Charged accelerating black hole in $f(R)$ gravity

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We obtain a charged accelerating AdS black hole solution in $f(R)$ gravity and investigate its thermodynamic behavior. We consider low-acceleration black holes that do not have an acceleration horizon, and we obtain the first law of thermodynamics for them. We further study the parameter space of charged, slowly accelerating, $f(R)$ AdS black holes before investigating the behavior of the free energy in both the canonical and grand canonical ensembles. We find a generalization of the reverse isoperimetric inequality, applicable to black holes in $f(R)$ gravity, that indicates that these black holes can become superentropic relative to their counterparts in Einstein gravity.

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I. INTRODUCTION

The well-known vacuum C-metric was introduced in Refs. [1,2] and then investigated in Refs. [3,4]. Later, a charged C-metric was discovered [5]. Mathematically, the C-metric is a solution of the Einstein(-Maxwell) field equations; physically, it can be interpreted as two uniformly accelerating black holes dragged by forces originating from conical singularities [6]. An accelerating black hole with not only charge but also rotation was discussed in Ref. [7].

The C-metric with a negative (positive) cosmological constant has been used as a test background to study gravitationally and electromagnetically radiative fields near timelike (spacelike) infinity [8,9]. However, in contrast to the flat or de Sitter (dS) C-metrics, the anti-de Sitter (AdS) C-metric can describe a pair of accelerated black holes only when the acceleration parameter A and the AdS radius l satisfy a certain restriction, originally thought to be $A > 1/l$ [10]. It has recently been shown that this condition must be corrected to be a more stringent one [11]. If this condition is not satisfied (which will be the case for sufficiently small acceleration A), then the AdS C-metric describes a single, uniformly accelerating black hole having an outer (and possibly inner) horizon [10–12].

For quite some time, the thermodynamics of accelerating black holes have been poorly understood (and indeed avoided) in the literature, most likely because they generically have two horizons, except for the slow-acceleration case noted above. However, it has recently been shown that it is possible to make sense of their thermodynamics in the slow-acceleration case, and a first law for these objects was derived [13]. Furthermore, by allowing the tension related to the conical deficits to vary, thermodynamics of the accelerating black hole was generalized to include a new potential named the thermodynamic length [14]. A holographic interpretation of the thermodynamics for slowly accelerating AdS black holes was subsequently given [11], and thermodynamic variables were found to be consistently calculated using both conformal and holographic methods. Conserved charges were obtained by choosing a canonical time coordinate and using a covariant phase space method in [15]; they were also checked by the Christodoulou-Ruffini mass formula [16]. An explication of the thermodynamic parameters of accelerating AdS black holes with charge and rotation was studied in [17], and these results were used to obtain a remarkable snapping-swallowtail phase transition for the charged case at a certain pressure [18]. For the rotating case this particular pressure “splits” into three values that have the effect of admitting reentrant phase transitions obtainable in two ways, either by varying the temperature at fixed pressure or by varying the pressure at fixed temperature [19].

Until now, all extant C-metrics describing accelerating black holes are solutions of the equations of motion in Einstein’s classical gravitational theory. It is of interest to find accelerating black hole solutions in modified gravity

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theories, such as the renowned $f(R)$ gravity [20]. A number of solutions whose origins are in high-energy physics [21–23] and cosmology [24,25] motivate the consideration of higher-order curvature invariants to the gravitational action. Amongst the various possibilities, $f(R)$ gravity is an excellent candidate for two reasons. First, it encapsulates the features of the higher-order gravity by, for example, taking a simple form

$$f(R) = \sum_i \alpha_i R^i, \quad (1)$$

where R is the Ricci scalar and the α_i are coefficients; other higher-order terms such as $R_{\mu\nu}R^{\mu\nu}$ are not considered [20]. Second, it may be the only higher-derivative theory that avoids the Ostrogradski instability [26].

In this paper, we obtain charged accelerating black hole solutions in $f(R)$ gravity and study their thermodynamics. The thermodynamics of charged black holes in $f(R)$ gravity have been studied previously [27], and phase transitions analogous to the van der Waals liquid-gas system between small and large black holes were observed. In our extension to this case, we find a number of new results. One is a generalization of the reverse isoperimetric inequality [28] that includes both conical deficits and the effects of $f(R)$ gravity. We find that $f(R)$ gravity can yield both subentropic and superentropic black holes, depending on the magnitude of $\eta = 1 + f'(R_0)$, where R_0 is the value of the Ricci scalar for the accelerating solution. Furthermore, we find that the parameter space of allowed solutions is either enlarged or diminished, depending on the size of η . Our investigation of the phase behavior of these black holes indicates that both the expected van der Waals transitions and snapping swallowtail behavior observed previously [18] occur here as well, but we also find new behavior in the fixed-potential ensemble, including Hawking-Page-type curves for unstable and stable black holes. We find that not only the lowest temperature attainable for the black hole but also the highest temperature that the small black hole can reach increases for increasing η . Furthermore, both kinds of temperatures depend on the acceleration of the black hole, as we shall demonstrate.

We begin by introducing a charged accelerating black hole solution in $f(R)$ gravity and by calculating its thermodynamic quantities in Sec. II. We analyze the parameter space and phase transition for the charged slowly accelerating $f(R)$ AdS black hole in Sec. III. The final section is devoted to our summary.

II. CHARGED ACCELERATING BLACK HOLE IN $f(R)$ GRAVITY

A. Solution

The action of $f(R)$ gravity containing the Maxwell term is

$$\mathcal{I} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \mathcal{L}, \quad (2)$$

with

$$\mathcal{L} = R + f(R) - F_{ab}F^{ab}. \quad (3)$$

Here R is the Ricci scalar, $f(R)$ is an auxiliary function of R , $F_{ab} = \nabla_a B_b - \nabla_b B_a$, and B_a is the gauge potential. The corresponding equations of motion are

$$\begin{aligned} R_{ab} - \frac{1}{2}Rg_{ab} - \frac{1}{2}f(R)g_{ab} + R_{ab}f'(R) - \nabla_b \nabla_a R f''(R) \\ + g_{ab} \nabla_c \nabla^c R f''(R) - \nabla_a R \nabla_b R f^{(3)}(R) \\ + g_{ab} \nabla_c R \nabla^c R f^{(3)}(R) = 2T_{ab} \end{aligned} \quad (4)$$

and

$$\nabla_b \nabla^b B^a - \nabla_b \nabla^a B^b = 0, \quad (5)$$

with

$$f'(R) = \frac{df(R)}{dR}, \quad (6)$$

$$T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}F_{cd}F^{cd}g_{ab}. \quad (7)$$

It is easy to check that the electromagnetic field is traceless, as

$$T^\mu{}_\mu = 0. \quad (8)$$

If we want to find a maximally symmetric solution, the Ricci scalar must be constant [20]. When the curvature scalar is constant, i.e., $R = R_0$ ($R_0 \neq 0$), according to Eq. (4), we can obtain [29,30]

$$R_0 - R_0 f'(R_0) + 2f(R_0) = 0. \quad (9)$$

Then Eq. (4) can be reexpressed as

$$\eta R_{ab} - \frac{\eta}{4} R_0 g_{ab} = 2T_{ab}, \quad (10)$$

where we have defined

$$\eta = 1 + f'(R_0). \quad (11)$$

According to the equation of motion (10), we can obtain a charged accelerating black hole solution

$$\begin{aligned} ds^2 = \frac{1}{\Omega^2} \left[-\frac{N(r)dt^2}{\alpha^2} + \frac{dr^2}{N(r)} \right. \\ \left. + r^2 \left(\frac{d\theta^2}{H(\theta)} + H(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} \right) \right], \end{aligned} \quad (12)$$

where

$$\Omega = 1 + Ar \cos \theta, \quad (13)$$

$$N(r) = (1 - A^2 r^2) \left(1 - \frac{2m}{r} + \frac{q^2}{\eta r^2} \right) - \frac{R_0 r^2}{12}, \quad (14)$$

$$H(\theta) = 1 + 2mA \cos \theta + \frac{q^2}{\eta} A^2 \cos^2 \theta, \quad (15)$$

$$\alpha = \sqrt{\Xi \left(1 + \frac{12A^2 \Xi}{R_0} \right)}, \quad (16)$$

$$\Xi = 1 + \frac{q^2 A^2}{\eta}. \quad (17)$$

In the black hole solution, the conformal factor Ω determines the conformal boundary r_b of the black hole and $r_b = -1/(A \cos \theta)$. Note that A , m , q are the acceleration, the mass parameter, and the electric parameter of the black hole, and K stands for the conical deficits of the black hole on the north and south poles. The parameter α is used to rescale the time coordinate so that we can get a normalized Killing vector at conformal infinity [11,12,17]. When $A = 0$ we recover the charged AdS black hole in $f(R)$ gravity [31].

Here, $R_0 < 0$, $R_0 = 0$, and $R_0 > 0$ correspond to asymptotically AdS, flat, and dS accelerating black holes, respectively. In the following, we only investigate the case $R_0 < 0$. When $A = 0$, $K = 1$, $R_0 < 0$, the solution reduces to the $f(R)$ black hole [29] for which we need $\eta > 0$ to assure the existence of inner and outer horizons [30]. For consistency, we only consider the case $\eta > 0$ for the accelerating $f(R)$ AdS black hole here. Then we can solve the equation of motion (5) for the gauge field to obtain the electromagnetic tensor

$$F_{ab} = (dB)_{ab}, \quad (18)$$

where

$$B_a = \frac{1}{\alpha} \left(\frac{q}{r_+} - \frac{q}{r} \right) (dt)_a, \quad (19)$$

and r_+ is the outer horizon.

When

$$f'(R_0) = 0, \quad R_0 = -\frac{12}{l^2}, \quad (20)$$

the solution is identical to the charged accelerating AdS black hole in Einstein gravity. Hereafter, we write $R_0 = -12/l^2$, so that the blackening factor of the black hole can be written as

$$N(r) = \left(1 - \frac{2m}{r} + \frac{q^2}{\eta r^2} \right) (1 - A^2 r^2) + \frac{r^2}{l^2} \quad (21)$$

and we note that η and l are independent parameters.

B. Thermodynamic first law of slowly accelerating $f(R)$ AdS black hole

Here, we investigate a slowly accelerating $f(R)$ AdS black hole. The condition for the slow acceleration will be discussed later. The mass of the slowly accelerating AdS black hole can be obtained by a conformal prescription [32,33]. We can choose a conformal factor $\bar{\Omega} = \eta^{-1} l \Omega r^{-1}$ to eliminate the divergence near the conformally infinite boundary. After a conformal transformation

$$\bar{g}_{ab} = \bar{\Omega}^2 g_{ab} \quad (22)$$

on the spacetime metric (12), we can obtain the conserved charge related to the Killing vector ξ^a as

$$\mathcal{Q}(\xi_c) = \frac{l}{8\pi} \lim_{\bar{\Omega} \rightarrow 0} \frac{l^2}{\bar{\Omega}} \bar{N}^a \bar{N}^b \bar{C}_{ab}^c \xi_c d\bar{S}^d, \quad (23)$$

where $\bar{N}_a$ is a normal covector of the conformal boundary determined by $\bar{\Omega}$, and $\bar{C}_{bcd}^a$ is the Weyl tensor of the conformal metric $\bar{g}_{ab}$. After letting $\bar{\Omega} \rightarrow 0$ in the conformal metric $\bar{g}_{ab}$, we can obtain the spacelike area element as

$$d\bar{S}_a = -\frac{\eta^2 l^2 \sin \theta d\theta d\phi}{\alpha K} (dt)_a. \quad (24)$$

Then the mass M of the slowly accelerating $f(R)$ AdS black hole can be calculated as

$$M = \mathcal{Q}(\partial_t) = \frac{\eta m (1 - A^2 l^2 \Xi)}{K \alpha}. \quad (25)$$

It should be noted that we have chosen the conformal factor so that the resulting mass of the slowly accelerating $f(R)$ AdS black hole becomes that of the $f(R)$ black hole [30] calculated via the quasilocal approach [34–36].

In contrast to the slowly accelerating AdS black hole in Einstein gravity, the entropy of the accelerating AdS black hole in $f(R)$ gravity cannot simply be calculated from the area law [37]; rather, the Wald method [38] must be used. This gives

$$S = -2\pi \oint d^2 x \sqrt{\hat{h}} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} = \frac{\eta \pi r_+^2}{K(1 - A^2 r_+^2)}, \quad (26)$$

where $\hat{h}$ is the determinant of the induced metric on the $t = \text{const}$ and $r = r_+$ hypersurface, and ϵ_{ab} is a normal bivector which satisfies $\epsilon_{ab} \epsilon^{ab} = -2$.

The temperature of the black hole can be obtained as

$$\begin{aligned} T &= \frac{f'(r_+)}{4\pi\alpha} \\ &= \frac{l^2(A^2r_+^2 - 1)^2(q^2 - \eta r_+^2) - \eta r_+^4(3 - A^2r_+^2)}{4l^2\pi\alpha\eta r_+^3(A^2r_+^2 - 1)} \\ &= \frac{r_+}{2\pi\alpha l^2(1 - A^2r_+^2)} + \frac{(A^2r_+^2 - 1)(q^2 - \eta m r_+)}{2\pi\alpha\eta r_+^3}. \end{aligned} \quad (27)$$

Then we find

$$TS = \frac{\eta m}{2\alpha K} - \frac{q^2}{2\alpha K r_+} + \frac{\eta r_+^3}{2\alpha l^2 K(1 - A^2r_+^2)^2}. \quad (28)$$

The electric charge of the black hole is

$$Q = \frac{1}{4\pi} \lim_{\Omega \rightarrow 0} \int *F = \frac{1}{4\pi} \int \frac{q}{K} \sin \theta d\theta d\phi = \frac{q}{K} \quad (29)$$

from Gauss' law. The conjugate electric potential is

$$\Phi = \frac{1}{4\pi Q\beta} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_a F^{ab} B_b = \frac{q}{\alpha r_+} \quad (30)$$

where n_a is a normal covector on the conformal boundary determined by Ω , h is the determinant of the induced metric on the conformal boundary, and β is related to the surface gravity κ at the event horizon r_+ by $\beta = 2\pi/\kappa$ [39].

Note that although a rescaling of q ,

$$\frac{q^2}{\eta} = Q_f^2, \quad (31)$$

renders the black hole solution (21) the same as that for the accelerated charged black hole in Einstein gravity [17], this parameter actually does not scale out of the thermodynamics. The reason for this is that the mass (25) and entropy (26) retain their η dependence, and in fact, the absence of η in Eqs. (29) and (30) is essential in satisfying the first law. We have checked that the gauge potential (19) does not contain a factor $1/\sqrt{\eta}$, or else Eq. (10) cannot be satisfied. Consequently, we disagree with the conserved electric charge and conjugate potential previously obtained for $f(R)$ black holes [27,30], where there are respective factors of $1/\sqrt{\eta}$ and $\sqrt{\eta}$.

The pressure of the slowly accelerating $f(R)$ AdS black hole can be defined as [27,30,40–46]

$$P = \frac{3}{8\pi l^2}. \quad (32)$$

Then the volume can be obtained as

$$V = \frac{4\pi\eta}{3K\alpha} \left[\frac{r_+^3}{(1 - A^2r_+^2)^2} + m l^4 A^2 \Xi \right] \quad (33)$$

by using the Smarr relation

$$M = 2(TS - PV) + \Phi Q. \quad (34)$$

The conical deficits at the north pole ($\theta_+ = 0$) and south pole ($\theta_- = \pi$) are

$$\delta_{\pm} = 2\pi \left(1 - \frac{H_{\pm}}{K} \right), \quad (35)$$

where

$$H_{\pm} = \Xi \pm 2mA. \quad (36)$$

Then the tensions related to the conical deficits on the north and south poles are [47–49]

$$\begin{aligned} \mu_{\pm} &= \frac{\delta_{\pm}}{8\pi} \\ &= \frac{1}{4} \left(1 - \frac{H_{\pm}}{K} \right) \end{aligned} \quad (37)$$

$$= \frac{1}{4} \left(1 - \frac{1 \pm 2mA + q^2 A^2 / \eta}{K} \right), \quad (38)$$

where $\delta_{\pm}$ represents conical deficits at the north and south poles, and $H_{\pm}$ stands for $H(\theta = 0)$ and $H(\theta = \pi)$. We can also define [50]

$$\Delta \equiv 1 - 2(\mu_+ + \mu_-) = \frac{A^2 q^2 + \eta}{\eta K}, \quad (39)$$

$$C \equiv \frac{\mu_- - \mu_+}{\Delta} = \frac{mA\eta}{A^2 q^2 + \eta}, \quad (40)$$

which, respectively, are average and differential conical deficits for the charged $f(R)$ accelerating black hole. We note that as $\eta \rightarrow 0$, we obtain a large average deficit Δ and a vanishingly small differential deficit C ; conversely, for large η , both Δ and C approach their values in the charge neutral case. Since both $\mu_{\pm}$ are dimensionless, these quantities, along with their thermodynamic conjugates $\lambda_{\pm}$ [14], do not enter the Smarr relation (34). There is a direct but rather cumbersome method to obtain these quantities from the first law [17]. Rather than proceeding in this manner, we can instead write the mass entirely in terms of the extrinsic thermodynamic parameters, obtaining

$$M^2 = \frac{\Delta\eta S}{4\pi} \left[\left(1 + \frac{8PS}{3\Delta\eta} + \frac{\pi Q^2}{\Delta S} \right)^2 - \frac{3\Delta\eta C^2}{2PS} \left(\frac{8PS}{3\Delta\eta} + 1 \right) \right]. \quad (41)$$

From this we can compute the temperature, electric potential, and thermodynamic volume as

$$T = \left(\frac{\partial M}{\partial S} \right)_{Q,P,\mu_{\pm}} = \frac{1}{24\pi\Delta\eta MS^2} [64P^2S^4 + 16\eta PS^2(\pi Q^2 + 2\Delta S) - 3\eta^2(\pi^2 Q^4 + \Delta^2 S^2(4C^2 - 1))], \quad (42)$$

$$\Phi = \left(\frac{\partial M}{\partial Q} \right)_{S,P,\mu_{\pm}} = \frac{Q(8PS^2 + 3\eta(\pi Q^2 + \Delta S))}{6\Delta MS}, \quad (43)$$

$$V = \left(\frac{\partial M}{\partial P} \right)_{S,Q,\mu_{\pm}} = \frac{27\Delta^3\eta^3 C^2 + 256P^3 S^3 + 96\eta P^2 S(\pi Q^2 + \Delta S)}{144\pi\Delta\eta MP^2}, \quad (44)$$

which, upon using the expressions (26), (29), and (32), can be shown to agree with those in Eqs. (27), (30), and (33), respectively.

The thermodynamic lengths conjugate to the tensions $\mu_{\pm}$ are therefore straightforwardly obtained:

$$\lambda_{\pm} = \left(\frac{\partial M}{\partial \mu_{\pm}} \right)_{S,Q,P} = \frac{1}{72\pi\Delta^2\eta MPS} [128P^3 S^4 + 96\pi\eta P^2 Q^2 S^2 - 18\eta^2 P(\Delta^2 S^2(1 \mp 2C)^2 - \pi^2 Q^4) \pm 27\Delta^3\eta^3 SC], \quad (45)$$

and it is then easy to check that the first law

$$dM = TdS + \Phi dQ + \lambda_+ d\mu_+ + \lambda_- d\mu_- + VdP \quad (46)$$

is satisfied.

The key advantage of the formulation (41) is that it elucidates the chemical structure of the accelerating black hole [50]. Indeed, using Eqs. (41) and (44) we obtain a generalization of the reverse isoperimetric inequality [28,51–53]

$$\left(\frac{3V}{4\pi} \right)^2 \geq \frac{1}{\eta\Delta} \left(\frac{S}{\pi} \right)^3 = \frac{\eta^2}{\Delta} \left(\frac{\mathcal{A}}{4\pi} \right)^3 \quad (47)$$

for accelerating $f(R)$ black holes, where $\mathcal{A}$ is the horizon area. The equality is saturated if $C = 0$, and it reduces to

that in Einstein gravity [50] if $\eta = 1$. A larger deficit yields a smaller Δ , and therefore a smaller entropy with respect to the volume, tending to make the black holes subentropic. However, a larger value of η works in a contrary manner, allowing for a larger entropy (but smaller area) relative to the volume, tending to make the black holes superentropic [54,55].

We can confirm our result by calculating the action with a Gibbons-Hawking boundary term and boundary counterterms [32,56–58],

$$I = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} [R + f(R) - F^2] + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-h} \eta \mathcal{K} - \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-h} \eta \mathcal{H}(l, \mathcal{R}, \nabla \mathcal{R}), \quad (48)$$

where

$$\mathcal{H}(l, \mathcal{R}, \nabla \mathcal{R}) = \frac{2}{l} + \frac{l}{2} \mathcal{R} - \frac{l^3}{2} \left(\mathcal{R}_{ab} \mathcal{R}^{ab} - \frac{3}{8} \mathcal{R}^2 \right) + \dots \quad (49)$$

In the bulk term, one should note that $R + f(R) = -6\eta/l^2$, $\mathcal{K}$ is the extrinsic curvature scalar of the boundary surface, and $\mathcal{R}$ is the intrinsic curvature of the conformal boundary described by the induced metric h_{ij} . One should note that the boundary here is the conformal boundary. After the calculation, one can obtain the Gibbs free energy G as

$$G = \frac{I}{\beta} = -\frac{\eta r_+^3}{2\alpha K l^2 (1 - A^2 r_+^2)^2} - \frac{q^2}{2\alpha K r_+} - \frac{\eta m}{2\alpha K} + \frac{\eta m(1 - A^2 l^2)}{\alpha K} - \frac{mA^4 l^2 q^2}{\alpha K} = M - TS - \Phi Q. \quad (50)$$

We have neglected the boundary term

$$I_b^{\text{em}} = \frac{1}{4\pi} \int_{\partial \mathcal{M}} d^3y \sqrt{|h|} n_a F^{ab} B_b \quad (51)$$

corresponding to the electromagnetic field in the action (48), which means that the system's electric potential is fixed but the electric charge is variable [59–62].

III. PHASE TRANSITION OF THE SLOWLY ACCELERATING $f(R)$ ADS BLACK HOLE

A. Parameter space

We discuss the permitted range of the parameters m , A , q , l , η for the slowly accelerating $f(R)$ AdS black hole. Following the convention in Ref. [18], we define

$$\tilde{m} = mA, \quad \tilde{q} = qA, \quad \tilde{A} = Al, \quad \tilde{r} = \frac{r}{l}, \quad (52)$$

which are dimensionless quantities; then $N(r)$ and $H(\theta)$ can be written as

$$N(\tilde{r}) = (1 - \tilde{A}^2 \tilde{r}^2) \left(1 - \frac{2\tilde{m}}{\tilde{A}\tilde{r}} + \frac{\tilde{q}^2}{\eta \tilde{A}^2 \tilde{r}^2} \right) + \tilde{r}^2, \quad (53)$$

$$H(\theta) = 1 + 2\tilde{m} \cos \theta + \frac{\tilde{q}^2}{\eta} \cos^2 \theta. \quad (54)$$

First, we must ensure the existence of an outer event horizon $\tilde{r}_+$ such that

$$N(\tilde{r}_+)|_{0 < \tilde{r}_+ < 1/\tilde{A}} = 0, \quad (55)$$

with $\tilde{r}_+ = r_+/l$, and the positivity of the blackening factor in the region between the outer horizon and the conformal boundary, which can be expressed as

$$N(\tilde{r})|_{\tilde{r}_+ < \tilde{r} < 1/\tilde{A}} > 0. \quad (56)$$

The condition (56) can be assured if we have

$$N'(\tilde{r}_+)|_{0 < \tilde{r}_+ < 1/\tilde{A}} > 0 = N(\tilde{r}_+)|_{0 < \tilde{r}_+ < 1/\tilde{A}}. \quad (57)$$

Second, we should make sure that

$$H(\theta)|_{-\pi \leq \theta \leq \pi} > 0, \quad (58)$$

which is equivalent to letting

$$\tilde{m} < \sqrt{\frac{\tilde{q}^2}{\eta}} \Theta\left(\frac{\tilde{q}^2}{\eta} - 1\right) + \frac{1}{2} \left(1 + \frac{\tilde{q}^2}{\eta}\right) \Theta\left(1 - \frac{\tilde{q}^2}{\eta}\right), \quad (59)$$

where Θ is the unit step function and $\Theta(0) = 1/2$.

Third, we must eliminate the acceleration horizon. The extreme condition for the emergence of the acceleration horizon is

$$N(\tilde{r})|_{\tilde{r}=-1/(\tilde{A} \cos \theta)} = N'(\tilde{r})|_{\tilde{r}=-1/(\tilde{A} \cos \theta)} = 0, \quad (60)$$

from which we can get $\tilde{m} = \tilde{m}(\tilde{q}, \theta, \eta)$ and $\tilde{A} = \tilde{A}(\tilde{q}, \theta, \eta)$.

Moreover, one should note that

$$1 - A^2 l^2 \Xi > 0 \quad (61)$$

and

FIG. 1. The permitted parameter spaces for $\eta = 0.2, 2$ and $\tilde{q} = 11/50$ are the white enclosed regions. The red, black, green, and blue lines, respectively, correspond to conditions (57), (59), (60), and (62).

in Eq. (16), which is equivalent to

$$\tilde{A} < \sqrt{\frac{\eta}{\eta + \tilde{q}^2}}. \quad (62)$$

The parameter space has been shown in Fig. 1 with white enclosed regions. We have chosen two different values for the characteristic quantity η of the charged accelerating black hole in $f(R)$ gravity. It is evident that a larger η is endowed with a larger physically reasonable parameter space. There is a point where the blue, green, and red curves intersect (we name it point X , as is the convention in Ref. [18]), which can be solved for, yielding

$$(\tilde{m}_X, \tilde{A}_X) = \left(\frac{\sqrt{\tilde{q}^2(\eta + \tilde{q}^2)}}{\eta}, \sqrt{\frac{\eta}{\eta + \tilde{q}^2}} \right). \quad (63)$$

Combining the expressions (57), (59), (60), (62), and (63) with the figures, when $\eta \rightarrow \infty$, we obtain

$$|\tilde{q}| < \tilde{m} < \frac{1}{2} \left(1 + \frac{\tilde{q}^2}{\eta} \right) \rightarrow \frac{1}{2}, \quad (64)$$

$$0 < \tilde{A} < 1, \quad (65)$$

whereas for $\eta \rightarrow 0$ [note that $\eta > 0$ for a charged accelerating $f(R)$ black hole], we have

$$|\tilde{q}| \sqrt{1 + \frac{\tilde{q}^2}{\eta}} < \tilde{m} < \frac{|\tilde{q}|}{\sqrt{\eta}}, \quad (66)$$

and

$$0 < \tilde{A} < \tilde{A}_X \rightarrow 0^+. \quad (67)$$

Equation (66) demands that $|\tilde{q}| < \tilde{m}$.

When all curves intersect into one point (we call it point E), which makes the permitted region vanish, we can obtain the coordinate of the extreme point as

$$(\tilde{q}_E, \tilde{m}_E, \tilde{A}_E) = \left(\sqrt{\frac{\eta}{3}}, \frac{2}{3}, \frac{\sqrt{3}}{2} \right). \quad (68)$$

When $\eta = 1$, the situation reduces to that studied previously [18] for charged accelerating black holes in Einstein gravity.

B. Phase transitions

We now consider the phase behavior of these slowly accelerating black holes. We begin with the canonical ensemble, where the electric charge of the black hole is fixed. The Helmholtz free energy for this ensemble, which can be obtained by including Eq. (51) in the computation of the action, is

$$F = M - TS. \quad (69)$$

Without loss of generality [18], we choose $K = \Xi + 2\tilde{m}$, so that $\mu_+ = 0$. The conical deficit is located entirely along the south pole axis $\theta = \pi$, with $\mu_- = \tilde{m}/K$. According to Eqs. (29), (32), (39), (40), and (63), the pressure corresponding to point X is

$$\begin{aligned} P_X &= \frac{3\tilde{m}^2}{8\pi Q^2(1 + \tilde{q}^2/\eta + 2\tilde{m})} \\ &= \frac{3\mu_-^2}{8\pi Q^2} \\ &= \frac{3(2C\Delta - \Delta + 1)^2}{128\pi Q^2}, \end{aligned} \quad (70)$$

from which we can see that the parameter η does not have a qualitatively different effect on the pressure of the intersection point X . As a result, when considering the phase structure of the slowly accelerating $f(R)$ AdS black hole, we see that it will be qualitatively similar to that in Einstein-AdS gravity [18].

FIG. 2. The free energy in terms of the temperature in the canonical ensemble with $\eta = 0.8$, $\mu_+ = 0$, $\mu_- = 0.05$, $Q = 0.8$, $P_1 = P_c = 1/(96\pi)$, $P_2 = P_1/4 = 25P_X/9$, $P_3 = P_1/16 = 25P_X/36$.

In other words, a slowly accelerating $f(R)$ AdS black hole will experience a snapping swallowtail process when the pressure of the black hole decreases below P_X , and the black hole will experience zeroth order, first order, second order, and reentrant phase transitions [18]. We depict the Helmholtz free energy in terms of the temperature in Fig. 2, from which we can see that the phase transition of the slowly accelerating $f(R)$ AdS black hole is van der Waals-like for $P > P_X$, and the branch corresponding to the small black hole with low temperature has snapped for $P < P_X$.

Turning to the grand canonical ensemble, where the electric potential of the black hole is fixed, we plot the Gibbs free energy (50) in terms of the temperature to see the phase structure. Results are shown in Fig. 3 for fixed η (top diagram), fixed μ_- (middle diagram), and fixed $\sqrt{\Delta\eta}$ (bottom diagram).

Setting $C = 0$ but $\Delta \neq 1$ (i.e., $\mu_- = \mu_+ \neq 0$), we obtain the zero-acceleration $f(R)$ black hole threaded by a cosmic string. In this case, by analyzing the derivative of the temperature with respect to the entropy

$$T'(S) = \frac{\Delta\eta(8PS + \Phi^2 - \Delta\eta)}{8\sqrt{\pi}(S\Delta\eta)^{3/2}}, \quad (71)$$

we find that $\sqrt{\Delta\eta} > \Phi$ yields curves similar to those in the top diagram in Fig. 3, whereas $\sqrt{\Delta\eta} \leq \Phi$ yields the Gibbs free energy monotonically decreasing with temperature (not illustrated here), a phenomenon noted previously for the zero-acceleration case without a cosmic string [63,64]. For $\Delta = 1$, $C = 0$ (i.e., $\mu_- = \mu_+ = 0$), and $\sqrt{\eta} > \Phi$, we obtain the uppermost (red) curve shown in the top diagram in Fig. 3, for which a standard Hawking-Page transition occurs at $G = 0$. We do not have an interpretation of what the transition at $G = 0$ means for $\mu_- > 0$.

For the charged slowly accelerating $f(R)$ black hole, we find that $T'(S) = 0$ always has a solution for $S > 0$. As a result, Hawking-Page(-like) curves for the Gibbs free energy are always present. From Fig. 3, we can see that the lowest temperature attainable by a charged slowly

FIG. 3. The Gibbs free energy G in terms of the temperature T for the charged slowly accelerating $f(R)$ AdS black hole with $\Phi = 0.8$, $\mu_+ = 0$, $P = 3/(8\pi)$. In the top diagram $\eta = 200$ and $\sqrt{\Delta\eta} > \Phi$; in the middle diagram $\mu_- = 0.1$ and $\sqrt{\Delta\eta} < \Phi$; in the bottom diagram $\sqrt{\Delta\eta} = \Phi$. Different values of Φ , μ_+ , and P do not quantitatively affect our results. To a numerical precision of 0.01, we find in the top diagram that the upper branches terminate at $T = 27.87, 19.64$ for $\mu_- = 0.1, 0.15$, respectively, and that the terminal temperature decreases for increasing μ_- . In the middle diagram, the upper branches terminate at $T = 0.30, 0.33, 0.36$ for $\eta = 0.65, 0.70, 0.75$, respectively, and the terminal temperature increases for increasing η . In the bottom diagram, the upper branches terminate for values of (T, G) at $(0.37, -0.026)$, $(0.38, -0.055)$, $(0.70, -0.465)$ individually.

accelerating $f(R)$ black hole increases for increasing μ_- and η . We also find that there exist curves that at low temperatures have $G > 0$ on portions of both branches, whereas at high temperatures $G < 0$ on both branches. The lower branch corresponds to stable black holes when $G < 0$. The upper branches terminate at some finite large temperature that is not visible in the diagrams (except for

the two terminal points shown in the bottom diagram of Fig. 3) and that increases for decreasing μ_- and for increasing η .

IV. SUMMARY

We have obtained the first accelerating black hole in modified gravity [$f(R)$ gravity] and investigated its thermodynamics, computing its mass, entropy, temperature, electric charge, electric potential, and tensions. We found a number of similarities to the charged accelerating case studied previously in Einstein gravity [18]. The structure of its thermodynamic phase transitions is similar to those in Einstein gravity, and the “snapping swallowtail” phenomenon was exhibited.

However, we also found a number of distinctions due to the parameter η that arises because the mass and entropy are both linearly dependent on this quantity. This means that it cannot be scaled out of the charge parameter, despite superficial appearances to the contrary. We also obtained a generalization of the reverse isoperimetric inequality (47), suitable for $f(R)$ gravity. While the dependence on the conical deficit is (as expected) the same as in Einstein gravity [50], we found that increasing η tends to allow the black holes to have more area relative to their thermodynamic volume, tending to make them superentropic. Furthermore, the allowed region of parameter space governing the phase transitions becomes η dependent, broadening the range of possibilities. We found that for increasing η the accessible parameter space is enlarged. Specifically, we gave the upper and lower bounds for the parameters $\tilde{m}$ and $\tilde{A}$ that characterize the parameter space when η varies between 0 and ∞ . We have analyzed the phase structures of the charged slowly accelerating $f(R)$ AdS black hole both in the canonical ensemble and in the grand canonical ensemble. In the fixed-charge ensemble, we found van der Waals phase transitions and snapping swallowtail behavior, similar to that in Einstein gravity [18]. In the ensemble where the electric potential is fixed, we found that the lowest temperature the black hole can attain increases for increasing μ_- and η . We also found that there are always two branches in the $G - T$ diagram for $\sqrt{\Delta\eta} > \Phi$, $\sqrt{\Delta\eta} = \Phi$, and even $\sqrt{\Delta\eta} < \Phi$, where the upper one terminates at a high temperature. The terminal temperature decreases for increasing μ_- and decreasing η .

Finally, we pause to provide a comparison to our results and those recently obtained for a new class of $f(R)$ spherically symmetric charged black holes [65]. These latter solutions have a dynamical Ricci scalar, in contrast to our solution in which the Ricci scalar is constant. Furthermore, our solution reduces to the standard one in Einstein gravity in the limit $\eta \rightarrow 1$, whereas the new black hole solutions do not have such a limit [65]. Finally, the entropy of these solutions is negative in certain regions of parameter space, a phenomenon observed elsewhere in higher curvature gravity

[66–71]—the physical interpretation of this remains unclear. By contrast, the entropy is positive for our class of solutions.

Note that $f(R)$ gravity enables us to study the inflationary and accelerated expansion epochs of our Universe. It can uniformly explain the early-time inflation as well as the dark energy epoch [72]. It has been testified by many applications in cosmology and astronomy [73,74]; it has also been constrained by realistic FRW cosmology [75] and stability considerations [76]. We note that $f(R)$ gravity is such a well-behaved modified theory that it can show the evolution of the Universe, and it can also give correct cosmological predictions [77,78]. It has been shown that $f(R)$ theory without a cosmological constant can reproduce the behavior of the Einstein-Hilbert action with a cosmological constant [79].

The charged accelerating black hole in $f(R)$ gravity should provide further insight into $f(R)$ gravity. The role of η needs to be better understood. For example, for $\eta > 1$, will these black holes exhibit thermodynamic instabilities similar to those found recently [80] for other superentropic

black holes? It is easy to extend our solution for the charged accelerating black hole in $f(R)$ gravity to one with rotation, and it is possible that interesting η -dependent phenomena may be present in this case. Likewise, the possibility that η itself could be a thermodynamic parameter merits investigation. We note also that $f(R)$ gravity is one class of Brans-Dicke theories [74,81], and it can be viewed as a scalar-tensor theory [74,82]. Considering this, it remains a subject of future investigation to analyze the metric perturbations of our solution by recasting the $f(R)$ gravity to the scalar-tensor theory, just like the procedures in Ref. [65]. Work on these topics is in progress.

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