Conformally soft theorem in gauge theory

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Asymptotic particle states in four-dimensional celestial scattering amplitudes are labeled by their $SL(2, \mathbb{C})$ Lorentz/conformal weights $(h, \bar{h})$ rather than the usual energy-momentum four-vector. These boost eigenstates involve a superposition of all energies. As such, celestial gluon (or photon) scattering cannot obey the usual (energetically) soft theorems. In this paper we show that tree-level celestial gluon scattering, in theories with sufficiently soft UV behavior, instead obeys *conformally soft* theorems involving $h \rightarrow 0$ or $\bar{h} \rightarrow 0$. Unlike the energetically soft theorem, the conformally soft theorem cannot be derived from low-energy effective field theory.

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I. INTRODUCTION

Four-dimensional (4D) Minkowskian scattering amplitudes are usually presented in a momentum-space basis, in which translations act trivially by phases, while the $SL(2, \mathbb{C})$ action of the Lorentz group is more nontrivial. The latter acts as the conformal group on the celestial two-sphere at null infinity. Recently there has been interest in and explicit construction of amplitudes in a conformal basis on the celestial sphere [1–16], in which the conformal action is trivial but translation invariance is hidden. We will use the symbol $\tilde{\mathcal{A}}$ to denote such amplitudes (and distinguish them from the usual amplitudes denoted $\mathcal{A}$) and simply refer to them as celestial amplitudes. The structure of celestial amplitudes has so far turned out to be surprisingly rich. Properties which are obscure in $\mathcal{A}$ are sometimes obvious in $\tilde{\mathcal{A}}$ and vice versa. Celestial amplitudes are of interest for many reasons, including a holographic representation of quantum gravity in flat space [1,2,4,17–23].

A central feature of the usual scattering amplitudes $\mathcal{A}$ in gravity or gauge theory is the existence of a variety of (energetically) soft theorems. These greatly constrain, and in some cases completely determine [24,25], all amplitudes. On the other hand, for celestial amplitudes there is no energy which can be taken to be soft. Instead of the energy, external particle states are characterized by their $SL(2, \mathbb{C})$ conformal dimensions $(h, \bar{h})$, which are roughly their boost weight plus or minus spin. This raises the question:

Is there an analog of the soft theorem which constrains celestial amplitudes?

The main result of this paper is to answer this question in the affirmative for non-Abelian gauge theories at tree level. The usual energetically soft theorem is replaced by the *conformally soft theorem*, in which the left or right conformal dimension (h or $\bar{h}$) of an external particle is taken to zero. The demonstration uses the energetically soft relations of $\mathcal{A}$ and is valid only when the UV behavior is sufficiently soft. Overlapping observations have been made in a number of papers including [2,15,26].

From some points of view this result is expected and obvious. However, on further inspection, our derivation encounters some subtleties and relies crucially on assumed good UV behavior of the momentum-space amplitudes $\mathcal{A}$. Moreover it does not fit into our usual effective field theory worldview. The energetically soft theorem is understood to be valid because very soft photons or gluons cannot probe the shorter-distance structure of the scattering region, and can sense only the external particle trajectories which extend to infinity. Conformally soft particles, on the other hand, involve arbitrarily high energies, and their scattering might have been expected to involve UV microphysics. The fact that this is *not* the case cannot be understood from low-energy reasoning, but seems natural from the celestial vantage point.

The constraints on $\mathcal{A}$ imposed by the various conformally soft theorems strongly resemble those of a current algebra on a two-dimensional (2D) conformal field theory, while for example supertranslation invariance (not discussed herein) provides an infinite number of a fundamentally new type of constraints which recursively relate operators with different dimensions [12,13,27]. One would like to combine all known constraints and construct the most general compatible scattering amplitude. One example of an interesting question, which has been raised for example in [24,25], is

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How does one characterize the most general 4D four-graviton scattering amplitude consistent with all known constraints?

The celestial picture provides a new framework for this question. Perhaps the answer is not impossibly far away.

In this paper we consider only tree-level amplitudes. Of course, the energetically soft theorem is generically corrected at the one-loop level in non-Abelian gauge theory. It would be very interesting to understand the nature of the loop corrections to the conformally soft theorem, whose form is highly constrained by the exact conformal symmetry of celestial amplitudes. See [28] for recent progress on celestial loop corrections.

This paper is organized as follows. In Sec. II we provide a general argument for the conformally soft theorem which applies to amplitudes with sufficiently soft UV behavior. This motivates a conjecture on the existence of perturbative quantum deformations of tree-level amplitudes in four-dimensional quantum field theory, which we present in Sec. III. The conformally soft theorem is explicitly verified in Sec. IV for the known tree-level celestial amplitudes, including four-point type I and heterotic string amplitudes as well as n -point maximally helicity violating (MHV) amplitudes in Yang-Mills theory.

II. CONFORMALLY SOFT FROM SOFT

In this section we derive a general identity for the scattering of conformally soft gluons in gauge theory. The main inputs are the (energetically) soft gluon theorem and assumed good UV behavior of momentum-space amplitudes.

The identity is expressed in terms of “celestial amplitudes,” denoted $\tilde{\mathcal{A}}$, of gluons in a conformal primary basis given by the Mellin transform of the standard momentum-space amplitudes $\mathcal{A}$ [5,6]

$$\tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i) = \left(\prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \right) \mathcal{A}_{J_1 \dots J_n}(\omega_i, z_i, \bar{z}_i), \quad (2.1)$$

where J_i label both the 4D and 2D helicities. We parametrize a null four-momentum by a sign $\epsilon_i = \pm 1$ (for outgoing and ingoing particles, respectively), a positive frequency ω_i , and a point on the celestial sphere $(z_i, \bar{z}_i)$ such that

$$p_i^\mu = \epsilon_i \omega_i q^\mu(z_i, \bar{z}_i), \quad (2.2)$$

where

$$q^\mu(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}). \quad (2.3)$$

The λ_i are related to the conformal dimensions $\Delta_i = 1 + i\lambda_i$ which in turn are related to $(h, \bar{h})$ by $(h, \bar{h}) = \frac{1}{2}(\Delta + J, \Delta - J)$. One may show [5,6] that the celestial amplitudes have the standard $SL(2, \mathbb{C})$ Lorentz transformation property

$$\begin{aligned} \tilde{\mathcal{A}}_{J_1 \dots J_n} & \left(\lambda_i, \frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}} \right) \\ &= \prod_{j=1}^n [(cz_j + d)^{\Delta_j + J_j} (\bar{c}\bar{z}_j + \bar{d})^{\Delta_j - J_j}] \tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i). \end{aligned} \quad (2.4)$$

The momentum-space amplitudes are conventionally normalized with inner product¹

$$(p_1, J_1; p_2, J_2) = (2\pi)^3 2p_1^0 \delta^{(3)}(\vec{p}_1 + \vec{p}_2) \delta_{J_1, -J_2}. \quad (2.5)$$

This implies the celestial inner product

$$\begin{aligned} & (\lambda_1, z_1, \bar{z}_1, J_1; \lambda_2, z_2, \bar{z}_2, J_2) \\ &= (2\pi)^4 \delta(\lambda_1 + \lambda_2) \delta^{(2)}(z_1 - z_2) \delta_{J_1, -J_2}. \end{aligned} \quad (2.6)$$

Let us consider the $\lambda_j \rightarrow 0$ limit, which for a positive (negative) helicity gluon corresponds to taking $\bar{h}$ (h) to zero

$$\begin{aligned} & \lim_{\lambda_j \rightarrow 0} i\lambda_j \tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i) \\ &= \left(\prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \right) \left(\int_0^\infty d\omega_j \lim_{\lambda_j \rightarrow 0} i\lambda_j \omega_j^{i\lambda_j - 1} \right) \\ & \quad \times \omega_j \mathcal{A}_{J_1 \dots J_n}(\omega_i, z_i, \bar{z}_i) \\ &= 2 \left(\prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \right) \int_0^\infty d\omega_j \delta(\omega_j) \omega_j \mathcal{A}_{J_1 \dots J_n}(\omega_i, z_i, \bar{z}_i). \end{aligned} \quad (2.7)$$

Importantly, the second equality is valid only under the assumption that the momentum-space amplitudes vanish sufficiently rapidly at large momenta. This is because we have used the representation of the Dirac delta function

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2} |x|^{\epsilon-1}, \quad (2.8)$$

which is only valid when integrated against functions which fall off at least as fast as x^{-b} for $b > \epsilon$ as $x \rightarrow \infty$.² To study the behavior of the integrand in (2.7), we change variables $\omega_k \rightarrow \omega_k \omega_j$ for every $k \neq j$ yielding

¹We use here the conventions of [6]. Helicities of all particles are labeled relative to the outgoing direction from the interaction.
²To see this, note the integral

$$\int_0^\infty dx \epsilon x^{\epsilon-1} \frac{1}{(x+a)^b} = \frac{1}{a^{b-\epsilon}} \frac{\epsilon \Gamma(b-\epsilon) \Gamma(\epsilon)}{\Gamma(b)}, \quad a, b > 0,$$

where $b > \text{Re}(\epsilon)$ in order for the integral to converge at the upper limit.

$$\begin{aligned}
& \tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i) \\
&= \left(\prod_{\substack{k=1 \\ k \neq j}}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \right) \int_0^\infty d\omega_j \omega_j^{i \sum_k \lambda_k - 1} \omega_j^n \\
&\quad \times \mathcal{A}_{J_1 \dots J_n}(\omega_j \omega_1, z_1, \bar{z}_1; \dots; \omega_j, z_j, \bar{z}_j; \dots). \quad (2.9)
\end{aligned}$$

Hence we can use the representation of the delta function (2.8) provided

$$\lim_{\omega_j \rightarrow \infty} \omega_j^n \mathcal{A}_{J_1 \dots J_n}(\omega_j \omega_1, z_1, \bar{z}_1; \dots; \omega_j, z_j, \bar{z}_j; \dots) = \mathcal{O}(\omega_j^{-\epsilon}). \quad (2.10)$$

Tree-level pure Yang-Mills amplitudes scale homogeneously under the rescaling of all momenta

$$\begin{aligned}
& \omega_j^n \mathcal{A}_{J_1 \dots J_n}^{\text{Yang-Mills}}(\omega_j \omega_1, z_1, \bar{z}_1; \dots; \omega_j, z_j, \bar{z}_j; \dots) \\
&= \mathcal{A}_{J_1 \dots J_n}^{\text{Yang-Mills}}(\omega_1, z_1, \bar{z}_1; \dots; 1, z_j, \bar{z}_j; \dots), \quad (2.11)
\end{aligned}$$

and hence marginally violate (2.10). However, in string theory the tree-level four-gluon amplitudes are exponentially suppressed at large momenta [29]

$$\lim_{\omega_j \rightarrow \infty} \omega_j^4 \mathcal{A}_{J_1 \dots J_4}^{\text{string}}(\omega_j \omega_1, z_1, \bar{z}_1; \dots; \omega_j, z_j, \bar{z}_j; \dots) \sim e^{-\alpha' \omega_j^2}, \quad (2.12)$$

so the use of this representation of the delta function is justified in this case.

Our approach will be to *define* the pure Yang-Mills celestial amplitudes as the $\alpha' \rightarrow 0$ limit of string amplitudes. More generally various types of UV and other singularities are encountered in celestial amplitudes at both tree and loop levels. This approach potentially provides a systematic regulator for all of these singularities.

In the theory so defined, we find that the $\lambda_j \rightarrow 0$ limit selects the residue of the pole in the momentum-space amplitude at $\omega_j = 0$, which is determined by the soft theorem. Using the (energetically) soft theorem for color-ordered partial amplitudes

$$\begin{aligned}
& \lim_{\omega_j \rightarrow 0} \omega_j \mathcal{A}_{J_1 \dots J_n}(\omega_i, z_i, \bar{z}_i) \\
&= -\frac{1}{2} \frac{z_{j-1j+1}}{z_{j-1j} z_{jj+1}} \mathcal{A}_{J_1 \dots J_{j-1} J_{j+1} \dots J_n}(\omega_i, z_i, \bar{z}_i), \quad (2.13)
\end{aligned}$$

we obtain the tree-level conformally soft theorem

$$\begin{aligned}
& \lim_{\lambda_j \rightarrow 0} i \lambda_j \tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i) \\
&= -\frac{1}{2} \frac{z_{j-1j+1}}{z_{j-1j} z_{jj+1}} \tilde{\mathcal{A}}_{J_1 \dots J_{j-1} J_{j+1} \dots J_n}(\lambda_i, z_i, \bar{z}_i). \quad (2.14)
\end{aligned}$$

III. A CONJECTURE

As we have just seen, the transformation from standard scattering amplitudes (in a momentum basis) to celestial amplitudes (in a conformal basis) involves an integral over all energies. Hence, the mere existence of the celestial amplitudes requires soft UV behavior even at tree level. In Yang-Mills theory, the UV behavior is just marginally soft enough. In the scalar theories studied in [3,7,16], the celestial amplitudes are also finite. In Einstein gravity, on the other hand, the Mellin transforms diverge and even the classical four-graviton celestial amplitudes do not exist, while they do appear to exist in classical string theory. It is intriguing that soft UV behavior is required even for the existence of the classical amplitudes. Of course, it is also known that there are a variety of restrictions on the UV behavior of classical scattering (such as the Froissart-Martin bound) required for a unitary quantum perturbation theory. Could these be the same restrictions? This motivates the following:

CONJECTURE: *A perturbative quantum deformation of a classical four-dimensional Minkowskian quantum field theory exists if and only if the celestial amplitudes $\tilde{\mathcal{A}}_{J_1 \dots J_n}(\lambda_i, z_i, \bar{z}_i)$ exist.*

This conjecture is consistent with the (rather small) number of examples which have been analyzed so far.

IV. EXPLICIT EXAMPLES

As a consistency check of our formal argument in Sec. II as well as to gain familiarity with celestial structures, in this section we verify that the conformally soft theorem holds for the known examples of tree-level gluon scattering in Yang-Mills and string theory.

A. Review of gluon amplitudes

In this subsection we review the expressions for celestial gluon amplitudes in both Yang-Mills and string theory [6,9,11].

The general formula for n -point MHV Yang-Mills amplitudes in momentum space is

$$\begin{aligned}
& \mathcal{A}_{--+\dots+}(p_1, \dots, p_n) \\
&= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \dots \langle n1 \rangle} \delta^{(4)} \left(\sum_{i=1}^n p_i \right) \\
&= \frac{1}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \dots z_{n1}} \frac{\omega_1 \omega_2}{\omega_3 \dots \omega_n} \delta^{(4)} \left(\sum_{i=1}^n \epsilon_i \omega_i q_i \right), \quad (4.1)
\end{aligned}$$

where in the last equality, we used $p_i = \epsilon_i \omega_i q_i$ as in (2.2) and

$$\langle ij \rangle = -2\epsilon_i \epsilon_j \sqrt{\omega_i \omega_j} z_{ij}. \quad (4.2)$$

Here and hereafter powers of the gauge (or string) coupling are absorbed into the wave function normalization. To study the three-point amplitude, we work in (2,2) signature, where the amplitude is nonvanishing. z_i and $\bar{z}_i$ are then independent real variables, enabling us to rewrite the momentum-conserving delta function as

$$\begin{aligned} \delta^{(4)}\left(\sum_{i=1}^3 \epsilon_i \omega_i q_i\right) &= \frac{\text{sgn}(z_{23} z_{31})}{4\omega_3^2 z_{23} z_{31}} \delta\left(\omega_1 - \frac{\omega_3 \epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right) \delta\left(\omega_2 - \frac{\omega_3 \epsilon_3 z_{31}}{\epsilon_2 z_{12}}\right) \\ &\times \delta(\bar{z}_{13}) \delta(\bar{z}_{23}), \end{aligned} \quad (4.3)$$

where the form above assumes $z_{ij} \neq 0$. An alternate form may be found by instead assuming $\bar{z}_{ij} \neq 0$,

$$\begin{aligned} \delta^{(4)}\left(\sum_{i=1}^3 \epsilon_i \omega_i q_i\right) &= \frac{\text{sgn}(\bar{z}_{23} \bar{z}_{31})}{4\omega_3^2 \bar{z}_{23} \bar{z}_{31}} \delta\left(\omega_1 - \frac{\omega_3 \epsilon_3 \bar{z}_{23}}{\epsilon_1 \bar{z}_{12}}\right) \delta\left(\omega_2 - \frac{\omega_3 \epsilon_3 \bar{z}_{31}}{\epsilon_2 \bar{z}_{12}}\right) \\ &\times \delta(z_{13}) \delta(z_{23}). \end{aligned} \quad (4.4)$$

For $--+$ scattering we use (4.3) in which case the resulting celestial amplitude is

$$\begin{aligned} \tilde{\mathcal{A}}_{--+}(\lambda_i, z_i, \bar{z}_i) &= -\pi \delta(\lambda_1 + \lambda_2 + \lambda_3) \frac{z_{12}^3}{z_{23}^2 z_{31}^2} \left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{31}}{\epsilon_2 z_{12}}\right)^{1+i\lambda_2} \\ &\times \text{sgn}(z_{23} z_{31}) \delta(\bar{z}_{13}) \delta(\bar{z}_{23}) \Theta\left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right) \Theta\left(\frac{\epsilon_3 z_{31}}{\epsilon_2 z_{12}}\right). \end{aligned} \quad (4.5)$$

Here Θ vanishes (equals unity) for negative (positive) argument.

Similarly, to determine the celestial four-point amplitude, the trick is to rewrite the momentum-conserving delta function

$$\begin{aligned} \delta^{(4)}\left(\sum_{i=1}^4 \epsilon_i \omega_i q_i\right) &= \frac{1}{4\omega_3} \delta\left(\omega_1 - \frac{\omega_3 \epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}}\right) \delta\left(\omega_2 - \frac{\omega_3 \epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right) \\ &\times \delta\left(\omega_4 - \frac{\omega_3 \epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right) \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}), \end{aligned} \quad (4.6)$$

from which it directly follows that

$$\begin{aligned} \tilde{\mathcal{A}}_{--+}(\lambda_i, z_i, \bar{z}_i) &= \frac{\pi}{2} \delta\left(\sum_{i=1}^4 \lambda_i\right) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}}\right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right)^{1+i\lambda_2} \\ &\times \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right)^{-1+i\lambda_4} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\ &\times \Theta\left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}}\right) \Theta\left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right) \Theta\left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right). \end{aligned} \quad (4.7)$$

The factor of $\delta(\sum \lambda_i)$ arises from the scale invariance of Yang-Mills amplitudes in momentum space and generalizes to the n -point amplitude.

In momentum space, the four-gluon amplitudes in type I and heterotic string theory differ from the Yang-Mills amplitude by a multiplicative factor [30,31]

$$\begin{aligned} \mathcal{A}_{--+}^{\text{type I}} &= \frac{\Gamma(1-s)\Gamma(1-u)}{\Gamma(1-s-u)} \mathcal{A}_{--+}^{\text{Yang-Mills}}, \\ \mathcal{A}_{--+}^{\text{heterotic}} &= -\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(s)\Gamma(t)\Gamma(u)} \mathcal{A}_{--+}^{\text{Yang-Mills}}, \end{aligned} \quad (4.8)$$

where $s = p_{12}$, $u = p_{23}$, $t = p_{13}$ with

$$p_{ij} = -\alpha'(p_i + p_j)^2 = 4\alpha' \epsilon_i \epsilon_j \omega_i \omega_j z_{ij} \bar{z}_{ij}. \quad (4.9)$$

As string amplitudes are not scale invariant, the $\delta(\sum \lambda_i)$ factor no longer appears. As shown in [11], the celestial four-gluon amplitudes in type I and heterotic string theory are obtained by the replacements

$$\begin{aligned} (2\pi) \delta\left(\sum_{i=1}^4 \lambda_i\right) &\rightarrow \int_0^\infty d\omega \omega^{i(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - 1} \\ &\times \left(\frac{\Gamma(1-s(\omega))\Gamma(1-u(\omega))}{\Gamma(1-s(\omega)-u(\omega))}\right) \\ &\equiv F_I \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} (2\pi) \delta\left(\sum_{i=1}^4 \lambda_i\right) &\rightarrow \int_0^\infty d\omega \omega^{i(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - 1} \\ &\times \left(-\frac{\Gamma(-s(\omega))\Gamma(-t(\omega))\Gamma(-u(\omega))}{\Gamma(s(\omega))\Gamma(t(\omega))\Gamma(u(\omega))}\right) \\ &\equiv F_H \end{aligned} \quad (4.11)$$

in (4.7), respectively. Here

$$\begin{aligned}
s(\omega) &= 4\alpha'\omega^2 \frac{z_{23}\bar{z}_{13}}{z_{42}\bar{z}_{14}} z_{34}\bar{z}_{34}, & u(\omega) &= 4\alpha'\omega^2 \frac{z_{13}\bar{z}_{34}}{z_{12}\bar{z}_{42}} z_{23}\bar{z}_{23}, \\
t(\omega) &= 4\alpha'\omega^2 \frac{z_{23}\bar{z}_{34}}{z_{12}\bar{z}_{14}} z_{13}\bar{z}_{13}.
\end{aligned} \quad (4.12)$$

A salient feature of celestial string amplitudes with five or more gluons is—assuming soft high-energy behavior—that they appear to be finite at generic values of (λ_i, z_i) . This contrasts with the momentum-space amplitudes, which vanish when momentum is not conserved and are infinite when it is (due to the delta function). For four gluons, however, all celestial amplitudes develop a kinematic delta-function singularity arising from translation invariance, which restricts the 2D conformal cross ratio to lie on the real axis. This can be seen directly from (4.6). For three gluons further kinematic singularities appear in (2,2) signature, while in (3,1) the amplitudes vanish. In the $\alpha' \rightarrow 0$ Yang-Mills limit, the celestial amplitudes are no longer generically finite due to the appearance of the $\delta(\sum \lambda_i)$ factor. While the regulation and properties of momentum-space delta-function singularities are long understood, the delta-function singularities of celestial amplitudes are rather different and remain to be fully understood.

B. Four-point Yang-Mills amplitude

In this section we explicitly study the conformally soft limit of the four-point Yang-Mills gluon amplitude (4.7). Taking $\lambda_4 \rightarrow 0$ and using the identity (2.8) introduces an additional delta function on the right-hand side of (4.7),

$$\begin{aligned}
&\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) \\
&= \delta\left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right) \left[\pi \delta\left(\sum_{i=1}^3 \lambda_i\right) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}}\right)^{1+i\lambda_1} \right. \\
&\quad \times \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right)^{1+i\lambda_2} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\quad \left. \times \Theta\left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}}\right) \Theta\left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right) \Theta\left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right) \right]. \quad (4.13)
\end{aligned}$$

Assuming

$$z_{23} \neq 0, \quad (4.14)$$

where we are taking z_i and $\bar{z}_i$ to be independent real variables, the additional delta function can be written as

$$\delta\left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}}\right) = \text{sgn}(\bar{z}_{14} z_{42} z_{23}) \frac{\bar{z}_{14} z_{42}}{z_{23}} \delta(\bar{z}_{13}). \quad (4.15)$$

Then, Eq. (4.13) simplifies to

$$\begin{aligned}
&\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) \\
&= \text{sgn}(\bar{z}_{14} z_{42} z_{23}) \frac{\bar{z}_{14} z_{42}}{z_{23}} \delta(\bar{z}_{13}) \\
&\quad \times \left[\frac{1}{4} (2\pi) \delta\left(\sum_{i=1}^3 \lambda_i\right) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right)^{1+i\lambda_1} \right. \\
&\quad \times \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right)^{1+i\lambda_2} \delta(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\quad \left. \times \Theta\left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right) \Theta\left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}}\right) \right], \quad (4.16)
\end{aligned}$$

where we find the argument of one of the step functions vanishes and we take $\Theta(0) = \frac{1}{2}$.

To further simplify, we focus on the other delta function by assuming³

$$z_{13} \neq 0, \quad z_{24} \neq 0, \quad \text{and} \quad \bar{z}_{34} \neq 0, \quad (4.17)$$

so that

$$\delta(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) = \frac{\text{sgn}(z_{13} z_{24} \bar{z}_{34})}{z_{13} z_{24} \bar{z}_{34}} \delta(\bar{z}_{12}). \quad (4.18)$$

Using this, the amplitude (4.16) further simplifies as

$$\begin{aligned}
&\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) \\
&= \frac{\text{sgn}(z_{23} z_{31})}{z_{31} z_{23}} \delta(\bar{z}_{12}) \delta(\bar{z}_{13}) \\
&\quad \times \left[\frac{1}{4} (2\pi) \delta\left(\sum_{i=1}^3 \lambda_i\right) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{31}}{\epsilon_2 z_{12}}\right)^{1+i\lambda_2} \right. \\
&\quad \left. \times \Theta\left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}}\right) \Theta\left(\frac{\epsilon_3 z_{31}}{\epsilon_2 z_{12}}\right) \right]. \quad (4.19)
\end{aligned}$$

Comparing with our expression for the three-point celestial amplitude (4.5), we verify the anticipated relation (2.14)

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) = -\frac{1}{2} \frac{z_{31}}{z_{34} z_{41}} \tilde{\mathcal{A}}_{--}(\lambda_i, z_i, \bar{z}_i). \quad (4.20)$$

C. Four-point type I amplitude

In this subsection we study the conformally soft limit of the four-point type I gluon amplitude

³We make such assumptions here because in this section we are interested in the conformally soft singularity structure of the amplitude as opposed to the collinear singularity structure. We do not study the regime in which the singularities overlap.

$$\begin{aligned}
& \tilde{\mathcal{A}}_{--++}^{\text{type I}}(\lambda_i, z_i, \bar{z}_i) \\
&= \frac{1}{4} F_I \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right)^{1+i\lambda_2} \\
&\quad \times \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right)^{-1+i\lambda_4} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\quad \times \Theta \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right) \Theta \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right) \Theta \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right), \quad (4.21)
\end{aligned}$$

with F_I given in (4.10). Performing the same limit as in pure Yang-Mills (4.13), again we obtain an additional delta function on the right-hand side of (4.21)

$$\begin{aligned}
& \lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}^{\text{type I}}(\lambda_i, z_i, \bar{z}_i) \\
&= \delta \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right) \left[\frac{1}{2} (F_I|_{\lambda_4=0}) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right)^{1+i\lambda_1} \right. \\
&\quad \times \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right)^{1+i\lambda_2} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\quad \left. \times \Theta \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right) \Theta \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right) \Theta \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right) \right]. \quad (4.22)
\end{aligned}$$

Assuming as before that $z_{23} \neq 0$ so (4.15) holds, we find the right-hand side of (4.22) reduces to the same expression as in pure Yang-Mills (4.16), since from (4.10) and (4.12) one readily finds

$$\begin{aligned}
F_I|_{\substack{\lambda_4=0 \\ z_{13}=0}} &= \int_0^\infty d\omega \omega^{i(\lambda_1+\lambda_2+\lambda_3)-1} \\
&= 2\pi\delta(\lambda_1 + \lambda_2 + \lambda_3). \quad (4.23)
\end{aligned}$$

Then, by following the remaining steps that were applied to the pure Yang-Mills amplitude, one concludes

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}^{\text{type I}}(\lambda_i, z_i, \bar{z}_i) = -\frac{1}{2} \frac{z_{31}}{z_{34} z_{41}} \tilde{\mathcal{A}}_{--+}(\lambda_i, z_i, \bar{z}_i). \quad (4.24)$$

The amplitude appearing on the right-hand side is the three-point celestial amplitude (4.5), which is identical in Yang-Mills and type I string theory.

D. Four-point heterotic amplitude

From the previous section, we learned that in order to show that the four-point amplitude in heterotic string theory obeys the same Ward identity as the four-point type I and Yang-Mills amplitudes, we first need to determine

$$F_H|_{\substack{\lambda_4=0 \\ z_{13}=0}} = \int_0^\infty d\omega \omega^{i(\lambda_1+\lambda_2+\lambda_3)-1} \left(-\frac{\Gamma(-u(\omega))}{\Gamma(u(\omega))} \right). \quad (4.25)$$

This readily follows from Eqs. (4.11) and (4.12) and the fact

$$\lim_{x \rightarrow 0} \frac{\Gamma(x)}{\Gamma(-x)} = -1. \quad (4.26)$$

Using this and following the steps outlined in the previous section, we arrive at

$$\begin{aligned}
& \lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}^{\text{heterotic}}(\lambda_i, z_i, \bar{z}_i) \\
&= \text{sgn}(\bar{z}_{14} z_{42} z_{23}) \frac{\bar{z}_{14} z_{42}}{z_{23}} \delta(\bar{z}_{13}) \left[\frac{1}{2} (F_H|_{\substack{\lambda_4=0 \\ z_{13}=0}}) \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \right. \\
&\quad \times \left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}} \right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right)^{1+i\lambda_2} \delta(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\quad \left. \times \Theta \left(\frac{\epsilon_3 z_{23}}{\epsilon_1 z_{12}} \right) \Theta \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right) \Theta(0) \right]. \quad (4.27)
\end{aligned}$$

Assuming a noncollinear configuration such that (4.18) applies, the right-hand side of (4.27) reduces to the expression for pure Yang-Mills (4.19) since

$$F_H|_{\substack{\lambda_4=0 \\ z_{13}=0 \\ z_{12}=0}} = (2\pi)\delta(\lambda_1 + \lambda_2 + \lambda_3). \quad (4.28)$$

Thus, the heterotic four-gluon amplitude likewise obeys the Ward identity

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \tilde{\mathcal{A}}_{--++}^{\text{heterotic}}(\lambda_i, z_i, \bar{z}_i) = -\frac{1}{2} \frac{z_{31}}{z_{34} z_{41}} \tilde{\mathcal{A}}_{--+}(\lambda_i, z_i, \bar{z}_i). \quad (4.29)$$

E. Permuting external legs

In momentum space, the four-point MHV amplitude is symmetric under the exchange of 3 and 4, and consequently the soft limits ω_3 and ω_4 can both be readily obtained from the momentum-space expression (4.1). On the other hand, the four-point celestial amplitudes were written in Sec. IV A in a form that elucidates their behavior in the $\lambda_4 \rightarrow 0$ conformally soft limit, but obscures their behavior in the $\lambda_3 \rightarrow 0$ conformally soft limit. Of course, one can start from the momentum-space amplitudes and derive a different expression for the celestial amplitudes in which the $\lambda_3 \rightarrow 0$ conformally soft limit can easily be taken. However, one of the goals of this paper is to understand everything directly in the conformal basis, so in this subsection we explain how to permute 3 and 4 without reference to the momentum-space representation. We shall see the permutation symmetry is related to scaling properties of the amplitudes. The general form of the four-gluon celestial amplitude is

$$\begin{aligned}
& \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) \\
&= \frac{1}{4} F(\lambda_i, \hat{s}, \hat{u}) \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right)^{1+i\lambda_1} \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right)^{1+i\lambda_2} \\
&\times \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right)^{-1+i\lambda_4} \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\times \Theta \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right) \Theta \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right) \Theta \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right), \quad (4.30)
\end{aligned}$$

where

$$F(\lambda_i, \hat{s}, \hat{u}) = \int_0^\infty d\omega \omega^{-1+i} \sum_{i=1}^4 \lambda_i f(\omega^2 \hat{s}, \omega^2 \hat{u}), \quad (4.31)$$

$\omega^2 \hat{s} = s$, and $\omega^2 \hat{u} = u$, with s and u given in (4.12) and related to t by $s + t + u = 0$.

Now suppose we perform a change of variables $\omega \rightarrow \omega\alpha$, where $\alpha > 0$. We find

$$F(\lambda_i, \hat{s}, \hat{u}) = \alpha^i \sum_{i=1}^4 \lambda_i F(\lambda_i, \alpha^2 \hat{s}, \alpha^2 \hat{u}), \quad (4.32)$$

and in the full amplitude,

$$\begin{aligned}
& \tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) \\
&= \frac{1}{4} F(\lambda_i, \alpha^2 \hat{s}, \alpha^2 \hat{u}) \left(\alpha \frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right)^{1+i\lambda_1} \\
&\times \left(\alpha \frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right)^{1+i\lambda_2} \left(\alpha \frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right)^{-1+i\lambda_4} \alpha^{-1+i\lambda_3} \\
&\times \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \\
&\times \Theta \left(\frac{\epsilon_3 z_{23} \bar{z}_{34}}{\epsilon_1 z_{12} \bar{z}_{14}} \right) \Theta \left(\frac{\epsilon_3 z_{13} \bar{z}_{34}}{\epsilon_2 z_{12} \bar{z}_{42}} \right) \Theta \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right). \quad (4.33)
\end{aligned}$$

Then, if we choose

$$\alpha = \left(\frac{\epsilon_3 \bar{z}_{13} z_{23}}{\epsilon_4 \bar{z}_{14} z_{42}} \right)^{-1}, \quad (4.34)$$

which is guaranteed to be positive by one of the step functions, we obtain a form of the amplitude in which we can easily study the limit $\lambda_3 \rightarrow 0$.

F. n -point MHV Yang-Mills amplitude

The n -point MHV gluon amplitudes were derived in terms of generalized hypergeometric functions by Schreiber, Volovich, and Zlotnikov [9]. In the Appendix we verify that these amplitudes are related by the conformally soft relation (2.14). In this section, we present an alternate route to the same end. We first present a new formula for the n -point MHV celestial amplitude which manifestly exhibits all poles in λ_i for $i = 3, \dots, n$, allowing a streamlined analysis of the conformally soft limit. Moreover, for $n > 4$, the absence of poles in λ_1, λ_2 implies $\lim_{\lambda_i \rightarrow 0} \lambda_i \tilde{\mathcal{A}}_{--+\dots+}^{(n)}(\lambda_i, z_i, \bar{z}_i) = 0$ for $i = 1, 2$, which is consistent with the vanishing of the momentum-space soft limit in these cases due to $\mathcal{A}_{--+\dots+}^{(n-1)}(\omega_i, z_i, \bar{z}_i) = 0$. In the $n = 4$ case, the conformally soft limit of the negative helicity gluons in (4.7) yields the expected conformally soft formula.

The Mellin transform of the n -point MHV amplitude can be written

$$\begin{aligned}
\tilde{\mathcal{A}}_{--+\dots+}^{(n)}(\lambda_i, z_i, \bar{z}_i) &= \frac{1}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \dots z_{n1}} \prod_{i=1}^n \left(\int_0^\infty d\omega_i \omega_i^{i\lambda_i} \right) \frac{\omega_1 \omega_2}{\omega_3 \dots \omega_n} \delta^{(4)} \left(\sum_{i=1}^n \epsilon_i \omega_i q_i \right) \\
&= \frac{1}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \dots z_{n1}} \prod_{i=1}^n \left(\int_0^\infty d\omega_i \omega_i^{i\lambda_i - J_i} \right) \int \frac{d^4 y}{(2\pi)^4} e^{-i \sum \omega_i (\epsilon_i y \cdot q_i - i\epsilon)} \\
&= \frac{1}{(-2)^{n-4}} \frac{1}{(2\pi)^4} \frac{z_{12}^3}{z_{23} \dots z_{n1}} \int d^4 y \prod_{i=1}^n \frac{\Gamma(i\lambda_i + 1 - J_i)}{[i\epsilon_i y \cdot q_i + \epsilon]^{i\lambda_i + 1 - J_i}}. \quad (4.35)
\end{aligned}$$

In the last line we used the formula

$$\int_0^\infty e^{-a\omega} \omega^z d\omega = \frac{\Gamma(z+1)}{a^{z+1}}. \quad (4.36)$$

We see explicitly from the Γ functions that $\tilde{\mathcal{A}}_{--+\dots+}^{(n)}(\lambda_i, z_i, \bar{z}_i)$ has poles in λ_i for $3 \leq i \leq n$ (but not in λ_1 or λ_2) so the conformally soft limit picks out the residue of the pole. Since the factor $z_{12}^3/(z_{23} \dots z_{n1})$ supplies the appropriate soft factor $z_{i-1+i}/(z_{i-1} z_{ii+1})$,

the amplitude obeys the conformally soft theorem so long as the remaining part becomes independent of $(z_i, \bar{z}_i)$ in the limit $\lambda_i \rightarrow 0$. Indeed, in this limit, the factor associated with the i th particle $[i\epsilon_i y \cdot q_i + \epsilon]^{-i\lambda_i - 1 + J_i} \rightarrow 1$, so the conformally soft formula (2.14) holds.

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APPENDIX: SCHREIBER-VOLOVICH-ZLOTNIKOV MHV AMPLITUDES

In this Appendix we analyze the soft limits of MHV amplitudes using the representation of the n -point celestial amplitude in terms of generalized hypergeometric functions given by Schreiber, Volovich, and Zlotnikov [9]. They found⁴

$$\begin{aligned} \tilde{\mathcal{A}}_{-+\dots+}^{(n)}(\lambda_i, z_i, \bar{z}_i) \\ = \frac{2\pi}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \cdots z_{n1}} \frac{1}{|U_{1,n}|} \\ \times \delta\left(\sum_{j=1}^n \lambda_j\right) \hat{\phi}_n(\{\alpha\}, x) \prod_{i,j} \Theta\left(-\frac{U_{i,j}}{U}\right), \end{aligned} \quad (\text{A1})$$

where $i = 1, \dots, n-4, j = n-3, \dots, n$, and

$$\begin{aligned} \hat{\phi}_n(\{\alpha\}, x) \\ = \frac{c_2}{c_1} \int_{u_1, \dots, u_3, 1 - \sum_{a=1}^3 u_a \geq 0} \prod_{j=1}^n P_j^{\alpha_j} \frac{dP_{n-3} \wedge dP_{n-2} \wedge dP_{n-1}}{P_{n-3} P_{n-2} P_{n-1}}, \\ P_j = x_{0j} + x_{1j}u_1 + x_{2j}u_2 + x_{3j}u_3, \\ \frac{c_2}{c_1} = \frac{\Gamma(2+i\lambda_1)\Gamma(2+i\lambda_2)\prod_{j=3}^{n-4}\Gamma(i\lambda_j)}{\Gamma(1-i\lambda_1)\prod_{i=1}^3\Gamma(1-i\lambda_{n-i})} \end{aligned} \quad (\text{A2})$$

are dual integral representations of hypergeometric functions. The parameters α_i and x_{ij} are given by

$$\begin{aligned} \alpha_1 = 1, \quad \alpha_2 = -2 - i\lambda_2, \quad \alpha_3 = -i\lambda_3, \dots, \\ \alpha_{n-4} = -i\lambda_{n-4}, \quad \alpha_{n-i} = -i\lambda_{n-i} + 1, \quad \alpha_n = -i\lambda_n, \\ x_{0j} = \frac{U_{j,n}}{U_{1,n}}, \quad x_{ij} = \frac{U_{j,n-4+i}}{U_{1,n-4+i}} - \frac{U_{j,n}}{U_{1,n}}, \\ x_{0n} = -\frac{U}{U_{1,n}} = -x_{in}, \quad x_{01} = 1, \\ x_{1n-3} = -\frac{U}{U_{1,n-3}}, \quad x_{2n-2} = -\frac{U}{U_{1,n-2}}, \quad x_{3n-1} = -\frac{U}{U_{1,n-1}}, \\ \text{for } i = 1, 2, 3, \quad j = 2, \dots, n-4, \end{aligned} \quad (\text{A3})$$

with

⁴As in [9], we restrict our attention to a subset of the allowed configurations on the celestial sphere, for which the amplitudes can be expressed in terms of generalized hypergeometric functions.

$$U = \epsilon_{n-3}\epsilon_{n-2}\epsilon_{n-1}\epsilon_n \det(q_{n-3}q_{n-2}q_{n-1}q_n),$$

$$U_{i,j} = \epsilon_i \epsilon_j \epsilon_{n-3} \epsilon_{n-2} \epsilon_{n-1} \epsilon_n \det(q_{n-3} \cdots q_{j \rightarrow i} \cdots). \quad (\text{A4})$$

First, notice that the factor c_2/c_1 has poles in λ_j for $j = 3, \dots, n-4$, so the conformally soft limit of any of these gluons will be given by the residue. Since, as discussed previously, the factor $z_{12}^3/(z_{23} \cdots z_{n1})$ gives the appropriate soft factor, these amplitudes obey the conformally soft theorem so long as $\hat{\phi}_n$ becomes independent of $(z_j, \bar{z}_j)$ in the limit $\lambda_j \rightarrow 0$. The only dependence on $(z_j, \bar{z}_j)$ in $\hat{\phi}_n$ appears in $P_j^{-i\lambda_j}$, which behaves as $P_j^{-i\lambda_j} \rightarrow 1$ in the conformally soft limit. Hence, $\hat{\phi}_n$ becomes independent of $(z_j, \bar{z}_j)$ and the celestial amplitude obeys the conformally soft theorem.

To obtain the conformally soft theorem for gluons $j = n-3, \dots, n$, we work with an expression for the Mellin-transformed amplitude before the change of variables that gives (A1)

$$\begin{aligned} \tilde{\mathcal{A}}_{-+\dots+} = \frac{1}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \cdots z_{n1}} \prod_{a=1}^{n-4} \int_0^\infty d\omega_a \omega_a^{i\lambda_a} \frac{\omega_1 \omega_2}{\omega_3 \cdots \omega_{n-4}} \\ \times \prod_{b=n-3}^n (\omega_b^*)^{i\lambda_b-1} \Theta(\omega_b^*), \end{aligned} \quad (\text{A5})$$

where

$$\omega_j^* = -\frac{1}{U} \sum_{i=1}^{n-4} \omega_i U_{i,j}. \quad (\text{A6})$$

To take the conformally soft limit on the n th leg, we use the identity (2.8) so that the factor $(\omega_n^*)^{i\lambda_n-1}$ is turned into $\delta(\omega_n^*)$. As before, we need to show that the n -point amplitude becomes proportional to an $(n-1)$ -point amplitude. To this end, we can use $\delta(\omega_n^*)$ to perform one of the ω_a integrals. For example, we can solve $\omega_n^* = 0$ for ω_{n-4} ,

$$\omega_{n-4} = -\sum_{i=1}^{n-5} \frac{U_{i,n}}{U_{n-4,n}} \omega_i, \quad (\text{A7})$$

and perform the integral with respect to ω_{n-4} . We obtain an expression similar to (A5)

$$\begin{aligned} \lim_{\lambda_n \rightarrow 0} i\lambda_n \tilde{\mathcal{A}}_{-+\dots+} = \frac{1}{(-2)^{n-4}} \frac{z_{12}^3}{z_{23} \cdots z_{n1}} \frac{1}{|U_{n-4,n}|} \\ \times \prod_{a=1}^{n-5} \int_0^\infty d\omega_a \omega_a^{i\lambda_a} \frac{\omega_1 \omega_2}{\omega_3 \cdots \omega_{n-5}} \\ \times \prod_{b=n-4}^{n-1} (\omega_b^*)^{i\lambda_b-1} \Theta(\omega_b^*), \end{aligned} \quad (\text{A8})$$

where ω_{n-4}^* is given by the right-hand side of (A7) and

$$\omega_b^* = -\frac{1}{U} \sum_{i=1}^{n-5} \left(U_{i,b} - \frac{U_{n-4,b} U_{i,n}}{U_{n-4,n}} \right) \omega_i, \quad (A9)$$

$$b = n-3, n-2, n-1.$$

To recognize that the right-hand side of (A8) is proportional to an $(n-1)$ -point amplitude, we put it in the form (A5). Introducing analogues of (A4) where $n-3, \dots, n$ are replaced with $n-4, \dots, n-1$,

$$U' = \epsilon_{n-4} \epsilon_{n-3} \epsilon_{n-2} \epsilon_{n-1} \det(q_{n-4} q_{n-3} q_{n-2} q_{n-1}),$$

$$U'_{i,j} = \epsilon_i \epsilon_j \epsilon_{n-4} \epsilon_{n-3} \epsilon_{n-2} \epsilon_{n-1} \det(q_{n-4} \cdots q_{j \rightarrow i} \cdots), \quad (A10)$$

and using the identity

$$U_{i,b} U_{n-4,n} - U_{i,n} U_{n-4,b} = -U U'_{i,b}, \quad (A11)$$

we find (A8) can be written as

$$\lim_{\lambda_n \rightarrow 0} i \lambda_n \tilde{A}_{--+\cdots+} = -\frac{1}{2} \frac{1}{(-2)^{n-5}} \frac{z_{n-1,1}}{z_{n-1,n} z_{n,1}} \frac{z_{1,2}^3}{z_{2,3} \cdots z_{n-1,1}} |U'|$$

$$\times \prod_{a=1}^{n-5} \int_0^\infty d\omega_a \omega_a^{i\lambda_a} \frac{\omega_1 \omega_2}{\omega_3 \cdots \omega_{n-5}}$$

$$\times \prod_{b=n-4}^{n-1} (\omega_b^*)^{i\lambda_b-1} \Theta(\omega_b^*), \quad (A12)$$

where now

$$\omega_j^* = -\frac{1}{U'} \sum_{i=1}^{n-5} \omega_i U'_{i,j}, \quad (A13)$$

so that the right-hand side of (A12) is readily identified as proportional to an $(n-1)$ -point amplitude. The conformally soft theorem of (A5) for the gluons $j = n-3, n-2, n-1$ follows from similar arguments.

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