary propagator $K(y, z, \bar{z}; 0, 0)$ has to satisfy the requirement

$$K(y, z, \bar{z}; 0, 0)|_{y \rightarrow 0} = y^{2h}G(z, 0) = y^{2h}z^{-2h}e^{iQ\bar{z}}. \quad (3.4)$$

Another basic requirement for the bulk-boundary propagator is to satisfy the Klein-Gordon equation

$$(\square - m^2)K(y, z, \bar{z}; 0, 0) = 0. \quad (3.5)$$

As the AdS₃ spacetime has the isometry group $SL(2, R) \times SL(2, R)$, the solutions to the above equation may respect the symmetry without imposing boundary conditions. With the Brown-Henneaux boundary condition, the isometry group is kept intact such that the boundary CFT has the full conformal symmetry. This is also reflected in the fact that in the Brown-Henneaux case, the right-hand side of the above equation could have a delta-function source. On the contrary, the CSS boundary conditions obviously break the full isometry group to

$SL(2, R) \times U(1)$. To solve the Klein-Gordon equation, we need to take the boundary conditions appropriately. For a general primary field, it is not clear how to impose the boundary condition. Because of the plane wave form of the $\bar{z}$ dependence, it is not possible to have a delta-function source. So we only require the Klein-Gordon equation to be satisfied when the bulk and boundary point are separated.

The above two requirements cannot determine the bulk-boundary propagator completely. To solve the problem, we make the ansatz that the dependence of the bulk-boundary propagator on $\bar{z}$ is simply $e^{iQ\bar{z}}$, considering the form of the WCFT correlators. A natural explanation is that we consider only the Kac-Moody $U(1)$ primary, so to transform correctly under the level zero $U(1)$ generator, this form of $\bar{z}$ dependence is needed. With this ansatz, we can expand the propagator in a power series

$$K(y, z, \bar{z}; 0, 0) = y^{2h} z^{-2h} e^{iQ\bar{z}} \sum_{n=0}^{\infty} C_n \left(\frac{y^2}{z} \right)^n, \quad (3.6)$$

where C_n 's are the coefficients to be determined. Substituting the expression into the Klein-Gordon equation, we can solve the equation order by order and get the relations for the coefficients. From the zeroth order, we obtain the mass equation, which is the same as the usual AdS/CFT correspondence

$$m^2 = 2h(2h - 2). \quad (3.7)$$

And from the orders above 0, we obtain a recursive relation for the coefficients

$$C_n = \frac{iQ}{n} C_{n-1}. \quad (3.8)$$

They are the expansion coefficients of an exponential function. Finally we obtain the bulk-boundary propagator

$$K(y, z, \bar{z}; 0, 0) = y^{2h} z^{-2h} e^{iQ\bar{z}} e^{\frac{iQy^2}{z}} = y^{2h} z^{-2h} e^{iQ\frac{z\bar{z}+y^2}{z}}. \quad (3.9)$$

We can compare it to the usual AdS_3 bulk-boundary propagator

$$K(y, z, \bar{z}; 0, 0) = \left(\frac{y}{y^2 + z\bar{z}} \right)^{2h}. \quad (3.10)$$

The AdS/CFT bulk-boundary propagator only depends on the geodesic distance between the bulk and the boundary points. For the AdS/WCFT bulk-boundary propagator, this is not the case.

In the usual AdS/CFT correspondence, one may reconstruct a bulk field in terms of a boundary primary and its global descendants by requiring it to reproduce the right bulk-boundary propagator. It is the global bulk field in the HKLL reconstruction [17,18]

$$\phi(y, z, \bar{z}) = \sum_{n=0}^{\infty} y^{2h+2n} \frac{(-1)^n}{n!(2h)_n} L_{-1}^n \bar{L}_{-1}^n \mathcal{O}, \quad (3.11)$$

where $\mathcal{O}$ is the boundary primary operator and ϕ is the bulk field.

In the AdS/WCFT correspondence, we can have a similar reconstruction. Comparing the structure of Virasoro-Kac-Moody algebra with two copies of the Virasoro algebra, we see that the $U(1)$ generator in the Kac-Moody sector replaces the Virasoro generator in the antiholomorphic sector. This leads us to replace the global antiholomorphic Virasoro generator in the AdS/CFT bulk field with the global $U(1)$ generator to get the AdS/WCFT bulk field in the HKLL proposal. This gives us

$$\phi(y, z, \bar{z}) = \sum_{n=0}^{\infty} y^{2h+2n} \frac{(-iQ)^n}{n!(2h)_n} L_{-1}^n \mathcal{O}. \quad (3.12)$$

One may check that this bulk field indeed gives the right bulk-boundary propagator.

Using the expression for a global bulk field, we can calculate the bulk-bulk propagator between two points $(y_1, z_1, \bar{z}_1)$ and $(y_2, z_2, \bar{z}_2)$. It is done by first calculating the correlator of the two primaries, then acting on the Virasoro operators. As the holomorphic Virasoro generators can be expressed as holomorphic derivatives, the bulk-bulk propagator can be computed,

$$\begin{aligned} \langle \phi_1 \phi_2 \rangle &= y_1^{2h} y_2^{2h} e^{iQ\bar{z}_{12}} \\ &\times \sum_{n_1=0, n_2=0}^{\infty} \frac{(-iQy_1^2)^{n_1} (-iQy_2^2)^{n_2} \partial_{z_1}^{n_1} \partial_{z_2}^{n_2} \left(\frac{1}{z_{12}^{2h}} \right)}{n_1! (2h)_{n_1} n_2! (2h)_{n_2}} \\ &= \frac{y_1^{2h} y_2^{2h}}{z_{12}^{2h}} e^{iQ\bar{z}_{12}} \\ &\times \sum_{n_1=0, n_2=0}^{\infty} \frac{(2h)_{n_1+n_2}}{n_1! n_2! (2h)_{n_1} (2h)_{n_2}} \left(\frac{iQy_1^2}{z_{12}} \right)^{n_1} \left(\frac{iQy_2^2}{z_{12}} \right)^{n_2}. \end{aligned} \quad (3.13)$$

It is a bivariate hypergeometric function and cannot be simplified to a closed form. It satisfies some nice properties. First, the bulk-bulk propagator simplifies to the bulk-boundary propagator as one point approaches the boundary. Secondly, we can check that the bulk-bulk propagator satisfies the Klein-Gordon equation.

B. Three-point Witten diagrams

As a consistent check, we compute the bulk three-point Witten diagram, which should capture the three-point correlation functions in holographic WCFT. From global symmetry consideration, the most general form of three-point functions of primary operators in WCFT is [27]

$$\begin{aligned} & \langle O(z_1, \bar{z}_1) O(z_2, \bar{z}_2) O(z_3, \bar{z}_3) \rangle \\ &= C \delta_{Q_1+Q_2+Q_3,0} \frac{e^{\frac{1}{3}(Q_1-Q_2)\bar{z}_{12}} e^{\frac{1}{3}(Q_2-Q_3)\bar{z}_{23}} e^{\frac{1}{3}(Q_3-Q_1)\bar{z}_{31}}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}}, \end{aligned} \quad (3.14)$$

where C is a constant that depends on the theory. The form of the three-point function should be reproduced from the bulk computation of the Witten diagrams.

$$G \propto \iiint y^{2h_1+2h_2+2h_3-1} e^{iQ_1\bar{z}_{14}+iQ_2\bar{z}_{24}+iQ_3\bar{z}_{34}} z_{14}^{-2h_1} z_{24}^{-2h_2} z_{34}^{-2h_3} e^{\frac{iQ_1y^2}{z_{14}} + \frac{iQ_2y^2}{z_{24}} + \frac{iQ_3y^2}{z_{34}}} dy dz_4 d\bar{z}_4. \quad (3.15)$$

We find that the integration of y and $\bar{z}$ can be performed independently. The $\bar{z}$ integration gives

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{iQ_1\bar{z}_{14}+iQ_2\bar{z}_{24}+iQ_3\bar{z}_{34}} d\bar{z}_4 \\ &= \delta_{Q_1+Q_2+Q_3,0} e^{\frac{1}{3}(Q_1-Q_2)\bar{z}_{12}+\frac{1}{3}(Q_2-Q_3)\bar{z}_{23}+\frac{1}{3}(Q_3-Q_1)\bar{z}_{31}}. \end{aligned} \quad (3.16)$$

The integration of y is a little more subtle, but we can allow y to have a small imaginary part and send the imaginary part to 0 to obtain a convergent result. The contribution of y integration is proportional to $(\frac{iQ_1}{z_{14}} + \frac{iQ_2}{z_{24}} + \frac{iQ_3}{z_{34}})^{-h_1-h_2-h_3}$. Finally we have to integrate over z_4 , and the result is proportional to

$$G \propto \int_{-\infty}^{\infty} z_{14}^{-2h_1} z_{24}^{-2h_2} z_{34}^{-2h_3} \left(\frac{iQ_1}{z_{14}} + \frac{iQ_2}{z_{24}} + \frac{iQ_3}{z_{34}} \right)^{-h_1-h_2-h_3} dz_4. \quad (3.17)$$

The integral is difficult to perform, but we can make use of the charge neutrality condition, expressing one charge in terms of the other two,

$$G \propto \int_{-\infty}^{\infty} z_{14}^{-2h_1} z_{24}^{-2h_2} z_{34}^{-2h_3} \left(\frac{iQ_1 z_{13}}{z_{14} z_{34}} + \frac{iQ_2 z_{23}}{z_{24} z_{34}} \right)^{-h_1-h_2-h_3} dz_4. \quad (3.18)$$

Expanding the result, each term in the expansion has the form

$$\begin{aligned} & \int_{-\infty}^{\infty} z_{14}^{-2h_1} z_{24}^{-2h_2} z_{34}^{-2h_3} \left(\frac{z_{13}}{z_{14} z_{34}} \right)^m \left(\frac{z_{23}}{z_{24} z_{34}} \right)^{-h_1-h_2-h_3-m} dz_4 \\ & \propto \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}}. \end{aligned} \quad (3.19)$$

Combining the above results, we have the correlation function via the Witten diagram

The Witten diagram calculation for the three-point function of a primary is to integrate over a bulk point the product of three bulk-boundary propagators between it and the boundary points. Now we have to use the bulk-boundary propagators in AdS/WCFT. The three-point vertex depends on the particular theory and is not important if we only care about the form of the three-point function. The expression for the Witten diagram is

$$G \propto \delta_{Q_1+Q_2+Q_3,0} \frac{e^{\frac{1}{3}(Q_1-Q_2)\bar{z}_{12}} e^{\frac{1}{3}(Q_2-Q_3)\bar{z}_{23}} e^{\frac{1}{3}(Q_3-Q_1)\bar{z}_{31}}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}}, \quad (3.20)$$

which is the same as the three-point functions in a WCFT up to a constant.

To summarize, by making an ansatz to respect the boundary conditions we obtain a bulk-boundary propagator that satisfies the bulk Klein-Gordon equation and matches the boundary correlator as the bulk point approaches the boundary. And we reconstruct globally a bulk field that reproduces the right form of the bulk-boundary propagator. By using the bulk field we obtain the bulk-bulk propagator that can be expressed in a bivariate hypergeometric function. Using the bulk-boundary propagator we calculate the Witten diagrams for three-point functions and find that they match with the one in WCFT.

IV. LOCAL BULK RECONSTRUCTION IN AdS/WCFT

In this section, we study the local bulk reconstruction in AdS/WCFT. We study the bulk protofield in two different ways. One is to impose the bulk primary conditions to define the bulk protofield. This allows us to solve the protofield and express it in terms of boundary quasiprimary operators. The other one is to use the bulk-boundary OPE blocks, which allows us to compute the correlators of the bulk-boundary propagator with products of the stress tensors.

A. Algebraic definition of the bulk protofield in AdS/WCFT

In AdS₃/CFT₂, the conformal symmetry can be used to define an AdS protofield as an exact linear combination of Virasoro descendants of a primary operator [19]. The AdS protofield agrees with the gravitational perturbation theory when it is expanded to all orders in the large c limit. It captures the information of gravitational background non-perturbatively. The requirement for the bulk protofield is the bulk primary conditions

$$L_n \Phi(y, 0, 0)|0\rangle = 0, \quad \bar{L}_n \Phi(y, 0, 0)|0\rangle = 0, \quad \forall n > 2. \quad (4.1)$$

By requiring the bulk primary conditions and requiring the global part of the protofield to match the global bulk field ϕ , we can determine the AdS protofield exactly. The bulk primary conditions come from the symmetry considerations. That is to consider the transformation of the bulk protofield under the action of Virasoro transformation. There is one unique extension of the boundary Virasoro transformations preserving the Fefferman-Graham (FG) gauge. And the resulting bulk Virasoro transformations that leave the bulk point invariant give rise to the bulk primary conditions.

The protofield in the AdS/WCFT correspondence can be constructed in a similar way. Instead of the AdS₃ gravity with the Brown-Henneaux boundary condition that is dual to a two-dimensional (2D) CFT, we consider the AdS₃ gravity with the CSS boundary condition. The CSS boundary condition is a Dirichlet-Neumann boundary condition. A generic solution with respect to the CSS boundary condition could be of the form (2.7). And it has been known that a generic Banados solutions can be related to the vacuum AdS metric through a diffeomorphism [28]. For the metric (2.7), we can find a diffeomorphism to relate it to the vacuum AdS metric as well. Let the $(w, \bar{w}, u)$ coordinates be the vacuum AdS₃ coordinates

$$ds^2 = \frac{du^2 + dw d\bar{w}}{u^2}. \quad (4.2)$$

The diffeomorphism is

$$\begin{aligned} w &= f(z) - \frac{2y^2 f'^2 \bar{f}''}{4f' \bar{f}' + y^2 f'' \bar{f}''}, \\ \bar{w} &= f(\bar{z}) - \frac{2y^2 \bar{f}'^2 f''}{4f' \bar{f}' + y^2 f'' \bar{f}''}, \\ u &= y \frac{4(f\bar{f})^{\frac{3}{2}}}{4f' \bar{f}' + y^2 f'' \bar{f}''}. \end{aligned} \quad (4.3)$$

Here the functions $f(z)$ and $f(\bar{z})$ satisfy the equations

$$S(f, z) = \frac{12}{c} L(z), \quad f(\bar{z}) = e^{\sqrt{-\frac{24}{c}} \bar{L}(\bar{z} - B(z))}, \quad (4.4)$$

where $S(f, z)$ stands for the Schwarzian derivative

$$S(f, z) = \frac{f''' f' - \frac{3}{2} f''^2}{(f')^2}. \quad (4.5)$$

The translation along $\bar{z}$ leaves $\bar{L}$ invariant, and the exponential map changes $\bar{L}$ by a constant. The exponential map is needed to relate AdS₃ with the CSS boundary conditions to the AdS vacuum. However there are ambiguities in the

exponential map. The exponential factor in the map can differ by an arbitrary constant time, but this does not change the discussion below.

We then look for the bulk generalizations of boundary Virasoro-Kac-Moody transformations. The bulk Virasoro-Kac-Moody transformations need to satisfy two requirements.

(1) They must agree with the Virasoro-Kac-Moody transformations on the boundary.

(2) They preserve the metric form in the FG gauge.

We suppose that the bulk Virasoro-Kac-Moody transformation is of the form

$$(y, z, \bar{z}, L, B) \rightarrow (\tilde{y}, \tilde{z}, \tilde{\bar{z}}, \tilde{L}, \tilde{B}). \quad (4.6)$$

And we use F to denote the transformations (4.3) obtained above. To preserve the metric form in the FG gauge, we can map an asymptotic AdS space characterized by (L, B) to the AdS vacuum and then map the AdS vacuum to another asymptotic AdS characterized by $(\tilde{L}, \tilde{B})$. The bulk Virasoro-Kac-Moody transformations have to satisfy

$$F(y, z, \bar{z}, L, B) = \tilde{F}(\tilde{y}, \tilde{z}, \tilde{\bar{z}}, \tilde{L}, \tilde{B}). \quad (4.7)$$

Also it has to agree with the infinitesimal boundary Virasoro-Kac-Moody transformations

$$\begin{aligned} \epsilon L_m(y, z, \bar{z})|_{y \rightarrow 0} &= \epsilon(0, z^{m+1}, 0), \\ \epsilon P_n(y, z, \bar{z})|_{y \rightarrow 0} &= \epsilon(0, 0, z^n), \end{aligned} \quad (4.8)$$

where L_m is the infinitesimal Virasoro transformation

$$L_m(y, z, \bar{z}) = (\delta_m y, \delta_m z, \delta_m \bar{z}), \quad (4.9)$$

and P_n is the infinitesimal Kac-Moody transformation

$$P_n(y, z, \bar{z}) = (\delta_n y, \delta_n z, \delta_n \bar{z}). \quad (4.10)$$

The solution for the bulk Virasoro-Kac-Moody transformations is

$$\delta_m y = \frac{1}{2}(m+1)y z^m, \quad (4.11)$$

$$\delta_m z = \frac{z^{m-1}((m^2 + m + z^2 S(z))\bar{S}(\bar{z})y^4 - 4z^2)}{y^4 S(z)\bar{S}(\bar{z}) - 4}, \quad (4.12)$$

$$\delta_m \bar{z} = \frac{2m(m+1)y^2 z^{m-1}}{y^4 S(z)\bar{S}(\bar{z}) - 4}, \quad (4.13)$$

$$\delta_n y = 0, \quad \delta_n z = 0, \quad \delta_n \bar{z} = z^n, \quad (4.14)$$

where the antiholomorphic Schwarzian is a constant. We can check that the bulk Virasoro-Kac-Moody transformations agree with Virasoro-Kac-Moody transformation on

the boundary. Moreover, the bulk Virasoro-Kac-Moody transformations preserve the form of the AdS metric with the CSS boundary conditions, keeping the antiholomorphic $\bar{L}$ unchanged.

The bulk scalar is invariant under coordinate transformation $\Phi(y, z, \bar{z}) = \Phi(y', z', \bar{z}')$. The Virasoro-Kac-Moody transformation acts on a scalar as

$$\begin{aligned} L_m \Phi(y, z, \bar{z}) &= (\delta_m y \partial_y + \delta_m z \partial_z + \delta_m \bar{z} \partial_{\bar{z}}) \Phi(y, z, \bar{z}), \\ P_n \Phi(y, z, \bar{z}) &= (\delta_n y \partial_y + \delta_n z \partial_z + \delta_n \bar{z} \partial_{\bar{z}}) \Phi(y, z, \bar{z}). \end{aligned} \quad (4.15)$$

It is easy to see that the point $(y, 0, 0)$ is invariant under L_m for $m \geq 2$ and P_n for $n \geq 1$. Using the transformation law on a scalar, we find the bulk primary condition

$$L_m \Phi(y, 0, 0) |0\rangle = 0, \quad m \geq 2, \quad (4.16)$$

$$P_n \Phi(y, 0, 0) |0\rangle = 0 \quad n \geq 1. \quad (4.17)$$

It is sufficient to determine the bulk protofield. The bulk Kac-Moody primary condition is easy to solve. And the bulk Virasoro primary condition can be used to solve the bulk protofield exactly. The solution can be written in an expansion of boundary operator

$$\Phi(y, 0, 0) |0\rangle = \sum_{N=0}^{\infty} y^{2h+2N} |\Phi_N\rangle, \quad (4.18)$$

where $|\Phi_N\rangle$ is a level N descendant of $\mathcal{O}$ in the holomorphic sector, as the antiholomorphic sector is trivial. The conditions (4.16) and (4.17) on $\Phi(y, 0, 0)$ are equivalent to

$$\begin{aligned} L_m |\Phi_N\rangle &= 0, \quad m \geq 2, \\ P_n |\Phi_N\rangle &= 0. \quad n \geq 1. \end{aligned} \quad (4.19)$$

We can solve it in terms of quasiprimaries and their global descendants. The result is

$$\begin{aligned} |\Phi_N\rangle &= \lambda_N \left(\frac{L_{-1}^N}{|L_{-1}^N \mathcal{O}|^2} + \frac{L_{-N}^{\text{quasi}}}{|L_{-N}^{\text{quasi}} \mathcal{O}|^2} \right. \\ &\quad \left. + \frac{L_{-1} L_{-N+1}^{\text{quasi}}}{|L_{-1} L_{-N+1}^{\text{quasi}} \mathcal{O}|^2} + \dots \right) |\mathcal{O}\rangle, \end{aligned} \quad (4.20)$$

where $\mathcal{O}$ is the boundary primary and the coefficient is determined such that the global part (the first term) agrees with the global reconstruction, and reproduces the right global bulk-boundary propagator. We have already obtained the bulk-boundary propagator of AdS/WCFT, so we can use it to determine the coefficient

$$\lambda_N = (-iQ)^N. \quad (4.21)$$

Then for the global part of Φ we recover the previous expression for the bulk field.

At low levels, we can solve the protofield explicitly. Obviously $|\Phi_0\rangle = |\mathcal{O}\rangle$, and

$$|\Phi_1\rangle = -iQ \frac{1}{2h} L_{-1} |\mathcal{O}\rangle, \quad (4.22)$$

and

$$\begin{aligned} |\Phi_2\rangle &= -Q^2 \frac{(2h+1)(c+8h)}{(2h+1)c+2h(8h-5)} \\ &\quad \times \left(L_{-1}^2 - \frac{12h}{c+8h} L_{-2} \right) |\mathcal{O}\rangle, \end{aligned} \quad (4.23)$$

where h is the conformal weight of $\mathcal{O}$. One remarkable fact is that in the large c limit, the terms L_{-1}^N dominate in Φ_N , suggesting that the theory is effectively free and only the global blocks survive.

The bulk protofield is a field that sees the gravitational background of the AdS physics to all orders. We can obtain a recursion relation for the correlators involving the stress tensor, the bulk protofield, and the boundary primary as in [19]. The recursion relations share the same structure as in [19].

With the reconstructed protofield and the boundary primary correlator, we can read the bulk-bulk propagator of two protoscalar fields directly. Using the fact that the correlator of distinct quasiprimaries vanishes, we can single out the contributions from a certain quasiprimary. The contribution to Φ from a level- n quasiprimary is

$$|\Phi^{(n)}\rangle = y^{2h+2n} \sum_{m=0}^{\infty} y^{2m} \lambda_{n+m} \frac{L_{-1}^m L_{-n}^{\text{quasi}} |\mathcal{O}\rangle}{|L_{-1}^m L_{-n}^{\text{quasi}} \mathcal{O}|^2}. \quad (4.24)$$

A useful identity for computing the propagator is

$$|L_{-1}^m L_{-n}^{\text{quasi}} \mathcal{O}|^2 = (2h+2n)_m m! |L_{-n}^{\text{quasi}} \mathcal{O}|^2. \quad (4.25)$$

The correlator of two quasiprimaries is

$$\langle L_{-n}^{\text{quasi}} \mathcal{O}_1 L_{-n}^{\text{quasi}} \mathcal{O}_2 \rangle = \frac{(-1)^n e^{iQ\bar{z}_{12}} |L_{-n}^{\text{quasi}} \mathcal{O}|^2}{z_{12}^{2h+2n}}. \quad (4.26)$$

So the contribution to the propagator of the protofield from a level- n quasiprimary is

$$\begin{aligned}
& \langle \Phi(y_1, z_1, \bar{z}_1)^{(n)} \Phi(y_2, z_2, \bar{z}_2)^{(n)} \rangle \\
&= \frac{(y_1 y_2)^{2h+2n} e^{iQ\bar{z}_{12}}}{|L_{-n}^{\text{quasi}} \mathcal{O}|^2} \sum_{m, m'=0}^{\infty} \frac{(iQ)^{2n} (-iQ y_1^2)^m (-iQ y_2^2)^{m'}}{m! m'! (2h_n)_m (2h_n)_{m'}} \\
&\quad \times \partial_{z_1}^m \partial_{z_2}^{m'} \left(\frac{1}{z_{12}^{2h+2n}} \right) \\
&= \frac{(y_1 y_2)^{2h+2n} (iQ)^{2n} e^{iQ\bar{z}_{12}}}{|L_{-n}^{\text{quasi}} \mathcal{O}|^2} \\
&\quad \times \sum_{m, m'=0}^{\infty} \frac{(\frac{iQ y_1^2}{z_{12}})^m (\frac{iQ y_2^2}{z_{12}})^{m'}}{m! m'!} \frac{(2h_n)_{m+m'}}{(2h_n)_m (2h_n)_{m'}}. \quad (4.27)
\end{aligned}$$

When $n = 0$, we get the global bulk-bulk scalar propagator, which is exactly what we got in the previous section. There is one interesting property of the result. The correlator of the n -level quasiprimary is proportional to the global bulk-bulk propagator with a different scaling dimension $2h + 2n$.

The expression (4.27) is a bivariate hypergeometric function that does not have a closed form. Recall that in the AdS/CFT correspondence, the bulk-bulk propagator has a simple dependence on the geodesic distance. But in the warped case the global symmetry is no longer manifest, and the geodesic distance is no longer an invariant, so one could expect that the bulk scalar propagator has no such simplification.

B. The bulk-boundary OPE blocks in AdS/WCFT

In a conformal field theory, the product of the primary operators can be expanded in terms of the primary modules. In 2D CFT the Virasoro block can be computed in a recursive way. In particular, in the large central charge limit, the vacuum block dominates [29,30]. For a warped CFT, it is expected that the vacuum module dominates in the large c limit as well [12,31].

Another way of studying the bulk protofield is by using the bulk-boundary OPE blocks. The basic idea is to take the bulk protofield $\Phi(y, z, \bar{z})$ as an operator in WCFT, and consider the OPE blocks involving $\Phi(y, z, \bar{z})$ and other primary operators. In particular, as we are interested in probing the nonperturbative information of the gravity, we focus on the scalar OPE block involving the vacuum module

$$\Phi(y, 0, 0) \mathcal{O}(z, \bar{z}) = y^{2h} z^{-2h} e^{iQ \frac{z\bar{z} + y^2}{z}} + \dots \quad (4.28)$$

The first term is simply the bulk-boundary propagator, coming from the identity operator. And the ellipsis consists of the contribution from the descendent operators in the vacuum module; it encodes the information of the correlators,

$$\langle \Phi(y, 0, 0) \mathcal{O}(z, \bar{z}) T(z_1) \cdots T(z_n) \rangle. \quad (4.29)$$

The bulk-boundary OPE block can be calculated by applying a diffeomorphism to the bulk-boundary propagator for pure AdS. In the last section, we have discussed the diffeomorphism between the pure AdS vacuum and the AdS solution (2.7) with nonvanishing stress tensor $T(z)$ and $U(1)$ current $B(z)$. The diffeomorphism is given by the relations (4.3) and (4.4). Now we take this diffeomorphism as an operator equation, and obtain an operator-valued AdS metric that depends on the operator $T(z)$ and $B(z)$. After applying this diffeomorphism to the bulk-boundary propagator, we obtain the vacuum sector of the bulk-boundary OPE block

$$\Phi(y, z_2, \bar{z}_2) \mathcal{O}(z_1, \bar{z}_1) = (w'(z_1) w'(\bar{z}_1))^h \left(\frac{u}{w_{12}} \right)^{2h} e^{iQ(\bar{w}_{12} + \frac{y^2}{w_{12}})}, \quad (4.30)$$

where $(u, w, \bar{w})$ are the pure AdS coordinates. To obtain the diffeomorphism, we have to solve the holomorphic function satisfying $S(f, z) = \frac{12}{c} T(z)$. We solve the equation order by order in $1/c$,

$$f(z) = z + \frac{1}{c} f_1(z) + \frac{1}{c^2} f_2(z) + \dots \quad (4.31)$$

The first two functions f_1 and f_2 satisfy the differential equations

$$\begin{aligned}
f_1'''(z) - 12T(z) &= 0, \\
f_1'''(z) f_1'(z) + 3f_1''(z)^2 - 2f_2'''(z) &= 0. \quad (4.32)
\end{aligned}$$

By integration, we get

$$\begin{aligned}
f_1(z) &= 6 \int_0^z dz' (z - z')^2 T(z'), \\
f_2(z) &= 72 \int_0^z dz' \int_0^{z'} dz'' (z - z')^2 (z - z'') T(z') T(z''). \quad (4.33)
\end{aligned}$$

Then we can expand (4.30) and get the bulk-boundary OPE block

$$\begin{aligned}
\log \Phi(y, 0, 0) \mathcal{O}(z, \bar{z}) &= \log \left(\frac{y^{2h}}{z^{2h}} e^{iQ(\bar{z} + \frac{y^2}{z})} \right) + \frac{h f_1'}{c} - \frac{2h f_1}{c z} \\
&\quad - iQ \frac{y^2 f_1}{c z^2} + \frac{h(2f_2' - f_1'^2)}{2c^2} - \frac{2h f_2}{c^2 z} \\
&\quad + \frac{h f_1^2}{c^2 z^2} - iQ \frac{y^2 f_2}{c^2 z^2} + iQ \frac{y^2 f_1^2}{c^2 z^3} + \dots \quad (4.34)
\end{aligned}$$

The bulk-boundary OPE blocks have no dependence on the antiholomorphic coordinate since only translations are allowed for $\bar{z}$. The order $1/c$ terms can be combined into the kernel

$$K_T = \frac{6}{c} \int_0^z dz' \left(2h \frac{z'(z-z')}{z} + iQ \frac{y^2(z-z')^2}{z^2} \right) T(z'). \quad (4.35)$$

In the $y \rightarrow 0$ limit, it reproduce the kernels for the “boundary-boundary” $\mathcal{O}(z)\mathcal{O}(0)$ OPE block [32]. At the next order we obtain the kernel

$$K_{TT} = \frac{72}{c^2} \int_0^z dz' \int_0^{z'} dz'' \left(\frac{h(z-z')^2 z''^2}{z^2} - iQ \frac{y^2(z-z')^2(z-z'')z''}{z^3} \right) T(z')T(z''). \quad (4.36)$$

Different from the usual AdS/CFT case, here are only the kernels involving the holomorphic stress tensors.

Using the bulk-boundary OPE block, we can calculate the correlators of the bulk-boundary propagator with products of local stress tensors. The simplest one is $\langle \Phi \mathcal{O} T \rangle$. It can be obtained using only the K_T kernel

$$\frac{\langle \Phi(y, 0, 0) \mathcal{O}(z, \bar{z}) T(z_1) \rangle}{\Phi(y, 0, 0) \mathcal{O}(z, \bar{z})} = \langle K_T T(z_1) \rangle = \frac{hz^2}{z_1^2(z_1 - z)^2} + iQ \frac{y^2 z}{z_1^3(z_1 - z)}. \quad (4.37)$$

From the correlators of the bulk-boundary propagator with one stress tensor, we can read the coefficients of Virasoro descendants in the expansion of the bulk protofield. We check that the coefficient in the lowest order agrees with the algebraic calculation. For example, let us consider the level-two quasiprimary operators. The only level-two quasiprimary operator is

$$L_{-2}^{\text{quasi}} = L_{-1}^2 - \frac{12h}{c} L_{-2}. \quad (4.38)$$

The quasiprimary operator contribution to the bulk protofield is

$$\delta\Phi_2 \approx \frac{(-iQ)^2 y^4}{2(2h)_2} \left(L_{-1}^2 - \frac{12h}{c} L_{-2} \right) \mathcal{O}(0). \quad (4.39)$$

We have obtained the correlator of bulk boundary OPE with one stress tensor. We subtract the contribution of the global bulk field. Expanding in the y coordinate, the first order is

$$\begin{aligned} & \langle (\Phi(y, 0, 0) - \Phi^{\text{global}}(y, 0, 0)) \mathcal{O}(z, \bar{z}) T(z_1) \rangle \\ &= -\frac{3h(iQ)^2 y^4 e^{iQ\bar{z}}}{(2h)_2 z_1^4 z^{2h}} + \dots \end{aligned} \quad (4.40)$$

The level-two quasiprimary operator has the following stress tensor correlator,

$$\langle (L_{-2} \mathcal{O}(0)) \mathcal{O}(z, \bar{z}) T(z_1) \rangle = \frac{ce^{iQ\bar{z}}}{2z^{2h} z_1^4}, \quad (4.41)$$

so we can read the coefficient of the quasiprimary in the protofield

$$\delta\Phi_2 \approx -\frac{(-iQ)^2 y^4}{2(2h)_2} \frac{12h}{c} L_{-2} \mathcal{O}(0). \quad (4.42)$$

This agrees with the result using the algebraic method.

V. CONCLUSION

The AdS/WCFT correspondence serves as one example of holographic duality beyond AdS/CFT. In this paper, we try to understand how to reconstruct the AdS field in terms of WCFT operators. By imposing a specific boundary condition, we get a vacuum bulk-boundary propagator that reproduces the WCFT correlator as the bulk point approaches the boundary. And we use it to globally reconstruct the bulk scalar and calculate the vacuum bulk-bulk propagator. Furthermore, we reconstruct locally a bulk protofield that knows the gravitational background nonperturbatively. We studied the bulk protofield in two different ways. One is to impose the bulk primary condition on the protofield, while the other is to use the bulk-boundary OPE block. The two ways lead to a consistent picture. By solving the bulk primary condition, the bulk protofield can be solved exactly.

The bulk protofield was constructed by fixing the gauge to a Fefferman-Graham one. There exists gauge redundancy in quantum gravity, which makes the definition of local operators ambiguous. In a recent work [33], it was shown that one could define bulk protofields with general gravitational dressing in AdS₃/CFT₂ correspondence. It would be interesting to study a similar issue in our construction.

In 2D CFT, the OPE blocks can be calculated using the $SL(2, R) \times SL(2, R)$ Wilson lines [32]. And the bulk-boundary OPE blocks can be calculated using bulk $SL(2, R) \times SL(2, R)$ Wilson lines. The Wilson line calculation is equivalent to the calculation using the AdS diffeomorphism [19]. It would be interesting to investigate if the Wilson line method still works in AdS/WCFT or in WCFT.

ACKNOWLEDGMENTS

We thank Peng-xiang Hao, Apolo Luis, and Wei Song for valuable discussions. The work is supported in part by NSFC Grants No. 11275010, No. 11325522, No. 11335012, and No. 11735001.

- [1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2**, 231 (1998); S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, *Phys. Lett. B* **428**, 105 (1998); E. Witten, anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [2] A. Strominger, The dS/CFT correspondence, *J. High Energy Phys.* **10** (2001) 034.
- [3] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, *Phys. Rev. D* **80**, 124008 (2009).
- [4] D. Anninos, W. Li, M. Padi, W. Song, and A. Strominger, Warped AdS(3) black holes, *J. High Energy Phys.* **03** (2009) 130.
- [5] G. Compère, W. Song, and A. Strominger, New boundary conditions for AdS₃, *J. High Energy Phys.* **05** (2013) 152.
- [6] J. D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, *Commun. Math. Phys.* **104**, 207 (1986).
- [7] A. Strominger, Black hole entropy from near horizon microstates, *J. High Energy Phys.* **02** (1998) 009.
- [8] C. Troessaert, Enhanced asymptotic symmetry algebra of AdS₃, *J. High Energy Phys.* **08** (2013) 044; S. G. Avery, R. R. Poojary, and N. V. Suryanarayana, An $sl(2, \mathbb{R})$ current algebra from AdS₃ gravity, *J. High Energy Phys.* **01** (2014) 144; L. Apolo and M. Porrati, Free boundary conditions and the AdS₃/CFT₂ correspondence, *J. High Energy Phys.* **03** (2014) 116; H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, Soft Heisenberg hair on black holes in three dimensions, *Phys. Rev. D* **93**, 101503 (2016); A. Pérez, D. Tempo, and R. Troncoso, Boundary conditions for general relativity on AdS₃ and the KdV hierarchy, *J. High Energy Phys.* **06** (2016) 103; D. Grumiller and M. Riegler, Most general AdS₃ boundary conditions, *J. High Energy Phys.* **10** (2016) 023; C. E. Valcárcel, New boundary conditions for (extended) AdS₃ supergravity, *Classical Quantum Gravity* **36**, 065002 (2019).
- [9] A. Castro, D. M. Hofman, and N. Iqbal, Entanglement entropy in warped conformal field theories, *J. High Energy Phys.* **02** (2016) 033.
- [10] W. Song, Q. Wen, and J. Xu, Modifications to holographic entanglement entropy in warped CFT, *J. High Energy Phys.* **02** (2017) 067.
- [11] S. Detournay, T. Hartman, and D. M. Hofman, Warped conformal field theory, *Phys. Rev. D* **86**, 124018 (2012).
- [12] L. Apolo and W. Song, Bootstrapping holographic warped CFTs or: How I learned to stop worrying and tolerate negative norms, *J. High Energy Phys.* **07** (2018) 112.
- [13] A. Castro, C. Keeler, and P. Szepietowski, Tweaking one-loop determinants in AdS₃, *J. High Energy Phys.* **10** (2017) 070.
- [14] B. Chen, P. X. Hao, and W. Song, Rényi mutual information in holographic warped CFTs, [arXiv:1904.01876](https://arxiv.org/abs/1904.01876).
- [15] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, AdS dynamics from conformal field theory, [arXiv:hep-th/9808016](https://arxiv.org/abs/hep-th/9808016).
- [16] D. Harlow, TASI lectures on the emergence of bulk physics in AdS/CFT, *Proc. Sci., TASI2017* (2018) 002 [[arXiv:1802.01040](https://arxiv.org/abs/1802.01040)].
- [17] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, Local bulk operators in AdS/CFT: A boundary view of horizons and locality, *Phys. Rev. D* **73**, 086003 (2006).
- [18] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, Holographic representation of local bulk operators, *Phys. Rev. D* **74**, 066009 (2006).
- [19] N. Anand, H. Chen, A. L. Fitzpatrick, J. Kaplan, and D. Li, An exact operator that knows its location, *J. High Energy Phys.* **02** (2018) 012.
- [20] H. Chen, A. L. Fitzpatrick, J. Kaplan, and D. Li, The AdS₃ propagator and the fate of locality, *J. High Energy Phys.* **04** (2018) 075.
- [21] D. M. Hofman and A. Strominger, Chiral Scale and Conformal Invariance in 2D Quantum Field Theory, *Phys. Rev. Lett.* **107**, 161601 (2011).
- [22] D. M. Hofman and B. Rollier, Warped conformal field theory as lower spin gravity, *Nucl. Phys. B* **897**, 1 (2015).
- [23] A. Castro, D. M. Hofman, and G. Srogi, Warped Weyl fermion partition functions, *J. High Energy Phys.* **11** (2015) 129.
- [24] L. Apolo, S. He, W. Song, J. Xu, and J. Zheng, Entanglement and chaos in warped conformal field theories, *J. High Energy Phys.* **04** (2019) 009.
- [25] T. Hartman, C. A. Keller, and B. Stoica, Universal spectrum of 2d conformal field theory in the large c limit, *J. High Energy Phys.* **09** (2014) 118.
- [26] G. Compère, W. Song, and A. Strominger, Chiral Liouville gravity, *J. High Energy Phys.* **05** (2013) 154.
- [27] W. Song and J. Xu, Correlation functions of warped CFT, *J. High Energy Phys.* **04** (2018) 067.
- [28] M. M. Roberts, Time evolution of entanglement entropy from a pulse, *J. High Energy Phys.* **12** (2012) 027.
- [29] T. Hartman, Entanglement entropy at large central charge, [arXiv:1303.6955](https://arxiv.org/abs/1303.6955).
- [30] B. Chen and J. J. Zhang, On short interval expansion of Rényi entropy, *J. High Energy Phys.* **11** (2013) 164; B. Chen, J. Long, and J. J. Zhang, Holographic Rényi entropy for CFT with W symmetry, *J. High Energy Phys.* **04** (2014) 041.
- [31] A. L. Fitzpatrick, J. Kaplan, and M. T. Walters, Virasoro conformal blocks and thermality from classical background fields, *J. High Energy Phys.* **11** (2015) 200.
- [32] A. L. Fitzpatrick, J. Kaplan, D. Li, and J. Wang, Exact Virasoro blocks from Wilson lines and background-independent operators, *J. High Energy Phys.* **07** (2017) 092.
- [33] H. Chen, J. Kaplan, and U. Sharma, AdS₃ reconstruction with general gravitational dressings, *J. High Energy Phys.* **07** (2019) 141.